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GAP PROBLEMS OF MATRIX MONOTONE FUNCTIONS AND MATRIX CONVEX FUNCTIONS AND THEIR APPLICATIONS

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1. Introduction

Let H be a Hilbert space and $B(H)$ be the algebra of all bounded linear operators on H. Let I be an open interval in the real line $\mathbf{R}$ and f be a real valued continuous function defined in I. For a pair of self-adjoint operators a and b on H with their spectra in I we say that the function f is a monotone operator function if $f(a) \leq f(b)$ whenever $a \leq b$. We say that it is a convex operator function if

$$f(\lambda a + (1-\lambda)b) \leq \lambda f(a) + (1-\lambda)f(b).$$

When the space H is infinite dimensional, these kinds of functions are usually called as operator monotone functions (resp. operator convex functions). When H is n-dimensional, that is, $B(H) = M_n$, n by n matrix algebra, they are called as matrix monotone functions of degree n, n-monotone in short (resp. matrix convex functions of degree n, n-convex in short). Denote these classes of functions as $P_\infty(I)$, and $P_n(I)$ (resp. as $K_\infty(I)$ and $K_n(I)$).

These classes of functions were introduced more than 70 years ago, 1934-1937, by Loewner [31] with the works of his two student, Dobsch [9] and Kraus [29]. Since then, the importance of these subjects has been recognized with large variety of applications to quantum mechanics, information theory, electric networks etc. Without regarding these things, however, we can see their importance in the theory of operators as well as in that of operator algebras. In fact, the famous Loewner-Heinz theorem (now developed to the Furuta inequality)

$0 \leq a \leq b$ implies $a^p \leq b^p$ for every $0 \leq p \leq 1$

simply means that the function t^p is an operator monotone function on the positive half line $[0, \infty)$ when p is ranging in the above interval. The function log t is operator monotone on $(0, \infty)$ but its inverse exponential function e^t is not even 2-monotone. The fact that the function tlog t (with assuming 0 as the value at the origin) is operator convex was proved by Umegaki. This result yielded a great impact to the early stage of information theory around the years 1960's.

Now, to start with note first that the classes $\{P_n(I)\}$ and $\{K_n(I)\}$ form naturally decreasing sequences down to the class of $P_\infty(I)$ and to that of $K_\infty(I)$ but the precise proofs of these facts are seldom seen in literature (though they are elementary). Thus, we state here proofs just for reference.

Proposition 1.1. *Let I be an interval, then we have*
(1) $P_\infty(I) = \bigcap P_n(I)$,
(2) $K_\infty(I) = \bigcap K_n(I)$.

Proof. Let $\{\xi_\lambda\}_{\lambda\in\Lambda}$ be a CNOS of the space H and $\{e_\lambda\}$ be the set of one dimensional projections on that CNOS. Consider then the set of finite subsets of Λ with its ordering by inclusions. For each element K of this set we can define the finite rank projection $p_K = \sum_{\lambda\in K} e_\lambda$, which makes a net converging to the identity in the strong operator topology. Take a pair of self-adjoint operators, a, b with their spectra in I and $a \leq b$. There exists then a finite interval $[\alpha, \beta]$ inside I such that $\alpha 1 \leq a, b \leq \beta 1$. Put

$$a_K = p_k a p_k + \alpha(1 - p_K), \quad b_K = p_K b p_K + \alpha(1 - p_K).$$

One may easily verify that a_K and b_K converge to a and b in the strong operator topology. Moreover, here spectrums of a_K and b_K are contained in $[\alpha, \beta]$, hence uniformly bounded. It follows that for any non-negative integer n, a_K^n converges to a^n as well as $b_K^n \to b^n$. Therefore for any polynomial q(t) we see that both $q(a_K)$ and $q(b_K)$ converge to $q(a)$ and $q(b)$.

Let f be a continuous function in I contained in the intersection of $\{P_n(I)\}$. By the Weierstrass approximation theorem the above arguments show first that $f(a_K)$ and $f(b_K)$ converge to $f(a)$ and $f(b)$ in the strong topology. On the other hand, spectral calculus for f leads us that

$$f(a_K) = f(p_K a p_K) + f(\alpha)(1 - p_K), \quad f(b_K) = f(p_K b p_K) + f(\alpha)(1 - p_K).$$

Note that here $f(p_K a p_K)) \leq f(p_K b p_K)$ by the assumption for f regarding those operators are the ones acting on the finite dimensional space $p_K H$. Hence taking limits we have that $f(a) \leq f(b)$, that is, f is operator monotone.

The assertion (2) is proved in a similar way.

We consider the above facts as piling structure of two sequences $\{P_n(I)\}$ and $\{K_n(I)\}$.

Now given an interval I, denote by $C^n(I)$ the class of all n-times continuously differentiable functions on I. In calculus, we then have a decreasing sequence $\{C^n(I)\}$ for which there exists a strict gap for each class $C^n(I)$, that is, we can always find n-times continuously differentiable functions which are not $n+1$-times continuously differentiable. There comes then the class of C^∞-functions and we meet next the class of analytic functions on I. We know then there is an exact gap for such classes. Notice the difference between this situation and the above result, which shows a particularity of non-commutative calculus here. As noticed above we regard those decreasing sequences $\{P_n(I)\}$ and $\{K_n(I)\}$ as non-commutative counter-parts corresponding to that of $\{C^n(I)\}$.

Until now many authors have been discussing about operator monotone functions and operator convex functions. Of course, we see in literatures results for n-monotone functions, notably found in Donoghue's book [12] as well as some results about n-convex functions starting from the paper by Kraus [29]. Actually speaking, however, in the picture as non-commutative calculus the most important basic parts corresponding to the piling structure of $\{C^n(I)\}$ is the problem of gaps between $P_n(I)$ and $P_{n+1}(I)$ as well as $K_n(I)$ and $K_{n+1}(I)$. It is therefore somewhat surprising that for almost seventy years since the time of the introduction of these notions, the most relevant literature had been simply asserting the existence of the gaps for every n without any example for $n \geq 3$.

2. Criteria for n-monotonicity and n-convexity and the local property theorem

To begin with we first introduce the notion of divided differences and regularization process. Let $t_1, t_2, t_3, \ldots$ be a sequence of distinct points. We write those divided differences with respect to a function f as

$$[t_1, t_2]_f = \frac{f(t_1) - f(t_2)}{t_1 - t_2} \quad \text{and inductively,}$$

$$[t_1, t_2, \ldots, t_{n+1}]_f = \frac{[t_1, t_2, \ldots, t_n] - [t_2, t_3, \ldots, t_{n+1}]}{t_1 - t_{n+1}}.$$

When f is sufficiently smooth, we can define

$$[t_1, t_1]_f = f'(t_1), \quad \text{and then inductively such as}$$

$$[t_1, t_1, t_2]_f = \frac{f'(t_1) - [t_1, t_2]}{t_1 - t_2}.$$

When there appears no confusion, we omit the index f. In this way we see that (n+1)-th divided difference $[t_0, t_0, \ldots, t_0]$ is $f^{(n)}(t_0)/n!$, which is nothing but the n-th coefficient of the Taylor expansion of $f(t)$ at the point t_0. An important property of divided differences is that they are permutation free so that one may find another forms of the definition for divided differences of arbitrary orders.

Throughout this note all functions should be continuous real valued but in calculation we often assume that relevant functions are smooth enough. This is because of the following so-called regularization process of those functions. Let $\varphi(t)$ be an even C^∞-function defined on $\mathbf{R}$. We also require that it is positive, supported on the interval $[-1, 1]$ and with the integral being one, that is, a mollifier. Let $f(t)$ be a continuous function on (α, β), then we form its regularization $f_\varepsilon(t)$ for a small positive ε by

$$f_\varepsilon(t) = 1/\varepsilon \int \varphi(\frac{t-s}{\varepsilon}) f(s) ds = \int \varphi(s) f(t - \varepsilon s) ds.$$

The regularization is uniformly continuous on any closed subinterval of (α, β) and converges to f uniformly on such subinterval when ε goes to zero. Moreover $f_\varepsilon(t)$ becomes a C^∞-function, and important points are the facts that when f is monotone or convex at some level (such as n-monotone or n-convex) on the interval, f_ε becomes monotone or convex at the same level on the interval $(\alpha + \varepsilon, \beta - \varepsilon)$.

In the following, we call a function f in the class C^n, written as $f \in C^n$ when it is n-times continuously differentiable.

Now we state the criteria of n-monotone functions. There are two criteria; one global (combinatorial) and the other local. They have, however, a funny history in the theory, somewhat peculiar facts in this field.

Given a function f in the interval I and an n-tuple $\{t_1, t_2, \ldots, t_n\}$ (not necessarily assumed to be distinct) from I the following matrix

$$L_n^f(t_1, t_2, \ldots, t_n) = ([t_i, t_j]_f)_{i,j=1}^n$$

is called the Loewner matrix for a function f. For the reference function f we follow the rule as in divided differences. In the following we often write as L_n^f instead of $L_n^f(t_1, t_2, \ldots, t_n)$.

CriterionI_a. Let f be a class C^1-function on the open interval $I = (\alpha, \beta)$. Then f is n-monotone if and only if for an arbitrary n-tuple $\{t_1, t_2, \ldots, t_n\}$ in I its Loewner matrix is positive semidefinite.

For the proof of this result we just refer [27, Theorem 6.6.36].

Comparing with this criterion the next local criterion for n-monotonicity is quite useful.

CriterionI_b. Let f be a functions in C^{2n-1} on the above open interval I. Then f is n-monotone if and only if the following $n \times n$ Hankel matrix

$$M_n(f;t) = \left(\frac{f^{(i+j-1)}(t)}{(i+j-1)!} \right)$$

is positive semi-definite for every $t \in I$.

In fact, to show that the exponential function e^t is not even 2-monotone one needs a little computation by I_a but if we follow the above criterion the assertion is trivial. These two criterions are now established facts. There appear then serious troubles about the relation between *Criterion*I_a and I_b. Assuming enough smoothness as mentioned above, for the implication from I_a to I_b we make use of the method using the so-called extended Loewner matrix $L_n^{ef}(t_1, t_2, \ldots, t_n)$ defined for an n-tuple $\{t_1, t_2, \ldots, t_n\}$ in I as

$$L_n^{ef}(t_1.t_2. \ldots, t_n) = ([t_1, t_2, \ldots, t_i, t_1, t_2, \ldots, t_j])_{i,j=1}^n.$$

As in the case of Loewner matrix we write often as L_n^{ef} instead of $L_n^{ef}(t_1, t_2, \ldots .t_n)$.

