Elliptic curves over $\mathbb{Q}(\sqrt{2})$ with good reduction outside $\sqrt{2}$

Takaaki KAGAWA

The elliptic curves defined over $k = \mathbb{Q}(\sqrt{2})$ with good reduction outside $\sqrt{2}$ are determined. All such curves have a k-rational point of order 2 and thus, following R. G. E. Pinch's idea, the determination of such curves is reduced to solving some diophantine equations over k.

There are exactly 400 isomorphism classes of such curves and their isogeny classes are classified.

Key words: Elliptic curves - Good reduction

E-mail: kagawa@se.ritsumei.ac.jp (T. Kagawa)

Department of Mathematical Sciences, Ritsumeikan University
Kusatsu, Shiga, 525–8577, Japan

1 Introduction

Let k be a quadratic field and S a finite set of prime ideals of k. We consider the problem of determining the elliptic curve defined over k with good reduction outside S. When k is imaginary and S consists of at most two primes, or k is real and $S = \emptyset$, we have obtained many results. For these results, we refer to [10], [12], [13], [14], [19] when k is complex and to [6], [7] [8] and the papers cited in them when k is real.

When k is real and $S \neq \emptyset$, the only case that has been treated is the case when $k = \mathbb{Q}(\sqrt{5})$ and $S = \{(2)\}$. As a matter of fact, Pinch proved the following result in his unpublished manuscript [16]:

Theorem A. *Let $k = \mathbb{Q}(\sqrt{5})$. Then there are exactly 368 k-isomorphism classes of elliptic curves defined over k with good reduction outside 2. (Here, and in what follows, we say that "good reduction outside π" for simplicity in case S consists of the only one principal prime ideal (π).)*

Following the argument of the proof of Theorem A, we prove the following theorem.

Theorem 1. *Let $k = \mathbb{Q}(\sqrt{2})$. There are exactly 400 k-isomorphism classes of elliptic curves defined over k with good reduction outside $\sqrt{2}$. They are listed in the tables at the end of this paper.*

Note that all curves have additive reduction. In particular, there are no elliptic curves with everywhere good reduction over $\mathbb{Q}(\sqrt{2})$. (cf. [6], [8].)

2 j-invariant

In the rest of this paper, we denote the real quadratic field $\mathbb{Q}(\sqrt{2})$ by k. We always consider k as a subfield of the reals $\mathbb{R}$ and $\sqrt{2}$ to be positive, i.e. $\sqrt{2} = 1.4142\ldots$. We also denote the ring of integers of k, the group of units of k, the fundamental unit of k greater than 1 and the conjugation of $k/\mathbb{Q}$ by $\mathcal{O}_k$ $(= \mathbb{Z}[\sqrt{2}\,])$, $\mathcal{O}_k^\times$, ε $(= 1 + \sqrt{2}\,)$ and $'$ respectively. Note that the class number of k is 1 and the norm of ε is -1.

In the subsequent sections, we prove the following two theorems.

Theorem 2. *The only solutions of the equation*

$$\sqrt{2}\,x^2 = u + v \tag{1}$$

in $x \in \mathcal{O}_k$, $u, v \in \mathcal{O}_k^\times$, and corresponding $t = 64u/v$, $\bar{t} = 4096/t$ are the ones in Table 1, where w is an arbitrary unit of k.

Theorem 3. *The only solutions of*

$$x^2 = \sqrt{2}^{\,e} u + v \tag{2}$$

in $x \in \mathcal{O}_k$, $u, v \in \mathcal{O}_k^\times$, $e \in \mathbb{N} \cup \{0\}$, and corresponding $t = 64\sqrt{2}^{\,-e} v/u$, $\bar{t} = 4096/t$ are the ones in Table 2, where w is an arbitrary unit of k.

x	u	v	$64u/v$	$64v/u$
0	w	$-w$	-64	-64
w	$-w^2$	εw^2	$64\varepsilon'$	-64ε
w	εw^2	$-w^2$	-64ε	$64\varepsilon'$
w	w^2	$-\varepsilon' w^2$	64ε	$-64\varepsilon'$
w	$-\varepsilon' w^2$	w^2	$-64\varepsilon'$	64ε
$\sqrt{2}w$	εw^2	$-\varepsilon' w^2$	$64\varepsilon^2$	$64\varepsilon'^2$
$\sqrt{2}w$	$-\varepsilon' w^2$	εw^2	$64\varepsilon'^2$	$64\varepsilon^2$
$2w$	$\varepsilon^2 w^2$	$-\varepsilon'^2 w^2$	$-64\varepsilon^4$	$-64\varepsilon'^4$
$2w$	$-\varepsilon'^2 w^2$	$\varepsilon^2 w^2$	$-64\varepsilon'^4$	$-64\varepsilon^4$
$13\sqrt{2}w$	$\varepsilon^7 w^2$	$-\varepsilon'^7 w^2$	$64\varepsilon^{14}$	$64\varepsilon'^{14}$
$13\sqrt{2}w$	$-\varepsilon'^7 w^2$	$\varepsilon^7 w^2$	$64\varepsilon'^{14}$	$64\varepsilon^{14}$

Table 1: The solutions of $\sqrt{2}\,x^2 = u + v$

Using these theorems, we can determine the possible values of j-invariant as follows.

Let E be an elliptic curve defined over k with good reduction outside $\sqrt{2}$. By Corollary 1.14.1 of [13], E has a k-rational point P of order 2. (See also [8].) Let $\bar{E} := E/\langle P\rangle$. We also let $J(X) := (X+16)^3/X$. Then, according to Theorem 2.3 of [12], one of the following two cases occurs:

(a) There exists a solution (x, u, v) of (1) such that

$$\{j(E), j(\bar{E})\} = \{J(64u/v), J(64v/u)\}.$$

(b) There exists a solution (x, u, v, e) of (2) such that

$$\{j(E), j(\bar{E})\} = \{J(64\sqrt{2}^{-e}v/u), J(64\sqrt{2}^{e}u/v)\}.$$

We list the possible j-invariant and corrsponding $t \in k$ so that $j = J(t)$.

3 Proof of Theorem 2

To prove Theorem 2, we use the following lemma which is well-known. (See [2], [3], [11] and [18]).

Lemma 1. (a) *The only solution of* $X^4 - 2Y^2 = -1$ *in non-negative integers is* $(X, Y) = (1, 1)$.

(b) *The only solutions of* $X^2 - 2Y^4 = \pm 1$ *in non-negative integers are* $(X, Y) = (1, 0)$, $(1, 1)$ *and* $(239, 13)$.

(c) *The only solutions of* $X^2 - 8Y^4 = \pm 1$ *in non-negative integers are* $(X, Y) = (1, 0)$ *and* $(3, 1)$.

Let $U := N_{k/\mathbb{Q}}(u)$, $V := N_{k/\mathbb{Q}}(v)$ $(= \pm 1)$.

Suppose first that $U = -V$. By changing u and v if necessary, we may suppose that $U = -1$ and $V = 1$. Then taking norm of (1), we have $-2N_{k/\mathbb{Q}}(x)^2 = \mathrm{Tr}_{k/\mathbb{Q}}(uv^{-1})$, so that $uv^{-1} = -N_{k/\mathbb{Q}}(x)^2 + b\sqrt{2}$ for some $b \in \mathbb{Z}$. It follows from Lemma 1 (a) that $u = -\varepsilon v$, $x^2 = -v$ or $u = -\varepsilon' v$, $x^2 = v$. Consequently, there exists an unit w such that $(x, u, v) = (w, \varepsilon w^2, -w^2)$ or $(w, -\varepsilon' w^2, w^2)$.

e	x	u	v	$64\sqrt{2}^{-e}v/u$	$64\sqrt{2}^{e}u/v$
0	0	w	$-w$	-64	-64
0	$\sqrt{2}w$	w^2	w^2	64	64
0	$\sqrt{2}w$	εw^2	$\varepsilon' w^2$	$-64\varepsilon'^2$	$-64\varepsilon^2$
0	$\sqrt{2}w$	$\varepsilon' w^2$	εw^2	$-64\varepsilon^2$	$-64\varepsilon'^2$
1	εw	εw^2	εw^2	$32\sqrt{2}$	$64\sqrt{2}$
1	εw	$\varepsilon^2 w^2$	$-\varepsilon w^2$	$32\sqrt{2}\varepsilon'$	$-64\sqrt{2}\varepsilon$
1	w	$-w^2$	εw^2	$-32\sqrt{2}\varepsilon$	$64\sqrt{2}\varepsilon'$
1	w	εw^2	$-\varepsilon w^2$	$-32\sqrt{2}$	$-64\sqrt{2}$
1	$(3-\sqrt{2})w$	$-\varepsilon'^3 w^2$	εw^2	$32\sqrt{2}\varepsilon^4$	$64\sqrt{2}\varepsilon'^4$
1	$(5+4\sqrt{2})w$	$\varepsilon^5 w^2$	$-\varepsilon w^2$	$-32\sqrt{2}\varepsilon'^4$	$-64\sqrt{2}\varepsilon^4$
2	εw	εw^2	w^2	$-32\varepsilon'$	128ε
2	$\varepsilon' w$	$\varepsilon' w^2$	w^2	-32ε	$128\varepsilon'$
2	w	w^2	$-w^2$	-32	-128
3	εw	εw^2	$-w^2$	$16\sqrt{2}\varepsilon'$	$-128\sqrt{2}\varepsilon$
3	$\varepsilon' w$	$-\varepsilon' w^2$	$-w^2$	$-16\sqrt{2}\varepsilon$	$128\sqrt{2}\varepsilon'$
5	$(1+2\sqrt{2})w$	εw^2	w^2	$-8\sqrt{2}\varepsilon'$	$256\sqrt{2}\varepsilon$
5	$(1-2\sqrt{2})w$	$-\varepsilon' w^2$	w^2	$8\sqrt{2}\varepsilon$	$-256\sqrt{2}\varepsilon'$
5	$\varepsilon^2 w$	$\varepsilon^2 w^2$	w^2	$8\sqrt{2}\varepsilon'^2$	$256\sqrt{2}\varepsilon^2$
5	$\varepsilon'^2 w$	$-\varepsilon'^2 w^2$	w^2	$-8\sqrt{2}\varepsilon^2$	$-256\sqrt{2}\varepsilon'^2$
6	$3w$	w^2	w^2	8	512
6	$(5+4\sqrt{2})w$	$\varepsilon^3 w^2$	w^2	$-8\varepsilon'^3$	$512\varepsilon^3$
6	$(5-4\sqrt{2})w$	$\varepsilon'^3 w^2$	w^2	$-8\varepsilon^3$	$512\varepsilon'^3$
7	$(7+4\sqrt{2})w$	$\varepsilon^3 w^2$	w^2	$-4\sqrt{2}\varepsilon'^3$	$512\sqrt{2}\varepsilon^3$
7	$(7-4\sqrt{2})w$	$-\varepsilon'^3 w^2$	w^2	$4\sqrt{2}\varepsilon^3$	$-512\sqrt{2}\varepsilon'^3$
9	$(33+24\sqrt{2})w$	$\varepsilon^6 w^2$	w^2	$2\sqrt{2}\varepsilon'^6$	$1024\sqrt{2}\varepsilon^6$
9	$(33-24\sqrt{2})w$	$-\varepsilon'^6 w^2$	w^2	$-2\sqrt{2}\varepsilon^6$	$-1024\sqrt{2}\varepsilon'^6$

Table 2: The solutions of $x^2 = \sqrt{2}^e u + v$

Suppose next that $U = V$. Considering the conjugate of (1), we have

$$-\sqrt{2}\,x'^2 = u' + v' = \frac{U}{u} + \frac{V}{v} = \frac{U}{uv}(u+v) = \frac{U}{uv}\sqrt{2}\,x^2.$$

This implies that the principal ideal (x) is ambiguous and hence x is of the form $x = \sqrt{2}^n y u_1$, where $y \in \mathbb{Z}, u_1 \in \mathcal{O}_k^\times$ and $n = 0, 1$. Replacing x, u and v by x/u_1, u/u_1^2 and v/u_1^2 respectively, we may assume that $u_1 = 1$ and that $u + v = \sqrt{2}\,x^2 \in \sqrt{2}\,\mathbb{Z}$. Letting $u = a + b\sqrt{2}$ with $a, b \in \mathbb{Z}$ and noting $U = V$, we have $v = -a \pm b\sqrt{2}$. If $v = -a - b\sqrt{2}$, then $x = 0$. If $v = -a + b\sqrt{2}$, then we have $2^n y^2 = 2b$. If $n = 0$, then y is even and $u = a + 2(y/2)^2\sqrt{2}$, $x = \pm y$. Lemma 1 (c) therefore implies that

$$(x, u, v) = (0, \pm 1, \mp 1), (\pm 2, \varepsilon^2, -\varepsilon'^2), (\pm 2, -\varepsilon'^2, \varepsilon^2).$$

If $n = 1$, then $u = a + y^2\sqrt{2}$, $x = \pm\sqrt{2}y$. Lemma 1 (b) therefore implies that

$$(x, u, v) = (0, \pm 1, \mp 1), (\pm\sqrt{2}, \varepsilon, -\varepsilon'), (\pm\sqrt{2}, -\varepsilon', \varepsilon), (\pm 13\sqrt{2}, \varepsilon^7, -\varepsilon'^7), (\pm 13\sqrt{2}, -\varepsilon'^7, \varepsilon^7).$$

The proof of Theorem 2 is now complete.

