High-accuracy representation of propagation properties of hybrid modes in a Bragg fiber based on Bloch theorem in cylindrical coordinates

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We investigate the propagation properties of a Bragg fiber with high-accuracy analytical representation. In this study, electromagnetic waves in the cladding are treated as genuine cylindrical waves, that is, Hankel functions. We apply the Bloch theorem in the cylindrical coordinates to the electromagnetic fields in the periodically stratified cladding structure. Then, effective indices are actually calculated for TE, TM, and hybrid (HE, EH) modes through eigenvalue equations. We show that these results are distinctly close to those by the multilayer division method that gives more accurate solutions for cylindrically symmetric fiber structures than the results by the asymptotic expansion method, even for the lowest mode HE11.

1. INTRODUCTION
The study of photonic crystal fiber (PCF)—optical fiber with a periodic transverse microstructure—has recently become an interesting discussion and development topic among researchers [1]. In PCF, the refractive index is controlled by the design of its internal structure, such as the alignment of air holes. A PCF in which the mode guiding is due to photonic bandgap [2] is referred to as a photonic bandgap fiber (PBF) [3]. The PBF indicates preferable properties to the conventional optical fibers of index guiding. That is, since the hollow core is available in PBF, extremely low-loss propagation is expected, and a wavelength other than the present telecommunication wavelength can be used.

A Bragg fiber proposed by Yeh et al. [4] is composed of a hollow core and concentric periodic cladding, which results in the Bragg diffraction in the cladding to form a photonic bandgap. Even earlier, a few layered model was studied [5]. Since the proposal, various analyses have been presented [6–14] and demonstrations have been reported [15,16].

In theoretical researches, the Bragg fiber has been investigated both numerically and analytically. In numerical approaches, results with a transfer matrix method [7], a multilayer division method [17], a supercell method [19], and a finite-element method [11] were previously proposed. Although these numerical procedures give highly accurate solutions for a cylindrically symmetric fiber structure, they are inadequate for giving physical perspectives. To overcome this defect, analytical approaches are available, and the asymptotic expansion method was often used in previous works [6,9,13,14]. In this analytical method, electromagnetic waves in the cladding were regarded as plane waves except for the factor $1/\sqrt{r}$, while those in the core were strictly treated. In this scheme, the Bloch theorem [18] was also applied to electromagnetic waves in the periodic cladding structure. As a result, the cladding structure can be characterized by the Bloch wavenumber, which simplifies the mathematical representation in the cladding. These numerical and analytical results showed good agreement except for the region of small core radius $r_c$.

The Bloch theorem, which was originally applicable to the infinitely repeated periodic structure in Cartesian coordinates [18], is valid within the asymptotic expansion approximation. However, for small $r_c$, the cylindrical property of electromagnetic fields is significant in the cladding. The original Bloch theorem is no longer appropriate for such situations. To obtain higher analysis accuracy for small $r_c$, the Bloch theorem must be extended to include the cylindrical coordinates. The authors have found a proper representation for the present purpose, that is, method of approximate Bloch theorem in cylindrical coordinates. Let us abbreviate this as ABC method here. We have applied it to analysis of the TE mode of Bragg fiber [19]. Although this ABC method still possesses an approximation, it provides a higher degree of accuracy than the asymptotic expansion method.

In this paper, we extend the above-method [19] to hybrid modes in Bragg fiber. A high-accuracy analytical representation of the propagation properties for Bragg fiber will be derived, and we will actually calculate the propagation constant. The present method is expected to yield a more accurate solution than the asymptotic expansion method even for lowest mode HE11, and we will show the consistency of the present results in adequate degree with those by the multilayer division method [17].

This paper is organized as follows. In Section 2, the electromagnetic fields of the hybrid modes in Bragg fiber are shown with the representation matrix. Section 3 describes the relationships between the amplitude coefficients in the cladding from the viewpoints of electromagnetics and the Bloch theorem. Here, the eigenvalue equation of the cladding is also introduced. In Section 4, the eigenvalue equation of Bragg fiber is obtained. In Section 5, numerical results through the eigenvalue equations are given. In the last section, we conclude this paper with a summary.
2. ELECTROMAGNETIC FIELDS OF HYBRID MODES IN BRAGG FIBER

The Bragg fiber consists of a homogeneous core and periodically stratified cladding made of layers $a$ and $b$ with refractive indices $n_a$, $n_a$, and $n_b$ ($n_a > n_b > n_c$). In Fig. 1, the distribution of the refractive index in the Bragg fiber is sketched. The core radius and the thicknesses of cladding layers $a$ and $b$ are set to $r_c$, $a$, and $b$. We also introduce period $\Lambda = a + b$.

Let us consider the hybrid modes of wavelength $\lambda_0$ propagating to the $z$ direction through this structure. Hybrid modes are represented by electromagnetic fields $H_z$, $E_\theta$, $E_z$, and $H_\theta$. In layer $i (= c, a, b)$,

$$
\begin{pmatrix}
H_z \\
iE_\theta \\
E_z \\
iH_\theta
\end{pmatrix} = U_{i\theta}(\theta)D_i(r) \begin{pmatrix} A_i \\ B_i \\ C_i \\ D_i \end{pmatrix},
$$

where $A_i$, $B_i$, $C_i$, and $D_i$ are the amplitude coefficients in layer $i$. Factor $U_{i\theta} = \exp[i(\omega t - \beta z)]$ gives the temporal and $z$ dependences with propagation constant $\beta$ and angular frequency $\omega$.

