Hybrid modes in a Bragg fiber: general properties and formulas under the quarter-wave stack condition

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We present general properties and useful formulas of hybrid modes in a Bragg fiber by using an asymptotic expansion method. Electromagnetic fields, eigenvalue equations, mode designation, and cutoff are shown. Eigenvalue equations reduce to greatly simplified forms under the quarter-wave stack (QWS) condition. Results are helpful not only for analyzing various properties of the Bragg fiber but also for relating its properties to those in the usual step-index fiber, the circular metallic waveguide, and fundamental optics. It is shown that eigenvalue equations and electromagnetic fields for the Bragg fiber have a formal equivalence to those for the usual step-index fiber. Under the QWS condition the Bragg fiber is nearly the same as the circular metallic waveguide in several characteristics. Hybrid modes change their properties depending on the operation condition in the Bragg fiber. The HE_{11} mode is the fundamental mode. © 2005 Optical Society of America

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1. INTRODUCTION

The possibility of a low-loss and stable fiber by using silica has been pointed out about 40 years before. Basic characteristics such as eigenvalue equations and electromagnetic fields in a step-index optical fiber have been revealed in an exact way or under a weak-guidance approximation. Silica or plastic fibers are put into practical use in various engineering and scientific fields. The guiding principle of these fibers is the index guiding based on the total reflection at the core–cladding interface.

Photonic crystal fibers have recently received much attention as a waveguide for the next generation. Photonic crystal fibers are first classified into two groups. One group is made up of a holey fiber having a solid core and air-hole cladding. The holey fiber guides optical waves through the index guiding, and its properties can be understood using the effective V value. The other group is made up of photonic bandgap fibers characterized by the presence of periodicity. Photonic bandgap fibers exhibit the light localization due to the photonic bandgap. A full two-dimensional photonic bandgap was presented in a previous paper in which the fiber size must be designed appropriately. The Bragg fiber has been analyzed by the transfer-matrix, supercell, biorthonormal basis, and Galerkin methods. An asymptotic analysis method has been presented to get good results with moderate manipulations. Recently, an air–silica Bragg fiber has been demonstrated.

Theoretical studies on the Bragg fiber were mainly devoted to deriving eigenvalue equations and to elucidating electromagnetic fields. Characteristics for TE and TM modes have been presented in a previous paper in which an asymptotic expansion for the Hankel function was employed to express the cladding field. The method is applied to reveal several properties of hybrid modes in this paper. Electromagnetic fields are formulated in a form of a 4×4 matrix. Eigenvalue equations for hybrid modes in the Bragg fiber are derived, and they can be reduced to much simpler equations under the quarter-wave stack (QWS) condition.

2. FUNDAMENTAL EQUATIONS OF HYBRID MODES IN A CYLINDRICALLY SYMMETRIC STRUCTURE

A schematic of the Bragg fiber is shown in Fig. 1. A cylindrical coordinate system (r, θ, z) is used, with z as the propagation direction of light. The core radius is r_c, and its refractive index is n_c. The cladding has a periodic structure in which high- and low-index layers are alternately repeated. One layer of the cladding has thickness a and index n_a, and the other layer has thickness b and index n_b (n_a > n_b > n_c). The cladding period is \Lambda = a + b.
The TE mode has nonzero components of the matrix form:

\[ E_z = g_1(r) \cos(\nu \theta + \theta_0), \quad H_z = f_1(r) \sin(\nu \theta + \theta_0) \]

Here, \( \nu \) denotes the azimuthal mode number, and the result for \( \nu = 0 \) reduces to TE and TM modes. In addition, \( \theta_0 = \pi/2 \) corresponds to the TM mode and \( \theta_0 = 0 \) corresponds to the TE mode. Lateral electromagnetic components are represented using the axial electromagnetic components. Angular dependence of other components is described by

\[ \begin{align*}
    i E_y &= f_2(r) \sin(\nu \theta + \theta_0), \\
    i H_y &= g_2(r) \cos(\nu \theta + \theta_0), \\
    i H_r &= f_3(r) \sin(\nu \theta + \theta_0), \\
    i E_r &= g_3(r) \cos(\nu \theta + \theta_0).
\end{align*} \]

The TE mode has nonzero components of \( H_r, E_y, \) and \( H_z \), and the TM mode has nonzero components of \( E_r, H_y, \) and \( E_z \).

Because discussions are restricted to only the cylindrically symmetric structure, only the radial coordinate dependence parts, \( f_3(r) \) and \( g_3(r) \), are considered now. In that optical waves in the Bragg fiber are regarded as being confined in the core region owing to the interference between inward- and outward-traveling waves along the radial direction, it is useful to describe optical fields by using Hankel functions that have a physical meaning, as stated above. Electromagnetic fields in the core are described by Hankel functions, and those in the cladding are represented by an asymptotic expansion for a Hankel function at large arguments.

**B. Fundamental Equations for Electromagnetic Fields**

Assume that electromagnetic field components have a spatiotemporal factor of \( U_{ij} = \exp[i(\omega t - \beta z)] \), with \( \omega \) and \( \beta \) as the angular frequency and propagation constant, respectively. Tangential components, \( H_z, E_y, E_z, \) and \( H_y \), to be connected in each interface, are selected as fundamental components. Electromagnetic fields are represented in a matrix form:

\[
\begin{pmatrix}
    H_z \\
    i E_y \\
    E_z \\
    i H_y
\end{pmatrix} = U_{ij} D_i(r) \begin{pmatrix}
    A_i \\
    B_i \\
    C_i \\
    D_i
\end{pmatrix},
\]

with

\[
D_i(r) = \begin{pmatrix}
    d_{11} & d_{12} & 0 & 0 \\
    d_{21} & d_{22} & d_{23} & d_{24} \\
    0 & 0 & d_{33} & d_{34} \\
    d_{41} & d_{42} & d_{43} & d_{44}
\end{pmatrix}.
\]

Here, \( A_i - D_i \) stands for the amplitude coefficients of the \( i \)th layer, and they are, in general, complex. The subscript is set to be \( i = c \) for the core region. \( D_i(r) \) is referred to as the representation matrix. Each component in the core is expressed by

\[
d_{11} = d_{33} = H^{(2)}_\nu(\kappa_c r), \quad d_{12} = d_{34} = H^{(1)}_\nu(\kappa_c r),
\]

\[
\begin{align*}
    d_{21} &= - \frac{d_{43}}{Y_c} = - \frac{\omega \mu_0}{\kappa_c} H^{(2)}_\nu(\kappa_c r), \\
    d_{22} &= - \frac{d_{44}}{Y_c} = - \frac{\omega \mu_0}{\kappa_c} H^{(1)}_\nu(\kappa_c r), \\
    d_{23} &= - d_{41} = - \frac{\nu \beta}{\kappa_c} H^{(2)}_\nu(\kappa_c r), \\
    d_{24} &= - d_{42} = - \frac{\nu \beta}{\kappa_c} H^{(1)}_\nu(\kappa_c r), \\
    \kappa_c &= k_0 \sqrt{\frac{n_r^2 - (\beta k_0)^2}{n_r^2 - (\beta k_0)^2}}; \quad 0 \leq r \leq r_c.
\end{align*}
\]

Here, \( H^{(1)}_\nu = J_\nu + i N_\nu \) and \( H^{(2)}_\nu = J_\nu - i N_\nu \) stand for the Hankel functions of the first and second kinds, respectively. The prime indicates differentiation with respect to the argument. In addition, \( \kappa_c \) is the lateral propagation constant of the core, \( k_0 = \omega / c = 2\pi / \lambda_0 \) is the wavenumber of vacuum, \( c \) is the light velocity of vacuum, and \( \lambda_0 \) is the vacuum wavelength. \( Y_i = n_i e_i / \mu_i \) denotes the characteristic admittance in a medium possessing the refractive index \( n_i \), \( e_i \) denotes the dielectric permittivity of vacuum, and \( \mu_i \) denotes the magnetic permeability of vacuum. If \( \nu = 0 \) is prescribed in Eq. (3b), then \( 2 \times 2 \) minor matrices in the left-upper and the right-lower parts reduce to results for TE and TM modes.