$j = (t+16)^3/t$	t
$j_1 = 1728$	$-64,\ 8$
$j_2 = 287496$	$512,\ 2\sqrt{2}\varepsilon'^6,\ -2\sqrt{2}\varepsilon^6$
$j_3 = 41113158120 + 29071392966\sqrt{2}$	$1024\sqrt{2}\varepsilon^6$
$j'_3 = 41113158120 - 29071392966\sqrt{2}$	$-1024\sqrt{2}\varepsilon'^6$
$j_4 = 128$	-32
$j_5 = 10976$	$-4\sqrt{2}\varepsilon'^3,\ 4\sqrt{2}\varepsilon^3,\ -128$
$j_6 = 52151080 + 36872164\sqrt{2}$	$512\sqrt{2}\varepsilon^3$
$j'_6 = 52151080 - 36872164\sqrt{2}$	$-512\sqrt{2}\varepsilon'^3$
$j_7 = 8000$	$64,\ -8\varepsilon'^3,\ -8\varepsilon^3$
$j_8 = 26125000 + 18473000\sqrt{2}$	$512\varepsilon^3$
$j'_8 = 26125000 - 18473000\sqrt{2}$	$512\varepsilon'^3$
$j_9 = 512 + 384\sqrt{2}$	$-16\sqrt{2}\varepsilon$
$j_{10} = 86752 - 59408\sqrt{2}$	$128\sqrt{2}\varepsilon'$
$j'_9 = 512 - 384\sqrt{2}$	$16\sqrt{2}\varepsilon'$
$j'_{10} = 86752 + 59408\sqrt{2}$	$-128\sqrt{2}\varepsilon$
$j_{11} = 2432 + 384\sqrt{2}$	$-32\varepsilon,\ 8\sqrt{2}\varepsilon,\ 8\sqrt{2}\varepsilon'^2$
$j_{12} = 56032 - 38944\sqrt{2}$	$128\varepsilon'$
$j_{13} = 418576 - 274424\sqrt{2}$	$-256\sqrt{2}\varepsilon'$
$j_{14} = 2278112 + 1609752\sqrt{2}$	$256\sqrt{2}\varepsilon^2$
$j'_{11} = 2432 - 384\sqrt{2}$	$-32\varepsilon',\ -8\sqrt{2}\varepsilon',\ -8\sqrt{2}\varepsilon^2$
$j'_{12} = 56032 + 38944\sqrt{2}$	128ε
$j'_{13} = 418576 + 274424\sqrt{2}$	$256\sqrt{2}\varepsilon$
$j'_{14} = 2278112 - 1609752\sqrt{2}$	$-256\sqrt{2}\varepsilon'^2$
$j_{15} = 2816 + 1600\sqrt{2}$	$32\sqrt{2}$
$j_{16} = 8960 + 3104\sqrt{2}$	$64\sqrt{2}$
$j'_{15} = 2816 - 1600\sqrt{2}$	$-32\sqrt{2}$
$j'_{16} = 8960 - 3104\sqrt{2}$	$-64\sqrt{2}$
$j_{17} = 3712 + 2624\sqrt{2}$	$-32\sqrt{2}\varepsilon$
$j_{18} = 19136 - 13344\sqrt{2}$	$64\sqrt{2}\varepsilon'$
$j'_{17} = 3712 - 2624\sqrt{2}$	$32\sqrt{2}\varepsilon'$
$j'_{18} = 19136 + 13344\sqrt{2}$	$-64\sqrt{2}\varepsilon$
$j_{19} = 10048 + 5056\sqrt{2}$	-64ε
$j_{20} = 16064 - 11328\sqrt{2}$	$64\varepsilon'$
$j'_{19} = 10048 - 5056\sqrt{2}$	$-64\varepsilon'$
$j'_{20} = 16064 + 11328\sqrt{2}$	64ε
$j_{21} = 60992 + 43136\sqrt{2}$	$-64\varepsilon^2$
$j'_{21} = 60992 - 43136\sqrt{2}$	$-64\varepsilon'^2$
$j_{22} = 79808 + 55168\sqrt{2}$	$64\varepsilon^2$
$j'_{22} = 79808 - 55168\sqrt{2}$	$64\varepsilon'^2$
$j_{23} = 1217792 + 862784\sqrt{2}$	$32\sqrt{2}\varepsilon^4$
$j_{24} = 4654592 - 3289568\sqrt{2}$	$64\sqrt{2}\varepsilon'^4$
$j'_{23} = 1217792 - 862784\sqrt{2}$	$-32\sqrt{2}\varepsilon'^4$
$j'_{24} = 4654592 + 3289568\sqrt{2}$	$-64\sqrt{2}\varepsilon^4$
$j_{25} = 2310848 + 1635072\sqrt{2}$	$-64\varepsilon^4$
$j'_{25} = 2310848 - 1635072\sqrt{2}$	$-64\varepsilon'^4$
$j_{26} = 106917943560128 + 75602392581248\sqrt{2}$	$64\varepsilon^{14}$
$j'_{26} = 106917943560128 - 75602392581248\sqrt{2}$	$64\varepsilon'^{14}$

Table 3: The j-invariants of elliptic curves with good reduction outside $\sqrt{2}$

4 Proof of Theorem 3

4.1 The case $e = 0$

Let (x, u, v) be a solution of (2) with $e = 0$ other than the first four in Table 2. Then Proposition 2 in [4] implies that there exist $w, u_0 \in \mathcal{O}_k^\times$ and $y \in \mathbb{Z}$ such that $u = w^2 u_0$, $v = w^2 u_0'$ and $\mathrm{Tr}_{k/\mathbb{Q}}(u_0) = y^2$. But this is impossible, since $\mathrm{Tr}_{k/\mathbb{Q}}(u_0) \equiv 2 \pmod 4$.

4.2 The case $e = 1$

We first give a lemma which will be often used.

Lemma 2. *If $a + b\sqrt{2}$ $(a, b \in \mathbb{Z})$ is prime to $\sqrt{2}$ and a square in k, then $a \equiv 3 \pmod 8$, $b \equiv 2 \pmod 4$ or $a \equiv 1 \pmod 8$, $b \equiv 0 \pmod 4$.*

In the following, (x, u, v) denotes a solution of (2). We may assume that $v = \pm 1$ or $\pm\varepsilon$, since $(wx, w^2 u, w^2 v)$ is also a solution of (2) for any $w \in \mathcal{O}_k^\times$. Suppose that $e = 1$. Then $v = \pm\varepsilon$, since if $v = \pm 1$, then Lemma 2 implies that $\sqrt{2}\,u \equiv 0 \pmod 2$, which is impossible. Since $x^2 = \sqrt{2}\,u - \varepsilon$ implies that $(\varepsilon x')^2 = \sqrt{2}(-u'\varepsilon^2) + \varepsilon$, we may suppose that $v = \varepsilon$.

Let $u\varepsilon' = a + b\sqrt{2}$ with $a, b \in \mathbb{Z}$ and let $c = N_{k/\mathbb{Q}}(x)$ $(\in \mathbb{Z})$. Then we have

$$\begin{cases} a^2 - 2b^2 = N_{k/\mathbb{Q}}(u\varepsilon') = -N_{k/\mathbb{Q}}(u), \\ c^2 = N_{k/\mathbb{Q}}(\sqrt{2}\,u + \varepsilon) = 4b - 2N_{k/\mathbb{Q}}(u) - 1. \end{cases}$$

Eliminating b yields

$$8a^2 = c^4 + 2\{2N_{k/\mathbb{Q}}(u) + 1\}c^2 + 5 - 4N_{k/\mathbb{Q}}(u). \tag{3}$$

Suppose first that $N_{k/\mathbb{Q}}(u) = 1$. Then (3) implies that $(2c^2, 8ac)$ is an integral point of the elliptic curve $Y^2 = X^3 + 12X^2 + 4X$. Since this curve, which is isomorphic over $\mathbb{Q}$ to the curve 64A3 in Table 1 of [5], has only the (affine) $\mathbb{Q}$-rational points $(0, 0)$ and $(2, \pm 8)$, it follows that $u = -\varepsilon^2$, $x^2 = -\varepsilon^2$, which is impossible, or $u = -1$, $x = \pm 1$.

Suppose next that $N_{k/\mathbb{Q}}(u) = -1$. Then, according to (3), $X := (c + 1)/2$, $Y := (a + 1)/2$ $(\in \mathbb{Z})$ satisfy

$$\left(\frac{X(X-1)}{2}\right)^2 = \frac{Y(Y-1)}{2}. \tag{4}$$

The main theorem of [1] implies that $(a, b, c) = (\pm 1, 0, \pm 1), (\pm 3, 2, \pm 3), (\pm 17, 12, \pm 7)$, whence $u = -\varepsilon, \varepsilon, -\varepsilon^3, -\varepsilon', -\varepsilon^5, -\varepsilon'^3$. The cases $u = \varepsilon$ and $u = -\varepsilon'^3$ lead to $x = \pm\varepsilon$ and $x = \pm(3 - \sqrt{2})$, respectively. For the remaining values of u, $\sqrt{2}\,u + \varepsilon$ is not a square.

Remark. (a) When $N_{k/\mathbb{Q}}(u) = -1$, then the curve defined by (3), and hence by (4), has infinitely many rational points, since the curve is birationally equaivalent to the elliptic curve $y^2 = x^3 - x^2 - 9x + 9$ (192A2 in Table 1 of [5]) whose rank is 1.

(b) We reduced the case $e = 1$ to finding all integral points on certain elliptic curves (defined by quartics). While, in [16], Pinch reduces the equation $x^2 = u + v$ in $x \in \mathbb{Z}[(1 + \sqrt{5})/2]$, $u, v \in \mathbb{Z}[(1 + \sqrt{5})/2]^\times$ to solving simultaneous Pell equations (cf. [15]), his equation can be treated in the same way as we did.

4.3 The case $e = 2$

As before, we may assume that $v = \pm 1, \pm\varepsilon$. Then $v = \pm 1$, since if $v = \pm\varepsilon$, then $x^2 \equiv 1 + \sqrt{2} \pmod 2$, which is impossible by Lemma 2

Suppose first that $v = 1$. Then equation (1) implies that $\alpha := (x-1)/\sqrt{2}$ and $\alpha + \sqrt{2} = (x+1)/\sqrt{2}$ are units. Letting $\alpha = a + b\sqrt{2}$ with $a, b \in \mathbb{Z}$, we have $N_{k/\mathbb{Q}}(\alpha + \sqrt{2}) = N_{k/\mathbb{Q}}(\alpha) - 4b - 2$. Reducing this modulo 4 shows that one of $N_{k/\mathbb{Q}}(\alpha)$, $N_{k/\mathbb{Q}}(\alpha + \sqrt{2})$ is 1 and the other -1. If $N_{k/\mathbb{Q}}(\alpha) = 1$, then $b = 0, a = \pm 1$, leading to $(x, u) = (\varepsilon, \varepsilon), (\varepsilon', \varepsilon')$. If $N_{k/\mathbb{Q}}(\alpha) = -1$, then $b = -1, a = \pm 1$, leading to $(x, u) = (-\varepsilon, \varepsilon), (-\varepsilon', \varepsilon')$.

Suppose next that $v = -1$. From the inequalities $u = (x^2 + 1)/2 \geq 1/2$, $u' = (x'^2 + 1)/2 \geq 1/2$,

$$\cdots > \varepsilon^2 > \varepsilon > \varepsilon^0 = 1 > \varepsilon^{-1} = 0.414\cdots > \varepsilon^{-2} = 0.171\cdots > \cdots, \tag{5}$$

and

$$\cdots > \varepsilon'^{-2} > \varepsilon'^0 = 1 > 0 > \cdots > \varepsilon'^3 = -0.071\cdots > \varepsilon' = -0.414\cdots > \varepsilon'^{-1} > \cdots, \tag{6}$$

it follows that $u = 1$, whence $x = \pm 1$.

4.4 The case $e = 3$

As before, we may assume that $v = \pm 1$. Then $v = -1$, since if $v = 1$, then $x^2 \equiv 1 + 2\sqrt{2} \pmod 4$, which is impossible by Lemma 2. The inequalities $u = (x^2 + 1)/(2\sqrt{2}) \geq 1/(2\sqrt{2}) = 0.35355\cdots$, $u' = (x'^2 + 1)/(-2\sqrt{2}) \leq -1/(2\sqrt{2}) = -0.35355\cdots$ and (5), (6) imply that $(x, u) = (\pm\varepsilon, \varepsilon), (\pm\varepsilon', -\varepsilon')$.

4.5 The case $e \geq 4$

Following the argument in the proof of Proposition 1.6 in [16], we reduce equation (2) with $e \geq 4$ to two equations which we have solved.