The electromagnetic fields have angular dependence by a diagonalized matrix:

$$
\Omega(\theta) = \text{diag}(\sin(\nu \theta + \theta_m), \cos(\nu \theta + \theta_m)),
$$

where $\nu$ and $\theta_m$ are the azimuthal mode number and an arbitrary initial angle. Here, the representation matrix is given by

$$
D_i(r) = \begin{pmatrix}
d_{11}^{(i)} & d_{12}^{(i)} & 0 & 0 \\
d_{11}^{(i)} & d_{22}^{(i)} & d_{23}^{(i)} & d_{24}^{(i)} \\
0 & 0 & d_{33}^{(i)} & d_{34}^{(i)} \\
d_{41}^{(i)} & d_{42}^{(i)} & d_{43}^{(i)} & d_{44}^{(i)}
\end{pmatrix},
$$

where

$$
\begin{align*}
H_z^{(i)} &= d_{11}^{(i)} = H_v^{(i)}(\kappa_i r), & d_{12}^{(i)} &= d_{13}^{(i)} = d_{14}^{(i)} = 0, \\
H_z^{(i)} &= -d_{21}^{(i)} = -\frac{\omega \mu_0}{\kappa_i} H_c^{(i)}(\kappa_i r), & d_{22}^{(i)} &= d_{23}^{(i)} = d_{24}^{(i)} = 0, \\
H_z^{(i)} &= -d_{31}^{(i)} = -\omega \mu_0 \nu_0 H_v^{(i)}(\kappa_i r), & d_{32}^{(i)} &= d_{33}^{(i)} = d_{34}^{(i)} = 0, \\
H_z^{(i)} &= -d_{41}^{(i)} = -\frac{\nu_0}{\kappa_i^2} H_c^{(i)}(\kappa_i r), & d_{42}^{(i)} &= d_{43}^{(i)} = d_{44}^{(i)} = 0.
\end{align*}
$$

and the lateral propagation constant in layer $i$

$$
\kappa_i = \left[\left(n_i \kappa_0\right)^2 - \beta^2\right]^{1/2} \quad (i = c, a, b)
$$

is introduced. In addition, $\kappa_0 = \omega/c$, $c$, and $\mu_0$ denote the wavenumber, the light velocity, and the absolute permeability, respectively, in the vacuum. $Y_i = n_i/\sqrt{\epsilon_0/\mu_0}$ is the characteristic admittance in layer $i$ possessing refractive index $n_i$, and $\epsilon_0$ gives the dielectric permittivity of the vacuum. Hankel functions $H_v^{(i)}(\nu \beta) = \nu \beta + i N_v^{(i)}$ and $H_c^{(i)}(\nu \beta) = \nu \beta - i N_c^{(i)}$ are related to waves propagating inward and outward on a cross section under the definition of spatio-temporal factor $U_{i\theta}$ employed. Here, $J_c$ and $N_c$ indicate the Bessel and Neumann functions, respectively. The prime symbol signifies the derivative with respect to argument $\kappa_i r$.

The elements in the off-diagonal blocks of representation matrix $D_i(r)$, that is, $d_{23}^{(i)}$, $d_{24}^{(i)}$, $d_{41}^{(i)}$, and $d_{42}^{(i)}$, are associated with the coupling process between TE and TM components. For the TE or TM mode with $\nu = 0$, these four elements equal zero, and then representation matrix $D_i(r)$ is separated into two blocks:

$$
D_{i\mid\nu=0} = \begin{pmatrix}
d_{11}^{(i)} & d_{12}^{(i)} \\
d_{21}^{(i)} & d_{22}^{(i)} \\
0 & 0 \\
d_{41}^{(i)} & d_{42}^{(i)}
\end{pmatrix} \oplus \begin{pmatrix}
d_{33}^{(i)} & d_{34}^{(i)} \\
d_{43}^{(i)} & d_{44}^{(i)}
\end{pmatrix} = D_{i\oplus}^{(TE)} \oplus D_{i\oplus}^{(TM)},
$$

where $D_{i\oplus}^{(TE)}$ and $D_{i\oplus}^{(TM)}$ correspond to the representation matrices of TE and TM modes.

In the core region, $B_c = A_c$ and $D_c = C_c$ are necessary for guaranteeing that the electromagnetic fields are finite at the core center ($r = 0$). Accordingly, the electromagnetic fields of core ($0 \leq r \leq r_c$) are reduced to

$$
\begin{pmatrix}
H_z \\
iE_\theta \\
E_z \\
iH_\theta
\end{pmatrix} = U_{ic}(\theta) \begin{pmatrix} A_c \\ B_c \\ C_c \\ D_c \end{pmatrix}.
$$

Note here that the fields in the core are strictly represented without any approximation.
3. ELECTROMAGNETIC FIELDS AND EIGENVALUE EQUATION IN CLADDING WITH PERIODICITY

In the present scheme, electromagnetic fields in the cladding are treated under an approximation in which the factors of $O((\kappa r)^{-1})$ ($i = a, b$) are considered, although the minute terms of $O((\kappa r)^{-2})$ are neglected. Within this approximation, the Bloch theorem in the cylindrical coordinates [19] is available, and the electromagnetic fields are described with Hankel functions, unlike treatment in the asymptotic expansion method [6,9,13,14]. This approximation will be retained throughout this manuscript.

A. Relationship of Amplitude Coefficients of Adjacent Layers

In this section, we associate the amplitude coefficients of electromagnetic fields in adjacent layers with each other from the viewpoint of electromagnetics. At the interface of $m$th cladding layers $a$ and $b$ ($r = r_c + (m-1)\Lambda + a \equiv r_{ma}$), the electromagnetic fields in both sides are continuous; then

$$D_b(r_{ma}) \begin{pmatrix} A_m^a \\ B_m^a \\ C_m^a \\ D_m^a \end{pmatrix} = D_a(r_{ma}) \begin{pmatrix} A_m \\ B_m \\ C_m \\ D_m \end{pmatrix},$$

(8)

where complex amplitude coefficients $A_m^a - D_m^a$ and $A_m^b - D_m^b$ are used in place of $A_a - D_a$ and $A_b - D_b$ in the $m$th layer in Eq. (1). Taking Lommel’s formula [20] into consideration, we have the following relationship between the amplitude coefficients of layers $a$ and $b$:

$$\begin{pmatrix} A_m^a \\ B_m^a \\ C_m^a \\ D_m^a \end{pmatrix} = \mathcal{H}_{ab}(r_{ma}) \begin{pmatrix} A_m \\ B_m \\ C_m \\ D_m \end{pmatrix}.$$