In order to check positive semidefiniteness of these matrices, we use the determinants of principal submatrix, elementary facts known in linear algebra. Namely they are;

(A) An $n \times n$ self-adjoint matrix is positive semidefinite if and only if the determinants of its principal submatrix are all non-negative.

(B) An $n \times n$ self-adjoint matrix is positive definite if and only if the determinants of its leading principal submatrices are all positive.

The conclusion of I_b is obtained by the semidefiniteness of the extended Loewner matrix L_n^{ef} and then by considering the limiting case where all $\{t_i\}$ coincide, but the proof of the semidefiniteness of L_n^{ef} by means of (A) is rather difficult when n is a higher order. Hence we use the assertion (B) instead, which is often computable, by considering small perturbations of the functions $\varepsilon\varphi(t)$ where $L_n^{\varepsilon\varphi}$ is positive definite for every t in I (for instance a non-rational operator monotone function like log function), that is, considering the function $f + \varepsilon\varphi$. Note that here we have

$$L_n^{f+\varepsilon\varphi} = L_n^f + \varepsilon L_n^\varphi \quad \text{and } M_n(f + \varepsilon\varphi; t) = M_n(f; t) + \varepsilon M_n(\varphi; t).$$

To illustrate the idea to make use of the extended Loewner matrix and its determinant we show here the argument for 2 by 2 matrices. We emphasize however that when $n \geq 3$ the arguments are not so simple ; one needs to consider what kinds of steps (subtraction some row by some other row and for columns as well) we should take to reach the final form of the extended matrix.

Proof of the implication $I_a \to I_b$ for $n = 2$. Let t_1, t_2 be a pair of distinct points. Subtract first the second column by the first column and then for the second movement subtract the second row by the first row. We have then

$$detL_2^f = \begin{vmatrix} [t_1, t_1] & [t_1, t_2] \\ [t_2, t_1] & [t_2, t_2] \end{vmatrix} = (t_2 - t_1) \begin{vmatrix} [t_1, t_1] & [t_1, t_1, t_2] \\ [t_1, t_2] & [t_2, t_2, t_1] \end{vmatrix}$$

$$= (t_2 - t_1)^2 \begin{vmatrix} [t_1, t_1] & [t_1, t_1, t_2] \\ [t_1, t_2, t_1] & [t_1, t_2, t_1, t_2] \end{vmatrix} = (t_2 - t_1)^2 detL_2^{ef}.$$

Now if f is 2-monotone, $detL_2^f$ is non-negative, and by the above identity $detL_2^{ef}$ becomes non-negative, too. Moreover $f'(t)$ is non-negative and we have the inequality $f'(t_1)f'(t_2) \geq [t_1, t_2]^2$. Hence we see that

$$[t_1, t_2, t_1, t_2] = \frac{f'(t_1) + f'(t_2) - 2[t_1, t_2]}{(t_1 - t_2)^2} \geq 0.$$

It follows that the matrix L_2^{ef} is positive semidefinite, and considering the limit case $t_1 = t_2$ we get the conclusion I_b for $n = 2$.

The general relation between determinant of the Loewner matrix L_r^f of size r and that of its extended form L_r^{ef} for $\{t_1, t_2, \ldots, t_r\}$ is

$$detL_r^f = \prod_{i>j} (t_i - t_j)^2 detL_r^{ef}.$$

Hence they have the same sign provided that all t_k's are distinct.

For the converse implication we need the following local property theorem.

Local Property theorem. Let (α, β) and (γ, δ) be two overlapping open intervals, where $\alpha < \gamma < \beta < \delta$. Suppose a function f is n-monotone on these intervals, then f is n-monotone on the larger interval (α, δ)

Though its formulation looks very simple, this theorem is very deep and its proof is hard. Never-the-less, to our surprise, Loewner himself said in his paper [31, p.212, Theorem 5.6] that "the proof of this theorem is very easy, hence leave its proof to readers". Further more, when his student Dobsch used this result in [9], he said that the result had been already proved by Loewner. Fortunately, forty years later Donoghue gave a comprehensive proof in his book [12], which amounts almost fifty pages !(together with the theory of interpolation functions of complex variable). There remains however still some ambiguity at the last part of the proof of this theorem [12, Chap.14 Theorem 5], but we can adjust this last part of the proof. Thus, we can now assert that the theorem is an established one but since Donoghue's proof is too long (as a whole) we still look for a simple minded proof of this local property theorem for matrix monotone functions. On the other hand, the local property theorem for matrix convex functions is still far beyond our scope as we explain later.

The reason we need the theorem for the implication from I_b to I_a is the following. We first give the proposition, whose proof is somewhat found in [12].

Proposition 2.1. *Let f be a function in the class $C^{2n-1}(I)$. Suppose there exist an interior point t_0 such that $M_n(t_0; f) > 0$. Then there exists a positive number δ such that f is n-monotone in the subinterval $(t_0 - \delta, t_0 + \delta)$.*

Proof. We note first that, by the fact (B), determinant of each leading principal submatrix of $M_n(t_0; f)$ is positive. It follows by the continuous dependence of matrix entries to points that we can find a small positive δ such that determinants of all leading principal submatrices of the extended Loewner matrix L_n^{ef} are positive in the open interval $(t_0 - \delta, t_0 + \delta)$ inside I. Thus from the general relations between determinants of leading principal submatrix of L_n^f and those of the extended Loewner matrix L_n^{ef} we see that those of leading principal submatrices of L_n^f are positive provided that given n-tuple $\{t_k\}$ consists of distinct points. Thus, here the corresponding Loewner matrices are positive definite by (B). Since the set of such n-tuples is dense in the set of all n-tuples without restrictions, the matrix L_n^f becomes positive semidefinite in this open interval and then by I_a the function f becomes n-monotone in the interval.

Now we are back to the local property theorem and consider a closed interval J inside I and its open covering by the family of the above discussed intervals. We then have a covering of J of finite number. Apply then the local property theorem to conclude that f is n-monotone in J. It follows that f is n-monotone in the interval I.

As for the criterions of n-convexity of functions we are in a similar situation but we have a serious trouble lacking in the local property theorem of convexity ! Thus, we are in the situation as follows.

$CriterionII_a$ Let f be a function in C^2 in the open interval $I = (\alpha, \beta)$. Then f is n-convex if and only if for an arbitrary n-tuple $\{t_1, t_2, \ldots, t_n\}$ in I the Kraus matrix of size n,

$$K_n^f(t_l) = ([t_i, t_j, t_l])_{i,j=1}^n = ([t_i, t_l, t_j])_{i,j=1}^n \quad \text{is positive semidefinite.}$$

Here t_l is fixed where $1 \leq l \leq n$.

An expected local criterion is

$CriterionII_b$. Let f be a function in C^{2n} in the interval I, then f is n-convex if and only if the following Hankel matrix

$$K_n(f;t) = \left(\frac{f^{(i+j)}(t)}{(i+j)!}\right) \quad \text{is positive semidefinite for every } t \in I.$$

For the proof of $CriterionII_a$ we refer [27, Theorem 6.6.52 (1)]. The implication from II_a to II_b has been proved in a similar way as in the case of $CriterionI's$ by [20] and [21]. Since, however, because of the difference of the order of relevant divided differences computations become much more complicated to paraphraze the original determinant into the determinant of the extended Kraus matrix, $K_n^{ef}(s) = ([t_1, \ldots, t_i, s, t_1, \ldots, t_j])_{i,j=1}^n$ similar to L_n^{ef}.

On the other hand, the local property theorem for n-convex functions is proved only in the case $n = 2$ (as we shall see later) and at present we have been unable to prove the theorem even for 3- convex functions. For the moment, all we can say now is the following fact.

Proposition 2.2. *Suppose that $K_n(f;t_0) > 0$ at some point t_0 in I, then there exists a small open subinterval J in I containing t_0 in which the function f is n-convex.*

This is proved along the similar way as the above mentioned proposition for monotone functions through the relations between leading determinants of the Kraus matrix and those of the extended Kraus matrix $K_n^{ef}(s)$ defined above.

We do not invoke here the whole details of the proof of the implication $II_a \to II_b$ in [20] and [21]. The readers are advised to give a proof for the case $n = 2$ as we have shown the case of 2-monotone functions and hopefully to try the case for $n = 3$. General relation between determinant of the Kraus matrix $K_r^f(s)$ of size r and determinant of the extended Kraus matrix $K_r^{ef}(s)$ of the same size r,

$$K_r^{ef}(s) = ([t, \ldots, t_i, s, t_1, \ldots, t_j])_{i,j=1}^r$$

is

$$detK_r^f(s) = \prod_{k=1}^{r-1}\prod_{l=1}^{r-k}(t_{k+l} - t_l)^2 detK_r^{ef}(s).$$

Hence they have the same sign for $r = 2, 3, \ldots, n$ provided that those r-tuples consist of distinct points.

We have to notice here that in order to find a small perturbation $\epsilon\varphi$ to make the matrix $K_n(f + \epsilon\varphi; t)$ positive definite as in the case of $M_n(f; t)$ we use those polynomials found in Theorem 3.1 (2) below (together with their transferred ones into the specified interval here).

We shall prove later the local property of 2-convexity.

The following (old) observation is useful through this note.

Proposition 2.3. *(1) Let f be a function in C^1 and 2-monotone on the interval I. If the derivative f' vanishes at some point t_0, then f becomes a constant function.*

(2) Let f be a function in C^2 and 2-convex on the interval I. If the second derivative f" vanishes at some point t_0, then f is at most a linear function.

For the proof of (1) take an arbitrary point t and consider the Loewner matrix for the pair $\{t_0, t\}$. Then the assumption and nonnegativity of $detL_2^f$ imply that $[t_0, t]_f = 0$. Hence $f(t) = f(t_0)$.

As for the assertion (2), we also consider the pair $\{t_0, t\}$ and the Kraus matrix for this. Then the assumption and nonnegativity of the determinant of the Kraus matrix imply that $[t_0, t, t_0] = 0$, which shows the conclusion.

Therefore, in our discussions we may usually assume, if necessary, that $f'(t) > 0$ for every t in I as well as $f"(t) > 0$.

3. Gaps, truncated moment problems

As we have mentioned before both classes of matrix monotone functions $\{P_n(I)\}$ and matrix convex functions $\{K_n(I)\}$ form decreasing sequences down to the classes $P_\infty(I)$ and $K_\infty(I)$. Therefore, as in the case of the standard classes $C^n(I)$ there appear natural question about the piling structure of these sequences, that is, the existence of gaps between them first. Nowadays, we do not find examples of those functions in $C^n(I) \setminus C^{n+1}(I)$ (in the text books of elementary calculus), but there should have been surely big discussions in old days about these gaps until we have fixed the class C^n. On the contrary, actually so many papers on monotone operator functions have been published since the introduction of this concept by Loewner, and most papers (notably in Donoghue's book [12, p.84]) had asserted the existence of gaps for the sequence $\{P_n(I)\}$ for arbitrary n, but no explicit examples were given for $n \geq 3$ until we provided such examples in [18] (seventy years later after the article [31]!). Moreover, examples presented before as n-monotone functions had been only operator monotone functions (though surely served as examples for them for an arbitrary n) and for the gap between $P_2(I)$ and $P_3(I)$ only one example was known [48].

