Correction method for dispersion characteristics in Bragg fibers

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Correction methods are presented to calculate the propagation constant and group velocity dispersion of TE, TM, and hybrid modes in a Bragg fiber. The propagation constant is evaluated as a correction term from a solution derived under the quarter-wave stack condition. The group velocity dispersion is calculated by two kinds of correction methods on the basis of the above propagation constant. Both propagation constant and group velocity dispersion can be calculated using simplified algebraic equations without solving transcendental equations. Their errors and application limit are also shown. © 2009 Optical Society of America

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1. INTRODUCTION

It is useful for fiber designers to determine the propagation constant, namely eigenvalue, as a function of fiber structural parameters in a simple way and with an appropriate accuracy. In ordinary step-index optical fibers, this has been realized by using the weakly guiding approximation [1]. An expression for the dispersion relation was formulated as a function of the normalized frequency [2].

Photonic crystal fibers (PCFs) have a complex dielectric cross section compared with the ordinary step-index optical fiber. A microstructured optical fiber, namely holey fiber, has a solid core and microstructured air holes in the cladding [3]. The optical wave is confined to the core owing to the same index guiding as that in the ordinary step-index fiber. An effective normalized frequency began to be used as a function of specific parameters [4] such that its peculiar characteristics can be interpreted by familiar words in the framework of the ordinary step-index fiber. Similar trials have been made to improve an accuracy in approximations or to widen the applicability of the structure [5,6].

Another class of PCFs are photonic bandgap fibers (PBFs) that have a hollow core. Its optical wave is localized by the photonic bandgap originating from the periodicity of the cladding. The first kind of PBF has an air-hole cladding [7]. The second kind of PBF is a Bragg fiber [8] or an omniguide fiber [9] that has a cylindrical symmetry and a periodic cladding with alternately high and low indices. Since the Bragg fiber has a complicated structure, its simplification is needed to investigate various characteristics. The assumption of infinite periodic cladding allows us to use asymptotic expansion methods, reducing manipulations in the Bragg fiber [10–13]. However, we must still solve a transcendental equation in any case and need tedious computations.

This paper presents simplified correction methods by which the eigenvalue, or propagation constant, can algebraically be evaluated in short computational time without solving transcendental equations of the Bragg fiber. We make use of an eigenvalue under the quarter-wave stack (QWS) condition as a starting value [13]. Next, the propagation constant is corrected with the help of that derived by the asymptotic expansion method. The group velocity dispersion is calculated by two kinds of correction methods using the above propagation constant. These methods are available for a moderate deviation from the QWS condition.

2. DERIVATION OF SIMPLIFIED EIGENVALUE EQUATIONS

In the Bragg fiber, the thickness and position of the successive cladding layers must vary with their radii to minimize the radial flux. This structure requires a complicated numerical means to solve. Therefore, the present paper assumes a radially periodic cladding with equithickness that enables us to use the asymptotic expansion method. Although the approximate method is valid only for a core radius larger than or comparable to a wavelength [13], it provides many useful results.

A schematic of the Bragg fiber is shown in Fig. 1. A cylindrical coordinate system \((r, \theta, z)\) is used with \(z\) being the propagation direction of light. The core radius and its refractive index are \(r_c\) and \(n_c\), respectively, and \(n_c = 1.0\) is used throughout this paper. The cladding has a periodic structure where high and low index layers are alternately repeated. One layer of the cladding has thickness \(a\) and index \(n_a\), while the other layer has thickness \(b\) and index \(n_b\) \((n_a > n_b > n_c)\). The cladding with period \(\Lambda = a + b\) is assumed to extend infinitely.

A. Brief Description of Previous Eigenvalue Equations

Eigenvalue equations for the Bragg fiber consist of two kinds: the first equation reflects its periodic cladding structure; the second equation describes the field continuity at the core–cladding boundary, as usual. The propaga-
where the minute quantity is defined by

$$\delta_i = \frac{1}{2} \frac{(\beta/k_0)^2 - 1}{n_i - 1} \quad (|\delta| \ll 1).$$

The QWS condition corresponds to equalities, $\delta_a = \delta_b = 0$. Application of Eq. (2) to the second eigenvalue equation yields approximate eigenvalue equations as a function of $\delta_a$ and $\delta_b$ for TE, TM, and hybrid modes, as shown in Appendix B of [13].