For a fiber having a relatively large core radius \( r_c \), we can employ an asymptotic expansion for a Hankel function at large arguments to express cladding fields. It allows us to considerably simplify equations. Then electromagnetic fields in the cladding are approximated by...
with

\[ D_i(r) = \sqrt{2} \pi G_i(r) Q_i(r) \quad (i = a, b). \] (6b)

Here, \( a_i - d_i \) are the amplitude coefficients in the cladding. The boundary matrix \( G_i(r) \) and the displacement matrix \( Q_i(r) \) included in the representation matrix \( D_i(r) \) can be written as

\[ G_i(r) = \frac{1}{\sqrt{\kappa_i}} \begin{bmatrix} 1 & 1 & 0 & 0 \\ g_{21} & g_{22} & g_{23} & g_{24} \\ 0 & 0 & 1 & 1 \\ g_{41} & g_{42} & g_{43} & g_{44} \end{bmatrix}, \] (7)

where

\[ g_{21} = -g_{22} = i\omega \mu_2 \kappa_i, \]
\[ g_{23} = g_{24} = -g_{41} = -g_{42} = -\nu \beta \kappa_i^2 r, \]
\[ g_{43} = -g_{44} = -i\omega \mu_2 Y_i^2 / \kappa_i, \] (8)

and

\[ Q_i(r) = \begin{bmatrix} q_{11} & 0 & 0 & 0 \\ 0 & q_{22} & 0 & 0 \\ 0 & 0 & q_{33} & 0 \\ 0 & 0 & 0 & q_{44} \end{bmatrix}, \] (9)

where

\[ q_{11} = q_{33} = \exp(-i \kappa_i r), \quad q_{22} = q_{44} = \exp(i \kappa_i r), \] (10)
\[ \kappa_i = k_0 \sqrt{n_i^2 - (\beta / k_0)^2} \quad (i = a, b). \] (11)

Here, \( \kappa_i \) is the lateral propagation constant of the cladding and is a real value in the case of the guided mode. Although matrix components in \( G_i(r) \) include the radial coordinate \( r \) unlike the case of TE and TM modes, it is referred to as the boundary matrix. Separation of \( Q_i(r) \) from \( G_i(r) \) in expression (6b) helps us to understand the cladding fields semiquantitatively. The above approximation is valid only for \( \kappa_i r \gg 1 \).

### 3. AMPLITUDE COEFFICIENTS IN THE CLADDING

We also introduce a relative coordinate system for radial coordinates of the cladding (see Fig. 1): for layer a \( (0 \leq r_{a,m} \leq a) \),

\[ r_{a,m} = r - [r_c + (m - 1) \Lambda] \]

for layer b \( (0 \leq r_{b,m} \leq b) \), where \( m \) indicates the layer number counted from the most inner cladding layer. We put an amplitude coefficient set \( a_m \) to \( d_m \) for layers a with refractive index \( n_a \) and thickness \( a \) and \( a_m' \) to \( d_m' \) for layers b with index \( n_b \) and thickness \( b \).

The relationship between amplitude coefficients can be obtained from the boundary condition between cladding layers a and b. Amplitude coefficients in the \( m \)th layer b are represented in terms of amplitude coefficients in the \( m \)th layer a as

\[ \begin{bmatrix} a_m' \\ b_m' \\ c_m' \\ d_m' \end{bmatrix} = G_b^{-1}(r = r_{mA}) G_a(r = r_{mA}) Q_a(r_{a,m} = a) \begin{bmatrix} a_m \\ b_m \\ c_m \\ d_m \end{bmatrix}, \] (12)

where

\[ h_{11} = h_{22}' = (1 + \kappa_a / \kappa_b) \exp(-i \kappa_a a), \]
\[ h_{12} = h_{21}' = (1 - \kappa_a / \kappa_b) \exp(i \kappa_a a), \]
\[ h_{13} = -h_{23}' = h_{31}' = h_{41}' Y_b^2 = -h_{42}' Y_b^2 \]
\[ = \frac{i \nu \beta}{\kappa_a \omega \mu_0} \left( \frac{1}{\kappa_a} - \frac{1}{\kappa_b} \right) \exp(-i \kappa_a a), \]
\[ h_{14} = -h_{24}' = h_{32}' = h_{42}' Y_b^2 \]
\[ = \frac{i \nu \beta}{\kappa_a \omega \mu_0} \left( \frac{1}{\kappa_a} - \frac{1}{\kappa_b} \right) \exp(i \kappa_a a), \]
\[ h_{33} = h_{44}' = \left( 1 + \frac{n_a^2 \kappa_a}{n_b^2 \kappa_b} \right) \exp(-i \kappa_a a), \]
\[ h_{34} = h_{43}' = \left( 1 - \frac{n_a^2 \kappa_a}{n_b^2 \kappa_b} \right) \exp(-i \kappa_a a), \] (13)

and \( r_{mA} = r_c + m \Lambda - b \). The relationship of amplitude coefficients between nearest-neighbor a layers is approximated by

\[ \begin{bmatrix} a_{m+1} \\ b_{m+1} \\ c_{m+1} \\ d_{m+1} \end{bmatrix} = \begin{bmatrix} X_{TE} & Y_{TE} & s_{13} & s_{14} \\ X_{TE} & Y_{TE} & s_{23} & s_{24} \\ s_{31} & s_{32} & X_{TM} & Y_{TM} \\ s_{41} & s_{42} & X_{TM} & Y_{TM} \end{bmatrix} \begin{bmatrix} a_m \\ b_m \\ c_m \\ d_m \end{bmatrix}, \] (14)

where

\[ X_S = \begin{bmatrix} \cos(\kappa_b b) - i \left( \frac{\xi_b \kappa_b}{\xi_b \kappa_b} + \frac{\xi_b \kappa_a}{\xi_b \kappa_b} \right) \sin(\kappa_b b) \end{bmatrix} \exp(-i \kappa_a a), \]
\[ Y_S = \begin{bmatrix} i \left( \frac{\xi_b \kappa_b}{\xi_b \kappa_b} + \frac{\xi_b \kappa_a}{\xi_b \kappa_b} \right) \sin(\kappa_b b) \exp(i \kappa_a a) \end{bmatrix}. \]
If minute quantities larger than $\delta \eta$ are neglected in calculating the determinant in Eq. (19), we have

$$\begin{vmatrix}
X_{TE} - (\eta + \delta \eta') & Y_{TE} \\
Y_{TE}^* & X_{TE}^* - (\eta + \delta \eta')
\end{vmatrix} = 0. \tag{20}
$$