The authors believe that this way of the assertion without any evidence should be against the principle of Mathematics.

Here we shall provide abundance of examples in the gaps for arbitrary n and so far those polynomials in finite intervals belonging to gaps we shall clarify their structure.

Before going into our discussions we review general aspect of the existence problem for gaps depending on intervals. Let I and J be finite interval in the same forms (open, closed etc). There is then a linear transition function with a positive coefficient for t from I to J and the converse. Since this function together with its inverse is both operator monotone and operator convex, once we find functions belonging to the gap $P_n(I) \setminus P_{n+1}(I)$ for any n those transposed functions on J belong to the gap on J in the same order. Therefore so far finite intervals are concerned we may choose any convenient interval for which we usually employ the interval of the form $[0, \alpha)$. Relations between two infinite intervals are more or less the same. In fact, if they are in the same direction the transferring function is just a shift. When they are in the opposite direction it becomes a combination of a shift and the reflection. Anyway in both cases we can easily transfer gaps of the one interval to those of the other one. Therefore the rest is the case where the one is a finite interval, say $[0, 1)$, and the other is an infinite one, say $[0, \infty)$. For this relation we notice first that the function $\frac{1}{t}$ is known to be operator convex in the interval $(0, \infty)$. Hence the function $\frac{t}{1-t} : [0, 1) \to [0, \infty)$ is both operator monotone and operator convex. The inverse of this function, $\frac{t}{1+t} : [0, \infty) \to [0, 1)$ is also operator monotone but operator concave. It follows that though we can freely transfer gaps for matrix monotone functions each other between arbitrary intervals, we can not treat the case of matrix convex functions in the same way.

These things in mind, the following result solves the problem of the existence of gaps providing abundance of examples belonging to them.

Theorem 3.1. ([20],[39]) *Let I be a finite interval and let n and m be natural numbers with $n \geq 2$.*

(1) If $m \geq 2n-1$, there exists an n-monotone polynomial $p_m : I \to \mathbf{R}$ of degree m,

(2) If $m \geq 2n$ there exists an n-convex and n-monotone polynomials $p_m : I \to \mathbf{R}$ of degree m. Likewise there exists an n-concave and n-monotone polynomial $q_m : I \to \mathbf{R}$ of degree m,

(3) There are no n-monotone polynomials of degree m in I for $m = 2, 3, \ldots, 2n-2$,

(4) There are no n-convex polynomials of degree m in I for $m = 3, 4, \ldots, 2n-1$.

Sketch of the proof.

We first introduce the polynomial p_m of degree m given by

$$p_m(t) = b_1 t + b_2 t^2 + \ldots + b_m t^m,$$

where

$$b_k = \int_0^1 t^{k-1} dt = \frac{1}{k}.$$

Then the ℓth derivative $p_m^{(\ell)}(0) = \ell! b_\ell$ for $\ell = 1, 2, \ldots, 2n-1$, and consequently

$$M_n(p_m; 0) = \left(\frac{p_m^{(i+j-1)}(0)}{(i+j-1)!} \right)_{i,j=1}^n = (b_{i+j-1})_{i,j=1}^n.$$

Now take a vector $c = (c_1, c_2, \ldots, c_n)$ in an n-dimensional Hilbert space, then

$$(M_n(p_m; 0)c|c) = \sum_{i,j=1}^n b_{i+j-1} c_j \bar{c}_i = \int_0^1 \left| \sum_{i=1}^n c_i t^{i-1} \right|^2 dt.$$

From this we can say that the matrix $M_n(p_m; 0)$ is positive definite, and then by the continuity of entries, we can find a positive number α such that $M_n(p_m; t)$ is positive in the interval $[0, \alpha)$. Hence by the criterion I_b the polynomial $p_m(t)$ becomes n-monotone here. This shows the assertion (1).

The first half of the proof of (2) goes in a similar way but use both matrices $M_n(p_m; 0)$ and $K_n(p_m; 0)$. Here besides the calculation for $M_n(p_m; 0)$ as above we have

$$K_n(p_m; 0) = \left(\frac{p_m^{i+j}(0)}{(i+j)!} \right)_{i,j=1}^n = (b_{i+j})_{i,j=1}^n.$$

and

$$(K_n(p_m; 0)c|c) = \sum_{i,j=1}^n b_{i+j} c_j \bar{c}_i = \int_0^1 t \left| \sum_{i=1}^n c_i t^{i-1} \right|^2.$$

Thus, both matrices are positive definite. Hence by Proposition 2.1 and 2.2 (adjusting proofs there for half open intervals) we can find a positive number α such that p_m becomes both n-monotone and n-convex in the interval $[0, \alpha)$.

For the second assertion we consider the polynomial $q_m(t)$ of degree m whose coefficients $\{b_k\}$ are defined as

$$b_k = \int_{-1}^0 t^{k-1} dt = \frac{(-1)^{k-1}}{k}.$$

The corresponding computation for $M_n(q_m; 0)$ shows that it is still positive definite whereas $K_n(q_m; 0)$ becomes negative definite because of the range of the integration. Therefore, by the same reason as above there exists a positive number α such that q_m becomes n-monotone and n-concave in the interval $[0, \alpha)$.

Proof of (3). Let f_m be an n-monotone polynomial of degree m on I with $2 \leq m \leq 2n-2$. We may assume as above that I contains 0 . Write

$$f_m(t) = b_0 + b_1 t + \ldots + b_m t^m \quad \text{where } b_m \neq 0.$$

We have then

$$f_m^{(m-1)}(0) = (m-1)! b_{m-1}, \quad f_m^{(m)}(0) = m! b_m, \quad f_m^{(m+1)}(0) = 0.$$

Consider the matrix $M_n(f_m; 0)$. We have to check two cases where $m = 2k$, even and $m = 2k-1$, odd. Note first that in both cases $k+1 \leq n$. In the first case, the principal submatrix of $M_n(f_m; 0)$ consisting of the rows and columns with numbers k and $k+1$ is given by

$$\begin{pmatrix} b_{m-1} & b_m \\ b_m & 0 \end{pmatrix}$$

and it has determinant $-b_m^2 < 0$. In the latter case, we consider the principal submatrix consisting of rows and columns with numbers $k-1$ and $k+1$ given by

$$\begin{pmatrix} b_{m-2} & b_m \\ b_m & 0 \end{pmatrix}$$

and this matrix also has determinant $-b_m^2 < 0$. Since $M_n(f_m; 0)$ is supposed to be positive semidefinite by I_b we have in both cases a contradiction.

The assertion (4) is proved in a similar way using the matrix $K_n(f_m; 0)$ since we have now the implication $II_a \rightarrow II_b$.

It is to be noticed here that the above arguments also assure the existence of an n-monotone function f as well as an n-convex function g for which $M_n(f; t)$ and $K_n(g; t)$ are positive definite for every t in I (strictly n-monotone and strictly n-convex).

The above theorem provides for a finite interval I abundance of examples of polynomials belonging to the gaps, $P_n(I) \setminus P_{n+1}(I)$ and $K_n(I) \setminus K_{n+1}(I)$ for any natural number n. Namely those polynomials of degrees $2n-1$ and $2n$ constructed in (1) (resp. of degrees $2n$ and $2n+1$ constructed in (2)) are belonging to the gap for monotone functions (resp. for convex functions). Moreover, in the above proof we can replace the Lebesgue measure by another measures but those measure should have relatively fat supports. We do not give here details of this kind of discussions. Readers may however easily realize this situation once they try to use Dirac measures in the above calculation (cf.[39]).

Roughly speaking, those polynomials belonging to gaps have essentially the form described above. On the other hand, we are wondering whether there could be a way to describe how fat is the set of polynomials in the set of $P_n(I)$ and $K_n(I)$ for a finite interval I. In fact, take an analytic operator monotone function f (resp. analytic operator convex function g) in I and suppose that $M_n(f; t_0)$ (resp. $K_n(g; t_0)$) is positive definite. Consider the truncated polynomial $p_f^m(t)$ of degree over $2n-1$ (resp. $q_g^m(t)$ of degree over $2n$) from the Taylor expansions of f and g. We see then that $M_n(f;t_0) = M_n(p_f;t_0)$ (resp.$K_n(g;t_0) = K_n(q_g;t_0)$). Therefore, there exists a positive number δ_n (resp. γ_n) such that p_f (resp. q_g) is n-monotone (resp. n-convex) in the interval $(t_0-\delta_n, t_0+\delta_n)$ (resp. $(t_0-\gamma_n, t_0+\gamma_n)$). Here when degrees of those polynomials p_f^m and q_g^m go to ∞, they naturally converge to f and g respectively, but troubles are the facts that δ_n and γ_n are depending on n and t_0. This way of thinking, however, could give some image about the sets of polynomials inside in $P_n(I)$ and in $K_n(I)$.

Now as an immediate consequence of the theorem we have

Corollary 3.2. *Let I be a non-trivial infinite interval. Then for any natural number n the gap between $P_n(I)$ and $P_{n+1}(I)$ is not empty.*

The reason is clear from the general observation before although with transferring gaps for finite intervals to the functions on I we can no more expect to have polynomials, but rational functions instead.

For gaps of matrix convex functions we need further arguments but finally obtain the following

Proposition 3.3. *Let I be a non-trivial infinite interval. Then for any natural number n the gap between $K_n(I)$ and $K_{n+1}(I)$ is not empty.*

For the proof we need a lemma.

Lemma 3.4. *A non-negative n-concave function f defined in the interval $[0,\infty)$ is necessarily n-monotone.*

Proof. Take a pair of $n\times n$ matrices a, b such that $0 \leq a \leq b$. Then for $0 < \lambda < 1$ we can write as

$$\lambda b = \lambda a + (1-\lambda)\lambda(1-\lambda)^{-1}(b-a).$$

Hence by assumptions,

$$f(\lambda b) \geq \lambda f(a) + (1-\lambda)f(\lambda(1-\lambda)^{-1}(b-a)) \geq \lambda f(a).$$

Taking λ to go to 1, we have that $f(a) \leq f(b)$.