As before, we may assume that $v = \pm 1$. Then $v = 1$, since, if $v = -1$, then $x^2 \equiv -1 \pmod 4$, which is impossible by Lemma 2.

Letting $\alpha := (x-1)/2$ we have $\sqrt{2}^{e-4} u = \alpha(\alpha + 1)$, which shows that $\alpha \in \mathcal{O}_k$. Since α and $\alpha + 1$ are coprime, either α or $\alpha + 1$ is a unit of k. By replacing x by $-x$ if necessary, we may suppose that $\alpha = \sqrt{2}^{e-4} u_1$, $\alpha + 1 = u_2$ for some $u_1, u_2 \in \mathcal{O}_k^\times$. Eliminating α yields

$$u_2 u_1^{-1} - u_1^{-1} = \begin{cases} \left(\sqrt{2}^{(e-4)/2}\right)^2 & \text{if } e \text{ is even,} \\ \sqrt{2} \cdot \left(\sqrt{2}^{(e-5)/2}\right)^2 & \text{if } e \text{ is odd.} \end{cases}$$

We can therefore obtain the possible values of e, u_1, u_2 and hence of u, x from the solutions of $x^2 = u + v$ and $\sqrt{2}\, x^2 = u + v$.

The proof of Theorem 3 is now complete.

5 Equations for curves with good reduction outside $\sqrt{2}$

In the following tables, we list the elliptic curves defined over $k = \mathbb{Q}(\sqrt{2})$ with good reduction outside $\sqrt{2}$. The curves are sorted according to their conductors. Each block is subdivided into isogeny classes by a line. The columns of the tables give the following data for each curve E:

(1) The name of the curves. The curve with $'$ such as A5$'$ in Table 4 is the conjugate of the curve without $'$ such as A5.

(2) The coefficients a_1, a_2, a_3, a_4, a_6 of a global minimal equation for E. In this column, (a, b) means that $a + b\sqrt{2}$.

(3) The structure of the torsion group. In this column, n and (m, n) mean the cyclic group of order n and the direct product of the cyclic groups of order m and n, respectively.

(4) The discriminant Δ of the equation.

(5) The Kodaira symbol and the local indices for E at $\sqrt{2}$.

(6) The j-invariant j of E. Here j_i, j_i' are the ones in Table 3.

(7) The curves isogenous to E via an isogeny of degree 2. For example, the entry "$1, 3, 4$" for the curve A2 in Table 4 indicates that A2 is 2-isogenous over k to A1, A3 and A4.

From a value j of j-invariant in Table 3, following the method described in [12], we can obtain the defining equations of the elliptic curves with good reduction outside $\sqrt{2}$ and j-invariant j. Note that every elliptic curve defined over k admits a global minimal equation, since the class number of k is 1. (See Corollary 8.3 in [17], Chapter VIII.) But in general, the equations we have obtained are not global minimal. To compute global minimal equations, the author used Kida's program TECC (see [9]). Torsion subgroups, conductors and Kodaira symbols are also obtained by using TECC.

To see that two elliptic curves E_1, E_2 are isogenous or not, we use the following two well-known facts.

(1) If $\#E_1(\mathcal{O}_k/\mathfrak{p}) \neq \#E_2(\mathcal{O}_k/\mathfrak{p})$ for some prime ideal $\mathfrak{p}$ of k, then E_1 and E_2 are not isogenos over k.

(2) Let P be a k-rational point of E_1 of order 2. We can compute a defining equation $E_1/\langle P\rangle$ by the formula given in [20]. Using TECC, we can check that the curve defined by this equation and E_2 are isomorphic over k or not.

Table 4: Elliptic curves of conductor $(\sqrt{2})^5$ (1 isogeny class)

	a_1	a_2	a_3	a_4	a_6	Tors.	Δ	type	j	Isog.
A1	0	0	0	1	0	4	$-\sqrt{2}^{12}$	III*, 2	j_1	2
A2	0	0	0	-1	0	(4, 2)	$\sqrt{2}^{12}$	I_3^*, 4	j_1	1, 3, 4
A3	(0, 1)	1	0	-2	-3	(2, 2)	$\sqrt{2}^{6}$	III, 2	j_2	$2, 5, 5'$
A4	(0, 1)	1	(0, 1)	-3	0	(4, 2)	$\sqrt{2}^{6}$	III, 2	j_2	$2, 6, 6'$
A5	$(0, -1)$	1	0	$(-22, -15)$	$(-69, -46)$	2	$\sqrt{2}^{9}\varepsilon^{-6}$	I_0^*, 2	j_3	3
A5′	(0, 1)	1	0	$(-22, 15)$	$(-69, 46)$	2	$-\sqrt{2}^{9}\varepsilon^{6}$	I_0^*, 2	j_3'	3
A6	(0, 1)	1	(0, 1)	$(-23, -15)$	(46, 31)	4	$\sqrt{2}^{9}\varepsilon^{-6}$	I_0^*, 1	j_3	4
A6′	$(0, -1)$	1	$(0, -1)$	$(-23, 15)$	$(46, -31)$	4	$-\sqrt{2}^{9}\varepsilon^{6}$	I_0^*, 1	j_3'	4

Table 5: Elliptic curves of conductor $(\sqrt{2})^6$ (1 isogeny class)

	a_1	a_2	a_3	a_4	a_6	Tors.	Δ	type	j	Isog.
A1	0	$(-1, -1)$	0	$(-4, 4)$	$(-6, 4)$	(2, 2)	$\sqrt{2}^{12}\varepsilon^{-6}$	I_2^*, 2	j_7	$2, 3, 3'$
A2	0	(1, 1)	0	$(-4, 4)$	$(6, -4)$	(4, 2)	$\sqrt{2}^{12}\varepsilon^{-6}$	I_2^*, 4	j_7	$1, 4, 4'$
A3	(0, 1)	$(-1, 1)$	0	$(2, -2)$	$(-5, 3)$	2	$\sqrt{2}^{6}\varepsilon^{-9}$	II, 1	j_8	1
A3′	$(0, -1)$	$(-1, -1)$	0	(2, 2)	$(-5, -3)$	2	$-\sqrt{2}^{6}\varepsilon^{9}$	II, 1	j_8'	2
A4	$(0, -1)$	$(0, -1)$	0	$(2, -2)$	$(5, -3)$	4	$\sqrt{2}^{6}\varepsilon^{-9}$	II, 1	j_8	1
A4′	(0, 1)	(0, 1)	0	(2, 2)	(5, 3)	4	$-\sqrt{2}^{6}\varepsilon^{9}$	II, 1	j_8'	2

Table 6: Elliptic curves of conductor $(\sqrt{2})^7$ (2 isogeny classes)

	a_1	a_2	a_3	a_4	a_6	Tors.	Δ	type	j	Isog.
A1	0	$(-1, 1)$	0	$(3, -2)$	$(-7, 5)$	4	$-\sqrt{2}^{16}\varepsilon^{-6}$	I_5^*, 4	j_4	2
A2	0	$(1, -1)$	0	$(-6, 4)$	$(-14, 10)$	(2, 2)	$\sqrt{2}^{14}\varepsilon^{-6}$	III*, 2	j_5	1, 3, 4
A3	$(0, -1)$	(0, 1)	$(0, -1)$	$(-7, 4)$	$(8, -6)$	4	$\sqrt{2}^{7}\varepsilon^{-9}$	II, 1	j_6	2
A4	(0, 1)	$(-1, 1)$	(0, 1)	$(-7, -4)$	$(-9, -6)$	2	$\sqrt{2}^{7}\varepsilon^{9}$	II, 1	j_6'	2
A1′	0	$(-1, -1)$	0	(3, 2)	$(-7, -5)$	4	$-\sqrt{2}^{16}\varepsilon^{6}$	I_5^*, 4	j_4	$2'$
A2′	0	(1, 1)	0	$(-6, -4)$	$(-14, -10)$	(2, 2)	$\sqrt{2}^{14}\varepsilon^{6}$	III*, 2	j_5	$1', 3', 4'$
A3′	(0, 1)	$(0, -1)$	(0, 1)	$(-7, -4)$	(8, 6)	4	$\sqrt{2}^{7}\varepsilon^{9}$	II, 1	j_6'	$2'$
A4′	$(0, -1)$	$(-1, -1)$	$(0, -1)$	$(-7, 4)$	$(-9, 6)$	2	$\sqrt{2}^{7}\varepsilon^{-9}$	II, 1	j_6	$2'$

Table 7: Elliptic curves of conductor $(\sqrt{2})^8$ (4 isogeny classes)

	a_1	a_2	a_3	a_4	a_6	Tors.	Δ	type	j	Isog.
A1	0	0	0	(3, 2)	0	2	$-\sqrt{2}^{12}\varepsilon^{6}$	I_0^*, 1	j_1	2
A2	0	0	0	$(-3, -2)$	0	(2, 2)	$\sqrt{2}^{12}\varepsilon^{6}$	I_0^*, 2	j_1	$1, 3, 3'$
A3	0	0	0	$(-33, 22)$	$(-98, 70)$	(2, 2)	$\sqrt{2}^{18}\varepsilon^{-6}$	I_6^*, 4	j_2	2, 4, 5
A3′	0	0	0	$(-33, -22)$	$(-98, -70)$	(2, 2)	$\sqrt{2}^{18}\varepsilon^{6}$	I_6^*, 4	j_2	$2, 4', 5'$
A4	0	0	0	$(-513, -362)$	(6314, 4466)	2	$\sqrt{2}^{21}$	I_9^*, 2	j_3	3
A4′	0	0	0	$(-513, 362)$	$(6314, -4466)$	2	$-\sqrt{2}^{21}$	I_9^*, 2	j_3'	$3'$
A5	0	0	0	$(-513, -362)$	$(-6314, -4466)$	2	$\sqrt{2}^{21}$	I_9^*, 4	j_3	3
A5′	0	0	0	$(-513, 362)$	$(-6314, 4466)$	2	$-\sqrt{2}^{21}$	I_9^*, 4	j_3'	$3'$
B1	0	-1	0	1	-1	2	$-\sqrt{2}^{16}$	I_4^*, 2	j_4	2
B2	0	1	0	-2	-2	(2, 2)	$\sqrt{2}^{14}$	I_2^*, 4	j_5	$1, 3, 3'$
B3	0	(0, 1)	0	$(-139, 98)$	$(818, -579)$	2	$\sqrt{2}^{19}\varepsilon^{-15}$	I_7^*, 2	j_6	2
B3′	0	$(0, -1)$	0	$(-139, -98)$	(818, 579)	2	$\sqrt{2}^{19}\varepsilon^{15}$	I_7^*, 2	j_6'	2
C1	0	1	0	1	1	2	$-\sqrt{2}^{16}$	I_4^*, 4	j_4	2
C2	0	-1	0	-2	2	(2, 2)	$\sqrt{2}^{14}$	I_2^*, 4	j_5	$1, 3, 3'$
C3	0	$(0, -1)$	0	$(-139, 98)$	$(-818, 579)$	2	$\sqrt{2}^{19}\varepsilon^{-15}$	I_7^*, 8	j_6	2
C3′	0	(0, 1)	0	$(-139, -98)$	$(-818, -579)$	2	$\sqrt{2}^{19}\varepsilon^{15}$	I_7^*, 4	j_6'	2
D1	0	$(0, -1)$	0	-1	(0, 1)	(2, 2)	$\sqrt{2}^{12}$	I_0^*, 2	j_7	$1', 2, 3$
D1′	0	(0, 1)	0	-1	$(0, -1)$	(2, 2)	$\sqrt{2}^{12}$	I_0^*, 2	j_7	$1, 2', 3'$
D2	0	$(0, -1)$	0	$(-11, -10)$	(30, 23)	4	$\sqrt{2}^{18}\varepsilon^{-3}$	I_6^*, 4	j_8	1
D2′	0	(0, 1)	0	$(-11, 10)$	$(30, -23)$	4	$-\sqrt{2}^{18}\varepsilon^{3}$	I_6^*, 4	j_8'	$1'$

	a_1	a_2	a_3	a_4	a_6	Tors.	Δ	type	j	Isog.
D3	0	$(0,1)$	0	$(-11,-10)$	$(-30,-23)$	2	$\sqrt{2}^{18}\varepsilon^{-3}$	I_6^*, 2	j_8	1
D3′	0	$(0,-1)$	0	$(-11,10)$	$(-30,23)$	2	$-\sqrt{2}^{18}\varepsilon^{3}$	I_6^*, 2	j_8'	1′