In particular, the application of the exact QWS condition results in very simplified eigenvalue equations [13],

$$\kappa_c r_c = 2\pi r_c [n_i^2 - (\beta/k_0)^2]^{1/2} = U_{QWS},$$

with

$$U_{QWS} = \begin{cases} 
  j_{1,\mu} = j'_{0,\mu+1} & \text{TE}_{0,\mu} \text{ mode} \\
  j_{0,\mu} & \text{TM}_{0,\mu} \text{ mode} \\
  j'_{\nu,\mu} (\nu \neq 1) & \text{HE}_{\nu,\mu} \text{ mode} \\
  j_{\nu,\mu} (\nu \neq 1) & \text{EH}_{\nu,\mu} \text{ mode} 
\end{cases} \quad (4b)$$

Here, $j_{\nu,\mu}$ and $j'_{\nu,\mu}$ indicate the $\nu$th zeros of the Bessel function, $J_\nu$, and its derivative, $J'_\nu$, respectively. The $\nu$ and $\mu$ indicate the azimuthal and radial mode numbers, respectively. Equation (4a) does not include cladding parameters but includes only core parameters except for the propagation constant $\beta$ and the vacuum wavelength $\lambda_0$.

### B. Simplified Eigenvalue Equations for Non-Quarter-Wave Stack Condition

Let us consider eigenvalue equations for a slight shift from the QWS condition. Their solutions can be sought by setting

$$\kappa_c r_c = U_{QWS}(1 + \delta_c)$$

with

$$\delta_c = \frac{\delta(\kappa_c r_c)}{U_{QWS}}.$$

Here, $|\delta_c| \ll 1$ holds in this case. In addition, we use

$$r_c Q = \frac{U_{QWS}(1 + \delta_c)}{\kappa_c},$$

where $r_c Q$ stands for the core radius at the exact QWS condition.

Applying the Taylor expansion to Bessel functions around the QWS condition, as shown in Appendix A, we have expressions of the $\delta_c$. Results are summarized below,

$$\delta_c^{TE} \equiv \frac{a}{r_c Q} \left( \frac{b}{a} - \frac{n_a n_b}{n_c^2} \right)^{-1} \left( \delta_b + \frac{b}{a} \right),$$

for the TE$_{0,\mu}$ mode. The expression of $\delta_c$ for the TM mode consists of only the core radius and cladding layer thicknesses except for the $\delta_a$ and $\delta_b$. Equation (8) has much simpler form than the exact eigenvalue equation that is a transcendental equation.

For the TM$_{0,\mu}$ mode, we have

$$\delta_c^{TM} \equiv \left( \frac{\pi}{2} \right)^2 \frac{r_c Q}{a_0} \left( \frac{n_a}{n_c} \right)^2 \left( \frac{n_a^2 a}{n_c^2 b} - \frac{n_b^2 b}{n_a^2 a} \right)^{-1} \left( \delta_b + \frac{n_a^2 a}{n_c^2 b} \right).$$

The expression of $\delta_c$ for the TM mode includes the zero, $j_{0,\mu}$, of Bessel function and refractive indices, $n_a$, $n_b$, and $n_c$, in addition to parameters included in the TE mode case.

For the HE$_{\nu,\mu}$ mode one obtains

$$\delta_c^{HE} \equiv \frac{N_{HE}}{D_{HE}},$$

where
The δc for the HE mode includes the azimuthal mode number ν apparently unlike other modes. Equation (10a) formally takes on the same expression as that in Eq. (8) of the TE mode if the ν in Eq. (10a) is replaced by 0. The second term of the numerator shown in Eq. (10b) indicates an effect peculiar to the hybrid modes and directly proportional to \( \nu^2 \) in the right-hand side of the eigenvalue equation. A combination of Eq. (5) with (10a) gives the eigenvalue equation applicable to a slight deviation from the QWS condition for the HE mode.

For the EH mode, we obtain

\[
\delta_{c EH} = \left( \frac{\pi^2}{2} \frac{q_r}{a_{EH}} \frac{r_c}{n_a} \frac{b}{n_c} \right) \frac{2}{n_a} \left( \frac{n_a^2 - n_b^2}{n_b^2} \right) \left( \delta_b + \frac{n_c}{n_b} \delta_a \right)^{-1}. 
\]

This expression has a formal similarity to Eq. (9) of the TM mode when we put \( \nu = 0 \) in Eq. (11). A combination of Eq. (11) with (5) gives the eigenvalue equation for the EH mode.