Proof of the Proposition. Assuming that $I = [0,\infty)$ we prove the result in a concave version. Let f be an n-monotone and n-concave polynomial in $[0,1)$ of degree $2n$. By adding a suitable constant we may assume that f is non-negative. The composition function

$$g(t) = f(\frac{t}{1+t}), \qquad t \geq 0$$

is n-concave. Note that by (3) of the theorem f can not be $(n+1)$-monotone and so g can not be $(n+1)$-monotone either. Now suppose g is $(n+1)$-concave, then by the above lemma it becomes $(n+1)$-monotone, a contradiction.

We remark that the transferred function g is no more a polynomial but a rational function. In connection with this the following result shows that on an (non-trivial) infinite interval we seldom have matrix monotone (resp. convex) polynomials.

Proposition 3.5. *Let I be an infinite interval and n a natural number with $n \geq 2$.*
(1) An n-monotone polynomial on I is at most a linear function,
(2) An n-convex polynomial on I is at most a quadratic function.

For the proof we may assume that $I = [0, \infty)$. Besides, we need to make use of the following results, which will be proved in the next section in a complete way. That is, for a natural number $m \geq 2$ the function t^m is not 2-monotone and for $m \geq 3$ the function t^m is not 2-convex.

Now let $p(t)$ be an n-monotone polynomial of order m in I

$$p(t) = c_m t^m + c_{m-1} t^{m-1} + \ldots + c_1 t + c_0$$

and consider those matrices $0 \leq a \leq b$ in M_n. Take a positive number s, then $0 \leq sa \leq sb$. Hence $0 \leq p(sa) \leq p(sb)$, and $p(sa)/s^m \leq p(sb)/s^m$. Therefore, letting s go to infinite we have that $c_m a^m \leq c_m b^m$. Since, here the coefficient c_m is easily seen to be positive from the assumption we see that the function t^m becomes n-monotone. Thus $m \leq 1$.

On the other hand, if $p(t)$ is n-convex a similar argument lead us to conclude that $m \leq 2$.

Finally we discuss problems of successive orders of matrix functions and related results. At first for such problem with respect to matrix monotone functions we introduce a fractional transformation $T(t_0, f)$ with the result of Nayak [36].

Let I be an open interval and take a point t_0 in I. For a function $f \in C^2(I)$ such that $f'(t) > 0$ in I, we consider the transformation $T(t_0, f)$ defined as

$$T(t_0, f)(t) = \frac{[t_0, t_0, t]}{[t_0, t_0][t_0, t]} = -\frac{1}{f(t) - f(t_0)} + \frac{1}{f'(t_0)(t - t_0)} \qquad t \in I.$$

Nayak's result in [36] states then

Theorem 3.6. *The transform $T(t_0, f)$ is in $P_n(I)$ for all t_0 in I if and only if $f \in P_{n+1}(I)$.*

Next for a function $f \in C^3(I)$ such that $f"(t) > 0$ in I, consider the transformation $S(t_0, f)$ defined as

$$S(t_0, f)(t) = \frac{[t_0, t_0, t_0, t]}{[t_0, t_0, t_0][t_0, t_0, t]} \qquad t \in I.$$

The result in [20] is then

Theorem 3.7. *The transform $S(t_0, f)$ is in $P_n(I)$ for all t_0 in I if and only if $f \in K_{n+1}(I)$.*

Here if we consider the function $d_{t_0}(t) = [t_0, t]_f$ we have the relations,

$$[t_0, t]_{d_{t_0}} = [t_0, t_0, t]_f, \qquad [t_0, t_0, t]_{d_{t_0}} = [t_0, t_0, t_0, t]_f.$$

Hence we obtain

$$S(t_0, f) = T(t_0, d_{t_0}).$$

But the above Nayak's result is not directly applicable here since the function d_{t_0} depends on t_0.

The original function f is realized by these transformations in each of the following way.

$$f(t) = f(t_0) - \frac{1}{T(t_0, f)(t) - \frac{1}{f'(t_0)(t-t_0)}},$$

and

$$f(t) = f(t_0) + f'(t_0)(t - t_0) - \frac{t - t_0}{S(t_0, f)(t) - \frac{1}{[t_0,t_0,t_0]_f(t-t_0)}}$$

As combinatorial problems we may also think about the versions, convex - convex and convex - monotone. The second author feels that to find the formulation of the first combination would be meaningful in some sense but the second combination of successive orders seems to be not so worth-while because convexity is more complicated than monotonicity for matrix functions.

The above two results both indicate the next classes stepped up, that is, $P_{n+1}(I)$ and $K_{n+1}(I)$ by the behavior of proceeding class of n - monotone functions.

4. Characterizations of two convex functions

As we have mentioned already, a big difference between the theory of matrix monotone functions and that of matrix convex functions is at the point that for matrix convex functions we have not obtained the local property theorem yet, and hence the (expected) criterion II_b is not fully available. In this section, we shall prove the local property theorem for two convex functions, and naturally establish both global and local characterizations of them. Unfortunately, the theorem is still far beyond our present scope even in the case of 3 by 3 matrices.

We first look back old results for two monotone functions ([12, p.74]). Let f be a non-constant two monotone function in an open interval I and assume that $f \in C^3(I)$. We know then the matrix $M_2(f; t)$ is positive semidefinite in I. In this case we have a further characterization of f. That is,

Proposition 4.1. *With the above assumptions, f is 2-monotone if and only if*

$$f'(t) = 1/c(t)^2$$

where $c(t)$ is positive concave function in I.

Thus, essentially a 2-monotone function has the form of an indefinite integral.

We can prove this result directly but the following inequality provides more strong tool for our discussions.

We recall first the expansion formulae of divided differences, which are known since the old time of Hermite.

Proposition 4.2. *Divided differences can be expanded by iterated integrals in the following form;*

$$[t_0, t_1]_f = \int_0^1 f'((1-s_1)t_0 + s_1 t_1)ds_1,$$

$$[t_0, t_1, t_2]_f = \int_0^1 \int_0^{s_1} f"((1-s_1)t_0 + (s_1 - s_2)t_1 + s_2 t_2)ds_2 ds_1,$$

$$\vdots$$

$$\begin{aligned}[t_0, t_1, \ldots, t_n]_f &= \int_0^1 \int_0^{s_1} \ldots \int_0^{s_{n-1}} f^{(n)}((1-s_1)t_0 + (s_1 - s_2)t_1 \\ &+ \ldots + (s_{n-1} - s_n)t_{n-1} + s_n t_n)ds_n \ldots ds_1,\end{aligned}$$

where $f \in C^n(I)$ for an open interval I, and $t_0, t_1, \ldots, t_n$ are (not necessarily distinct) points in I.

By using these formulas, we obtain the inequality.

Proposition 4.3. *Let I be an open interval and n a natural number. For a function $f \in C^n(I)$, assume that its n-th derivative $f^{(n)}$ is strictly positive. If in addition the function*

$$c(t) = 1/f^{(n)}(t)^{1/(n+1)}$$

is convex, then we have the inequality for the divided difference

$$[t_0, t_1, \ldots, t_n]_f \geq \prod_{i=0}^{n} [t_i, t_i, \ldots, t_i]_f^{1/(n+1)}$$

for arbitrary $t_, t_1, \ldots, t_n$ in I, where the divided differences are of order n (that is, for $n+1$-tuples).

If the function $c(t)$ is concave, then the inequality is reversed.

The inequality for a very special case of the exponential function, that is, the case $c(t) = 1/\exp^{(n)}(t)^{1/(n+1)} = \exp(-t/(n+1))$ is found in literature since the above function is apparently convex.

For applications of this inequality we need

Proposition 4.4. *Let I be an open interval and f be the function in the class $C^{n+2}(I)$ for which $f^{(n)}(t)$ is strictly positive in I. Then the function $c(t)$ in the above setting becomes concave if and only if the following matrix*

$$\begin{pmatrix} f^{(n)}(t)/n! & f^{(n+1)}(t)/(n+1)! \\ f^{(n+1)}(t)/(n+1)! & f^{(n+2)}(t)/(n+2)! \end{pmatrix}.$$

is positive semidefinite.

From this observation, we see that when $n = 1$ the concavity of $c(t)$ is equivalent to the positive semidefiniteness of the matrix $M_2(f : t)$ and when $n = 2$ it is equivalent to the positive semidefiniteness of the matrix $K_2(f : t)$. Thus the first case concerns with 2-monotonicity of f through the criterion I_b , whereas the second case concerns with 2-convexity of f by the next characterization theorem of a 2-convex function. In case when $n \geq 3$ the above matrix is neither a principal submatrix of $M_n(f : t)$ nor of $K_n(f : t)$.

Theorem 4.5. ([20]) *Let I be an open interval, and take a function $f \in C^4(I)$ such that $f"(t) > 0$ for every $t \in I$. The following assertions are equivalent.*

(1) f is 2-convex,

(2) Determinant of the Kraus matrix K_2^f is non-negative for any pair $\{t_0, t_1\}$ in I,

(3) The matrix $K_2(f; t)$ is positive semidefinite for every $t \in I$,

(4) There exists a positive concave function $c(t)$ in I such that $f"(t) = c(t)^{-3}$ for every $t \in I$,

(5) The inequality

$$[t_0, t_0, t_0][t_1, t_1, t_1] - [t_0, t_1, t_1][t_0, t_0, t_1] \geq 0$$

is valid for all $t_0, t_1 \in I$.

An immediate consequence of this theorem is that at least for a function f satisfying the assumption of the theorem the local property theorem holds. Hence by the regularization process and by Proposition 2.3 (2) we can see the following result.

" Two convexity has the local property".

For the proof of the theorem, the equivalence of (1) and (2) is essentially contained in the criterion II_a, and the implication (2) to (3) is included in the implication of II_a to II_b. In connection with the local property, a main point of this theorem is to show the implication (3) $\rightarrow$ (2). This will be done through the way, (3) $\rightarrow$ (4) $\rightarrow$ (5) $\rightarrow$ (2). We have, however, already mentioned the equivalency of (3) and (4). The implication (4) $\rightarrow$ (5) is derived from the above inequality.

In the literature of concerning operator monotone functions and operator convex functions one usually assumes that the relevant interval I should be non-trivial. The reason of this fact is used to be explained by appealing to deep results of integral representations of those functions. The following observation shows, however, that 2-monotonicity and 2-convexity are simply at the turning points of these behavior. Namely we have

Proposition 4.6. *A 2-monotone function defined in the whole real line* $\mathbf{R}$ *must be constant. The same is true for a 2-convex function except the trivial case of a non-constant linear function.*

In fact, a positive concave function defined in $\mathbf{R}$ must be constant because of its geometrical picture, hence f is. On the other hand if there exists a point t_0 on which f' or f" vanishes, then by Proposition 2.3 f must be constant on $\mathbf{R}$ or a linear function in case f being 2-convex. Assumptions on the differentiability of f is absorbed in the regularity process.