Table 8: Elliptic curves of conductor $(\sqrt{2})^9$ (8 isogeny classes)

	a_1	a_2	a_3	a_4	a_6	Tors.	Δ	type	j	Isog.
A1	0	-1	0	$(-4,3)$	0	$(2,2)$	$\sqrt{2}^{10}\varepsilon^{-8}$	III, 2	j_{11}	$2,3,4$
A2	0	$(-1,-1)$	0	$(-5,6)$	$(-11,7)$	2	$-\sqrt{2}^{20}\varepsilon^{-1}$	I_7^*, 2	j_{12}	1
A3	0	$(0,1)$	0	$(-11,-6)$	$(-22,-15)$	2	$\sqrt{2}^{17}\varepsilon^{5}$	I_4^*, 2	j_{13}	1
A4	0	$(0,1)$	0	$(-21,-16)$	$(46,33)$	2	$\sqrt{2}^{17}\varepsilon^{2}$	II^*, 1	j_{14}	1
A1′	0	-1	0	$(-4,-3)$	0	$(2,2)$	$\sqrt{2}^{10}\varepsilon^{8}$	III, 2	j_{11}'	$2',3',4'$
A2′	0	$(-1,1)$	0	$(-5,-6)$	$(-11,-7)$	2	$\sqrt{2}^{20}\varepsilon$	I_7^*, 2	j_{12}'	1′
A3′	0	$(0,-1)$	0	$(-11,6)$	$(-22,15)$	2	$\sqrt{2}^{17}\varepsilon^{-5}$	I_4^*, 2	j_{13}'	1′
A4′	0	$(0,-1)$	0	$(-21,16)$	$(46,-33)$	2	$-\sqrt{2}^{17}\varepsilon^{-2}$	II^*, 1	j_{14}'	1′
B1	0	$(-1,-1)$	0	$(0,1)$	0	$(2,2)$	$\sqrt{2}^{10}\varepsilon^{-2}$	III, 2	j_{11}	$2,3,4$
B2	0	$(0,1)$	0	$(4,4)$	$(8,4)$	4	$-\sqrt{2}^{20}\varepsilon^{5}$	I_7^*, 4	j_{12}	1
B3	0	$(-1,1)$	0	$(-58,-42)$	$(-246,-174)$	2	$\sqrt{2}^{17}\varepsilon^{11}$	II^*, 1	j_{13}	1
B4	0	$(-1,-1)$	0	$(0,-4)$	$(8,4)$	4	$\sqrt{2}^{17}\varepsilon^{-4}$	I_4^*, 4	j_{14}	1
B1′	0	$(-1,1)$	0	$(0,-1)$	0	$(2,2)$	$\sqrt{2}^{10}\varepsilon^{2}$	III, 2	j_{11}'	$2',3',4'$
B2′	0	$(0,-1)$	0	$(4,-4)$	$(8,-4)$	4	$\sqrt{2}^{20}\varepsilon^{-5}$	I_7^*, 4	j_{12}'	1′
B3′	0	$(-1,-1)$	0	$(-58,42)$	$(-246,174)$	2	$\sqrt{2}^{17}\varepsilon^{-11}$	II^*, 1	j_{13}'	1′
B4′	0	$(-1,1)$	0	$(0,4)$	$(8,-4)$	4	$-\sqrt{2}^{17}\varepsilon^{4}$	I_4^*, 4	j_{14}'	1′
C1	0	1	0	$(-4,3)$	0	$(2,2)$	$\sqrt{2}^{10}\varepsilon^{-8}$	III, 2	j_{11}	$2,3,4$
C2	0	$(0,-1)$	0	$(4,4)$	$(-8,-4)$	2	$-\sqrt{2}^{20}\varepsilon^{5}$	I_7^*, 4	j_{12}	1
C3	0	$(0,-1)$	0	$(-11,-6)$	$(22,15)$	4	$\sqrt{2}^{17}\varepsilon^{5}$	I_4^*, 4	j_{13}	1
C4	0	$(0,-1)$	0	$(-21,-16)$	$(-46,-33)$	2	$\sqrt{2}^{17}\varepsilon^{2}$	II^*, 1	j_{14}	1
C1′	0	1	0	$(-4,-3)$	0	$(2,2)$	$\sqrt{2}^{10}\varepsilon^{8}$	III, 2	j_{11}'	$2',3',4'$
C2′	0	$(0,1)$	0	$(4,-4)$	$(-8,4)$	2	$\sqrt{2}^{20}\varepsilon^{-5}$	I_7^*, 4	j_{12}'	1′
C3′	0	$(0,1)$	0	$(-11,6)$	$(22,-15)$	4	$\sqrt{2}^{-5}\varepsilon^{-17}$	I_4^*, 4	j_{13}'	1′
C4′	0	$(0,1)$	0	$(-21,16)$	$(-46,33)$	2	$-\sqrt{2}^{17}\varepsilon^{-2}$	II^*, 1	j_{14}'	1′
D1	0	$(1,1)$	0	$(0,1)$	0	$(2,2)$	$\sqrt{2}^{10}\varepsilon^{-2}$	III, 2	j_{11}	$2,3,4$
D2	0	$(1,1)$	0	$(-5,6)$	$(11,-7)$	4	$-\sqrt{2}^{20}\varepsilon^{-1}$	I_7^*, 4	j_{12}	1
D3	0	$(1,-1)$	0	$(-58,-42)$	$(246,174)$	2	$\sqrt{2}^{17}\varepsilon^{11}$	II^*, 1	j_{13}	1
D4	0	$(1,1)$	0	$(0,-4)$	$(-8,-4)$	2	$\sqrt{2}^{17}\varepsilon^{-4}$	I_4^*, 2	j_{14}	1
D1′	0	$(1,-1)$	0	$(0,-1)$	0	$(2,2)$	$\sqrt{2}^{10}\varepsilon^{2}$	III, 2	j_{11}'	$2',3',4'$
D2′	0	$(1,-1)$	0	$(-5,-6)$	$(11,7)$	4	$\sqrt{2}^{20}\varepsilon$	I_7^*, 4	j_{12}'	1′
D3′	0	$(1,1)$	0	$(-58,42)$	$(246,-174)$	2	$\sqrt{2}^{17}\varepsilon^{-11}$	II^*, 1	j_{13}'	1′
D4′	0	$(1,-1)$	0	$(0,4)$	$(-8,4)$	2	$-\sqrt{2}^{17}\varepsilon^{4}$	I_4^*, 2	j_{14}'	1′

Table 9: Elliptic curves of conductor $(\sqrt{2})^{10}$ (16 isogeny classes)

	a_1	a_2	a_3	a_4	a_6	Tors.	Δ	type	j	Isog.
A1	0	0	0	2	0	2	$-\sqrt{2}^{18}$	I_4^*, 4	j_1	2
A2	0	0	0	-2	0	$(2,2)$	$\sqrt{2}^{18}$	I_4^*, 4	j_1	$1,3,3'$
A3	0	0	0	-22	$(0,28)$	$(2,2)$	$\sqrt{2}^{24}$	I_{10}^*, 4	j_2	$2,4,5'$
A3′	0	0	0	-22	$(0,-28)$	$(2,2)$	$\sqrt{2}^{24}$	I_{10}^*, 4	j_2	$2,4',5$
A4	0	0	0	$(-182,-120)$	$(1232,924)$	4	$\sqrt{2}^{27}\varepsilon^{-6}$	I_{13}^*, 4	j_3	3
A4′	0	0	0	$(-182,120)$	$(1232,-924)$	4	$-\sqrt{2}^{27}\varepsilon^{6}$	I_{13}^*, 4	j_3'	3′
A5	0	0	0	$(-182,-120)$	$(-1232,-924)$	2	$\sqrt{2}^{27}\varepsilon^{-6}$	I_{13}^*, 2	j_3	3′
A5′	0	0	0	$(-182,120)$	$(-1232,924)$	2	$-\sqrt{2}^{27}\varepsilon^{6}$	I_{13}^*, 2	j_3'	3
B1	0	0	0	$(6,4)$	0	2	$-\sqrt{2}^{18}\varepsilon^{6}$	I_4^*, 2	j_1	2
B2	0	0	0	$(-6,-4)$	0	$(2,2)$	$\sqrt{2}^{18}\varepsilon^{6}$	I_4^*, 4	j_1	$1,3,4$
B3	0	0	0	$(-66,44)$	$(280,-196)$	$(2,2)$	$\sqrt{2}^{24}\varepsilon^{-6}$	I_{10}^*, 4	j_2	$2,5,5'$
B4	0	0	0	$(-66,44)$	$(-280,196)$	$(2,2)$	$\sqrt{2}^{24}\varepsilon^{-6}$	I_{10}^*, 4	j_2	$2,6,6'$
B5	0	0	0	$(-66,4)$	$(616,-308)$	2	$\sqrt{2}^{27}\varepsilon^{-12}$	I_{13}^*, 2	j_3	3
B5′	0	0	0	$(-66,-4)$	$(616,308)$	2	$-\sqrt{2}^{27}\varepsilon^{12}$	I_{13}^*, 2	j_3'	3
B6	0	0	0	$(-66,4)$	$(-616,308)$	2	$\sqrt{2}^{27}\varepsilon^{-12}$	I_{13}^*, 4	j_3	4