(9)

with

$$\mathcal{H}_{ab}(r_{ma}) = D_b^{-1}(r_{ma})D_a(r_{ma}).$$

(10)

Here, for $i, j = a, b$,

$$[\mathcal{H}_{ij}(r)]_{11} = [\mathcal{H}_{ij}(r)]_{22} = f^{(TE)}_{ij}(r),$$

$$[\mathcal{H}_{ij}(r)]_{12} = [\mathcal{H}_{ij}(r)]_{21} = g^{(TE)}_{ij}(r),$$

$$[\mathcal{H}_{ij}(r)]_{13} = [\mathcal{H}_{ij}(r)]_{24} = f^{(TM)}_{ij}(r),$$

$$[\mathcal{H}_{ij}(r)]_{34} = [\mathcal{H}_{ij}(r)]_{43} = g^{(TM)}_{ij}(r).$$

(11)

Using the determinant of the representation matrix, namely, $\det[D_i(r)] = (4n_i k_i / \pi r^2)^2$, results in

$$\det[\mathcal{H}_{ij}(r)] = \left(\frac{\kappa_j}{\kappa_i}\right)^2 \frac{(\kappa_i n_i)^2}{(\kappa_j n_j)^2}.$$

(15)

This relationship guarantees the energy conservation of the electromagnetic field in each layer of the cladding, which was discussed in [19].

In a similar way, the continuity condition at the interface between $m$th layer $b$ and $(m+1)$th layer $a$ ($r = r_c + m\Lambda \equiv r_{mb}$) is

$$\begin{pmatrix} A_{m+1}^b \\ B_{m+1}^b \\ C_{m+1}^b \\ D_{m+1}^b \end{pmatrix} = \mathcal{H}_{be}(r_{mb}) \begin{pmatrix} A_m^a \\ B_m^a \\ C_m^a \\ D_m^a \end{pmatrix}.$$

(16)

In combining Eqs. (9) and (16), amplitude coefficients in $m$th and $(m+1)$th layers $a$ are related:

$$\begin{pmatrix} A_{m+1}^b \\ B_{m+1}^b \\ C_{m+1}^b \\ D_{m+1}^b \end{pmatrix} = S(r_{mb}; r_{ma}) \begin{pmatrix} A_m \\ B_m \\ C_m \\ D_m \end{pmatrix}.$$

(17)

where

$$f^{(S)}_{ij}(r) = \frac{i \pi \kappa_{ij}}{4} \frac{\kappa_j}{\kappa_i} \left(\kappa_j H^{(1)}_{ij}(k_j r) - H^{(1)}_{ij}(k_j r)H^{(2)}_{ij}(k_j r) - H^{(2)}_{ij}(k_j r)H^{(1)}_{ij}(k_j r)\right).$$

$$g^{(S)}_{ij}(r) = \frac{i \pi \kappa_{ij}}{4} \frac{\kappa_j}{\kappa_i} \left(\kappa_j H^{(1)}_{ij}(k_j r)H^{(2)}_{ij}(k_j r) - H^{(1)}_{ij}(k_j r)H^{(2)}_{ij}(k_j r)\right).$$

$$p_{ij}(r) = -\frac{i \pi \kappa_{ij}}{4 \alpha_0} \left(1 - \frac{\kappa_j^2}{\kappa_i^2}\right) H^{(1)}_{ij}(k_j r)h^{(2)}_{ij}(k_j r).$$

$$q_{ij}(r) = -\frac{i \pi \kappa_{ij}}{4 \alpha_0} \left(1 - \frac{\kappa_j^2}{\kappa_i^2}\right) h^{(1)}_{ij}(k_j r)H^{(1)}_{ij}(k_j r).$$

(12)

for $S = TE, TM$. The superscripts “TE” and “TM” indicate TE and TM components, respectively, in a hybrid mode. When $\nu = 0$, they are reduced to the representations of TE and TM modes, respectively. Hereafter, this notation will consistently be used. Note that $p_{ij}(r)$ and $q_{ij}(r)$ are minute factors that should be kept in the present approximation, while $f^{(S)}_{ij}(r)$ and $g^{(S)}_{ij}(r)$ are of the order of unity.

For the exchange of indices $i$ and $j$, we can see

$$f^{(S)}_{ji}(r) = \left(\frac{\kappa_j}{\kappa_i}\right)^2 \overline{f^{(S)}_{ij}(r)}, \quad g^{(S)}_{ji}(r) = -\left(\frac{\kappa_j}{\kappa_i}\right)^2 \overline{g^{(S)}_{ij}(r)},$$

$$p_{ji}(r) = \left(\frac{\kappa_j}{\kappa_i}\right)^2 p_{ij}(r), \quad q_{ji}(r) = -\left(\frac{\kappa_j}{\kappa_i}\right)^2 q_{ij}(r).$$

(14)
We only study the case of $F(\tau)$ since $G(\tau)$ can be treated in a similar way. Function $F(\tau)$ satisfies Bessel’s differential equation, which corresponds to a scalar wave function and is valid within each cladding layer. When function $\Phi(\tau) \equiv \sqrt{r}F(\tau)$ is introduced, it is reduced to the equation of a harmonic oscillator:

$$\frac{d^2\Phi(\tau)}{d\tau^2} + \kappa^2(\tau)\Phi(\tau) = 0,$$

where terms equal to and smaller than $r^{-2}$ under $(\kappa r)^2 \gg 1$ ($i = a, b$) are discarded. Here, we also assume only the case where $r$ is not so large ($0 < \tau \ll 1$). This assumption is sufficient for practical uses. Considering that the lateral propagation constant is a periodic function $[\tau(\tau + \Lambda) = \kappa(\tau)]$, we can apply the Bloch theorem to function $\Phi(\tau)$. For $F(\tau)$, we have [19]

$$\sqrt{r} + \Lambda F(\tau + \Lambda) = \exp(-i\kappa\Lambda)\sqrt{r}F(\tau),$$

$$F(\tau) = \exp(-i\kappa r)\frac{u(\tau)}{\sqrt{r}},$$

with the aid of periodic function $u(\tau + \Lambda) = u(\tau)$. Here, $\kappa$ is the Bragg wavenumber, which is generally a complex number. A similar relationship holds for $G(\tau)$. We call the above procedure the method of approximate Bloch theorem in cylindrical coordinates, namely, the ABC method.