We provide here further evidence to show that 2-monotonicity and 2-convexity are turning properties towards operator monotonicity and operator convexity. (See also the next section.) This is the property of our basic function t^p for a general exponent p on the positive half-line. It is well known that the function is operator monotone on $[0, \infty)$ if and only if $0 \le p \le 1$, which is nothing but the Lowener-Heinz theorem. On the other hand it becomes operator convex if and only if either $1 \le p \le 2$ or $-1 \le p \le 0$ ($t > 0$ in the latter case). The next result shows that these conditions are required already at the level two.

Proposition 4.7. *Consider the function*

$$f(t) = t^p \qquad t \in I$$

defined in any subinterval of the positive half-line. Then f is 2-monotone if and only if $0 \le p \le 1$, and it is 2-convex if and only if either $1 \le p \le 2$ or $-1 \le p \le 0$.

Proof. For the monotonicity, there is nothing to prove if f is at most linear, so we may assume that $p \neq 0$ and $p \neq 1$. Suppose f be 2-monotone, then by Proposition 4.1 we can write

$$f'(t) = 1/c(t)^2 \qquad t \in I,$$

where $c(t)$ is a positive concave function. Since $f'(t) > 0$ we see that $p > 0$, hence $c(t)$ is concave only for $0 < p \le 1$. One may alternatively consider the determinant

$$detM_2(f;t) = \begin{vmatrix} pt^{p-1} & \frac{p(p-1)t^{p-2}}{2} \\ \frac{p(p-1)t^{p-2}}{2} & \frac{p(p-1)(p-2)t^{p-3}}{6} \end{vmatrix} = -\frac{p^2(p-1)(p+1)t^{2p-4}}{12}$$

and note that the above matrix is positive semidefinite for $0 \le p \le 1$.

As for the convexity, the second derivative is written by Theorem 4.5 (4) on the form (use the same notation $c(t)$)

$$f''(t) = p(p-1)t^{p-2} = 1/c(t)^3.$$

Hence $c(t) = (p(p-1))^{-1/3}t^{(2-p)/3}$, and this function is concave only for $-1 \le p \le 0$ or $1 \le p \le 2$. One can also make use of positive semidefiniteness of the matrix $K_2(f;t)$ as in the above computation.

Recall that a C^∞ real function $f(t)$ on the half-axis $t > 0$ is said to be completely monotone if

$$(-1)^n f^{(n)}(t) \ge 0 \qquad \text{for all } n \ge 0.$$

The completely monotone functions are characterized by the theorems of Bernstein, one of which states that [12, p.13-14]

" If $f(t)$ is completely monotone, then it is the restriction to the positive half-axis of a function analytic in the right half-plain".

It is then proved (cf.[12, p.86-87]) that

" If $f(t)$ is an operator monotone function on the half-axis, then $f'(t)$ becomes completely monotone."

This result is used to give a proof of the Loewner's theorem.

Now by using the function $c(t)$ for 2-monotonicity and 2-convexity, we can sharpen the above result in the following way.

Theorem 4.8. ([21]) *Consider a function f defined in an interval of the form (α, ∞) for some real α,*

(1) If f is n-monotone and in the class C^{2n-1}, then

$$(-1)^k f^{(k+1)}(t) \ge 0 \qquad k = 0, 1, \ldots, 2n-2.$$

Therefore, the function f and its even derivatives up to order $2n-4$ are concave functions, and the odd derivatives up to order $2n-3$ are convex functions.

(2) If f is n-convex and in the class C^{2n}, then

$$(-1)^k f^{(k+2)}(t) \ge 0 \qquad k = 0, 1, \ldots, 2n-2.$$

Therefore, the function f and its even derivatives up to order $2n-2$ are convex functions, and the odd derivatives up to order $2n-3$ are concave functions.

5. On the class of 2-monotone functions

Recall that a two monotone function is continuously differentiable. We then ([44]) consider a subset of $P_2(I)$ defined as

$$K_2 = \{f \in P_2(I): \quad f(0) = 0, f'(0) = 1\}.$$

Here we note the well known fact that if a function f is 2-monotone and its derivative vanishes at some point in I, then it becomes constant ([9], [20, Lemma 1.1 (1)]). Hence if f is not such a trivial function

$f'(t) > 0$ for every $t \in I$. Thus the set K_2 determines completely the set $P_2(I)$. We shall show that this set is compact with respect to the pointwise convergence topology. Hansen and Pedersen [19] noticed first the importance of the set K_∞ defined in the set of operator monotone functions on I, that is,

$$K_\infty = \{f \in P_\infty(I) : \quad f(0) = 0, f'(0) = 1\}.$$

Thus the above assertion means that we can already define the corresponding compact subset for the class of 2-monotone functions.

We first point out the following fact, which is usually recognized as the result for operator monotone functions [6, Lemma 3.5].

Proposition 5.1. *Let $f(t)$ be a 2-monotone function on I, then for all real numbers λ with $-1 \le \lambda \le 1$, the function $g_\lambda(t) = (t+\lambda)f(t)$ becomes convex.*

The proof of this fact is obtained by a careful check of the proof of [6, Lemma 3.5].

The next lemma assures that Lemma 4.1 in [19] still holds for a 2-monotone function.

Lemma 5.2. *If $f \in K_2$, then*

$$f(t) \le t(1-t)^{-1} \quad \text{for } t \ge 0, \quad f(t) \ge t(1+t)^{-1} \quad \text{for } t \le 0$$

For the proof of this lemma the only point we have to notice is the fact that even if $f(t)$ is 2-monotone we get the assertion that those functions $(t \pm 1)f(t)$ become convex by the above proposition. Moreover since a 2-monotone function is continuously differentiable, we can follow the original proof of [19, Lemma 4.1] as word by word and reach the conclusion.

Now we can state the theorem in this section

Theorem 5.3. *The set K_2 is a compact subset of $P_2(I)$ with respect to the pointwise convergence topology.*

As we mentioned before every function f in $P_2(I)$ is expressed as $f(t) = \lambda g(t) + f(0)$ for some function g in K_2 and $\lambda > 0$. Indeed, set $\lambda = f'(0)$ and $g(t) = \dfrac{1}{\lambda}(f(t) - f(0))$. Then $g(0) = 0$ and $g'(0) = 1$. Hence $g \in K_2$. Then we can write $f(t) = \lambda g(t) + f(0)$. The same thing happens for $P_n(I)$. Therefore, we have the following corollary.

Corollary 5.4. *Put $K_n = \{f \in P_n(I) \mid f(0) = 0, f'(0) = 1\}$, then K_n determines $P_n(I)$ completely and the sequence $\{K_n\}$ forms a (strictly) decreasing compact subsets in $\{P_n(I)\}$.*

We next consider the results of the work by M. S. Moslehian, H. Najafi and M. Uchiyama [35]. In their paper they have proved the result that if $f(t)$ is an odd operator monotone function on I, then it is convex on the interval $(0,1)$, hence necessarily concave on $(-1,0)$. We shall show that this fact is also essentially true for a 2-monotone function on I. For the proof, once we obtain Theorem 5.3, the rest is rather straightforward.

Theorem 5.5. *Let f be a three times continuously differentiable odd 2-monotone function on I, then f is convex on $(0,1)$ and concave on $(-1,0)$.*

6. Double piling structure of matrix monotone functions and matrix convex functions

So far we have been discussing piling structures for matrix monotone functions and matrix convex functions in a separate way. There are however mixed pictures of those pilings as illustrated in the following well known result in [19] with respects to the old Jensen's inequality. We recall first the original Jensen's inequality and its operator theoretic version due to F.Hansen [15]. Let f be a convex continuous real function on an interval I. We have

$$f(\sum_1^n \lambda_i t_i) \leq \sum_1^n \lambda_i f(t_i)$$

for any convex combination $\{\lambda_i\}$ and points $\{t_i\}$ in I. And, if f is operator convex in the interval $[0,\infty)$ with $f(0)=0$ we have that

$$f(a^\star x a) \leq a^\star f(x) a$$

for any positive operator x and a contraction a. In fact, the following choice of a positive matrix x and a contraction matrix a shows that the latter leads the former. That is,

$$x = \begin{pmatrix} t_1 & 0 & 0 & \dots & 0 \\ 0 & t_2 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & t_n \end{pmatrix}, \qquad a = \begin{pmatrix} \sqrt{\lambda_1} & 0 & \dots & 0 \\ \sqrt{\lambda_2} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ \sqrt{\lambda_n} & 0 & \dots & 0 \end{pmatrix}.$$

Now the above mentioned result is the following theorem, in which all operators are bounded.

Theorem 6.1. ([19]). *Let $0 < \alpha \leq \infty$ and let f be a real valued continuous function in $I = [0,\alpha)$. Then the following assertions are equivalent.*

(1) f is operator convex and $f(0) \leq 0$,

(2) For an operator a with its spectrum in $[0,\alpha)$ and a contraction c , the inequality $f(c^\star ac) \leq c^\star f(a)c$ holds,

(3) For two operators a and b whose spectra are in $[0,\alpha)$ and others two c and d such that $c^\star c + d^\star d \le 1$, we have the inequality, $f(c^\star ac + d^\star bd) \le c^\star f(a)c + d^\star f(b)d$,

(4)For an operator a with the same spectrum condition and for any projection p, the inequality $f(pap) \le pf(a)p$ holds,

(5) The function $f(t)/t$ in $(0,\alpha)$ is operator monotone on the interval $(0,\alpha)$.

In order to show the structure of the above equivalencies we use the notation $(A)_m \prec (B)_n$, by which we mean that if the assertion (A) holds on the matrix algebra M_m the assertion (B) holds on M_n. A standard proof of the equivalencies then goes as follows:

$$(1)_{2n} \prec (2)_n \prec (5)_n \prec (4)_n, \quad (2)_{2n} \prec (3)_n \prec (4)_n, \text{and } (4)_{2n} \prec (1)_n.$$

We regard the theorem as a consequence of the seesaw game between piling structures of $\{P_n([0,\alpha))\}$ and $\{K_n([0,\alpha))\}$, that is, an aspect of bipiling structure. Thus it is quite natural to look for each implication at the fixed level n. The investigation of this bipiling structure seems however to go a long way, and we first concentrate the relationships between those assertions (1), (2) and (5) at the level n. Thus we consider the following three assertions.

(i) f is n-convex and $f(0) \le 0$.

(ii) For a positive semidefinite n-matrix a with the spectrum in $[0,\alpha)$ and a contraction matrix c, the inequality $f(c^\star ac) \le c^\star f(a)c$ holds.

(iii) The function $f(t)/t$ is n-monotone on the interval $(0,\alpha)$.

For each $n \in \mathbf{N}$, we have the following results for these assertions.