B6′	0	0	0	$(-66,-4)$	$(-616,-308)$	2	$-\sqrt{2}^{27}\varepsilon^{12}$	I_{13}^*, 4	j_3'	4
C1	0	$(0,1)$	0	1	0	2	$-\sqrt{2}^{10}$	II, 1	j_4	2
C2	0	$(0,1)$	0	-4	$(0,-4)$	$(2,2)$	$\sqrt{2}^{20}$	I_6^*, 4	j_5	$1,3,4$
C3	0	-1	0	$(-279,196)$	$(-2037,1440)$	2	$\sqrt{2}^{25}\varepsilon^{-15}$	I_{11}^*, 4	j_6	2
C4	0	1	0	$(-279,-196)$	$(2037,1440)$	4	$\sqrt{2}^{25}\varepsilon^{15}$	I_{11}^*, 4	j_6'	2
C1′	0	$(0,-1)$	0	1	0	2	$-\sqrt{2}^{10}$	II, 1	j_4	$2'$
C2′	0	$(0,-1)$	0	-4	$(0,4)$	$(2,2)$	$\sqrt{2}^{20}$	I_6^*, 4	j_5	$1',3',4'$
C3′	0	-1	0	$(-279,-196)$	$(-2037,-1440)$	2	$\sqrt{2}^{25}\varepsilon^{15}$	I_{11}^*, 4	j_6'	$2'$
C4′	0	1	0	$(-279,196)$	$(2037,-1440)$	4	$\sqrt{2}^{25}\varepsilon^{-15}$	I_{11}^*, 4	j_6	$2'$
D1	0	$(1,1)$	0	2	$(2,-1)$	2	$-\sqrt{2}^{10}\varepsilon^{-6}$	II, 1	j_4	2
D2	0	$(1,1)$	0	$(-13,10)$	$(27,-19)$	$(2,2)$	$\sqrt{2}^{20}\varepsilon^{-6}$	I_6^*, 4	j_5	$1,3,3'$
D3	0	$(1,1)$	0	$(-53,30)$	$(-141,105)$	2	$\sqrt{2}^{25}\varepsilon^{-9}$	I_{11}^*, 2	j_6	2
D3′	0	$(1,-1)$	0	$(-53,-30)$	$(-141,-105)$	2	$\sqrt{2}^{25}\varepsilon^{9}$	I_{11}^*, 2	j_6'	2
E1	0	$(-1,-1)$	0	2	$(-2,1)$	2	$-\sqrt{2}^{10}$	II, 1	j_4	2
E2	0	$(-1,-1)$	0	$(-13,10)$	$(-27,19)$	$(2,2)$	$\sqrt{2}^{20}\varepsilon^{-6}$	I_6^*, 4	j_5	$1,3,3'$
E3	0	$(-1,-1)$	0	$(-53,30)$	$(141,-105)$	2	$\sqrt{2}^{25}\varepsilon^{-9}$	I_{11}^*, 2	j_6	2
E3′	0	$(-1,1)$	0	$(-53,-30)$	$(141,105)$	2	$\sqrt{2}^{25}\varepsilon^{9}$	I_{11}^*, 2	j_6'	2
F1	0	1	0	-3	1	$(2,2)$	$\sqrt{2}^{18}$	I_4^*, 4	j_7	$2,3,3'$
F2	0	-1	0	-3	-1	$(2,2)$	$\sqrt{2}^{18}$	I_4^*, 4	j_7	$1,4,4'$
F3	0	1	0	$(-23,-20)$	$(69,40)$	2	$\sqrt{2}^{24}\varepsilon^{-3}$, 4	I_{10}^*, 4	j_8	1
F3′	0	1	0	$(-23,20)$	$(69,-40)$	2	$-\sqrt{2}^{24}\varepsilon^{3}$	I_{10}^*, 4	j_8'	1
F4	0	-1	0	$(-23,-20)$	$(-69,-40)$	2	$\sqrt{2}^{24}\varepsilon^{-3}$	I_{10}^*, 2	j_8	2
F4′	0	-1	0	$(-23,20)$	$(-69,40)$	2	$-\sqrt{2}^{24}\varepsilon^{3}$	I_{10}^*, 2	j_8'	2
G1	0	$(-1,1)$	0	$(-9,6)$	$(-7,5)$	$(2,2)$	$\sqrt{2}^{18}\varepsilon^{-6}$	I_4^*, 4	j_7	$1',2,3'$
G1′	0	$(-1,-1)$	0	$(-9,-6)$	$(-7,-5)$	$(2,2)$	$\sqrt{2}^{18}\varepsilon^{6}$	I_4^*, 4	j_7	$1,2',3$
G2	0	$(-1,1)$	0	$(11,-14)$	$(-83,65)$	2	$\sqrt{2}^{24}\varepsilon^{-9}$	I_{10}^*, 2	j_8	1
G2′	0	$(-1,-1)$	0	$(11,14)$	$(-83,-65)$	2	$-\sqrt{2}^{24}\varepsilon^{9}$	I_{10}^*, 2	j_8'	$1'$
G3	0	$(1,-1)$	0	$(11,-14)$	$(83,-65)$	2	$\sqrt{2}^{24}\varepsilon^{-9}$	I_{10}^*, 4	j_8	$1'$
G3′	0	$(1,1)$	0	$(11,14)$	$(83,65)$	2	$-\sqrt{2}^{24}\varepsilon^{9}$	I_{10}^*, 4	j_8'	1
H1	0	$(0,-1)$	0	$(-8,6)$	0	$(2,2)$	$\sqrt{2}^{16}\varepsilon^{-8}$	I_2^*, 4	j_{11}	$2,3,4$
H2	0	1	0	$(2,2)$	$(2,2)$	2	$-\sqrt{2}^{14}\varepsilon^{5}$	I_0^*, 1	j_{12}	1
H3	0	-1	0	$(-23,-12)$	$(-37,-32)$	2	$\sqrt{2}^{23}\varepsilon^{5}$	I_9^*, 4	j_{13}	1
H4	0	-1	0	$(-43,-32)$	$(175,124)$	2	$\sqrt{2}^{23}\varepsilon^{2}$	I_9^*, 4	j_{14}	1
H1′	0	$(0,1)$	0	$(-8,-6)$	0	$(2,2)$	$\sqrt{2}^{16}\varepsilon^{8}$	I_2^*, 4	j_{11}'	$2',3',4'$
H2′	0	1	0	$(2,-2)$	$(2,-2)$	2	$\sqrt{2}^{14}\varepsilon^{-5}$	I_0^*, 1	j_{12}'	$1'$
H3′	0	-1	0	$(-23,12)$	$(-37,32)$	2	$\sqrt{2}^{23}\varepsilon^{-5}$	I_9^*, 4	j_{13}'	$1'$
H4′	0	-1	0	$(-43,32)$	$(175,-124)$	2	$-\sqrt{2}^{23}\varepsilon^{-2}$	I_9^*, 4	j_{14}'	$1'$
I1	0	$(1,-1)$	0	-1	$(-1,1)$	$(2,2)$	$\sqrt{2}^{16}\varepsilon^{-2}$	I_2^*, 4	j_{11}	$2,3,4$
I2	0	$(-1,1)$	0	$(-2,2)$	$(6,-4)$	2	$-\sqrt{2}^{14}\varepsilon^{-1}$	I_0^*, 2	j_{12}	1
I3	0	$(-1,-1)$	0	$(-117,-82)$	$(-579,-409)$	2	$\sqrt{2}^{23}\varepsilon^{11}$	I_9^*, 2	j_{13}	1
I4	0	$(1,-1)$	0	$(-1,-10)$	$(15,7)$	4	$\sqrt{2}^{23}\varepsilon^{-4}$	I_9^*, 4	j_{14}	1
I1′	0	$(1,1)$	0	-1	$(-1,-1)$	$(2,2)$	$\sqrt{2}^{16}\varepsilon^{2}$	I_2^*, 4	j_{11}'	$2',3',4'$
I2′	0	$(-1,-1)$	0	$(-2,-2)$	$(6,4)$	2	$\sqrt{2}^{14}\varepsilon$	I_0^*, 2	j_{12}'	$1'$
I3′	0	$(-1,1)$	0	$(-117,82)$	$(-579,409)$	2	$\sqrt{2}^{23}\varepsilon^{-11}$, 2	I_9^*, 2	j_{13}'	$1'$
I4′	0	$(1,1)$	0	$(-1,10)$	$(15,-7)$	4	$-\sqrt{2}^{23}\varepsilon^{4}$	I_9^*, 4	j_{14}'	$1'$
J1	0	$(0,1)$	0	$(-8,6)$	0	$(2,2)$	$\sqrt{2}^{16}\varepsilon^{-8}$	I_2^*, 4	j_{11}	$2,3,4$
J2	0	-1	0	$(2,2)$	$(-2,-2)$	2	$-\sqrt{2}^{14}\varepsilon^{5}$	I_0^*, 1	j_{12}	1
J3	0	1	0	$(-23,-12)$	$(37,32)$	2	$\sqrt{2}^{23}\varepsilon^{5}$	I_9^*, 2	j_{13}	1
J4	0	1	0	$(-43,-32)$	$(-175,-124)$	2	$\sqrt{2}^{23}\varepsilon^{2}$	I_9^*, 4	j_{14}	1
J1′	0	$(0,-1)$	0	$(-8,-6)$	0	$(2,2)$	$\sqrt{2}^{16}\varepsilon^{8}$	I_2^*, 4	j_{11}'	$2',3',4'$
J2′	0	-1	0	$(2,-2)$	$(-2,2)$	2	$\sqrt{2}^{14}\varepsilon^{-5}$	I_0^*, 1	j_{12}'	$1'$
J3′	0	1	0	$(-23,12)$	$(37,-32)$	2	$\sqrt{2}^{23}\varepsilon^{-5}$	I_9^*, 2	j_{13}'	$1'$
J4′	0	1	0	$(-43,32)$	$(-175,124)$	2	$-\sqrt{2}^{23}\varepsilon^{-2}$	I_9^*, 4	j_{14}'	$1'$
K1	0	$(-1,1)$	0	-1	$(1,-1)$	$(2,2)$	$\sqrt{2}^{16}\varepsilon^{-2}$	I_2^*, 4	j_{11}	$2,3,4$
K2	0	$(1,-1)$	0	$(-2,2)$	$(-6,4)$	2	$-\sqrt{2}^{14}\varepsilon^{-1}$	I_0^*, 2	j_{12}	1
K3	0	$(1,1)$	0	$(-117,-82)$	$(579,409)$	4	$\sqrt{2}^{23}\varepsilon^{11}$	I_9^*, 4	j_{13}	1
K4	0	$(-1,1)$	0	$(-1,-10)$	$(-15,-7)$	2	$\sqrt{2}^{23}\varepsilon^{-4}$	I_9^*, 4	j_{14}	1

K1′	0	$(-1,-1)$	0	-1	$(1,1)$	$(2,2)$	$\sqrt{2}^{16}\varepsilon^{2}$	I_2^*, 4	j'_{11}	$2',3',4'$
K2′	0	$(1,1)$	0	$(-2,-2)$	$(-6,-4)$	2	$\sqrt{2}^{14}\varepsilon$	I_0^*, 2	j'_{12}	$1'$
K3′	0	$(1,-1)$	0	$(-117,82)$	$(579,-409)$	4	$\sqrt{2}^{23}\varepsilon^{-11}$	I_9^*, 4	j'_{13}	$1'$
K4′	0	$(-1,-1)$	0	$(-1,10)$	$(-15,7)$	2	$-\sqrt{2}^{23}\varepsilon^{4}$	I_9^*, 4	j'_{14}	$1'$

Table 10: Elliptic curves of conductor $(\sqrt{2})^{11}$ (16 isogeny classes)

	a_1	a_2	a_3	a_4	a_6	Tors.	Δ	type	j	Isog.
A1	0	1	0	$(3,2)$	$(3,2)$	2	$-\sqrt{2}^{18}\varepsilon^{4}$	III*, 2	j_9	2
A2	0	-1	0	$(-4,-2)$	$(4,2)$	2	$\sqrt{2}^{15}\varepsilon^{5}$	I_0^*, 2	j_{10}	1
A1′	0	1	0	$(3,-2)$	$(3,-2)$	2	$-\sqrt{2}^{18}\varepsilon^{-4}$	III*, 2	j'_9	$2'$
A2′	0	-1	0	$(-4,2)$	$(4,-2)$	2	$\sqrt{2}^{15}\varepsilon^{-5}$	I_0^*, 2	j'_{10}	$1'$
B1	0	$(0,-1)$	0	$(2,1)$	0	2	$-\sqrt{2}^{12}\varepsilon^{4}$	III, 2	j_9	2
B2	0	$(0,-1)$	0	$(-8,-4)$	$(8,8)$	2	$\sqrt{2}^{21}\varepsilon^{5}$	I_6^*, 2	j_{10}	1
B1′	0	$(0,1)$	0	$(2,-1)$	0	2	$-\sqrt{2}^{12}\varepsilon^{-4}$	III, 2	j'_9	$2'$
B2′	0	$(0,1)$	0	$(-8,4)$	$(8,-8)$	2	$\sqrt{2}^{21}\varepsilon^{-5}$	I_6^*, 2	j'_{10}	$1'$
C1	0	$(-1,1)$	0	1	$(-1,1)$	2	$-\sqrt{2}^{18}\varepsilon^{-2}$	III*, 2	j_9	2
C2	0	$(1,-1)$	0	$(-4,2)$	$(-8,6)$	2	$\sqrt{2}^{15}\varepsilon^{-1}$	I_0^*, 1	j_{10}	1
C1′	0	$(-1,-1)$	0	1	$(-1,-1)$	2	$-\sqrt{2}^{18}\varepsilon^{2}$	III*, 2	j'_9	$2'$
C2′	0	$(1,1)$	0	$(-4,-2)$	$(-8,-6)$	2	$\sqrt{2}^{15}\varepsilon$	I_0^*, 1	j'_{10}	$1'$
D1	0	$(1,1)$	0	$(1,1)$	1	2	$-\sqrt{2}^{12}\varepsilon^{-2}$	III, 2	j_9	2
D2	0	$(1,1)$	0	$(-9,6)$	$(15,-11)$	2	$\sqrt{2}^{21}\varepsilon^{-1}$	I_6^*, 4	j_{10}	1
D1′	0	$(1,-1)$	0	$(1,-1)$	1	2	$-\sqrt{2}^{12}\varepsilon^{2}$	III, 2	j'_9	$2'$
D2′	0	$(1,-1)$	0	$(-9,-6)$	$(15,11)$	2	$\sqrt{2}^{21}\varepsilon$	I_6^*, 4	j'_{10}	$1'$
E1	0	-1	0	$(3,2)$	$(-3,-2)$	2	$-\sqrt{2}^{18}\varepsilon^{4}$	III*, 2	j_9	2
E2	0	1	0	$(-4,-2)$	$(-4,-2)$	2	$\sqrt{2}^{15}\varepsilon^{5}$	I_0^*, 2	j_{10}	1
E1′	0	-1	0	$(3,-2)$	$(-3,2)$	2	$-\sqrt{2}^{18}\varepsilon^{-4}$	III*, 2	j'_9	$2'$
E2′	0	1	0	$(-4,2)$	$(-4,2)$	2	$\sqrt{2}^{15}\varepsilon^{-5}$	I_0^*, 2	j'_{10}	$1'$
F1	0	$(0,1)$	0	$(2,1)$	0	2	$-\sqrt{2}^{12}\varepsilon^{4}$	III, 2	j_9	2
F2	0	$(0,1)$	0	$(-8,-4)$	$(-8,-8)$	2	$\sqrt{2}^{21}\varepsilon^{5}$	I_6^*, 4	j_{10}	1
F1′	0	$(0,-1)$	0	$(2,-1)$	0	2	$-\sqrt{2}^{12}\varepsilon^{-4}$	III, 2	j'_9	$2'$
F2′	0	$(0,-1)$	0	$(-8,4)$	$(-8,8)$	2	$\sqrt{2}^{21}\varepsilon^{-5}$	I_6^*, 4	j'_{10}	$1'$
G1	0	$(1,-1)$	0	1	$(1,-1)$	2	$-\sqrt{2}^{18}\varepsilon^{-2}$	III*, 2	j_9	2
G2	0	$(-1,1)$	0	$(-4,2)$	$(8,-6)$	2	$\sqrt{2}^{15}\varepsilon^{-1}$	I_0^*, 1	j_{10}	1
G1′	0	$(1,1)$	0	1	$(1,1)$	2	$-\sqrt{2}^{18}\varepsilon^{2}$	III*, 2	j'_9	$2'$
G2′	0	$(-1,-1)$	0	$(-4,-2)$	$(8,6)$	2	$\sqrt{2}^{15}\varepsilon$	I_0^*, 1	j'_{10}	$1'$
H1	0	$(-1,-1)$	0	$(1,1)$	-1	2	$-\sqrt{2}^{12}\varepsilon^{-2}$	III, 2	j_9	2
H2	0	$(-1,-1)$	0	$(-9,6)$	$(-15,11)$	2	$\sqrt{2}^{21}\varepsilon^{-1}$	I_6^*, 2	j_{10}	1
H1′	0	$(-1,1)$	0	$(1,-1)$	1	2	$-\sqrt{2}^{12}\varepsilon^{2}$	III, 2	j'_9	$2'$
H2′	0	$(-1,1)$	0	$(-9,-6)$	$(-15,-11)$	2	$\sqrt{2}^{21}\varepsilon$	I_6^*, 2	j'_{10}	$1'$