The first and second terms in Eq. (20) are the same as eigenvalue equations for Eq. (19) where all $s_{ij}$’s are put to be zero. Consequently, Eq. (20) implies that the eigenvalue for the hybrid modes can be composed of those for TE and TM modes. Their eigenvalues $\eta_S$ are given by

$$\eta_S = \exp(-iK^2\lambda) = \text{Re}(X_S) \pm \sqrt{[\text{Re}(X_S)]^2 - 1},$$

for $S = \text{TE or TM}; \quad j = 1, 2, \tag{21}$

with the help of Eq. (17). Here, $K^2$ is the Bloch wavenumber. The upper and lower signs in the double-sign notation correspond to $j = 1$ and 2, respectively. Equations (21) are eigenvalue equations for the periodic medium and include the propagation constant $\beta$ implicitly. Accuracy in the eigenvalue is kept unchanged even if $s_{ij} = 0$ is used in Eq. (19). Then the determinant in the left-hand side of Eq. (19) is equal to $(|X_{TE}|^2 - |Y_{TE}|^2)(|X_{TM}|^2 - |Y_{TM}|^2) = 1$ with the help of Eq. (17).

Amplitude coefficients for layer a are evaluated by

$$\begin{pmatrix}
(a_m) \\
(b_m) \\
c_m \\
d_m
\end{pmatrix} = \exp(-iK\lambda) \begin{pmatrix}
(a_1) \\
(b_1) \\
c_1 \\
d_1
\end{pmatrix}, \tag{22}
$$

in the same way as those for TE and TM modes. Equations (22) and (23) are formally identical with expressions for TE and TM modes, respectively. However, amplitude coefficients, $a_m - d_m$, for hybrid modes are different from those for TE and TM modes because of different propagation constants.

Here, $\xi_m$ is a constant to be determined later from the boundary condition. Once coefficient $\xi_m$ for the inner cladding layer is determined, amplitude coefficients $a_1$ to $d_1$, are calculated from the latter parts in Eqs. (22) and (23). Then, amplitude coefficients for $m$th cladding layer a can be calculated using Eqs. (22) and (23), and those for cladding layer b can be calculated with the aid of Eq. (12). By the way, coefficient $\xi_m$ will be determined from the boundary condition at the core–cladding interface in the manner described in Section 5.
5. EIGENVALUE EQUATION FOR THE BRAGG FIBER

From the requirement that fields must be finite in the core, amplitude coefficients for the core must satisfy $B_c = A_c$ and $D_c = C_c$. As independent variables, $A_c$ and $C_c$ are used for the core, and $\xi_{1c}$ and $\xi_{3c}$ are used for the cladding. Then the boundary condition at the core-cladding interface results in an eigenvalue equation for the hybrid modes:

\[
\begin{bmatrix}
J_J'(\kappa_c r_c) + \frac{\kappa_c}{i\kappa_a} \Sigma_{1c}^TE_j
\end{bmatrix}
\begin{bmatrix}
J_J'(\kappa_c r_c) + \frac{\kappa_c n^2}{i\kappa_a} \Sigma_{1c}^TM_j
\end{bmatrix}

= \left(\frac{\nu \beta}{n_c \kappa_c k_0}\right)^2 \left[1 + \left(\frac{\kappa_c}{i\kappa_a}\right)^2\right],
\]  

(24)

where

\[
\Sigma_j^S = \frac{\exp(-iK_j^S A) - X_S - Y_S}{\exp(-iK_j^S A) - X_S + Y_S},
\]  

(25)

In Eq. (24) $\beta$ is tacitly included in the left-hand side while it is apparently included in the right-hand side. An expression similar to but a little bit different from Eq. (24) has been obtained, the treatment of which is helpful to improve accuracy, whereas the present treatment is effective in gaining physical insight, as described later.

From Eqs. (5) and (11) we have $(\beta/k_0)^2 = [n^2_c + n^2_c(\kappa_c/i\kappa_a)^2]/[1 + (\kappa_c/i\kappa_a)^2]$. Inserting this into Eq. (24), one obtains a new expression for the eigenvalue equation as

\[
\begin{bmatrix}
J_J'(\kappa_c r_c) + \frac{1}{i\kappa_a} \Sigma_{1c}^TE_j
\end{bmatrix}
\begin{bmatrix}
J_J'(\kappa_c r_c) + \frac{1}{i\kappa_a} \Sigma_{1c}^TM_j
\end{bmatrix}

= \nu^2 \left[1 + \left(\frac{\kappa_c}{i\kappa_a}\right)^2\right]^2 \left[\left(\frac{\kappa_c}{i\kappa_a}\right)^2 + \left(\frac{\kappa_c n^2}{i\kappa_a}\right)^2\right].
\]  

(26)

Equation (26) has a good symmetry between the left-hand and the right-hand sides. Equations (24) and (26) are transcendental equations. One can evaluate the propagation constant $\beta$ in the Bragg fiber by solving Eqs. (21) and (26) simultaneously. In particular, for $\nu=0$ the first and second square-bracket pairs in the left-hand side of Eq. (24) or (26) reduce to the result for TE and TM modes, respectively.

Amplitude coefficients in the case of $\nu \neq 0$ can be represented by

\[
\frac{C_c}{A_c} = \frac{-\omega \mu_0 \kappa_c}{\nu \beta [1 + (\kappa_c/i\kappa_a)^2]} \left[\frac{J_J'(\kappa_c r_c)}{J_J'(\kappa_c r_c) + \frac{\kappa_c}{i\kappa_a} \Sigma_{1c}^TM_j}\right],
\]  

(27)

\[
\frac{\xi_{1c}^{TE}}{2A_c} = \sqrt{\frac{\pi \kappa_c r_c}{2 \exp(-iK_{TE}^S A) - X_{TE} + Y_{TE}}}.
\]  

(28)

For the TE mode, one obtains $C_c = \xi_{1c}^{TM} = 0$ from Eqs. (27) and (29); hence $E_C = H_C = 0$. For the TM mode, we have $A_c = \xi_{1c}^{TE} = 0$ by combining Eqs. (27) and (28) with Eq. (24); hence $H_C = E_C = 0$.

6. SOME PROPERTIES

A. P PARAMETER FOR MODE DESIGNATION

Hybrid modes in the Bragg fiber are designated by a parameter

\[
P = -\frac{\omega \mu_0 H_C}{\beta E_c} = -\frac{\omega \mu_0 A_c}{\beta C_c}
\]

\[
= \frac{\nu [1 + (\kappa_c/i\kappa_a)^2]^{\frac{1}{2}}}{\kappa_c r_c \left[\frac{J_J'(\kappa_c r_c)J_J'(\kappa_c r_c)}{J_J'(\kappa_c r_c) + \frac{\kappa_c}{i\kappa_a} \Sigma_{1c}^TM_j}\right]}
\]

\[
= \frac{\nu [1 + (\kappa_c n^2_2(\kappa_c/i\kappa_a)^2)]^{\frac{1}{2}}}{\kappa_c r_c \left[\frac{J_J'(\kappa_c r_c)J_J'(\kappa_c r_c)}{J_J'(\kappa_c r_c) + \frac{\kappa_c}{i\kappa_a} \Sigma_{1c}^TM_j}\right]}
\]

(30)

in a manner similar to that in a step-index fiber. For $P \geq 1$, modes satisfying $P > 0$ are called the EH mode, and modes satisfying $P < 0$ are called the HE mode. By definition, we get $P = 0$ for the TM mode and $P = \infty$ for the TE mode.