Theorem 6.2. ([41])
a) The assertions $(ii)_n$ and $(iii)_n$ are equivalent,
b) $(i)_n \prec (ii)_{n-1}$.

For the proof of a) the implication $(ii)_n \to (iii)_n$ is known as we mentioned before as $(2) \prec (5)$. Hence we show the converse. Let a be positive semidefinite matrix with its spectrum in $[0,\alpha)$ and c be a contraction in M_n. We may assume that a is invertible. Take a positive number ε. We have then

$$a^{1/2}(cc^\star + \varepsilon)a^{1/2} \le (1+\varepsilon)a.$$

This implies the inequality

$$\frac{f(a^{1/2}(cc^\star + \varepsilon)a^{1/2})}{a^{1/2}(cc^\star + \varepsilon)a^{1/2}} \le \frac{f((1+\varepsilon)a)}{(1+\varepsilon)a}.$$

Hence producing the element $a^{1/2}(cc^\star + \epsilon)a^{1/2}$ from both sides and letting ε go to zero we get the inequality

$$a^{1/2}(cc^\star)a^{1/2}f(a^{1/2}cc^\star a^{1/2}) \le a^{1/2}cc^\star f(a)cc^\star a^{1/2}.$$

Note that here we have the identity,

$$c^\star a^{1/2}f(a^{1/2}cc^\star a^{1/2}) = f(c^\star ac)c^\star a^{1/2}$$

due to the general equality

$$xf(x^\star x) = f(xx^\star)x.$$

Therefore the above inequality leads to the form,

$$a^{1/2}cf(c^\star ac)c^\star a^{1/2} \le a^{1/2}cc^\star f(a)cc^\star a^{1/2}.$$

It follows that

$$cf(c^\star ac)c^\star \le cc^\star f(a)cc^\star.$$

Thus taking a vector ξ in the underlying space H_n we have

$$(f(c^\star ac)c^\star\xi, c^\star\xi) \le ((c^\star f(a)c)c^\star\xi, c^\star\xi).$$

Now consider the orthogonal decomposition of H_n such that $H_n = [\text{Range } c^\star] \oplus [\text{Ker } c]$ and write $\xi = \xi_1 + \xi_2$. Then

$$\begin{array}{rcl}
(f(c^\star ac)\xi, \xi) & = & (f(c^\star ac)\xi_1 + f(0)\xi_2, \xi_1 + \xi_2) \\
& = & (f(c^\star ac)\xi_1, \xi_1) + (f(c^\star ac)\xi_1, \xi_2) + f(0)\|\xi_2\|^2 \\
& = & (f(c^\star ac)\xi_1, \xi_1) + f(0)\|\xi_2\|^2 \\
& \le & (f(c^\star ac)\xi_1, \xi_1) \\
& \le & (c^\star f(a)c\xi, \xi_1) = (c^\star f(a)c\xi, \xi).
\end{array}$$

Thus, the inequality $f(c^\star ac) \le c^\star f(a)c$ holds.

In the above computation, we have used the fact that $f(0) \le 0$, which is derived from the monotonicity of $g(t)$. For, from the assumption we have the inequality $f(t) \le \frac{tf(t_0)}{t_0}$ for every $0 < t \le t_0$ and we obtain the condition $f(0) \le 0$.

We skip the proof of the assertion b). Actually we can shorten the difference between (i) and (ii) at most one, but we still have not completely clarified their relations yet except $n = 2$. When $n = 1$, apparently the assertion (i) implies (ii) but the converse does not hold. In fact, the function $f(t) = -t^3 + 2t^2 - t$ gives a counter-example for this converse at the interval $[0, 1)$. Moreover, there are many examples of 2-convex polynomials in $[0, \alpha)$ with $f(0) \le 0$ satisfying the assertion (ii) (as well as (iii)) but we do not know in this case whether or not (i) implies (ii) in general.

In particular, if we consider $\text{Gap}(n) = \min\{k - n \mid (\text{iii})_k \to (\text{i})_n\}$, we do not know whether the set $\{\text{Gap}(n)\}_{n\in\mathbf{N}}$ is bounded or not.

To discuss the precise argument for the assertion we introduce the another class $Q(I)$ of functions on an interval $I \subset \mathbf{R}$. Let I be an interval of the real line and $n \in \mathbf{N}$ with $n \ge 2$. The class $Q_n(I)$ is defined as the class of all real C^1 functions f on I such that for each $\lambda_1, \lambda_2, \ldots, \lambda_n \in I$ the corresponding Loewner matrix $L_f = ([\lambda_i, \lambda_j]_f)$ is an almost positive matrix. Here an $n \times n$ Hermitian matrix A is said to be *almost positive* if

$$(x \mid Ax) \geq 0$$

for all $x \in H^n = \{x = (x_1, \ldots, x_n) \in \mathbf{C}^n \mid \sum_{i=1}^n x_i = 0\}$.

Note that $P_n(I) \subset Q_n(I)$ for each $n \in \mathbf{N}$. Since for each $n \in \mathbf{N}$ and an interval (a, b) there is an example of a n-monotone and n-convex polynomial on (a, b) ([20, Proposition 1.3]), we know that $Q_n(a, b) \cap K_n(a, b) \neq \emptyset$.

Since a real C^1 function f belongs to $Q_2(a, b)$ if and only if the derivative f' is convex on (a, b) ([12, XV Lemma 3]), we know, then, that

(1) If f is 2-convex, $f' \in Q_2(a, b)$,
(2) $e^t \in Q_2(a, b) \backslash \{f' \mid f \in K_2(a, b) \cap P_2(a, b)\}$.

Moreover, the following basic observations about $Q_2(a, b)$ holds:

Proposition 6.3. *Suppose that $f(t)$ is 2-convex in $[0, \alpha)$ with $f(0) \leq 0$. Let $g(t) = \frac{f(t)}{t}$. Then the function $g(t)$ belongs to the class $Q_2(0, \alpha)$.*

Now besides those equivalencies mentioned above there are other problems of equivalencies, which we also have to investigate on the level of matrix monotone functions as well as matrix convex functions. For the moment, we have been making some progress ([26]). We illustrate such assertions,

(6) When $\alpha = \infty$ and $f(t) \leq 0$ all the way, the above five assertions are equivalent to the assertion that $-f$ is operator monotone,

(7) This is equivalent to the assertion that when $f \geq 0$ operator monotonicity of f is equivalent to the operator concavity.

If a continuous function f is positive on $(0, \infty)$ the following facts are known ([19]) about the next four assertions.

(a) f is operator monotone,
(b) $t/f(t)$ is operator monotone,
(c) f is operator concave,
(d) $1/f(t)$ is operator convex.

Then the first three are equivalent whereas they imply the last assertion (d).

7. Monotone operator functions and convex operator functions on C*- algebras

In this section we shall briefly sketch the results in [38] and in [40]. Let A be a C*-algebra and I the positive half-line $[0, \infty)$. We consider the class of all real continuous functions defined in I which are monotone (resp. convex) on the algebra A as the sets $P_A(I)$ and $K_A(I)$. Note that the C*-algebra A is located in a corner of $B(H)$ for a (presumably infinite dimensional) Hilbert space H. In this sense concepts of matrix monotone functions and operator monotone functions (resp. matrix convex functions and operator convex functions) should be regarded as the classes following full scaling of the order of operators, whereas the classes $P_A(I)$ and $K_A(I)$ mean to consider the classes following a local scaling with respect to the order of an algebra A in a corner of $B(H)$. Thus, the first main problem here is the question whether these classes yield another classes of functions out of $P_n(I)$ and $P_\infty(I)$ (resp. $K_n(I)$ and $K_\infty(I)$). We call those function A-monotone and A- convex respectively. We shall show that these are not the case, namely $P_A(I)$ coincides either with one of those $P_n(I)$'s or with $P_\infty(I)$ (resp. those $K_n(I)$'s or $K_\infty(I)$). We show the precise conditions when $P_A(I) = P_n(I)$ for some n (resp. $K_A(I) = K_n(I)$ for some n) or $P_A(I) = P_\infty(I)$ (resp. $K_A(I) = K_\infty(I)$). In the following theorem we do not specify the C*-algebra A to be unital or nonunital.

We recall here that a C*-algebra is said to be n-homogeneous if every irreducible representation of the algebra is n-dimensional. A C*algebra is said to be n-subhomogeneous if the highest dimension among all its irreducible representations is n.

Theorem 7.1. (1) *$P_A(I) = P_\infty(I)$ if and only if either the set of dimensions of finite dimensional irreducible representations of A is unbounded or A has an infinite dimensional irreducible representation. The condition for $K_A(I) = K_\infty(I)$ is the same.*

(2) *$P_A(I) = P_n(I)$ for some positive integer n if and only if A is n-subhomogeneous. The condition for $K_A(I) = K_n(I)$ for some n is the same.*

Proofs of the theorem are based on the following lemma.

Lemma 7.2. (i) *If A has an irreducible representation of dimension n, then any A- monotone (resp. A - convex) function becomes n-monotone (resp. n-convex), that is, $P_A(I) \subseteq P_n(I)$ (resp. $K_A(I) \subseteq K_n(I)$).*

(ii) *If $dim\pi \leq n$ for any irreducible representation π of A, then $P_n(I) \leq P_A(I)$.*

(iii) *If the set of dimensions of finite dimensional irreducible representations of A is unbounded, then every A-monotone (resp. A - convex) function is operator monotone (resp. operator convex), that is, $P_A(I) = P_\infty(I)$ (resp. $K_A(I) = K_\infty(I)$).*

(iv) *If A has an infinite dimensional irreducible representation, then $P_A(I) = P_\infty(I)$ and $K_A(I) = K_\infty(I)$.*

A key point of the first three assertions is just lifting of monotonicity and convexity in M_n, the image of an n-dimensional irreducible representation π. Note that a pair of self-adjoint matrices $\{a, b\}$ in M_n with $a \leq b$ can be lifted up to A as a pair of self-adjoint operators, $\{c, d\}$

such that $c \leq d$. Besides the function calculus for a function f and the operation π commute, that is, $\pi(f(a)) = f(\pi(a))$. For the proof of the assertion (iv), we apply Kadison's transitivity theorem in a form stated in Takesaki's book [49, Theorem 4.8] to an infinite dimensional irreducible representation and reduce the problem to the case of finite dimensional irreducible representations of some C*-subalgebras of A, and obtain the conclusions.

We remark that there exists a C*-algebra which has an irreducible representation in an arbitrary high dimension but which does not have an infinite dimensional irreducible representation, thus showing a strict difference between the assertions (iii) and (iv). The $c_0 - sum$ of those matrix algebras $\{M_n, n = 1, 2, \ldots\}$ serves a such example. Furthermore according to the structure of irreducible representations we can provide many examples of C*-algebras serving each condition.