By the present process, we can calculate the propagation constant without solving transcendental eigenvalue equations for non-QWS condition. Note that these formulas cannot predict the existence of the photonic band edges [12] unlike the transcendental equations. The photonic band edges correspond to a value of \( \beta k_0 \) at which a particular mode is no longer guided.

### 3. CALCULATION METHODS OF DISPERSION CHARACTERISTICS BY CORRECTION

Correction methods are presented here to calculate the propagation constant and the group velocity dispersion with the help of the above results.

#### A. Dispersion Relation

Let us introduce two kinds of correction methods to consider a case where an operation point is slightly shifted from the QWS condition. We choose the propagation constant \( \beta_Q \) versus core radius \( r_c \) under the QWS condition as a starting data in both cases, as shown in Fig. 2. Cladding thicknesses, \( a \) and \( b \), are set such that Eq. (1) holds for prescribed values of \( n_a \), \( n_c \), and \( n_b \).

In a first method the propagation constant is assumed to be fixed at \( \beta_Q \) and the core radius deviates from the \( r_c \) by a small value \( \delta r_c \). This is called a horizontal correction here. In this case, we have \( \delta (k_r r_c) = k_0 (\beta_Q/k_0)^{1/2} \delta r_c \) due to the QWS condition for the HE mode. The group velocity dispersion is defined by

\[
D = \frac{1}{c \lambda_0} k_0 \frac{d^2 \beta}{dr_c^2}; \tag{14}
\]

where \( c \) is the light velocity in vacuum. We differentiate the propagation constant with respect to the vacuum wavenumber \( k_0 \) using a numerical derivative formula within the least-squares approximation [14]. We use a five-points approximation. Since fiber structure parameters must be unchanged in numerical calculation, we make use of Eq. (13) for the vertical correction that

\[
\frac{\sigma}{\beta} = \frac{1}{\beta} \left( \frac{U_{QWS}}{r_c} \right)^{2/3}; \tag{13}
\]

Equation (13) is valid for an inequality \( |\delta \beta| < |\beta| \). Application of Eq. (4a) to this equation leads to \( U_{QWS}/2 \pi = (1 + |\delta_r|)^{1/2} \approx r_c / \lambda_0 \). The lower application limit for \( r_c / \lambda_0 \) is given by \( U_{QWS}/2 \pi + C \), where \( U_{QWS}/2 \pi \) corresponds to the guiding limit, \( \beta = 0 \), of each mode under the QWS condition, and \( C = 0.08 \) or thereabouts is practically available for most lower-order modes. Thus, there exists the lower application limit in the vertical correction method. This does not restrict an application of the present method because a large value of \( \beta / k_0 \) is important for practical use.

#### B. Calculation Method of Group Velocity Dispersion

The group velocity dispersion is defined by

\[
D = \frac{1}{c \lambda_0} k_0 \frac{d^2 \beta}{dr_c^2} \tag{14}
\]
fixes the core radius. Moreover, cladding layer thicknesses must be fixed in calculation. In the following methods we employ expressions given in Eqs. (3), (8), (9), (10a)–(10c), and (11). Then we keep fiber refractive indices unchanged.

C. Group Velocity Dispersion for the Quasi-Quarter-Wave Stack Condition

As a first correction method, we impose the QWS condition on the central point of five calculation points. This is called a quasi-QWS condition here. Five wavenumbers are set as \( k_p = k_0 + p \Delta k \), where \( p \) is an integer ranging from 2 to \(-2\), and \( \Delta k \) denotes the wavenumber spacing. Both Eqs. (1) and (4a) must be satisfied at the central calculation point, \( p = 0 \). We keep the cladding structure at the remaining four points, too, by adjusting the \( n_t = \beta/k_0 \) value in Eq. (1) after the correction.

The procedure of calculation is as follows:

(i) Parameter \( n_{ta,p} \) corresponding to the cladding a layer thickness \( a \) is determined so as to satisfy

\[
\frac{\pi}{2k_0(n_a^2 - (\beta Q/k_p)^2)^{1/2}} = \frac{\pi}{2k_p(n_a^2 - n_{ta,p}^2)^{1/2}}.
\]  

(15)

The central and right portions correspond to values before and after the correction, respectively. Parameter \( n_{tb,p} \) corresponding to the cladding layer b thickness \( b \) is given by expressions where both \( a \) and subscripts \( a \) are replaced with \( b \) in Eq. (15).