Table 11: Elliptic curves of conductor $(\sqrt{2})^{12}$ (12 isogeny classes)

	a_1	a_2	a_3	a_4	a_6	Tors.	Δ	type	j	Isog.
A1	0	0	0	$(2,2)$	0	2	$-\sqrt{2}^{18}\varepsilon^{3}$	I_2^*, 2	j_1	2
A2	0	0	0	$(-2,-2)$	0	2	$\sqrt{2}^{18}\varepsilon^{3}$	I_2^*, 4	j_1	1
A1′	0	0	0	$(2,-2)$	0	2	$\sqrt{2}^{18}\varepsilon^{-3}$	I_2^*, 2	j_1	$2'$
A2′	0	0	0	$(-2,2)$	0	2	$-\sqrt{2}^{18}\varepsilon^{-3}$	I_2^*, 4	j_1	$1'$
B1	0	0	0	$(-1,1)$	0	2	$-\sqrt{2}^{12}\varepsilon^{-3}$	II, 1	j_1	2
B2	0	0	0	$(1,-1)$	0	2	$\sqrt{2}^{12}\varepsilon^{-3}$	II, 1	j_1	1
B1′	0	0	0	$(-1,-1)$	0	2	$\sqrt{2}^{12}\varepsilon^{3}$	II, 1	j_1	$2'$
B2′	0	0	0	$(1,1)$	0	2	$-\sqrt{2}^{12}\varepsilon^{3}$	II, 1	j_1	$1'$
C1	0	1	0	$(-3,-2)$	$(1,2)$	2	$\sqrt{2}^{18}\varepsilon^{-1}$	I_2^*, 2	j_{21}	2
C2	0	-1	0	$(-3,2)$	$(-1,2)$	2	$-\sqrt{2}^{18}\varepsilon$	I_2^*, 2	j'_{21}	1
C1′	0	1	0	$(-3,2)$	$(1,-2)$	2	$-\sqrt{2}^{18}\varepsilon$	I_2^*, 2	j'_{21}	$2'$

C2′	0	−1	0	(−3, −2)	(−1, −2)	2	$\sqrt{2}^{18}\varepsilon^{-1}$	I_2^*, 2	j_{21}	1′
D1	0	(0, −1)	0	(−1, −1)	(2, 1)	2	$\sqrt{2}^{12}\varepsilon^{-1}$	II, 1	j_{21}	1′
D1′	0	(0, 1)	0	(−1, 1)	(2, −1)	2	$-\sqrt{2}^{12}\varepsilon$	II, 1	j'_{21}	1
E1	0	(−1, 1)	0	−1	(13, −9)	2	$\sqrt{2}^{18}\varepsilon^{-7}$	I_2^*, 4	j_{21}	1′
E1′	0	(−1, −1)	0	−1	(13, 9)	2	$-\sqrt{2}^{18}\varepsilon^{7}$	I_2^*, 4	j'_{21}	1
F1	0	(1, 1)	0	(0, 1)	(−4, 3)	2	$\sqrt{2}^{12}\varepsilon^{-7}$	II, 1	j_{21}	2
F2	0	(−1, −1)	0	(−8, 7)	(−16, 11)	2	$-\sqrt{2}^{12}\varepsilon^{-5}$	II, 1	j'_{21}	1
F1′	0	(1, −1)	0	(0, −1)	(−4, −3)	2	$-\sqrt{2}^{12}\varepsilon^{7}$	II, 1	j'_{21}	2′
F2′	0	(−1, 1)	0	(−8, −7)	(−16, −11)	2	$\sqrt{2}^{12}\varepsilon^{5}$	II, 1	j_{21}	1′
G1	0	(0, 1)	0	(−1, −1)	(−2, −1)	2	$\sqrt{2}^{12}\varepsilon^{-1}$	II, 1	j_{21}	1′
G1′	0	(0, −1)	0	(−1, 1)	(−2, 1)	2	$-\sqrt{2}^{12}\varepsilon$	II, 1	j'_{21}	1
H1	0	(1, −1)	0	−1	(−13, 9)	2	$\sqrt{2}^{18}\varepsilon^{-7}$	I_2^*, 4	j_{21}	1′
H1′	0	(1, 1)	0	−1	(−13, −9)	2	$-\sqrt{2}^{18}\varepsilon^{7}$	I_2^*, 4	j'_{21}	1

Table 12: Elliptic curves of conductor $(\sqrt{2})^{13}$ (48 isogeny classes)

	a_1	a_2	a_3	a_4	a_6	Tors.	Δ	type	j	Isog.
A1	0	1	0	(1, −1)	(1, −1)	2	$\sqrt{2}^{14}\varepsilon^{-3}$	III, 2	j_{15}	2
A2	0	−1	0	(−2, 1)	(2, −1)	2	$\sqrt{2}^{13}\varepsilon^{-3}$	II, 1	j_{16}	1
A1′	0	1	0	(1, 1)	(1, 1)	2	$-\sqrt{2}^{14}\varepsilon^{3}$	III, 2	j'_{15}	2′
A2′	0	−1	0	(−2, −1)	(2, 1)	2	$\sqrt{2}^{13}\varepsilon^{3}$	II, 1	j'_{16}	1′
B1	0	(0, 1)	0	(2, −2)	(−4, 2)	2	$\sqrt{2}^{20}\varepsilon^{-3}$	III*, 2	j_{15}	2
B2	0	(0, −1)	0	(−4, 2)	(−4, 4)	2	$\sqrt{2}^{19}\varepsilon^{-3}$	I_2^*, 2	j_{16}	1
B1′	0	(0, −1)	0	(2, 2)	(−4, −2)	2	$-\sqrt{2}^{20}\varepsilon^{3}$	III*, 2	j'_{15}	2′
B2′	0	(0, 1)	0	(−4, −2)	(−4, −4)	2	$\sqrt{2}^{19}\varepsilon^{3}$	I_2^*, 2	j'_{16}	1′
C1	0	(1, 1)	0	(−1, −1)	(−3, −2)	2	$\sqrt{2}^{14}\varepsilon^{3}$	III, 2	j_{15}	2
C2	0	(−1, −1)	0	(−2, −1)	(4, 3)	2	$\sqrt{2}^{13}\varepsilon^{3}$	II, 1	j_{16}	1
C1′	0	(1, −1)	0	(−1, 1)	(−3, 2)	2	$-\sqrt{2}^{14}\varepsilon^{-3}$	III, 2	j'_{15}	2′
C2′	0	(−1, 1)	0	(−2, 1)	(4, −3)	2	$\sqrt{2}^{13}\varepsilon^{-3}$	II, 1	j'_{16}	1′
D1	0	(−1, 1)	0	(−3, −4)	(−5, −3)	2	$\sqrt{2}^{20}\varepsilon^{3}$	III*, 2	j_{15}	2
D2	0	(1, −1)	0	(−5, −4)	(7, 5)	2	$\sqrt{2}^{19}\varepsilon^{3}$	I_2^*, 4	j_{16}	1
D1′	0	(−1, −1)	0	(−3, 4)	(−5, 3)	2	$-\sqrt{2}^{20}\varepsilon^{-3}$	III*, 2	j'_{15}	2′
D2′	0	(1, 1)	0	(−5, 4)	(7, −5)	2	$\sqrt{2}^{19}\varepsilon^{-3}$	I_2^*, 4	j'_{16}	1′
E1	0	−1	0	(1, −1)	(−1, 1)	2	$\sqrt{2}^{14}\varepsilon^{-3}$	III, 2	j_{15}	2
E2	0	1	0	(−2, 1)	(−2, 1)	2	$\sqrt{2}^{13}\varepsilon^{-3}$	II, 1	j_{16}	1
E1′	0	−1	0	(1, 1)	(−1, −1)	2	$-\sqrt{2}^{14}\varepsilon^{3}$	III, 2	j'_{15}	2′
E2′	0	1	0	(−2, −1)	(−2, −1)	2	$\sqrt{2}^{13}\varepsilon^{3}$	II, 1	j'_{16}	1′
F1	0	(0, −1)	0	(2, −2)	(4, −2)	2	$\sqrt{2}^{20}\varepsilon^{-3}$	III*, 2	j_{15}	2
F2	0	(0, 1)	0	(−4, 2)	(4, −4)	2	$\sqrt{2}^{19}\varepsilon^{-3}$	I_2^*, 2	j_{16}	1
F1′	0	(0, 1)	0	(2, 2)	(4, 2)	2	$-\sqrt{2}^{20}\varepsilon^{3}$	III*, 2	j'_{15}	2′
F2′	0	(0, −1)	0	(−4, −2)	(4, 4)	2	$\sqrt{2}^{19}\varepsilon^{3}$	I_2^*, 2	j'_{16}	1′
G1	0	(−1, −1)	0	(−1, −1)	(3, 2)	2	$\sqrt{2}^{14}\varepsilon^{3}$	III, 2	j_{15}	2
G2	0	(1, 1)	0	(−2, −1)	(−4, −3)	2	$\sqrt{2}^{13}\varepsilon^{3}$	II, 1	j_{16}	1
G1′	0	(−1, 1)	0	(−1, 1)	(3, −2)	2	$-\sqrt{2}^{14}\varepsilon^{-3}$	III, 2	j'_{15}	2′
G2′	0	(1, −1)	0	(−2, 1)	(−4, 3)	2	$\sqrt{2}^{13}\varepsilon^{-3}$	II, 1	j'_{16}	1′
H1	0	(1, −1)	0	(−3, −4)	(5, 3)	2	$\sqrt{2}^{20}\varepsilon^{3}$	III*, 2	j_{15}	2
H2	0	(−1, 1)	0	(−5, −4)	(−7, −5)	2	$\sqrt{2}^{19}\varepsilon^{3}$	I_2^*, 4	j_{16}	1
H1′	0	(1, 1)	0	(−3, 4)	(5, −3)	2	$-\sqrt{2}^{20}\varepsilon^{-3}$	III*, 2	j'_{15}	2′
H2′	0	(−1, −1)	0	(−5, 4)	(−7, 5)	2	$\sqrt{2}^{19}\varepsilon^{-3}$	I_2^*, 4	j'_{16}	1′
I1	0	(0, 1)	0	(−46, −33)	(130, 92)	2	$\sqrt{2}^{14}\varepsilon^{13}$	III, 2	j_{17}	2
I2	0	1	0	(0, 1)	(0, 1)	2	$-\sqrt{2}^{13}\varepsilon^{2}$	II, 1	j_{18}	1
I1′	0	(0, −1)	0	(−46, 33)	(130, −92)	2	$-\sqrt{2}^{14}\varepsilon^{-13}$	III, 2	j'_{17}	2′
I2′	0	1	0	(0, −1)	(0, −1)	2	$\sqrt{2}^{13}\varepsilon^{-2}$	II, 1	j'_{18}	1′
J1	0	−1	0	(−93, −66)	(461, 326)	2	$\sqrt{2}^{20}\varepsilon^{13}$	III*, 2	j_{17}	2
J2	0	(0, 1)	0	(0, 2)	4	2	$-\sqrt{2}^{19}\varepsilon^{2}$	I_2^*, 4	j_{18}	1
J1′	0	−1	0	(−93, 66)	(461, −326)	2	$-\sqrt{2}^{20}\varepsilon^{-13}$	III*, 2	j'_{17}	2′
J2′	0	(0, −1)	0	(0, −2)	4	2	$\sqrt{2}^{19}\varepsilon^{-2}$	I_2^*, 4	j'_{18}	1′