Note also that a C*-algebra is commutative if and only if it has only one dimensional irreducible representations.

It would be worth-while to state a little history in connection with this fact (we omit its detailed references). In 1955, Ogasawara proved that if $0 \leq a \leq b$ implies $a^2 \leq b^2$ in A then A is commutative. Pedersen in his book gives the extended version that if the assumption implies $a^p \leq b^p$ for some $p > 1$ then A becomes commutative. Furthermore in 1998 Wu showed that if the exponential function e^t is A - monotone then A is commutative. The readers here may now easily realize that this is the problem of A - monotonicity of the functions t^p and e^t whose standard monotonicity has been already discussed (Proposition 4.7 and other remark that e^t is not 2-monotone). Thus along the line discussed here we can characterize the commutativity of a C*-algebra A in terms of the conditions for A- monotone functions, where the result covers all previous results (cf. [42]).

The next result shows that there exist appropriate C*-subalgebras in A corresponding to each situation of its irreducible representations.

Theorem 7.3. (1) *If A has an n-dimensional irreducible representation, then for any positive integer $m \leq n$ there exists an m-homogeneous C*-subalgebra.*

(2) *If A has an infinite dimensional irreducible representation π, then*

(2*a*) *for any positive integer m there exists an m-homogeneous C*-subalgebra.*

(2*b*) *There exists an ∞ - homogeneous C*-subalgebra if and only if A is not residually finite-dimensional.*

Here we call A residually finite-dimensional if A has sufficiently many finite dimensional irreducible representations. By an ∞- homogeneous C-algebra we mean a C*-algebra having only infinite dimensional irreducible representations.*

(3) *If the set of dimensions of finite dimensional irreducible representations is unbounded, then for any positive integer m there exists an m-homogeneous C^*-subalgebra.*

8. Appplications

In this section we introduce several applications of previous discussions, which are contained in [25], [23], [24], [26], [22], [45].

8.1. **Generalized Powers-Størmer inequality.** Powers-Størmer inequality (see, for example, [47, Lemma 2.4], [46, Theorem 11.19]) asserts that for $s \in [0, 1]$ the following inequality

$$2\,\mathrm{Tr}(A^s B^{1-s}) \geq \mathrm{Tr}(A + B - |A - B|) \tag{1}$$

holds for any pair of positive matrices A, B. This is a key inequality to prove the upper bound of Chernoff bound, in quantum hypothesis testing theory [5]. This inequality was first proven in [5], using an integral representation of the function t^s. After that, N. Ozawa gave a much simpler proof for the same inequality, using fact that for $s \in [0, 1]$ function $f(t) = t^s$ $(t \in [0, +\infty))$ is an operator monotone. Recently, Y. Ogata in [37] extended this inequality to standard von Neumann algebras. The motivation of this paper is that if the function $f(t) = t^s$ is replaced by another operator monotone function, then $\mathrm{Tr}(A+B-|A-B|)$ may get smaller upper bound that is used in quantum hypothesis testing. Based on N. Ozawa's proof we formulate Powers-Størmer's inequality for an arbitrary operator monotone function on $[0, +\infty)$ in the context of general C^*-algebras.

The following comes from the standard argument.

Lemma 8.1. *Let f be a $2n$-monotone, continuous function on $[0, \infty)$ such that $f((0, \infty)) \subset (0, \infty)$, and let g be a Borel function on $[0, \infty)$ defined by $g(t) = \begin{cases} \frac{t}{f(t)} & (t \in (0, \infty)) \\ 0 & (t = 0) \end{cases}$. Then for any pair of positive matrices $A, B \in M_n$ with $A \leq B$, $g(A) \leq g(B)$*

Theorem 8.2. *Let Tr be a canonical trace on M_n and f be a $2n$-monotone function on $[0, \infty)$ such that $f((0, \infty)) \subset (0, \infty)$. Then for any pair of positive matrices $A, B \in M_n$*

$$\mathrm{Tr}(A) + \mathrm{Tr}(B) - \mathrm{Tr}(|A - B|) \leq 2\,\mathrm{Tr}(f(A)^{\frac{1}{2}} g(B) f(A)^{\frac{1}{2}}), \tag{2}$$

where $g(t) = \begin{cases} \frac{t}{f(t)} & (t \in (0, \infty)) \\ 0 & (t = 0) \end{cases}$.

Proof. Let A, B be any positive matrices in M_n.

For operator $(A - B)$ let us denote by $P = (A - B)^+$ and $Q = (A - B)^-$ its positive and negative part, respectively. Then we have

$$A - B = P - Q \quad \text{and} \quad |A - B| = P + Q, \tag{3}$$

from that it follows that

$$A + Q = B + P. \tag{4}$$

On account of (4) the inequality (9) is equivalent to the following

$$\mathrm{Tr}(A) - \mathrm{Tr}(f(A)^{\frac{1}{2}} g(B) f(A)^{\frac{1}{2}}) \le \mathrm{Tr}(P).$$

Since $B + P \ge B \ge 0$ and $B + P = A + Q \ge A \ge 0$, we have $g(A) \le g(B + P)$ by Lemma 8.1 and

$$\begin{aligned}
&\mathrm{Tr}(A) - \mathrm{Tr}(f(A)^{\frac{1}{2}} g(B) f(A)^{\frac{1}{2}}) \\
&\quad = \mathrm{Tr}(f(A)^{\frac{1}{2}} g(A) f(A)^{\frac{1}{2}}) - \mathrm{Tr}(f(A)^{\frac{1}{2}} g(B) f(A)^{\frac{1}{2}}) \\
&\quad \le \mathrm{Tr}(f(A)^{\frac{1}{2}} g(B+P) f(A)^{\frac{1}{2}}) - \mathrm{Tr}(f(A)^{\frac{1}{2}} g(B) f(A)^{\frac{1}{2}}) \\
&\quad = \mathrm{Tr}(f(A)^{\frac{1}{2}} (g(B+P) - g(B)) f(A)^{\frac{1}{2}}) \\
&\quad \le \mathrm{Tr}(f(B+P)^{\frac{1}{2}} (g(B+P) - g(B)) f(B+P)^{\frac{1}{2}}) \\
&\quad = \mathrm{Tr}(f(B+P)^{\frac{1}{2}} g(B+P) f(B+P)^{\frac{1}{2}}) \\
&\quad - \mathrm{Tr}(f(B+P)^{\frac{1}{2}} g(B) f(B+P)^{\frac{1}{2}}) \\
&\quad \le \mathrm{Tr}(B+P) - \mathrm{Tr}(f(B)^{\frac{1}{2}} g(B) f(B)^{\frac{1}{2}}) \\
&\quad = \mathrm{Tr}(B+P) - \mathrm{Tr}(B) \\
&\quad = \mathrm{Tr}(P).
\end{aligned}$$

Hence, we have the conclusion.

□

Remark 8.3. (i) *When given positive matrices A, B in M_n satisfies the condition $A \le B$, the inequality (9) becomes*

$$\mathrm{Tr}(A) \le \mathrm{Tr}(f(A)^{\frac{1}{2}} g(B) f(A)^{\frac{1}{2}}).$$

(ii) *The 2-monotonicity of f is needed to guarantee the inequality (9). Indeed, let $f(t) = t^3$ and $n = 1$. Then, for any $a, b \in (0, \infty)$, the inequality (9) would imply*

$$a \le f(a)^{\frac{1}{2}} g(b) f(a)^{\frac{1}{2}},$$

that is,

$$\frac{a}{f(a)} \le \frac{b}{f(b)}.$$

Since $\frac{t}{f(t)}$ is, however, not 1-monotone, the latter inequality is impossible.

Remark 8.4. *If we replace the n-convexity in Theorem 6.2(i) by the n-concavity, we can get the same assertions. Therefore, we know that Theorem 8.11 also holds under the condition that f is a $(n+1)$-concave function on $[0,\infty)$* ([26, Thorem 2.2]).

Remark 8.5. *For matrices $A, B \in M_n^+$ let us denote*

$$Q(A,B) = \min_{s\in[0,1]} \operatorname{Tr}(A^{(1-s)/2}B^sA^{(1-s)/2}) \tag{5}$$

and

$$Q_{\mathcal{F}_{2n}}(A,B) = \inf_{f\in\mathcal{F}_{2n}} \operatorname{Tr}(f(A)^{\frac{1}{2}}g(B)f(A)^{\frac{1}{2}}), \tag{6}$$

where $\mathcal{F}_{2n}$ is the set of all $2n$-monotone functions on $[0,+\infty)$ satisfy condition of the Theorem 8.11 and $g(t) = tf(t)^{-1}$ $(t \in [0,+\infty))$.

Note that the function $f(t) = t^s$ $(t \in [0,+\infty))$ satisfies the conditions of Theorem 8.11. Since the class of $2n$-monotone functions is large enough [39]*, we know that $Q_{\mathcal{F}_{2n}}(A,B) \le Q(A,B)$. Hence, we hope on finding another $2n$-monotone function f on $[0,+\infty)$ such that*

$$\operatorname{Tr}(f(A)^{\frac{1}{2}}g(B)f(A)^{\frac{1}{2}}) < Q(A,B). \tag{7}$$

If we can find such a function, then we may get smaller upper bound than what is used in quantum hypothesis testing [5]*. For example, considering the trace distance $T(A,B) = \dfrac{\operatorname{Tr}(|A-B|)}{2}$, we might have the following better estimate*

$$\frac{1}{2}\operatorname{Tr}(A+B) - Q_{\mathcal{F}_{2n}}(A,B) \le T(A,B) \le \sqrt{\{\frac{1}{2}\operatorname{Tr}(A+B)\}^2 - Q_{\mathcal{F}_{2n}}(A,B)^2}.$$

(*See the estimate* (6) *in* [5].)

8.2. **Interpolation functions.** A function $f\colon \mathbf{R}_+ \to \mathbf{R}_+$ is called an *interpolation function of order n* ([1]) if for any $T, A \in M_n$ with $A > 0$ and $T^*T \le 1$

$$T^*AT \le A \quad \Longrightarrow \quad T^*f(A)T \le f(A).$$

We denote by $\mathcal{C}_n$ the class of all interpolation functions of order n on $\mathbf{R}_+$.

Let $\mathcal{P}(\mathbf{R}_+)$ be the set of all Pick functions on $\mathbf{R}_+$, and $\mathcal{P}'$ the set of all positive Pick functions on $\mathbf{R}_+$, i.e., functions of the form

$$h(s) = \int_{[0,\infty]} \frac{(1+t)s}{s+t} d\rho(t), \quad s > 0,$$

where ρ is some positive Radon measure on $[0,\infty]$.