(ii) We set \( \delta_{a,p} \) and \( \delta_{b,p} \) for Eq. (3) where the \( n_t \) is replaced with \( n_{ta,p} \) and \( n_{ta,p} \), respectively, and the \( \beta/k_p \) is replaced with \( \beta Q/k_p \). Here, \( \beta Q/k_p \) is the propagation constant under the QWS for each \( k_p \).

(iii) Parameter \( \delta_{a,p} \) and \( \delta_{b,p} \) into Eqs. (8), (9), (10a)–(10c), and (11) respectively.

(iv) The final propagation constant \( \beta_{Q,p} \) for \( k_p \) is obtained by Eq. (13) in which the \( \delta_{c} \) and \( \beta_{Q} \) are replaced with \( \delta_{c,p} \) and \( \beta_{Q,p} \), respectively.

A method employing the propagation constant thus evaluated is called a single correction method hereafter.

D. Group Velocity Dispersion for General Cases

Let us consider a case where a fiber structure does not satisfy the QWS condition. In this case, we use two stage processes. In a first stage, the corrected propagation constant \( \beta_{Q,p} \) for \( k_p \) is used in a way nearly similar to the single correction method. However, in the step (ii), the \( \beta_{Q,p}/k_p \) is used in place of \( \beta_{Q,p}/k_p \) to improve an accuracy.

In the second stage, the fiber is characterized by the \( n_t \). This stage proceeds as follows:

(i) Parameter \( n_{ta,p} \) is determined so as to satisfy

\[
\frac{\pi}{2k_0(n_a^2 - (\beta Q/k_p)^2)^{1/2}} = \frac{\pi}{2k_p(n_a^2 - n_{ta,p}^2)^{1/2}}.
\]  

(16)

The \( n_{tb,p} \) for cladding layer b is determined in the same way as that for the cladding layer a.

(ii) We set \( \delta_{a,p} \) and \( \delta_{b,p} \) for Eq. (3) where the \( n_t \) is replaced with \( n_{ta,p} \) and \( n_{ta,p} \), respectively, and the \( \beta/k_0 \) is replaced with \( \beta_{Q,p}/k_0 \).
The dispersion relation of the TE and TM modes is plotted in Fig. 4 for several cladding high indices $n_a$. Only the horizontal correction is shown to avoid the complexity. Fiber parameters used are $n_a=1.5$ and $n_t=0.8$. As the $n_a$ increases, we see the improvement in agreement between the correction and the asymptotic expansion methods. For example, relative errors of the $\text{TE}_{01}$ mode are $0.037\%$ and $0.031\%$ for $n_a=3.5$ and $4.5$, respectively, at $\beta/k_0=0.1$. Those for the vertical correction are $(1.0\times 10^{-3})\%$ and $(5.0\times 10^{-4})\%$ at the lower application limit, although these graphs are not shown here. A decrease in the error for large $n_a$ is caused by a fact that the increase in the $n_a$ leads to the improvement in the optical power confinement factor $[16]$.

Figure 5 shows the dispersion relation of the TE and TM modes for several $n_t$ values. Only the horizontal correction is shown. Change in the dispersion relation by the $n_t$ is small. Relative errors have a tendency that they increase for large $n_t$ but they decrease for large mode number. For example, the $\text{TE}_{01}$ mode has errors of $0.045\%$, $0.40\%$, and $3.9\%$ for $n_t=0.7$, 0.8, and 0.9, respectively, at $\beta/k_0=0.1$.

B. Group Velocity Dispersion for Quasi-Quarter-Wave Stack Condition

Shown in Fig. 6(a) is a comparison in the group velocity dispersion of the TE mode under the quasi-QWS condition, and shown in Fig. 6(b) are the relative errors between the single correction and asymptotic expansion methods. Fiber parameters are $n_a=2.5$, $n_b=1.5$, and $\lambda_0 = 1.0 \mu m$. Cladding layer thicknesses are set such that the QWS holds for the above parameters. We see a good agreement in results by the two methods. One finds from Fig. 6(b) that relative errors are less than $5.0\%$ above a core radius of $2.0 \mu m$ for any modes, although they increase for small core radius. Relative errors of the $\text{TE}_{01}$ mode are less than $0.60\%$ except for extremely small core radius. Higher-order modes show a somewhat large error unlike the dispersion relation shown above.