The ratio of amplitude coefficients can be represented by $A_c/C_c = -(\beta/\omega \mu_0) P$ by using Eq. (30).

B. EXPRESSIONS FOR ELECTROMAGNETIC FIELDS IN TERMS OF P

If the $P$ parameter is employed, then electromagnetic field components in the Bragg fiber formally take on an expression identical to those in the step-index fiber. Results are summarized by

\[
E_z = \alpha \Phi^{(1)}_z \cos(\nu \theta),
\]

(31a)

\[
H_z = -\frac{\beta}{\omega \mu_0} \alpha P \Phi^{(1)}_z \sin(\nu \theta),
\]

(31b)

\[
iE_r = \frac{\beta}{\kappa} \left[\frac{1 - P}{2} \Phi^{(2)}_{\nu+1} - \frac{1 + P}{2} \Phi^{(2)}_{\nu-1}\right] \cos(\nu \theta),
\]

(31c)

\[
iE_\phi = -\frac{\beta}{\alpha} \left[\frac{1 - P}{2} \Phi^{(1)}_{\nu+1} + \frac{1 + P}{2} \Phi^{(1)}_{\nu-1}\right] \sin(\nu \theta),
\]

(31d)
Here, $\alpha$ is the ratio of amplitude coefficients between the core and the cladding, and $\kappa$ is the lateral propagation constant. For the step-index fiber, $\kappa_2 = [(n_2 k_0)^2 - \beta^2]^{1/2}$ denotes the lateral propagation constant in the core, $\gamma_2 = [\beta^2 - (n_2 k_0)^2]^{1/2}$ denotes the lateral propagation constant in the cladding, $n_1$ denotes the core refractive index, and $n_2$ denotes the cladding refractive index. Each parameter for the Bragg and step-index fibers is listed in Table 1 and the following equations:

$\alpha_{\text{Bragg}} = \sqrt{\frac{\pi \kappa_0 r_c}{2 \lambda}} J_j(\kappa_j r_c)$, \hspace{1cm} (32)

\[
M_z = M_z^\text{TE} \pm \frac{\nu}{i \kappa_j} M_z^\text{TM},
\]

\[
N_z = M_z^\text{TM} \pm \frac{\nu}{i \kappa_j} M_z^\text{TE},
\]

Table 1. Parameters in Electromagnetic Fields for Bragg and Step-Index Fibers

<table>
<thead>
<tr>
<th>Items</th>
<th>Bragg Fiber</th>
<th>Step-Index Fiber</th>
</tr>
</thead>
<tbody>
<tr>
<td>Core–Cladding</td>
<td>Core Cladding</td>
<td>Core Cladding</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>$\alpha_{\text{Bragg}}$</td>
<td>$1$</td>
</tr>
<tr>
<td>$P$</td>
<td>Eq. (30)</td>
<td>Eq. (30)</td>
</tr>
<tr>
<td>$\psi^{(1)}$</td>
<td>$J_j(\kappa_j r_c)$</td>
<td>$M_z^\text{TE}$</td>
</tr>
<tr>
<td>$\psi^{(2)}$</td>
<td>$J_j(\kappa_j r_c)$</td>
<td>$M_z^\text{TM}$</td>
</tr>
<tr>
<td>$\psi_{-1}$</td>
<td>$J_{-1}(\kappa_j r_c)$</td>
<td>$M_-$</td>
</tr>
<tr>
<td>$\psi_{+1}$</td>
<td>$J_{+1}(\kappa_j r_c)$</td>
<td>$-M_+$</td>
</tr>
<tr>
<td>$\psi^{(3)}$</td>
<td>$J_j(\kappa_j r_c)$</td>
<td>$N_-$</td>
</tr>
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<td>$J_{+1}(\kappa_j r_c)$</td>
<td>$-N_+$</td>
</tr>
<tr>
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</tr>
<tr>
<td>$n_1$</td>
<td>$n_c$</td>
<td>$n_2$</td>
</tr>
</tbody>
</table>

Each parameter corresponds to that in Eqs. (31).

Here, $J_j(z)$ and $K_j(z)$ indicate the Bessel and modified Bessel functions of order $\nu$ and argument $z$, respectively. Double-sign notation corresponds to each other in both sides of Eqs. (33)–(35). The subscript $i$ applies to $a$ or $b$. If the discrepancy between TE and TM modes in the Bragg fiber can be neglected, then it is unnecessary to distinguish the superscript $1$ from $2$ of $\Phi$ in Eqs. (31a)–(31f) because $M_z = N_z$ holds in that case.

The mechanism for the optical confinement is different in each fiber: The step-index fiber makes use of index guiding, whereas the Bragg fiber employs the photonic bandgap. However, in that both optical fibers have cylindrical symmetry, they show a similar expression.

C. Formal Equivalence in Eigenvalue Equations between the Bragg and Step-Index Fibers

If $\kappa_c$ and $i \kappa_\alpha$ correspond to lateral propagation constants, $\kappa_1$ and $\gamma_2$, for the core and cladding, respectively, in the step-index fiber and, further, if the ratio $\Sigma_j$ corresponds to $K_j(\gamma_j r_c)/J_j(\kappa_j r_c)$ in the step-index fiber, then we see that eigenvalue Eq. (26) formally agrees with that for the step-index fiber.

The ratio of electromagnetic components, $i E_{\theta}/H_z$, in the cladding side at the core–cladding interface is expressed by

$$\frac{i E_{\theta}}{H_z} = \begin{cases} \frac{\omega \mu_0}{\kappa_a} \Sigma_j & \text{for the Bragg fiber} \\ \frac{\omega \mu_0}{\gamma_2} K_j(\gamma_j r_c) & \text{for the step-index fiber} \end{cases},$$

by using Eqs. (31b) and (31d). The ratio $i E_{\theta}/H_z$ in the core side is described by

$$\frac{i E_{\theta}}{H_z} = \begin{cases} \frac{\omega \mu_0}{\kappa_c} J_j(\kappa_j r_c) & \text{for the Bragg fiber} \\ \frac{\omega \mu_0}{\gamma_2} J_j(\gamma_j r_c) & \text{for the step-index fiber} \end{cases},$$

by using Eq. (3a). An equality in the above two equations leads to an expression within the first square-bracket pair of the left-hand side of Eq. (24). If $i H_{\theta}$ and $E_z$ are replaced by $i H_{\theta}$ and $E_z$, then this procedure yields an expression within the second square-bracket pair of the same equation.

D. Complex Conjugation for Amplitude Coefficients

The complex-conjugate property between amplitude coefficients holds for the hybrid modes because diagonal components in the hybrid modes are composed of those identical with TE and TM modes. After some manipulations, we can get

$$b_m = a_m, \quad d_m = c_m, \quad b'_m = a'_m, \quad d'_m = c'_m$$

for all positive integers $m$ of hybrid modes by using the mathematical induction method. The amplitude coeffi-
7. USEFUL FORMULAS UNDER THE QUARTER-WAVE STACK CONDITION

It will be shown in this section that eigenvalue equation (24) or (26) is much simplified under the quarter-wave stack (QWS) condition.