Let us derive the representation of the Bloch theorem in the Bragg fiber. In the cladding structure, the lateral propagation constant is a periodic function with respect to radial position $r$. Here, we consider the innermost position of $m$th and $(m + 1)$th layers $a$, that is, $r = r_{(m-1)a}$ and $r = r_{ma}$. Substituting $H_z$ in $m$th and $(m + 1)$th layers $a$ to Eq. (22), we obtain the representation of the Bloch theorem for the longitudinal component:

$$H_z^{(m+1)}(r_{ma}) = \exp(-i\kappa\Lambda)\frac{r_{(m-1)a}}{r_{ma}}H_z^{(m)}(r_{(m-1)a}),$$

where superscript $(m)$ means that magnetic field $H_z$ exists in $m$th cladding layer $a$. It is natural that a Bloch wavenumber is associated with a propagation mode in the Bragg fiber. That is, the identical Bloch wavenumber $\kappa$ should be shared among all electromagnetic components in a hybrid mode. Accordingly, an expression for $E_z(\tau)$ is represented by replacing $H_z^{(m)}(\tau)$ by $E_z^{(m)}(\tau)$ in Eq. (25). Using the Maxwell equation in the cylindrical coordinates

$$i\epsilon_\phi = \frac{1}{\kappa^2} \left( \frac{1}{r} \frac{\partial^2 E_z}{\partial \phi^2} - \alpha_\phi \frac{\partial H_z}{\partial \phi} \right), \tag{26}$$

we can approximately derive the second representation of the Bloch theorem corresponding to the lateral component:

$$E_z^{(m+1)}(r_{ma}) = \exp(-i\kappa\Lambda)\frac{r_{(m-1)a}}{r_{ma}}E_z^{(m)}(r_{(m-1)a}). \tag{27}$$

This relationship is formally similar to that of the longitudinal component (25). With an analogous procedure, a representation of $H_\phi(\tau)$ is derived, where $E_z^{(m)}(\tau)$ is replaced by $H_\phi^{(m)}(\tau)$ in Eq. (27).
With amplitude coefficients, the representations of the Bloch theorem are described in the following matrix form:

$$
\sqrt{r_{mB}}D_a(r_{mB})\begin{pmatrix}
A_{m+1} \\
B_{m+1} \\
C_{m+1} \\
D_{m+1}
\end{pmatrix} = \exp(-i\tilde{K}\Lambda)\sqrt{r_{mB}}D_a(r_{mB}) = \exp(-i\tilde{K}\Lambda)\begin{pmatrix}
A_m \\
B_m \\
C_m \\
D_m
\end{pmatrix},
$$

(28)

where $D_a(r)$ is the representation matrix introduced in Eq. (3). Premultiplying both sides by $D_a^{-1}(r_{mB})$, we obtain a sequence between the amplitude coefficients in $n$th and $(m+1)$th layers as follows:

$$
\begin{pmatrix}
A_{m+1} \\
B_{m+1} \\
C_{m+1} \\
D_{m+1}
\end{pmatrix} = \exp(-i\tilde{K}\Lambda)\Theta_{(m+1),m}\begin{pmatrix}
A_m \\
B_m \\
C_m \\
D_m
\end{pmatrix},
$$

(29)

where

$$
\Theta_{m',m} = \sqrt{r_{mB}^{-1}}D_a^{-1}(r_{mB})D_a(r_{mB}).
$$

(30)

The $\Theta$ matrix is immediately found to be $\Theta_{m,m} = 1$ for $m = m'$. From the definition in Eq. (30), we can derive a reduction formula on $\Theta$ matrix:

$$
\Theta_{m',m}^{\prime} = \Theta_{m',m}.
$$

(31)

When $m' = m$, we obtain the inverse element, $\Theta_{m',m'} = \Theta_{m,m}^{-1}$. Under the present approximation, matrix $\Theta_{(m+1),m}$ is reduced to a diagonalized matrix:

$$
\Theta_{(m+1),m} \simeq \text{diag}(\exp(i\kappa_{a}^m), \exp(-i\kappa_{a}^m), \exp(i\kappa_{a}^m), \exp(-i\kappa_{a}^m)),
$$

(32)

where we utilized Lommel's formula $H_0^{(1)}(z)H_1^{(2)}(z) - H_1^{(1)}(z)H_0^{(2)}(z) = -4i/\pi z$ [20] and asymptotic expansion of Hankel functions for large arguments that keeps terms of the order of $(\kappa_{a}^m r)^{-1}$. The $\Theta$ matrix introduced here resembles that in [19], where the relationship between the amplitude coefficients of adjacent layers $a$ was introduced using the longitudinal component $M_a$ only, and appropriate matrix $M_a$ was introduced instead of the present $D_a^{-1}(r_{mB})$. Present $\Theta$ matrix and that in [19] are consistent with each other within the present approximation.