K1	0	$(-1,-1)$	0	$(-7,-5)$	$(17,12)$	2	$\sqrt{2}^{14}\varepsilon^{7}$	III, 2	j_{17}	2
K2	0	$(-1,1)$	0	$(-4,3)$	$(10,-7)$	2	$-\sqrt{2}^{13}\varepsilon^{-4}$	II, 1	j_{18}	1
K1′	0	$(-1,1)$	0	$(-7,5)$	$(17,-12)$	2	$-\sqrt{2}^{14}\varepsilon^{-7}$	III, 2	j'_{17}	2′
K2′	0	$(-1,-1)$	0	$(-4,-3)$	$(10,7)$	2	$\sqrt{2}^{13}\varepsilon^{4}$	II, 1	j'_{18}	1′
L1	0	$(1,-1)$	0	$(-15,-12)$	$(33,23)$	2	$\sqrt{2}^{20}\varepsilon^{7}$	III*, 2	j_{17}	2
L2	0	$(-1,-1)$	0	$(-9,8)$	$(-19,13)$	2	$-\sqrt{2}^{19}\varepsilon^{-4}$	I_2^*, 2	j_{18}	1
L1′	0	$(1,1)$	0	$(-15,12)$	$(33,-23)$	2	$-\sqrt{2}^{20}\varepsilon^{-7}$	III*, 2	j'_{17}	2′
L2′	0	$(-1,1)$	0	$(-9,-8)$	$(-19,-13)$	2	$\sqrt{2}^{19}\varepsilon^{4}$	I_2^*, 2	j'_{18}	1′
M1	0	$(0,-1)$	0	$(-46,-33)$	$(-130,-92)$	2	$\sqrt{2}^{14}\varepsilon^{13}$	III, 2	j_{17}	2
M2	0	-1	0	$(0,1)$	$(0,-1)$	2	$-\sqrt{2}^{13}\varepsilon^{2}$	II, 1	j_{18}	1
M1′	0	$(0,1)$	0	$(-46,33)$	$(-130,92)$	2	$-\sqrt{2}^{14}\varepsilon^{-13}$	III, 2	j'_{17}	2′
M2′	0	-1	0	$(0,-1)$	$(0,1)$	2	$\sqrt{2}^{13}\varepsilon^{-2}$	II, 1	j'_{18}	1′
N1	0	1	0	$(-93,-66)$	$(-461,-326)$	2	$\sqrt{2}^{20}\varepsilon^{13}$	III*, 2	j_{17}	2
N2	0	$(0,-1)$	0	$(0,2)$	-4	2	$-\sqrt{2}^{19}\varepsilon^{2}$	I_2^*, 4	j_{18}	1
N1′	0	1	0	$(-93,66)$	$(-461,326)$	2	$-\sqrt{2}^{20}\varepsilon^{-13}$	III*, 2	j'_{17}	2′
N2′	0	$(0,1)$	0	$(0,-2)$	-4	2	$\sqrt{2}^{19}\varepsilon^{-2}$	I_2^*, 4	j'_{18}	1′
O1	0	$(1,1)$	0	$(-7,-5)$	$(-17,-12)$	2	$\sqrt{2}^{14}\varepsilon^{7}$	III, 2	j_{17}	2
O2	0	$(1,-1)$	0	$(-4,3)$	$(-10,7)$	2	$-\sqrt{2}^{13}\varepsilon^{-4}$	II, 1	j_{18}	1
O1′	0	$(1,-1)$	0	$(-7,5)$	$(-17,12)$	2	$-\sqrt{2}^{14}\varepsilon^{-7}$	III, 2	j'_{17}	2′
O2′	0	$(1,1)$	0	$(-4,-3)$	$(-10,-7)$	2	$\sqrt{2}^{13}\varepsilon^{4}$	II, 1	j'_{18}	1′
P1	0	$(-1,1)$	0	$(-15,-12)$	$(-33,-23)$	2	$\sqrt{2}^{20}\varepsilon^{7}$	III*, 2	j_{17}	2
P2	0	$(1,1)$	0	$(-9,8)$	$(19,-13)$	2	$-\sqrt{2}^{19}\varepsilon^{-4}$	I_2^*, 2	j_{18}	1
P1′	0	$(-1,-1)$	0	$(-15,12)$	$(-33,23)$	2	$-\sqrt{2}^{20}\varepsilon^{-7}$	III*, 2	j'_{17}	2′
P2′	0	$(1,-1)$	0	$(-9,-8)$	$(19,13)$	2	$\sqrt{2}^{19}\varepsilon^{4}$	I_2^*, 2	j'_{18}	1′
Q1	0	1	0	$(-9,2)$	$(-17,6)$	2	$\sqrt{2}^{20}\varepsilon^{-5}$	III*, 2	j_{23}	2
Q2	0	-1	0	$(-27,18)$	$(-65,46)$	2	$\sqrt{2}^{19}\varepsilon^{-1}$	I_2^*, 4	j_{24}	1
Q1′	0	1	0	$(-9,-2)$	$(-17,-6)$	2	$-\sqrt{2}^{20}\varepsilon^{5}$	III*, 2	j'_{23}	2′
Q2′	0	-1	0	$(-27,-18)$	$(-65,-46)$	2	$\sqrt{2}^{19}\varepsilon$	I_2^*, 4	j'_{24}	1′
R1	0	$(0,1)$	0	$(-4,1)$	$(-2,2)$	2	$\sqrt{2}^{14}\varepsilon^{-5}$	III, 2	j_{23}	2
R2	0	$(0,-1)$	0	$(-13,9)$	$(-32,23)$	2	$\sqrt{2}^{13}\varepsilon^{-1}$	II, 1	j_{24}	1
R1′	0	$(0,-1)$	0	$(-4,-1)$	$(-2,-2)$	2	$-\sqrt{2}^{14}\varepsilon^{5}$	III, 2	j'_{23}	2′
R2′	0	$(0,1)$	0	$(-13,-9)$	$(-32,-23)$	2	$\sqrt{2}^{13}\varepsilon$	II, 1	j'_{24}	1′
S1	0	$(-1,-1)$	0	$(-19,-12)$	$(59,43)$	2	$\sqrt{2}^{20}\varepsilon$	III*, 2	j_{23}	2
S2	0	$(1,1)$	0	-9	$(-5,3)$	2	$\sqrt{2}^{19}\varepsilon^{5}$	I_2^*, 2	j_{24}	1
S1′	0	$(-1,1)$	0	$(-19,12)$	$(59,-43)$	2	$-\sqrt{2}^{20}\varepsilon^{-1}$	III*, 2	j'_{23}	2′
S2′	0	$(1,-1)$	0	-9	$(-5,-3)$	2	$\sqrt{2}^{19}\varepsilon^{-5}$	I_2^*, 2	j'_{24}	1′
T1	0	$(1,-1)$	0	$(-9,-7)$	$(-15,-10)$	2	$\sqrt{2}^{14}\varepsilon$	III, 2	j_{23}	2
T2	0	$(-1,1)$	0	$(-4,-1)$	$(-2,-1)$	2	$\sqrt{2}^{13}\varepsilon^{5}$	II, 1	j_{24}	1
T1′	0	$(1,1)$	0	$(-9,7)$	$(-15,10)$	2	$-\sqrt{2}^{14}\varepsilon^{-1}$	III, 2	j'_{23}	2′
T2′	0	$(-1,-1)$	0	$(-4,1)$	$(-2,1)$	2	$\sqrt{2}^{13}\varepsilon^{-5}$	II, 1	j'_{24}	1′
U1	0	-1	0	$(-9,2)$	$(17,-6)$	2	$\sqrt{2}^{20}\varepsilon^{-5}$	III*, 2	j_{23}	2
U2	0	1	0	$(-27,18)$	$(65,-46)$	2	$\sqrt{2}^{19}\varepsilon^{-1}$	I_2^*, 4	j_{24}	1
U1′	0	-1	0	$(-9,-2)$	$(17,6)$	2	$-\sqrt{2}^{20}\varepsilon^{5}$	III*, 2	j'_{23}	2′
U2′	0	1	0	$(-27,-18)$	$(65,46)$	2	$\sqrt{2}^{19}\varepsilon$	I_2^*, 4	j'_{24}	1′
V1	0	$(0,-1)$	0	$(-4,1)$	$(2,-2)$	2	$\sqrt{2}^{14}\varepsilon^{-5}$	III, 2	j_{23}	2
V2	0	$(0,1)$	0	$(-13,9)$	$(32,-23)$	2	$\sqrt{2}^{13}\varepsilon^{-1}$	II, 1	j_{24}	1
V1′	0	$(0,1)$	0	$(-4,-1)$	$(2,2)$	2	$-\sqrt{2}^{14}\varepsilon^{5}$	III, 2	j'_{23}	2′
V2′	0	$(0,-1)$	0	$(-13,-9)$	$(32,23)$	2	$\sqrt{2}^{13}\varepsilon$	II, 1	j'_{24}	1′
W1	0	$(1,1)$	0	$(-19,-12)$	$(-59,-43)$	2	$\sqrt{2}^{20}\varepsilon$	III*, 2	j_{23}	2
W2	0	$(-1,-1)$	0	-9	$(5,-3)$	2	$\sqrt{2}^{19}\varepsilon^{5}$	I_2^*, 2	j_{24}	1
W1′	0	$(1,-1)$	0	$(-19,12)$	$(-59,43)$	2	$-\sqrt{2}^{20}\varepsilon^{-1}$	III*, 2	j'_{23}	2′
W2′	0	$(-1,1)$	0	-9	$(5,3)$	2	$\sqrt{2}^{19}\varepsilon^{-5}$	I_2^*, 2	j'_{24}	1′
X1	0	$(-1,1)$	0	$(-9,-7)$	$(15,10)$	2	$\sqrt{2}^{14}\varepsilon$	III, 2	j_{23}	2
X2	0	$(1,-1)$	0	$(-4,-1)$	$(2,1)$	2	$\sqrt{2}^{13}\varepsilon^{5}$	II, 1	j_{24}	1
X1′	0	$(-1,-1)$	0	$(-9,7)$	$(15,-10)$	2	$-\sqrt{2}^{14}\varepsilon^{-1}$	III, 2	j'_{23}	2′
X2′	0	$(1,1)$	0	$(-4,1)$	$(2,-1)$	2	$\sqrt{2}^{13}\varepsilon^{-5}$	II, 1	j'_{24}	1′

Table 13: Elliptic curves of conductor $(\sqrt{2})^{14}$ (48 isogeny classes)