A. Eigenvalue Equations under the Quarter-Wave Stack Condition

In a periodic structure having high and low refractive indices, Bragg diffraction takes place efficiently under the QWS condition. Namely, the optical power hardly penetrates into the cladding. This phenomenon resembles the performance in the metallic hollow waveguide in which its field value is zero at its surface and only TE and TM modes are supported. It is natural that TE and TM modes are accompanied with the TE mode under the QWS condition.

In summary, eigenvalue equations (45)–(47) can always be derived from the zeros of the Bessel function under the QWS condition. In addition, these eigenvalue equations apparently include only the core information, \( \kappa_u \), and \( r_c \). Namely, it is possible to designate the mode by using only the core information.

B. Classification of Hybrid Modes

The sign of the \( P \) parameter must be evaluated to distinguish HE from EH modes in Eq. (47). The numerator of \( P \) in the first line of Eq. (30) is always positive. The \( \Sigma_1^{TE} \) in Eq. (47) approaches zero under the QWS condition. For hybrid modes satisfying \( J_1(\kappa_u r_c) = 0 \) in Eq. (47), the \( P \) value tends to go toward infinity. This shows the same property as that of the TE mode. In contrast, for hybrid modes satisfying \( J_1(\kappa_u r_c) = 0 \), the \( P \) value goes to zero; this resembles the property of the TM mode.

The above tendency can be explained as follows: The optical wave is most efficiently confined in the core under the QWS condition. Namely, the optical power hardly penetrates into the cladding. This phenomenon resembles the performance in the metallic hollow waveguide in which its field value is zero at its surface and only TE and TM modes are supported.

\[
\Sigma_1^{TM} = \frac{\pi}{2} \left( \delta_b + \delta_a a \right)
\]

\[
J_0(\kappa_u r_c) = J_1(\kappa_u r_c) = 0 \quad \text{TE mode under QWS} \quad (45)
\]

by using Eq. (B1). Equation (B4) leads to the eigenvalue equation for the TM mode:

\[
J_0(\kappa_u r_c) = 0 \quad \text{TM mode under QWS} \quad (46)
\]

if \( n_u^2 a^2 \neq n_b^2 b \) is satisfied.

For hybrid modes (\( \nu \geq 1 \)), one obtains the eigenvalue equation

\[
J_1(\kappa_u r_c)J_1(\kappa_u r_c) = 0 \quad \nu \geq 1 \quad \text{hybrid modes under QWS} \quad (47)
\]

from Eq. (B6) if both \( a \neq b \) and \( n_u^2 a^2 \neq n_b^2 b \) are satisfied. In particular, for \( \nu = 1 \), Eq. (47) produces \( J_1(\kappa_u r_c) = J_1(\kappa_u r_c) = 0 \). This means that the hybrid mode with \( \nu = 1 \) is always satisfied.

\[
\kappa_u a = \kappa_u b = \pi/2, \quad (40)
\]

which includes the cladding parameters alone. Assume that fiber parameters are prescribed such that the QWS condition holds at \( \beta/k_0 = n_i \), where \( n_i \) is called the tentative index here. If cladding parameters are slightly shifted from the QWS condition, we can write that

\[
\kappa i = \frac{\pi}{2} (1 - \delta) \quad (i = a, b), \quad (41)
\]

where

\[
\delta i = \frac{1}{2} \frac{(\beta/k_0)^2 - n_i^2}{n_i^2}, \quad (42)
\]

Here, \( \delta i \ll 1 \). Then, one can express parameters \( \Sigma_j^S \) included in the eigenvalue equation or \( P \) parameter as

\[
\Sigma_j^{TE} = \frac{1}{2} \left( \delta_b - \delta_a a \right) + \frac{1}{2} \left( \delta_b - \delta_a a \right) \left( \frac{b - a}{a - b} \right) + \frac{1}{2} \left( \delta_b - \delta_a a \right) \left( \frac{b - a}{a - b} \right) + \frac{1}{2} \left( \delta_b - \delta_a a \right) \left( \frac{b - a}{a - b} \right) \left( \frac{b - a}{a - b} \right), \quad (43)
\]

accompanied with the TE mode under the QWS condition.
are inclined to be propagated through the Bragg fiber under the QWS condition.

Next, let us consider a case in which an operation condition is slightly shifted from the QWS condition. Numerical simulation tells us the following fact: In the case of a mode satisfying \( J_r(\kappa r_c) = 0 \) in Eq. (47), we have \( P < 0 \) for \( \beta/k_0 > n_t \), whereas we have \( P > 0 \) for \( \beta/k_0 < n_t \). The parameter differs with \( \beta/k_0 \), unlike the step-index fiber. For the other case in which \( J_r(\kappa r_c) = 0 \) is satisfied, we get an opposite sign of \( P \) against the former case.

If the phenomenon is judged from the property in \( n_t < \beta/k_0 \), then the lowest-order mode in the Bragg fiber becomes the HE\(11\) mode, which is the same as that in the step-index fiber. Accordingly, we designate the mode by using the \( P \) value in the \( n_t < \beta/k_0 \) region. The hybrid mode satisfying \( J_r(\kappa r_c) = 0 \) is called the HE-like mode, whereas the hybrid mode satisfying \( J_r(\kappa r_c) = 0 \) is called the EH-like mode. Hereafter, -like is abbreviated for simplicity. The mode designation is listed in Table 2. The dispersion characteristics under the QWS condition can be summarized as follows:

\[
\kappa r_c = 2\pi \frac{r_c}{\lambda_0} \sqrt{n_c^2 - (\beta/k_0)^2} = U_{\text{QWS}}, \tag{48a}
\]

with

\[
U_{\text{QWS}} = \begin{cases} J_{j_{\nu,\mu}} \text{ mode} & \text{TE}_{0,\mu} \text{ mode} \\ J_{j_{\nu,\mu}}(v \geq 1) \text{ mode} & \text{T}_{M,0,\mu} \text{ mode} \\ J'_{j_{\nu,\mu}}(v \geq 1) \text{ mode} & \text{HE-like mode} \\ J'_{j_{\nu,\mu}}(v \geq 1) \text{ mode} & \text{EH-like mode} \end{cases} \tag{48b}
\]

Here, \( J_{j_{\nu,\mu}} \) and \( J'_{j_{\nu,\mu}} \) indicate the \( \mu \)th zeros of \( J_\nu \) and \( J'_\nu \), respectively. Equation (48a) is a simple equation, whereas eigenvalue Eq. (26) is a transcendental equation. It turns out that we can get the eigenvalue equation including only the core information. The result for the TE mode in Eq. (48a) agrees with a previous one obtained by a cavity model.22