C. Eigenvalue Equation of Components in Cladding Region

In the preceding sections, we obtained the sequences between the amplitude coefficients from the viewpoints of electromagnetics and the Bloch theorem. In this section, let us equate these two sequences. From Eqs. (17) and (29), one obtains

$$
S(r_{mB}; r_{mA})\begin{pmatrix}
A_m \\
B_m \\
C_m \\
D_m
\end{pmatrix} = \exp(-i\tilde{K}\Lambda)\Theta_{(m+1),m}\begin{pmatrix}
A_m \\
B_m \\
C_m \\
D_m
\end{pmatrix}.
$$

(33)

Premultiplying both sides by $\Theta_{(m+1),m}^{-1}$, we have an eigenvalue equation in the cladding:

$$
\tilde{S}(r_{mB}; r_{mA})\begin{pmatrix}
A_m \\
B_m \\
C_m \\
D_m
\end{pmatrix} = \exp(-i\tilde{K}\Lambda)\begin{pmatrix}
A_m \\
B_m \\
C_m \\
D_m
\end{pmatrix}.
$$

(34)

where we set

$$
\tilde{S}(r_{mB}; r_{mA}) \equiv \Theta_{(m+1),m}^{-1}S(r_{mB}; r_{mA})
$$

and

$$
\tilde{X}_m^{(S)} = \exp(-i\kappa_a^m)X_m^{(S)} \quad \text{and} \quad \tilde{Y}_m^{(S)} = \exp(-i\kappa_a^m)Y_m^{(S)} \quad (S = \text{TE}, \text{TM}).
$$

The expressions of elements in the off-diagonal blocks, that is, $\tilde{s}$, $\tilde{t}$, $\tilde{u}$, and $\tilde{v}$, are abbreviated because they are minute terms constructed by the multiplication of $\tilde{s}$, $\tilde{t}$, $\tilde{u}$, and $\tilde{v}$ are negligible in accordance with the present approximation. Since $\det(\Theta_{(m+1),m}) = 1$ from the definition in Eq. (30), $\det(\tilde{S}(r_{mB}; r_{mA})) = 1$ also holds strictly. On the other hand, relationship $|\tilde{X}_m^{(S)}|^2 - |\tilde{Y}_m^{(S)}|^2 \approx 1$ is valid approximately.

To have a nontrivial solution for eigenvalue Eq. (34), $\det(\tilde{S}(r_{mB}; r_{mA})) = 0$ is necessary, where $\tilde{\eta} = \exp(-i\tilde{K}\Lambda)$. When minute terms constructed by the multiplication of $\tilde{s}$, $\tilde{t}$, $\tilde{u}$, and $\tilde{v}$ are neglected in accordance with the present approximation, we obtain the following relationship:

$$
[\tilde{\eta}^2 - 2\text{Re}(\tilde{X}_m^{(S)})\tilde{\eta} + (|\tilde{X}_m^{(S)}|^2 - |\tilde{Y}_m^{(S)}|^2)]|\tilde{\eta}|^2 - 2\text{Re}(\tilde{X}_m^{(S)})\tilde{\eta} + (|\tilde{X}_m^{(S)}|^2 - |\tilde{Y}_m^{(S)}|^2) = 0,
$$

(35)

which is separated into two parts: one for the parameters of the TE component, and the other of the TM component. The form of Eq. (36) is similar to that of the asymptotic expansion method [14]. It follows that Eq. (36) results in four different eigenvalues:

$$
\tilde{\eta}_j^{(S)} = \exp(-i\tilde{X}_j^{(S)}) \quad \text{where} \quad S = \text{TE or TM}, \quad \text{and} \quad j = 1(2)
$$

(37)

where $S = \text{TE}$ or TM, and index $j = 1(2)$ indicates the positive (negative) sign in the double sign notation. Strictly speaking,
these eigenvalues, that is, the Bloch wavenumbers, depend on layer indication \( m \) through its component \( \text{Re}(\tilde{X}_m^{(S)}) \). However, as referred to below in Section 5 (Fig. 2), its dependence on \( m \) is not so practically marked. Then we can adopt values \( K_j \) as Bloch wavenumbers over the entire cladding region. This assumption is consistent with the fact that the corresponding value is independent of \( m \) in the asymptotic expansion method [14].

The eigenvector for each eigenvalue \( \tilde{\eta}_j^{(S)} \) is calculated by Eqs. (34) and (37), and four different eigenvalues are obtained mathematically. Note that either eigenvalue \( \tilde{\eta}_1^{(S)} \) or \( \tilde{\eta}_2^{(S)} \) is valid for the propagation mode, because one of electromagnetic fields corresponding to them diverges, and the other converges to zero at the infinity [13]. This means that only one of eigenvectors corresponding to either eigenvalue can simultaneously exist in the cladding. As a result, the superposition of two eigenvectors forms an electromagnetic field in the cladding of Bragg fiber:

\[
\begin{pmatrix}
A_m \\
B_m \\
C_m \\
D_m
\end{pmatrix}
= \tilde{\alpha}_m^{(S)}
\begin{pmatrix}
P_{m,j}^{(TE)} \\
Q_{m,j}^{(TE)} \\
R_{m,j}^{(TE)} \\
S_{m,j}^{(TE)}
\end{pmatrix}
+ \tilde{\beta}_m^{(S)}
\begin{pmatrix}
P_{m,j}^{(TM)} \\
Q_{m,j}^{(TM)} \\
R_{m,j}^{(TM)} \\
S_{m,j}^{(TM)}
\end{pmatrix},
\]  