	a_1	a_2	a_3	a_4	a_6	Tors.	Δ	type	j	Isog.
A1	0	0	0	$(0,1)$	0	2	$-\sqrt{2}^{15}$	III, 2	j_1	$1'$
A1′	0	0	0	$(0,-1)$	0	2	$\sqrt{2}^{15}$	III, 2	j_1	1
B1	0	0	0	$(0,2)$	0	2	$-\sqrt{2}^{21}$	III*, 2	j_1	$1'$
B1′	0	0	0	$(0,-2)$	0	2	$\sqrt{2}^{21}$	III*, 2	j_1	1
C1	0	0	0	$(2,1)$	0	2	$-\sqrt{2}^{15}\varepsilon^{3}$	III, 2	j_1	2
C2	0	0	0	$(-2,-1)$	0	2	$\sqrt{2}^{15}\varepsilon^{3}$	III, 2	j_1	1
C1′	0	0	0	$(2,-1)$	0	2	$-\sqrt{2}^{15}\varepsilon^{-3}$	III, 2	j_1	$2'$
C2′	0	0	0	$(-2,1)$	0	2	$\sqrt{2}^{15}\varepsilon^{-3}$	III, 2	j_1	$1'$
D1	0	0	0	$(4,2)$	0	2	$-\sqrt{2}^{21}\varepsilon^{3}$	III*, 2	j_1	2
D2	0	0	0	$(-4,-2)$	0	2	$\sqrt{2}^{21}\varepsilon^{3}$	III*, 2	j_1	1
D1′	0	0	0	$(4,-2)$	0	2	$-\sqrt{2}^{21}\varepsilon^{-3}$	III*, 2	j_1	$2'$
D2′	0	0	0	$(-4,2)$	0	2	$\sqrt{2}^{21}\varepsilon^{-3}$	III*, 2	j_1	$1'$
E1	0	0	0	$(4,3)$	0	2	$-\sqrt{2}^{15}\varepsilon^{6}$	III, 2	j_1	$1'$
E1′	0	0	0	$(4,-3)$	0	2	$\sqrt{2}^{15}\varepsilon^{-6}$	III, 2	j_1	1
F1	0	0	0	$(8,6)$	0	2	$-\sqrt{2}^{21}\varepsilon^{6}$	III*, 2	j_1	$1'$
F1′	0	0	0	$(8,-6)$	0	2	$\sqrt{2}^{21}\varepsilon^{-6}$	III*, 2	j_1	1
G1	0	$(1,-1)$	0	$(-7,4)$	$(1,-1)$	2	$\sqrt{2}^{21}\varepsilon^{-5}$	III*, 2	j_{19}	2
G2	0	$(-1,1)$	0	$(-11,8)$	$(-13,9)$	2	$-\sqrt{2}^{21}\varepsilon^{-4}$	III*, 2	j_{20}	1
G1′	0	$(1,1)$	0	$(-7,-4)$	$(1,1)$	2	$\sqrt{2}^{21}\varepsilon^{5}$	III*, 2	j'_{19}	$2'$
G2′	0	$(-1,-1)$	0	$(-11,-8)$	$(-13,-9)$	2	$\sqrt{2}^{21}\varepsilon^{4}$	III*, 2	j'_{20}	$1'$
H1	0	$(1,1)$	0	$(-3,3)$	$(3,-2)$	2	$\sqrt{2}^{15}\varepsilon^{-5}$	III, 2	j_{19}	2
H2	0	$(-1,-1)$	0	$(-5,5)$	$(-9,6)$	2	$-\sqrt{2}^{15}\varepsilon^{-4}$	III, 2	j_{20}	1
H1′	0	$(1,-1)$	0	$(-3,-3)$	$(3,2)$	2	$\sqrt{2}^{15}\varepsilon^{5}$	III, 2	j'_{19}	$2'$
H2′	0	$(-1,1)$	0	$(-5,-5)$	$(-9,-6)$	2	$\sqrt{2}^{15}\varepsilon^{4}$	III, 2	j'_{20}	$1'$
I1	0	1	0	$(-5,-2)$	$(3,2)$	2	$\sqrt{2}^{21}\varepsilon$	III*, 2	j_{19}	2
I2	0	-1	0	$(-1,2)$	$(1,2)$	2	$-\sqrt{2}^{21}\varepsilon^{2}$	III*	j_{20}	1
I1′	0	1	0	$(-5,2)$	$(3,-2)$	2	$\sqrt{2}^{21}\varepsilon^{-1}$	III*, 2	j'_{19}	$2'$
I2′	0	-1	0	$(-1,-2)$	$(1,-2)$	2	$\sqrt{2}^{21}\varepsilon^{-2}$	III*, 2	j'_{20}	$1'$
J1	0	$(0,1)$	0	$(-2,-1)$	$(-2,-2)$	2	$\sqrt{2}^{15}\varepsilon$	III, 2	j_{19}	2
J2	0	$(0,-1)$	0	$(0,1)$	-2	2	$-\sqrt{2}^{15}\varepsilon^{2}$	III, 2	j_{20}	1
J1′	0	$(0,-1)$	0	$(-2,1)$	$(-2,2)$	2	$\sqrt{2}^{15}\varepsilon^{-1}$	III, 2	j'_{19}	$2'$
J2′	0	$(0,1)$	0	$(0,-1)$	-2	2	$\sqrt{2}^{15}\varepsilon^{-2}$	III, 2	j'_{20}	$1'$
K1	0	$(-1,1)$	0	$(-7,4)$	$(-1,1)$	2	$\sqrt{2}^{21}\varepsilon^{-5}$	III*, 2	j_{19}	2
K2	0	$(1,-1)$	0	$(-11,8)$	$(13,-9)$	2	$-\sqrt{2}^{21}\varepsilon^{-4}$	III*, 2	j_{20}	1
K1′	0	$(-1,-1)$	0	$(-7,-4)$	$(-1,-1)$	2	$\sqrt{2}^{21}\varepsilon^{5}$	III*, 2	j'_{19}	$2'$
K2′	0	$(1,1)$	0	$(-11,-8)$	$(13,9)$	2	$\sqrt{2}^{21}\varepsilon^{4}$	III*, 2	j'_{20}	$1'$
L1	0	$(-1,-1)$	0	$(-3,3)$	$(-3,2)$	2	$\sqrt{2}^{15}\varepsilon^{-5}$	III, 2	j_{19}	2
L2	0	$(1,1)$	0	$(-5,5)$	$(9,-6)$	2	$-\sqrt{2}^{15}\varepsilon^{-4}$	III, 2	j_{20}	1
L1′	0	$(-1,1)$	0	$(-3,-3)$	$(-3,-2)$	2	$\sqrt{2}^{15}\varepsilon^{5}$	III, 2	j'_{19}	$2'$
L2′	0	$(1,-1)$	0	$(-5,-5)$	$(9,6)$	2	$\sqrt{2}^{15}\varepsilon^{4}$	III, 2	j'_{20}	$1'$
M1	0	-1	0	$(-5,-2)$	$(-3,-2)$	2	$\sqrt{2}^{21}\varepsilon$	III*, 2	j_{19}	2
M2	0	1	0	$(-1,2)$	$(-1,-2)$	2	$-\sqrt{2}^{21}\varepsilon^{2}$	III*, 2	j_{20}	1
M1′	0	-1	0	$(-5,2)$	$(-3,2)$	2	$\sqrt{2}^{21}\varepsilon^{-1}$	III*, 2	j'_{19}	$2'$
M2′	0	1	0	$(-1,-2)$	$(-1,2)$	2	$\sqrt{2}^{21}\varepsilon^{-2}$	III*, 2	j'_{20}	$1'$
N1	0	$(0,-1)$	0	$(-2,-1)$	$(2,2)$	2	$\sqrt{2}^{15}\varepsilon$	III, 2	j_{19}	2
N2	0	$(0,1)$	0	$(0,1)$	2	2	$-\sqrt{2}^{15}\varepsilon^{2}$	III, 2	j_{20}	1
N1′	0	$(0,1)$	0	$(-2,1)$	$(2,-2)$	2	$\sqrt{2}^{15}\varepsilon^{-1}$	III, 2	j'_{19}	$2'$
N2′	0	$(0,-1)$	0	$(0,-1)$	2	2	$\sqrt{2}^{15}\varepsilon^{-2}$	III, 2	j'_{20}	$1'$
O1	0	$(0,1)$	0	$(-32,23)$	$(34,-24)$	2	$\sqrt{2}^{15}\varepsilon^{-13}$	III, 2	j_{22}	2
O2	0	$(0,1)$	0	$(-32,-23)$	$(-34,-24)$	2	$\sqrt{2}^{15}\varepsilon^{13}$	III, 2	j'_{22}	1
O1′	0	$(0,-1)$	0	$(-32,-23)$	$(34,24)$	2	$\sqrt{2}^{15}\varepsilon^{13}$	III, 2	j'_{22}	$2'$
O2′	0	$(0,-1)$	0	$(-32,23)$	$(-34,24)$	2	$\sqrt{2}^{15}\varepsilon^{-13}$	III, 2	j_{22}	$1'$
P1	0	-1	0	$(-65,46)$	$(-31,22)$	2	$\sqrt{2}^{21}\varepsilon^{-13}$	III*, 2	j_{22}	$1'$
P1′	0	-1	0	$(-65,-46)$	$(-31,-22)$	2	$\sqrt{2}^{21}\varepsilon^{13}$	III*, 2	j'_{22}	1

Q1	0	$(-1,1)$	0	$(-5,3)$	$(3,-2)$	2	$\sqrt{2}^{15}\varepsilon^{-7}$	III, 2	j_{22}	$1'$
Q1′	0	$(-1,-1)$	0	$(-5,-3)$	$(3,2)$	2	$\sqrt{2}^{15}\varepsilon^{7}$	III, 2	j'_{22}	1
R1	0	$(-1,-1)$	0	$(-11,8)$	$(3,-1)$	2	$\sqrt{2}^{21}\varepsilon^{-7}$	III*, 2	j_{22}	2
R2	0	$(1,-1)$	0	$(-11,-8)$	$(-3,-1)$	2	$\sqrt{2}^{21}\varepsilon^{7}$	III*, 2	j'_{22}	1
R1′	0	$(-1,1)$	0	$(-11,-8)$	$(3,1)$	2	$\sqrt{2}^{21}\varepsilon^{7}$	III*, 2	j'_{22}	$2'$
R2′	0	$(1,1)$	0	$(-11,8)$	$(-3,1)$	2	$\sqrt{2}^{21}\varepsilon^{-7}$	III*, 2	j_{22}	$1'$
S1	0	1	0	$(-65,46)$	$(31,-22)$	2	$\sqrt{2}^{21}\varepsilon^{-13}$	III*, 2	j_{22}	$1'$
S1′	0	1	0	$(-65,-46)$	$(31,22)$	2	$\sqrt{2}^{21}\varepsilon^{13}$	III*, 2	j'_{22}	1
T1	0	$(1,-1)$	0	$(-5,3)$	$(-3,2)$	2	$\sqrt{2}^{15}\varepsilon^{-7}$	III, 2	j_{22}	$1'$
T1′	0	$(1,1)$	0	$(-5,-3)$	$(-3,-2)$	2	$\sqrt{2}^{15}\varepsilon^{7}$	III, 2	j'_{22}	1
U1	0	1	0	$(-13,-6)$	$(-21,-18)$	2	$\sqrt{2}^{21}\varepsilon^{-2}$	III*, 2	j_{25}	2
U2	0	-1	0	$(-13,6)$	$(21,-18)$	2	$-\sqrt{2}^{21}\varepsilon^{2}$	III*, 2	j'_{25}	1
U1′	0	1	0	$(-13,6)$	$(-21,18)$	2	$-\sqrt{2}^{21}\varepsilon^{2}$	III*, 2	j'_{25}	$2'$
U2′	0	-1	0	$(-13,-6)$	$(21,18)$	2	$\sqrt{2}^{21}\varepsilon^{-2}$	III*, 2	j_{25}	$1'$
V1	0	$(0,-1)$	0	$(-6,-3)$	$(-6,-2)$	2	$\sqrt{2}^{15}\varepsilon^{-2}$	III, 2	j_{25}	$1'$
V1′	0	$(0,1)$	0	$(-6,3)$	$(-6,2)$	2	$-\sqrt{2}^{15}\varepsilon^{2}$	III, 2	j'_{25}	1
W1	0	$(-1,1)$	0	$(-15,8)$	$(-33,21)$	2	$\sqrt{2}^{21}\varepsilon^{-8}$	III*, 2	j_{25}	$1'$
W1′	0	$(-1,-1)$	0	$(-15,-8)$	$(-33,-21)$	2	$-\sqrt{2}^{21}\varepsilon^{8}$	III*, 2	j'_{25}	1
X1	0	$(1,1)$	0	$(-7,5)$	$(15,-12)$	2	$\sqrt{2}^{15}\varepsilon^{-8}$	III, 2	j_{25}	2
X2	0	$(-1,1)$	0	$(-7,-5)$	$(-15,-12)$	2	$-\sqrt{2}^{15}\varepsilon^{8}$	III, 2	j'_{25}	1
X1′	0	$(1,-1)$	0	$(-7,-5)$	$(15,12)$	2	$-\sqrt{2}^{15}\varepsilon^{8}$	III, 2	j'_{25}	$2'$
X2′	0	$(-1,-1)$	0	$(-7,5)$	$(-15,12)$	2	$\sqrt{2}^{15}\varepsilon^{-8}$	III, 2	j_{25}	$1'$
Y1	0	$(0,1)$	0	$(-6,-3)$	$(6,2)$	2	$\sqrt{2}^{15}\varepsilon^{-2}$	III, 2	j_{25}	$1'$
Y1′	0	$(0,-1)$	0	$(-6,3)$	$(6,-2)$	2	$-\sqrt{2}^{15}\varepsilon^{2}$	III, 2	j'_{25}	1
Z1	0	$(1,-1)$	0	$(-15,8)$	$(33,-21)$	2	$\sqrt{2}^{21}\varepsilon^{-8}$	III*, 2	j_{25}	$1'$
Z1′	0	$(1,1)$	0	$(-15,-8)$	$(33,21)$	2	$-\sqrt{2}^{21}\varepsilon^{8}$	III*, 2	j'_{25}	1
a1	0	1	0	$(-563,-239)$	$(4369,4063)$	2	$\sqrt{2}^{15}\varepsilon^{-7}$	III, 2	j_{26}	2
a2	0	-1	0	$(-563,239)$	$(-4369,4063)$	2	$\sqrt{2}^{15}\varepsilon^{7}$	III, 2	j'_{26}	1
a1′	0	1	0	$(-563,239)$	$(4369,-4063)$	2	$\sqrt{2}^{15}\varepsilon^{7}$	III, 2	j'_{26}	$2'$
a2′	0	-1	0	$(-563,-239)$	$(-4369,-4063)$	2	$\sqrt{2}^{15}\varepsilon^{-7}$	III, 2	j_{26}	$1'$
b1	0	$(0,1)$	0	$(-1126,-478)$	$(16252,8738)$	2	$\sqrt{2}^{21}\varepsilon^{-7}$	III*, 2	j_{26}	$1'$
b1′	0	$(0,-1)$	0	$(-1126,478)$	$(16252,-8738)$	2	$\sqrt{2}^{21}\varepsilon^{7}$	III*, 2	j'_{26}	1
c1	0	$(1,1)$	0	$(-2645,-1843)$	$(71213,50286)$	2	$\sqrt{2}^{15}\varepsilon^{-1}$	III, 2	j_{26}	$1'$
c1′	0	$(1,-1)$	0	$(-2645,1843)$	$(71213,-50286)$	2	$\sqrt{2}^{15}\varepsilon$	III, 2	j'_{26}	1
d1	0	$(-1,1)$	0	$(-5291,-3688)$	$(206435,146113)$	2	$\sqrt{2}^{21}\varepsilon^{-1}$	III*, 2	j_{26}	2
d2	0	$(1,1)$	0	$(-5291,3688)$	$(-206435,146113)$	2	$\sqrt{2}^{21}\varepsilon$	III*, 2	j'_{26}	1
d1′	0	$(-1,-1)$	0	$(-5291,3688)$	$(206435,-146113)$	2	$\sqrt{2}^{21}\varepsilon$	III*, 2	j'_{26}	$2'$
d2′	0	$(1,-1)$	0	$(-5291,-3688)$	$(-206435,-146113)$	2	$\sqrt{2}^{21}\varepsilon^{-1}$	III*, 2	j_{26}	$1'$
e1	0	$(0,-1)$	0	$(-1126,-478)$	$(-16252,-8738)$	2	$\sqrt{2}^{21}\varepsilon^{-7}$	III*, 2	j_{26}	$1'$
e1′	0	$(0,1)$	0	$(-1126,478)$	$(-16252,8738)$	2	$\sqrt{2}^{21}\varepsilon^{7}$	III*, 2	j'_{26}	1
f1	0	$(-1,-1)$	0	$(-2645,-1843)$	$(-71213,-50286)$	2	$\sqrt{2}^{15}\varepsilon^{-1}$	III, 2	j_{26}	$1'$
f1′	0	$(-1,1)$	0	$(-2645,1843)$	$(-71213,50286)$	2	$\sqrt{2}^{15}\varepsilon$	III, 2	j'_{26}	1

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