Let us evaluate the \( P \) value for the slight shift from the QWS condition. Application of Eqs. (43) and (66) to the middle expression of Eq. (30) produces an approximate expression for \( P \) in terms of the first order in \( \delta_a \) and \( \delta_b \) after a considerable amount of manipulation. Only the result is shown here:

\[
P_{\text{HE}} = \frac{r_{\text{co}}}{\sqrt{a}} \left( \begin{array}{c}
a_j J_{j_{\nu,\mu}} \\
j_{\nu,\mu} \end{array} \right) \left( \begin{array}{c} n_a \\
n_c \end{array} \right)^2 \left[ 1 - \left( \frac{2a_j}{\pi r_c} \right)^2 \right]^{-1} \left( \begin{array}{c}
n_j^2 a \\
n_j^2 b \end{array} \right) \left( \delta_b + \delta_a \right) \tag{49}
\]

for the HE mode, and

\[
P_{\text{EH}} = \frac{r_{\text{co}}}{\sqrt{a}} \left( \begin{array}{c} n_{j_{\nu,\mu}} \\
j_{\nu,\mu} \end{array} \right) \left( \begin{array}{c} n_a \\
n_c \end{array} \right)^2 \left[ 1 - \left( \frac{2a_{j_{\nu,\mu}}}{\pi r_c} \right)^2 \right]^{-1} \left( \begin{array}{c}
n_j^2 a \\
n_j^2 b \end{array} \right) \left( \delta_b + \delta_a \right) \tag{50}
\]

for the EH mode, where \( r_{\text{co}} \) denotes the core radius under the QWS condition. The first-order terms of \( \delta_a \) and \( \delta_b \) are omitted from the numerator of expression (49) and from the denominator of expression (50) for clarity of the sign change. We see that \( (n_j^2 a/n_j^2 b - n_j^2 a/n_j^2 b) > 0 \). The term \((2a_{j_{\nu,\mu}}/\pi r_c)^2\) within the square brackets of the above equations is originally \((\kappa/\kappa_2)^2 \) and \(0 < (\kappa/\kappa_2)^2 = (n_t/n_2)^2\). Consequently, two square-bracket terms are always positive. For the HE mode, we have \( P < 0 \) for \( \beta/k_0 > n_t \) \((\delta_a > 0 \text{ and } \delta_b > 0)\), \( P > 0 \) for \( \beta/k_0 < n_t \) \((\delta_a < 0 \text{ and } \delta_b < 0)\), and \( P = 0 \) at QWS, as can be seen from expression (49). For the EH mode, we get an opposite sign against the HE mode case for the same condition.

Under the QWS condition, amplitude coefficients of HE and EH modes take on a particular form, as shown in Appendix C.

Information about the core and that about cladding are related to each other via \( \beta/k_0 \). If the QWS condition holds, fiber parameters must simultaneously satisfy the holds, fiber parameters must simultaneously satisfy the following:

\[
\left( \frac{1}{2a} \right)^2 \left( \frac{U_{\text{QWS}}}{\pi r_c} \right)^2 \lambda_0^2 = 4(n_t^2 - n_2^2), \tag{51a}
\]

and

\[
\left( \frac{1}{a^2 - b^2} \right)^2 \lambda_0^2 = 16(n_2^2 - n_2^2), \tag{51b}
\]

with the aid of Eqs. (40) and (48a).

C. Some Properties Deduced from Zeros of Bessel Functions

It is found from Eq. (48b) that characteristics between \( \text{EH}_{1\mu} \) and \( \text{TE}_{0\mu} \) modes exactly agree with each other under the QWS condition. When \( \nu \) is fixed, the zeros interlace according to the inequalities \( j'_{\nu+1} > j'_{\nu} > j_{\nu} \) \( \text{HE}_{1\mu} \); this means that \( \text{HE}_{\nu\mu} \) and \( \text{EH}_{\nu\mu} \) modes appear alternately with increasing \( \kappa r_c \) for a fixed \( \nu \).

In addition, we have the inequalities \( j'_{1,1} \text{, } j'_{0,1} \text{, } j'_{2,1} < j_{1,1} \text{, } j_{0,1} \text{, } j_{1,1} \); this indicates that the \( \text{HE}_{11} \), \( \text{TM}_{01} \), \( \text{HE}_{21} \), and \( \text{EH}_{11} = \text{TE}_{01} \) modes appear in the order from the lowest-order mode. Hence, the \( \text{HE}_{11} \) mode is the fundamental mode in the Bragg fiber.

We can obtain additional properties of higher-order modes. An agreement between \( j'_{\nu\mu} \) and \( j_{\nu+1,\mu} \) is improved with increasing \( \mu \), say, \( \mu \geq 3 \). The \( j_{2,\mu} \) is nearly equal to \( j_{1,\mu} \), whereas \( j_{3,\mu} \) is also close to \( j_{0,\mu+1} \) for \( \mu \geq 3 \). These facts secure resemblance in characteristics between higher-
order modes ($\mu \geq 3$): between HE$_{1\mu}$ and TM$_{2\mu}$ modes, between HE$_{2\mu}$ and TE$_{0\mu}$ modes, and between EH$_{2\mu}$ and TM$_{0,\mu+1}$ modes, respectively.

D. Correspondence to the Circular Metallic Waveguide

The dispersion characteristics for HE and EH modes in Eq. (48b) agree exactly with those for TE and TM modes, respectively, in the circular metallic waveguide,

which has a hollow region inside a perfectly conducting wall. This is quantitatively obvious for two reasons: The QWS condition implies that an optical wave is efficiently confined in the core, which is similar to the phenomenon that a wave is exactly confined in the hollow region in the circular metallic waveguide. HE and EH modes in the Bragg fiber reduce to TE and TM modes, respectively, under the QWS condition, judging from the $P$ value, as shown in Subsection 7.B.

Resemblance between the Bragg fiber and the circular metallic waveguide is exemplified as follows: Degeneracy between TM$_{1\mu}$ and TE$_{0\mu}$ modes corresponds to that between TM$_{1\mu}$ and TE$_{0\mu}$ modes in the circular metallic waveguide. The fundamental mode is the HE$_{11}$ mode for the former, whereas the TE$_{11}$ mode is that for the latter. For the HE mode under the QWS condition, we have $E_z = 0$, $H_z = - (\beta / \omega \mu_0) E_r$ and $E_r = (\omega \mu_0 / \beta) H_z$ for every $r$ and $E_\theta = H_r = 0$ at $r = r_c$ by Eqs. (31a)-(31f). For the EH mode, one obtains $H_z = 0$, $E_z = - (\omega \mu_0 Y_1^0 / \beta) E_\theta$, and $E_r = (\beta / \omega \mu_0 Y_1^0) H_\theta$ for every $r$ and $E_\theta = E_r = H_z = 0$ at $r = r_c$ in the same fashion. The vanishing condition is in accordance with the boundary condition in the circular metallic waveguide. The discrepancy between them is in that the optical wave penetrates into the cladding in the Bragg fiber, whereas a wave exactly vanishes at the conducting wall of the circular metallic waveguide.

E. Cutoff under the Quarter-Wave Stack Condition

Cutoff is obtained from the condition $\Re(X_3) = \pm 1$,

which corresponds to a condition giving edges of the photonic bandgap. $\Re(X_3)$ under the QWS condition is given by an equality that the first term of expression (A1) is equal to $-1$. From these conditions we have

$$\frac{(\beta / k_0)^2}{\eta_a n_a^2 - \eta_b n_b^2} = (\zeta_a n_a)^2 - (\zeta_b n_b)^2.$$

Because $\zeta_a = \zeta_b = 1$ holds for the TE mode, the cutoff is obtained in spite of $\beta / k_0$ when

$$n_a = n_b \quad \text{TE mode.} \quad (52)$$

This is in agreement with the physical image that an absence of the index difference in the cladding does not contribute to the optical confinement. The relation $n_a = n_b$, for TE mode cutoff, is equivalent to a relation $a = b$ under the QWS condition. If the QWS condition is not satisfied, that is, $\beta / k_0 \neq n_r$, then the cutoff may arise for $n_a \neq n_b$.