(38)

where \( \tilde{\alpha}_m^{(S)} \) gives the weight of the corresponding vector, and

\[
\begin{align*}
\tilde{P}_{m,j}^{(TE)} &= \tilde{Y}_m^{(TE)}[(\tilde{X}_m^{(TE)} - \tilde{\eta}_j^{(TE)})(\tilde{X}_m^{(TM)})^{-1} - (\tilde{Y}_m^{(TM)})(\tilde{X}_m^{(TM)})^{-1}] \\
\tilde{Q}_{m,j}^{(TE)} &= (\tilde{\eta}_j^{(TE)} - \tilde{X}_m^{(TE)})(\tilde{X}_m^{(TM)})^{-1} - (\tilde{Y}_m^{(TM)})(\tilde{X}_m^{(TM)})^{-1} \\
\tilde{R}_{m,j}^{(TE)} &= \tilde{Y}_m^{(TE)}[\tilde{\tau}(\tilde{X}_m^{(TM)})^{-1} - \tilde{\eta}_j^{(TE)}] + \tilde{\nu}(\tilde{Y}_m^{(TM)}) \\
\tilde{S}_{m,j}^{(TE)} &= \tilde{Y}_m^{(TE)}[\tilde{\nu}(\tilde{X}_m^{(TM)})^{-1} - \tilde{\eta}_j^{(TE)}] + \tilde{\tau}(\tilde{Y}_m^{(TM)}) \\
\tilde{P}_{m,j}^{(TM)} &= [-\tilde{\eta}_j^{(TE)}(\tilde{X}_m^{(TM)})^{-1} + \tilde{\eta}_j^{(TE)}] - [\tilde{Y}_m^{(TM)}]^{-1} \\
\tilde{Q}_{m,j}^{(TM)} &= -[\tilde{\eta}_j^{(TE)}(\tilde{X}_m^{(TM)})^{-1} - \tilde{\eta}_j^{(TE)}] + [\tilde{Y}_m^{(TM)}]^{-1} \\
\tilde{R}_{m,j}^{(TM)} &= [\tilde{\tau}(\tilde{X}_m^{(TM)})^{-1} - \tilde{\eta}_j^{(TE)}] + [\tilde{\nu}(\tilde{Y}_m^{(TM)})]^{-1} \\
\tilde{S}_{m,j}^{(TM)} &= [-\tilde{\nu}(\tilde{X}_m^{(TM)})^{-1} + \tilde{\nu}(\tilde{Y}_m^{(TM)})]^{-1} \\
\end{align*}
\]

(39)

Note that \( \tilde{P}_{m,j}^{(TE)}, \tilde{S}_{m,j}^{(TE)}, \tilde{P}_{m,j}^{(TM)}, \text{and} \tilde{Q}_{m,j}^{(TM)} \) are minute factors that should be kept in the present approximation since they are of the same order as \( \tilde{s}, \tilde{t}, \tilde{u}, \text{and} \tilde{v} \). Adjacent \( \tilde{z}_m^{(S)}'s \) are associated with each other from the boundary condition.

Amplitude coefficients in \( n^\text{th} \) cladding layer \( a \) can be indicated with those of innermost (\( m = 1 \)) layer \( a \) as

\[
\begin{pmatrix}
A_m \\
B_m \\
C_m \\
D_m
\end{pmatrix}
= \sum_{S=TE,TM} \tilde{z}_m^{(S)} \exp[-i(m-1)\tilde{K}_j^{(S)} A] \tilde{\Theta}_m, \begin{pmatrix}
P_{1,j}^{(S)} \\
Q_{1,j}^{(S)} \\
R_{1,j}^{(S)} \\
S_{1,j}^{(S)}
\end{pmatrix},
\]

(40)

where we repeatedly exploited relationships (29) and (30). Coefficient \( \tilde{z}_1^{(S)} \) is determined later from the boundary condition at the core-cladding interface. It is clear from (40) that no common factor can be taken outside as a unique Bloch wavenumber. We can see that electromagnetic waves of hybrid modes in the cladding have two components characterized by two Bloch wavenumbers \( \tilde{K}_j^{(TE)} \) and \( \tilde{K}_j^{(TM)} \).

4. EIGENVALUE EQUATION FOR BRAGG FIBER

The electromagnetic fields of the core and innermost cladding layer \( a \) must be continuous at the core-cladding interface \( (r = r_c) \). With Eqs. (1), (7), and (38), at \( m = 1 \), the boundary condition can be written in matrix form:

\[
\begin{pmatrix}
\sigma_{11} & \sigma_{12} & 0 & \sigma_{14} \\
\sigma_{21} & \sigma_{22} & \sigma_{23} & \sigma_{24} \\
0 & \sigma_{32} & \sigma_{33} & \sigma_{34} \\
\sigma_{41} & \sigma_{42} & \sigma_{43} & \sigma_{44}
\end{pmatrix}
\begin{pmatrix}
A_c \\
\tilde{z}_1^{(TE)} \\
\tilde{z}_1^{(TM)} 
\end{pmatrix}
= 0,
\]

(41)

where
\[ \sigma_{11} = 2J_x(k_a r_c), \quad \sigma_{21} = -\frac{1}{Y_c} \sigma_{43} = -2 \frac{\omega \mu_0 J_x'(k_a r_c)}{k_a}, \]
\[ \sigma_{23} = -\sigma_{41} = -2 \frac{\mu \beta}{k_a^2 r_c} J_y(k_a r_c), \quad \sigma_{44} = -E^{(S)}, \]
\[ \sigma_{42} = \frac{\omega \mu_0}{k_a} \tilde{F}_y^{(S)} + \frac{\mu \beta}{k_a^2 r_c} \tilde{F}_x^{(S)}, \quad \sigma_{43} = \tilde{F}_z^{(S)}, \]
\[ \sigma_{44} = -\frac{\mu \beta}{k_a r_c} \tilde{F}_y^{(S)} - Y_c \frac{\omega \mu_0}{k_a} \tilde{H}_z^{(S)}, \quad l = \begin{cases} 2 & (S = \text{TE}) \\ 4 & (S = \text{TM}) \end{cases} \] (42)

\[ \tilde{E}_x^{(S)} = \tilde{F}_x^{(S)} H_c^{(2)}(k_a r_c) + \tilde{F}_y^{(S)} H_c^{(1)}(k_a r_c), \]
\[ \tilde{F}_x^{(S)} = \tilde{F}_x^{(S)} H_c^{(2)}(k_a r_c) + \tilde{F}_y^{(S)} H_c^{(1)}(k_a r_c), \]
\[ \tilde{G}_y^{(S)} = \tilde{F}_x^{(S)} H_c^{(2)}(k_a r_c) + \tilde{F}_y^{(S)} H_c^{(1)}(k_a r_c), \]
\[ \tilde{H}_z^{(S)} = \tilde{F}_x^{(S)} H_c^{(2)}(k_a r_c) + \tilde{F}_y^{(S)} H_c^{(1)}(k_a r_c). \] (43)