Denote by $\mathcal{P}'_n$ the set of all strictly positive n-monotone functions on $(0,\infty)$. Let us recall a well-known characterization of functions in $\mathcal{C}_n$ that actually is due to Ameur [2] and Ameur, Kaijser, and Sergei [4] (see also [12]).

The following useful characterization of a function in $\mathcal{C}_n$ is due to Donoghue (see [11], [10]), and to Ameur (see [1]).

Theorem 8.6. ([4, Corollary 2.4]) *A function $f\colon \mathbf{R}_+ \to \mathbf{R}_+$ belongs to $\mathcal{C}_n$ if and only if for every n-set $\{\lambda_i\}_{i=1}^n \subset \mathbf{R}_+$ there exists a function h from $\mathcal{P}'$ such that $f(\lambda_i) = h(\lambda_i)$ for $i = 1, \dots, n$.*

As a consequence, there is a 'local' integral representation of every function in $\mathcal{C}_n$ as follows.

Corollary 8.7. *Let A be a positive definite matrix in M_n and $f \in \mathcal{C}_n$. Then there exists a positive Radon measure ρ on $[0,\infty]$ such that*

$$f(A) = \int_{[0,\infty]} A(1+s)(A+s)^{-1} d\rho(s).$$

Remark 8.8. *The following properties can be found in* [1], [2],[3], [13], [31] *or* [4], *:*

(i) $P' = \cap_{n=1}^{\infty} P'_n$, $P' = \cap_{n=1}^{\infty} \mathcal{C}_n$;
(ii) $\mathcal{C}_{n+1} \subseteq \mathcal{C}_n$;
(iii) $P'_{n+1} \subseteq \mathcal{C}_{2n+1} \subseteq \mathcal{C}_{2n} \subseteq P'_n$, $P'_n \subsetneq \mathcal{C}_n$
(iv) $\mathcal{C}_{2n} \subsetneq P'_n$ [42];
(v) *A function $f\colon \mathbf{R}_+ \to \mathbf{R}_+$ belongs to $\mathcal{C}_n$ if and only if $\dfrac{t}{f(t)}$ belongs to $\mathcal{C}_n$* [4, Proposition 3.5].

8.3. **Petz trace inequlity.** Applying Theorem 8.6 we give the following generalized Petz trace inequality.

Theorem 8.9. *Let $f \in \mathcal{C}_{2n}$. For positive definite matrices K and L in M_n, let Q the projection onto the range of $(K-L)_+$. We have, then,*

$$\operatorname{Tr}(QL(f(K) - f(L))) \geq 0. \tag{8}$$

Corollary 8.10. *Let $f \in \mathcal{P}'_{n+1}$. For positive definite matrices K and L in M_n, let Q be the projection onto the range of $(K-L)_+$. We have, then,*

$$\operatorname{Tr}(QL(f(K) - f(L))) \geq 0.$$

Proof. It is suffices to mention that $\mathcal{P}'_{n+1} \subset \mathcal{C}_{2n}$ by Remark 8.8. The conclusion follows from Theorem 8.9. ∎

Using Theorem 8.9 we get a generalized Powers-Størmer type inequality.

Theorem 8.11. *Let f be a function from $(0,\infty)$ into itself such that $tf(t) \in \mathcal{C}_{2n}$. Then for any pair of positive definite matrices $A, B \in M_n$,*

$$\mathrm{Tr}(A^2) + \mathrm{Tr}(B^2) - \mathrm{Tr}(|A^2 - B^2|) \leq 2\,\mathrm{Tr}(Af(A)^{\frac{1}{2}}g(B)f(A)^{\frac{1}{2}}), \tag{9}$$

where $g(t) = \dfrac{t}{f(t)}$, $t \in (0,\infty)$.

Corollary 8.12 ([5])**.** *Let A, B be positive definite matrices, then for all $0 \leq s \leq 1$*

$$\mathrm{Tr}(A + B - |A - B|) \leq 2\,\mathrm{Tr}(A^{1-s}B^s).$$

8.4. **Matrix means.** In the paper [34] Kubo and Ando developed an axiomatic theory of operator means. This theory has found a number of applications in operator theory and quantum information theory.

Restricting the definition of operator means from [34] on the set of positive matrices of order n, we can consider *matrix means* of positive matrices of order n.

Definition 8.13. *A binary operation σ on M_n^+, $(A,B) \mapsto A\sigma B$ is called a matrix connection of order n (or n-connection) if it satisfies the following properties:*

(I) *$A \leq C$ and $B \leq D$ imply $A\sigma B \leq C\sigma D$.*
(II) *$C(A\sigma B)C \leq (CAC)\sigma(CBC)$.*
(III) *$A_n \downarrow A$ and $B_n \downarrow B$ imply $A_n\sigma B_n \downarrow A\sigma B$*

where $A_n \downarrow A$ means that $A_1 \geq A_2 \geq \dots$ and A_n converges strongly to A.

A mean is a normalized connection, i.e. $1\sigma 1 = 1$. An operator connection means a connection of every order. A n-semi-connection is a binary operation on M_n^+ satisfying the conditions (II) and (III).

In [34], there is an affine order-isomorphism from the set of connections onto the set of operator monotone functions. In this section, we describe the similar relation between the connections of order n and $\mathcal{C}_n \supsetneq \mathcal{C}_{2n}$.

Applying the representation in Corollary 8.7, we give a 'local' integral formula for a connection of order n corresponding to a n-monotone function on $(0,\infty)$ (hence, an interpolation function of order n) via the formula

$$A\sigma B = A^{\frac{1}{2}}f(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}}.$$

Furthermore, this 'local' formula also establishes, for each interpolation function f of order $2n$, a connection σ of order n corresponding to the given interpolation function f.

Therefore, it shows that the map σ from the n-connections to the interpolation functions of order n, defined by $\Sigma(\sigma) = f_\sigma$, is injective with the range containing the interpolation functions of order $2n$.

Theorem 8.14. *For any natural number n there is an injective map Σ from the set of matrix connections of order n to $P_n^{'} \supset \mathcal{C}_{2n}$ associating each connection σ to the function f_σ such that $f_\sigma(t)I_n = I_n\sigma(tI_n)$ for $t > 0$. Furthermore, the range of this map contains $\mathcal{C}_{2n}$.*

Via the usual embedding of M_n into M_{n+1}, it is straightforward to check that the classes of connections of order n is decreasing. It is natural to ask the following question: *Is there a matrix mean σ_n of the order n on M_n such that σ_n is not of order $n+1$?*

The following observation gives partially affirmative data to the above question.

Proposition 8.15.

(1) *For any $n \geq 2$ there is a matrix mean σ_n of order n which is not of order $n+2$.*
(2) *There is a matrix mean σ_1 of order 1 which is not of order 2.*

Remark 8.16. *From the second proof of Proposition 8.15, we highlight the inclusion: For each natural number n,*

$$\mathcal{C}_{2(n+1)} \subseteq \Sigma_{n+1} \subseteq P_{n+1}^{'} \subseteq \mathcal{C}_{2n+1} \subseteq \mathcal{C}_{2n} \subseteq \Sigma_n \subseteq P_n^{'}.$$

As the same in [34], we can recall some notations and properties of connections as follows. Let σ be a n-connection. The *transpose* σ', the *adjoint* σ^* and the *dual* $\sigma^\perp$ of σ are defined by

$$A\sigma' B = B\sigma A, \quad A\sigma^* B = (A^{-1}\sigma B^{-1})^{-1}, \quad \sigma^\perp = \sigma'^*.$$

A connection is called *symmetric* if it equals to its transpose. Denoted by Σ_n^{sym} the set of n-monotone representing functions of symmetric n-connections, i.e., Σ_n^{sym} is the image of the set of all symmetric n-connections via the canonical map in Theorem 8.14.

Then, using the same argument as in [34], we can state the following properties for any n-connection:

(1) $\sigma + \sigma'$ and $\sigma(:)\sigma'$ are symmetric.
(2) $\omega_l(\sigma)\omega_r = \sigma$; $\omega_r(\sigma)\omega_l = \sigma'$, where $A\omega_l B = A$ and $A\omega_r B = B$.
(3) The n-monotone representing function of the n-connection $\sigma(\tau)\rho$ is $f(x)g[h(x)/f(x)]$, where f, g, h are the representing functions of σ, τ, ρ in Theorem 8.14, respectively.
(4) σ is symmetric if and only if its n-monotone representing function f is *symmetric*, that is, $f(x) = xf(x^{-1})$.

Proposition 8.17. *Let $f(x), g(x), h(x)$ belong to Σ_n. Then the following statements hold true:*

(i) $k(x) = xf(x^{-1})$, $f^*(x) = f(x^{-1})^{-1}$, $\frac{x}{f(x)}$, $f(x)g[h(x)/f(x)]$, $af(x) + bg(x)$ *all belong to* Σ_n;

(ii) $f(x)+k(x), \frac{f(x)k(x)}{f(x)+k(x)}$ *all belong to* Σ_n^{sym}.

Proof. By the hypothesis, there are n-connections σ, τ, ρ such that their representing functions are $f(x), g(x), h(x)$, respectively. Then the statements follow from the the fact that the functions $k(x) = xf(x^{-1})$, $f^*(x) = f(x^{-1})^{-1}$, $\frac{x}{f(x)}$, $af(x) + bg(x)$, $f(x)g[h(x)/f(x)]$, $f(x) + k(x)$, $\frac{f(x)k(x)}{f(x)+k(x)}$ are the representing functions of n-connections σ', σ^*, $\sigma^\perp$, $a\sigma + b\tau$, $\sigma(\tau)\rho$, $\sigma + \sigma'$, $\sigma(:)\sigma'$, respectively. $\square$

Corollary 8.18.

$$\mathcal{C}_{2n} \subseteq \Sigma_n \subsetneq P'_n.$$

But if we restrict our attention to the class of the symmetric, we get the following equality.

Theorem 8.19.

$$\Sigma_n^{sym} \ = \ P_n'^{sym},$$

where $P_n'^{sym}$ *is the set of all symmetric functions in* P'_n.

9. Concluding remarks

Regarding the present subject as non-commutative calculus we are still at the beginning stage of the theory. Besides the development of theory itself, the whole aspects of operator algebras are expanding and are coming to be more and more basic machines in mathematics. Now we meet non-commutative topology, non-commutative geometry and algebraic geometry etc. in which operator algebras are used as tools, not as the research object as in the case of Baum-Connes conjecture. In this point of view, since the calculus is the most classical starting part of analysis, theory of matrix monotone functions and that of matrix convex function are expected to play the basic role as non-commutative calculus as well as the theory of operator monotone functions and that of operator convex functions. Besides the local property theorem for matrix convex functions, there are actually many problems to be left out such as the bipiling structure.

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