In the case of the TM mode, the cutoff arises for $n_a^2 a = n_b^2 b$ under the QWS condition. If $\zeta_a \neq \zeta_b (n_a \neq n_b)$ holds, then we can get the propagation constant $\beta_{cut}$ at the cutoff from

$$\frac{1}{(\beta_{cut}/k_0)^2} = \frac{1}{n_a^2} + \frac{1}{n_b^2} \quad \text{TM and hybrid modes.} \quad (53)$$

Equation (53) means that TM modes may be cut off even for $n_a \neq n_b$ and that $(\beta_{cut}/k_0)^2$ is obtained by a harmonic mean of $n_a^2$ and $n_b^2$. This result reflects the fact that the TM mode is more easily cut off than the TE mode. TM modes are guided for $0 < \beta / k_0 < \beta_{cut}/k_0$. TM modes have the upper cutoff if $n_a$ and $n_b$ are situated in the shaded region of Fig. 2. The minimum value of $\beta_{cut}/k_0$ is found to be $n_a / \sqrt{2}$ when both $n_a$ and $n_b$ approach $n_c$. The cutoff for hybrid modes is identical with that for the TM mode under the present approximation. If lower cladding index $n_b$ exceeds $\sqrt{2} n_c$, the hybrid and TM modes never have the upper cutoff.

Normalized propagation constant $\beta_{cut}/k_0$ in Eq. (53) agrees exactly with that derived from the Brewster’s angle in fundamental optics. The result is satisfied at a plane boundary and is approximately valid for the whole cladding boundary under the present analysis. This is because the $P$ component is not included in a reflected wave at the Brewster’s angle; that is, the TM mode is not reflected back into the core side at all.

The limit of single-mode operation is determined by the TM$_01$ mode. The minimum core radius, $r_{c, min}$, for the single-mode operation can be obtained by

$$r_{c, min} = \frac{\lambda_0 j_{0,1}}{2 \pi n_c}. \quad (54)$$

For example, we have $r_{c, min} = 0.593 \ \mu m$ for $\lambda_0 = 1.55 \ \mu m$ and $n_c = 1.0$ by using $j_{0,1} = 2.4048$.

F. Effective Field Decay Rate in the Cladding

The electromagnetic field in the cladding for the step-index fiber is approximated by

$$K_j(\gamma_f) \equiv \sqrt{\frac{\pi}{2 \gamma_f}} \ exp(- \gamma_f) \quad (55)$$

for large arguments. Here, $\gamma_f(>0)$ indicates the lateral propagation constant in the cladding, which represents the decay rate in the field. On the other hand, the envelope of the cladding field in the Bragg fiber is represented by

![Fig. 2. Classification of cutoff under the QWS condition as a function of cladding indices, $n_a$ and $n_b$. Thick dashed curve, a part of the curve, $1/(n_a/n_c)^2 + 1/(n_b/n_c)^2 = 1$.](image)
This result is valid for any combination of expressions (51a) and (51b). The HE_{11} mode is the lowest-order mode and TM_{01} is the first higher-order mode, as pointed out in Subsection 7.E. Hence we can see that the cladding field is constant for any radial coordinate r at cutoff except for the factor of 1/r.

G. Dispersion Relation

Generalized dispersion relations are illustrated in Fig. 3 for the TE, TM, and hybrid modes under the QWS condition. The graph is calculated using Eq. (48a). If the operating wavelength \( \lambda_0 \) is varied, then the dispersion relation is easily obtainable from Fig. 3 for any wavelength, say, \( \lambda_0 = 1.55 \) or 10.6 \( \mu \)m. Cladding parameters, such as \( n_a, n_b, \alpha, \) and \( b, \) must be changed so as to satisfy Eqs. (51a) and (51b). This result is valid for any combination of \( n_a \) and \( n_b, \) as long as \( n_a \) and \( n_b \) are located above the shaded region in Fig. 2. Then there is no upper cutoff. It is evident that the HE_{11} mode is the lowest-order mode and TM_{01} is the first higher-order mode, as pointed out in Subsection 7.C. This is a remarkable contrast to the usual step-index optical fiber in which the TE_{01} mode is the first higher-order mode. The QWS condition leads to a property that the EH_{11} mode degenerates into the TE_{01} mode.

Figure 4 shows \( \beta/k_0 \) versus the \( r_c/\lambda_0 \) relation for \( n_a = 1.4 \) and \( n_b = 1.2. \) In this case, only the TE modes have no upper cutoff because we have \( \beta_{\text{core}}/k_0 = 0.9111 \) from Eq. (53). If a combination of \( n_a/\lambda_0 \) and \( n_b/\lambda_0 \) exists in the shaded region of Fig. 2, we can get nearly similar curves except that only the \( \beta_{\text{core}}/k_0 \) value is changed.

The dispersion relation has been calculated for an air-core Bragg fiber. In the first example, \( r_c = 7.5 \mu \)m, \( n_a = 3.5, \) \( n_b = 2.0, \) \( a = 0.11 \mu \)m, and \( b = 0.21 \mu \)m, we see that Eq. (51a) is satisfied at \( \lambda_0 = 1.4760 \mu \)m, for which we get \( \beta/k_0 = 0.9984 \) for the HE_{11} mode and \( \beta/k_0 = 0.9968 \) for the TM_{01} mode by extrapolation of Fig. 2(b) in the cited paper. One obtains \( \beta/k_0 = 0.9983 \) for the HE_{11} mode and \( \beta/k_0 = 0.9972 \) for the TM_{01} mode for the same parameters from Eq. (48a). An excellent agreement about \( \beta/k_0 \) can be seen, although we get \( \kappa_a = 1.5078 \) and \( \kappa_b = 1.5492 \) for the HE_{11} mode. In the second example of \( r_c = 0.16 \mu \)m, \( n_a = 4.6, \) \( n_b = 1.6, \) \( a = 0.33 \mu \)m, and \( b = 0.67 \mu \)m, Eq. (51a) holds only for the HE_{11} mode at \( \lambda_0 = 6.041 \mu \)m. Then Eq. (48a) yields a combination of \( \omega_N = 0.1655 \) and \( \beta_N = 0.0770 \) in a scale of Fig. 3 of the literature, where \( \omega_N = \lambda_0 \lambda_0 \) is an angular frequency normalized by \( 2\pi/c/\lambda \) and \( \beta_N = (\beta/k_0)_{\text{QWS}} \) is a propagation constant normalized by \( 2\pi/c/\lambda. \) One can read \( \omega_N = 0.1669 \) and \( \beta_N = 0.077 \) as the closest point corresponding to the above result, leading to a relative error of about 0.8%. In this case we have \( \kappa_a = 1.5078 \) and \( \kappa_b = 1.0686, \) where the latter value is vastly different from \( \pi/2. \)

The present theory is valid for \( k_r \gg 1, \) typically \( (r_c/\lambda_0) \times [n_a^2-(\beta/k_0)^2]^{1/2} > 0.159 \) with \( [n_b^2-(\beta/k_0)^2]^{1/2} \) usually being of the order of unity. Numerical results for the first and second examples produce \( r_c/\lambda_0 = 5.08 \) and 0.331, respectively. High accuracy is obtained for the second example in spite of a small \( r_c/\lambda_0 \) value and the deviation of the cladding b layer from the QWS condition. For Fig. 5 in Ref. 17, there exists no wavelength near which an exact or approximate QWS condition is satisfied for fiber parameters prescribed.
8. SUMMARY

Propagation properties of hybrid modes in the Bragg fiber were derived in a 4×4 matrix form. An asymptotic expansion for Hankel functions was used to express electromagnetic fields in the periodic cladding. Eigenvalue equations consist of two equations: One is Eq. (21) for the periodic cladding, and the other is Eq. (24) or (26) derived from the boundary condition. The solution for the hybrid modes is formed by those for TE and TM modes under an approximate approximation.