Since \( \tilde{R}_{11}^{(S)} \), \( \tilde{S}_{11}^{(S)} \), \( \tilde{F}_{11}^{(S)} \), and \( \tilde{G}_{11}^{(S)} \) are minute factors that should be kept in the present approximation, \( \tilde{E}_x^{(TE)} \), \( \tilde{H}_z^{(TE)} \), and \( \tilde{G}_y^{(TE)} \) are also minute factors of the same order.

From the condition for a nontrivial solution in Eq. (41), the eigenvalue equation of the Bragg fiber can be derived as
\[ \left[ J_x'(k_a r_c) k_a + \frac{\omega \mu_0}{k_a} \Sigma_j^{(TE)} \right] J_x(k_a r_c) + k_a n_a^2 \Sigma_j^{(TM)} = \left( \frac{\nu \beta}{n_a k_a E_0} \right)^2 \left[ 1 + \left( \frac{k_a}{n_a} \right)^2 \right]^2, \] (44)

where
\[ \Sigma_j^{(TE)} = -i \tilde{G}_j^{(TE)} \tilde{E}_j^{(TE)} \quad \text{and} \quad \Sigma_j^{(TM)} = -i \tilde{F}_j^{(TM)} \tilde{E}_j^{(TM)}, \] (45)

and suffix \( j = 1, 2 \) means the plus and minus cases in Eq. (37).

Solving this multivalued transcendental equation produces propagation constant \( \beta \), which is obtained as a real number. This implies that no decay of the electromagnetic field exists along propagation direction \( z \) because we assume an infinite periodic structure in the cladding in the present ABC method.

The eigenvalue equation obtained here is formally similar to that of the asymptotic expansion method in [14]. However, \( \Sigma_j^{(S)} \) consists of the components of eigenvectors in high-accuracy representation and accordingly gives a more accurate solution than the asymptotic expansion case.

The amplitude coefficients in Eq. (41) are related:
\[ C_e = -\frac{\omega \mu_0 \kappa E_x(k_a r_c) + \kappa_e \Sigma_j^{(TE)}}{\nu \beta}, \]
\[ \lambda_e = \frac{1}{Y_c} \frac{\nu \beta}{1 + (k_a/n_a)^2} \] (46)
\[ \frac{\tilde{z}^{(TE)}}{2A_e} = \frac{J_x(k_a r_c)}{\tilde{E}_j^{(TE)}}, \]
\[ \times \frac{\omega \mu_0 \kappa E_x(k_a r_c) + \kappa_e \Sigma_j^{(TM)}}{\nu \beta + (k_a/n_a)^2} \]
\[ \times \left[ \frac{E_x'(k_a r_c)}{J_x(k_a r_c)} + \frac{\kappa_e \Sigma_j^{(TE)}}{i k_a} \right]. \] (47)

5. NUMERICAL RESULTS ON PROPAGATION CONSTANT

In this section, we study dependence of the Bloch wavenumber on the cladding layer indication \( m \) and show that it can be regarded as a constant over the entire cladding region in practice. Then we show the numerical solutions of propagation constant \( \beta \) by simultaneously solving eigenvalue Eqs. (37) and (44) and compare them with those calculated with the multilayer division and asymptotic expansion methods.

A. Uniformity of Bloch Wavenumber Over Entire Cladding Region

The cladding structure in the present scheme is characterized by the Bloch wavenumber \( \vec{K}^{(S)} \) (\( S = \text{TE}, \text{TM} \)), which is represented in Eq. (37), where \( \bar{X}_m^{(S)} \) and \( \bar{G}_m^{(S)} \) are approximately equal to each other given the condition \( \bar{X}_m^{(S)} \). Strictly speaking, the \( \vec{K}^{(S)} \) also depends upon \( m \) by \( \text{Re}(X_m^{(S)}) \) for hybrid modes, as is the case of the TE01 mode discussed in [19]. The factor \( \text{Re}(X_m^{(S)}) \) contains the propagation constant \( \beta \) implicitly, which is not obtainable at the step of Eq. (37). To evaluate the dependence in this step, we employ \( \beta \) under the quasi-wave condition (QWS) condition. It is calculated through Eqs. (48a) and (48b) in [14] and is substituted into Eq. (37) to obtain the Bloch wavenumber. The thickness of cladding layer \( a \) is determined under the QWS condition [14] as
\[ \frac{X_0}{a} = 2 \left[ \frac{4(n_a^2 - n_c^2)}{U_{QWS} \pi} \right]^{1/2} \left( \frac{k_0}{r_c} \right)^{1/2}, \] (50)

where \( U_{QWS} \) is given by the zeros of Bessel function or its derivative, depending on each mode. The thickness of cladding layer \( b \) can be obtained by replacing \( a \) and \( n_a \) with \( b \) and \( n_b \), respectively. The factor \( \text{Re}(X_m^{(S)}) \) is calculable with the aid of Eqs. (19) and (35).