A formal equivalence to the step-index optical fiber was explained for eigenvalue equations and electromagnetic fields in Subsections 6.B and 6.C. If the \( P \) parameter is introduced, then electromagnetic fields for the Bragg fiber are represented by the same form as those in the step-index fiber. The field decay rate in the cladding was well related to that of the step-index fiber in the QWS condition, as described in Subsection 7.F.

If the QWS condition is used, then we can get various useful results. Eigenvalue equations are much simplified to Eq. (48a). They are composed of the zeros of Bessel functions, and information about the core is separated from that about the cladding therein. Characteristics under the QWS condition coincide with those in the circular metallic waveguide, as explained in Subsection 7.D. The cutoff condition has an excellent correspondence to that considered by the Brewster’s angle, as stated in Subsection 7.E.

An excellent optical confinement to the core can be explained by the complex-conjugate relation between amplitude coefficients in the individual layer. Although hybrid modes in the Bragg fiber can be designated by the \( P \) parameter, their property changes depending on an operation condition, unlike the step-index fiber. Dispersion curves for TE, TM, and hybrid modes have been shown and have been compared with previous results. The HE_{11} mode is the fundamental mode in the Bragg fiber. Single-mode operation is limited by the TM_{01} mode.

APPENDIX A: APPROXIMATE EXPRESSION FOR PARAMETERS IN EQ. (25) NEAR THE QUARTER-WAVE STACK CONDITION

Parameters \( X_S \) and \( Y_S \) can be expressed as

\[
X_S = -\left[ \frac{\xi b}{\xi a} \left( \frac{\xi a}{\xi a} + \frac{\xi a}{\xi b} \right) \right] - \frac{1}{2} (\delta_b - \delta_a) \left( \frac{\xi b}{\xi a} + \frac{\xi b}{\xi b} \right)
- \frac{1}{2} (\delta_b - \delta_a) \left( \frac{\xi b}{\xi a} + \frac{\xi b}{\xi b} \right),
\]

(\text{A1})

\[
Y_S = \left[ \frac{\xi b}{\xi a} \left( \frac{\xi a}{\xi a} + \frac{\xi a}{\xi b} \right) \right] + \frac{1}{2} (\delta_b - \delta_a) \left( \frac{\xi b}{\xi a} + \frac{\xi b}{\xi b} \right)
- \frac{1}{2} (\delta_b - \delta_a) \left( \frac{\xi b}{\xi a} + \frac{\xi b}{\xi b} \right).
\]

(\text{A2})

by one’s inserting expression (41) into Eqs. (15). Use of the above expressions in Eq. (21) yields

\[
\exp(-iK_T A) = -\left[ \frac{1}{2} \left( \frac{\xi b}{\xi a} + \frac{\xi a}{\xi b} \right) \right] - \frac{1}{2} (\delta_b - \delta_a) \left( \frac{\xi b}{\xi a} + \frac{\xi b}{\xi b} \right) \pm \frac{1}{2} \left[ \frac{1}{2} \left( \frac{\xi b}{\xi a} + \frac{\xi a}{\xi b} \right) \right],
\]

(\text{A3})

\[
\exp(-iK_T A) = \frac{1}{2} \left( \frac{n^2 a}{n^2 b} + \frac{n^2 b}{n^2 a} \right) + \frac{1}{2} (\delta_b - \delta_a) \left( \frac{n^2 a}{n^2 b} - \frac{n^2 b}{n^2 a} \right)
- \frac{n^2 b}{n^2 a} \pm \left( \frac{1}{2} \left( \frac{n^2 a}{n^2 b} - \frac{n^2 b}{n^2 a} \right) \right)
\times \left( \frac{n^2 a}{n^2 b} + \frac{n^2 b}{n^2 a} \right).
\]

(\text{A4})

The upper and lower signs correspond to \( j=1 \) and \( 2 \) in the above double-sign notation. Eigenvalue \( \exp(-iK_T A) \) is always a real number because \( |\text{Re}(X_S)| > 1 \) holds for guided modes.

APPENDIX B: APPROXIMATE EXPRESSIONS FOR EIGENVALUE EQUATIONS NEAR THE QUARTER-WAVE STACK CONDITION

Terms up to the first order in \( \delta_b \) and \( \delta_a \) are taken into consideration in the following equations. For the TE mode, substituting expression (43) into the first term of the left-hand side of Eq. (24), we obtain

\[
\frac{\pi}{2} \left[ J_0(\kappa_a r_c) - \frac{1}{a} \left( \frac{a}{b} - \frac{a}{b} \right) \right] \left( \delta_b - \delta_a \right) = 0,
\]

(\text{B1})

\[
\left( \frac{\pi}{2} \right)^2 J_0(\kappa_a r_c) - \frac{1}{a} \left( \delta_b + \delta_a \right) = 0
\]

(\text{B2})

from real and imaginary parts of an eigenvalue equation. Proportional terms are retained in the above and following equations for reference. For the TM mode, we have

\[
\frac{\pi}{2} \left( \frac{\xi a}{\xi b} \right)^2 J_0(\kappa_a r) \left( \delta_b + \delta_a \right) = 0,
\]

(\text{B3})

\[
- \left[ J_0(\kappa_c r_a) \frac{\xi a}{\xi b} \right] \left( \frac{n^2 a}{n^2 b} - \frac{n^2 b}{n^2 a} \right) = 0
\]

(\text{B4})

from the second term of Eq. (24) in a similar way.

For hybrid modes \( (\nu \gg 1) \), one obtains two equations:
from Eq. (24) after a considerable amount of calculation.

APPENDIX C: AMPLITUDE COEFFICIENTS OF HE AND EH MODES UNDER THE QUARTER-WAVE STACK CONDITION

When the QWS condition is exactly satisfied, amplitude coefficients reduce to a particular form with the help of results in Appendix B. For HE modes, we get

\[ C_{1,2} = D_1 = \frac{n_2}{n_1}, \quad a_1 = b_1 = a_1 = b_1, \quad a_2 = b_2 = \frac{n_2}{n_1}, \quad \beta_{1,2} = \frac{n_1 n_2}{n_1 n_2} \]

For EH modes, one obtains

\[ A_{1,2} = B_{1,2} = 0, \quad a_1 = b_1 = 1, \quad c_1 = d_1 = 1 = \frac{n_1 n_2}{n_1 n_2} \]

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