The dependence of \( \text{Re}(X_m^{(S)}) \) on \( m \) is indicated in Fig. 2 for the HE11 mode, the lowest mode. For reference of \( X_m^{(S)} \), we
suppose that of \( m = 1000 \), which can be regarded as a saturated value. Under the QWS condition, the \( HE_{11} \) mode is guided when \( r_c/\lambda_0 \geq U_{\text{QWS}}/2m \), and we study the cases of \( r_c/\lambda_0 \approx 0.3, 0.6, \) and \( 1.0 \) with \( n_c = 1.0, \) \( n_a = 2.5, \) and \( n_b = 1.5 \). Here, we also study the cases of \( n_a = 3.5 \) and \( 4.5 \) while \( n_c \) and \( n_b \) are fixed at the same value as above. It is found from these results that the relative difference of each case is at most 1% at innermost cladding layer \( (m = 1) \) and it is negligible. Even for the worst case where \( n_b \) is extremely large, the relative difference is sufficiently small (\( \approx 1.1\% \)) and becomes smaller than that for large \( r_c/\lambda_0 \). Therefore, the Bloch wavenumber \( k_1^{(\text{TE})} \) can substantially be regarded as a constant over the entire cladding region, and the eigenvalue Eq. \( (44) \) is derived with use of the value at \( m = 1 \). The dependence of \( \text{Re}(X_{m}^{(\text{TM})}) \) is omitted here, as it is similar to that of \( \text{Re}(X_{m}^{(\text{TE})}) \).

**B. Numerical Comparison on Propagation Constant Among Several Methods**

In this section, we solve eigenvalue Eqs. \( (37) \) and \( (44) \) simultaneously with numerical method. The results are compared with the asymptotic expansion method \[14\] and multilayer division (MLD) method \[17\], which numerically gives a highly accurate solution for a cylindrically symmetric fiber system. We introduce tentative index \( n_t = \beta/k_0 \), and the thicknesses of cladding layers \( a \) and \( b \) are determined to satisfy the QWS condition at \( n_t \):

\[
a = \frac{\pi}{2k_0 \sqrt{n_a^2 - n_t^2}}, \quad b = \frac{\pi}{2k_0 \sqrt{n_b^2 - n_t^2}}. \tag{51}
\]

Note that thicknesses \( a \) and \( b \) are determined with the tentative index instead of \( U_{\text{QWS}} \), and that this procedure is different from that in Section 5.4. Since eigenvalue Eq. \( (44) \) is a function of \( r_c/\lambda_0 \) and \( \beta/k_0 \), wavelength \( \lambda_0 \) can arbitrarily be employed, and the cladding parameters are determined according to the wavelength. For example, when \( \lambda_0 = 1.55 \) [\( \mu \text{m} \)] and \( n_t = 0.8 \), we have \( a = 0.1636 \) [\( \mu \text{m} \)] and \( b = 0.3054 \) [\( \mu \text{m} \)]. In the MLD method, the number of cladding pairs is assumed to be \( N = 40 \), which gives a sufficiently high-accuracy solution for the real part of the propagation constant. This assumption is substantially regarded as a case of an infinite number of pairs, which is a similar situation to the present scheme. Using the above parameter settings, we evaluate the propagation constant through eigenvalue equations.

Figure 3 shows effective index \( n_{\text{eff}} = \beta/k_0 \) as a function of \( r_c/\lambda_0 \) for two lower orders of TE, TM, and hybrid (HE and EH) modes, respectively. The results are compared among the present, MLD, and asymptotic expansion methods, with solid lines, various symbols, and dashed lines, respectively. In this evaluation, the refractive indices of the core and cladding layers \( a \) and \( b \) are set to \( n_c = 1.0, \) \( n_a = 2.5, \) and \( n_b = 1.5 \), and the tentative index \( n_t = 0.8 \) is the same as above. The results of the \( EH_{11} \) and \( TE_{01} \) modes are almost degenerated with each other. We found that, for the present and asymptotic expansion methods \[14\], the results of \( TM_{01} \) and the upper modes agree quite well with each other, and they are also consistent with the result calculated by the MLD method. This indicates that the asymptotic expansion method gives highly correct results for such cases. For the \( HE_{11} \) mode, on the other hand, the results by the present and the MLD methods \[17\] show excellent agreement, except for the vicinity of \( \beta \approx 0 \), while an appreciable difference appears between them and the curve of the asymptotic expansion method. In the present scheme, the approximation referred to at the top of Section 3 is consistently used in the cladding region. However, the present analysis provides reasonable results in practice, even for such extremely small \( r_c \) regions as the lowest mode \( HE_{11} \).

The dependence of the effective index on \( n_a \) is illustrated in Fig. 4 for the lowest mode \( HE_{11} \). Here, the cases of \( n_a = 2.5, 3.5, \) and \( 4.5 \) are studied, while the refractive indices of core and cladding layer \( b \) are fixed at \( n_c = 1.0 \) and \( n_b = 1.5 \). For \( n_a = 3.5 \) and \( 4.5 \) cases, we can see excellent agreement between the present and MLD methods even in the neighborhood of \( \beta \approx 0 \). This is because an increase in the refractive
index contrast improves the optical confinement. On the other hand, a slight discrepancy remains between their results and the asymptotic expansion method.

On the basis of the above results, the present scheme yields solution in higher accuracy, especially for the HE_{11} mode, than the asymptotic expansion method.

6. SUMMARY

The propagation properties of hybrid modes in a Bragg fiber was studied analytically, and the eigenvalue equations and the field amplitude coefficients were derived. In the present ABC method, the electromagnetic waves in the cladding are treated as genuine cylindrical waves, and the Bloch theorem in the cylindrical coordinates was also taken into account. For the validity of this theorem, we require a small azimuthal mode number (0 \leq \nu \lesssim 3) and an approximation that ignores minute terms equal to and smaller than \nu^{-2} under \nu \gg 1 \ (i = a, b). This approximation gives more accurate results than that employed in the asymptotic expansion method, especially for the HE_{11} mode.

Propagation constant \beta was calculated by the eigenvalue equations, and the result of the present scheme was compared with that of the multilayer division method that gives numerical solutions in high-accuracy for cylindrically symmetric fiber structure. We found that they are adequately consistent with each other even for the lowest HE_{11} mode, except for the vicinity of \beta \approx 0.

REFERENCES

20. M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables (Dover, New York, 1965), Chap. 9.