To check the accuracy of present correction method, we show results of the $\text{TE}_{01}$ mode calculated by a multilayer division method $[17]$ by open circles in Fig. 6(a). The method employs the exact Hankel functions for cladding...
fields. So, it gives a more accurate result than the present one, although it requires more computational time. The number of cladding pairs is prescribed to be 30, which is regarded as infinity in evaluating the real part of the complex propagation constant from which the group velocity dispersion is calculated. Relative errors between the present and multilayer division methods are 0.097%, 0.37%, 0.24%, 0.23%, and 0.22% for core radii of 2.0, 4.0, 6.0, 8.0, and 10.0 µm, respectively. These small errors support the validity of the single correction method at least above $r_c/\lambda_0=2.0/1.0=2.0$.

Cladding high index dependence of the group velocity dispersion of the TE mode and its relative errors are illustrated in Figs. 7(a) and 7(b) under the quasi-QWS condition. Fiber parameters are the same as in Fig. 6 except for the $n_a$. A comparison between Figs. 6(b) and 7(b) reveals that a relatively small error is still retained, although errors increase with increasing the $n_a$. The maximum errors are about 1.5% and 1.7% for $n_a=3.5$ and 4.5, respectively, for the $T_{01}$ mode except for extremely small core radius. Relative errors decrease for higher-order modes.

Figures 8(a) and 8(b) show the group velocity dispersion of the HE mode under the quasi-QWS condition and its relative errors between the single correction and asymptotic expansion methods, respectively. Fiber parameters are $n_a=2.5$, $n_b=1.5$, and $\lambda_0=1.0$ μm. Ten modes are plotted in the order from the lowest-order mode. One can see a good agreement between the two methods for the HE$_{01}$ mode ($\mu \neq 1$). This is because the HE$_{01}$ mode ($\mu \neq 1$) has a good optical power confinement factor and their group velocity dispersion goes toward zero for large core radius [15]. Relative errors of the HE mode are larger than that of the TE mode, as can be seen from Figs. 6(b) and 8(b). Results for the HE$_{01}$ mode are abbreviated because the single correction method gives poor results for this mode group.

It should be noted that the single correction method does not give a good accuracy for the group velocity dispersion of the TM, HE$_{01}$, and EH modes. The reason can be explained in the following way: the correction method makes use of the Taylor expansion within the first order in minute quantity, as shown in Appendix A, and does not express a quadratic change with respect to the wavenumber. This leads to a tendency that the group velocity dispersion goes toward zero for large core radius. This tendency holds for modes that show a good optical power confinement factor [15]. Since the TM, HE$_{01}$, and EH modes do not exhibit a good optical power confinement factor, the present correction method fails to give a good accuracy for their group velocity dispersion.
C. Group Velocity Dispersion for General Cases

The group velocity dispersion of the TE mode is shown in Fig. 9(a) with \( n_t = 0.8 \). Other parameters are the same as in Fig. 6 except for \( n_t = 0.8 \). Figure 9(b) shows relative errors between the double correction and asymptotic expansion methods. Relative errors are large for higher-order modes. A comparison between Figs. 6(b) and 9(b) shows that relative errors somewhat increase for this general case, although a good accuracy is kept for the TE01 mode. For example, the TE01 mode has an error of less than 1.0% above a core radius of 2.3 \( \mu \text{m} \). To check an accuracy of the present double correction method, we also show results of the TE01 mode calculated by the multilayer division method [17]. Relative errors between the present and multilayer division methods are 3.4%, 0.33%, 0.51%, 0.41%, and 0.34% for core radii of 2.0, 4.0, 6.0, 8.0, and 10.0 \( \mu \text{m} \), respectively. These small errors support the relevance of the double correction method.

Figure 10 shows cladding high index dependence of the group velocity dispersion of the TE mode with \( n_t = 0.8 \). The group velocity dispersion decreases with increasing the \( n_a \) for the same core radius. The discrepancy between the double correction and asymptotic expansion methods is reduced with the increase of the \( n_a \). Relative errors are large for higher-order modes and are less than 10% even for the TE05 mode, although these data are not shown here. Although the HE\( \mu \) mode (\( \mu \neq 1 \)) shows good results in the value’s tendency using the double correction method, there exist cases where relative errors amount to several tens of percent. Hence, we show neither results of the hybrid modes nor those of the TM mode here.

![Fig. 8. Group velocity dispersion of the HE mode under the quasi-QWS condition as a function of core radius. (a) Group velocity dispersion. (b) Relative errors. Parameters are the same as in Fig. 6.](image1)

![Fig. 9. Comparison in the group velocity dispersion of the TE mode with \( n_t = 0.8 \). (a) Group velocity dispersion. (b) Relative errors. \( n_a = 2.5, n_b = 1.5, \) and \( \lambda_a = 1.0 \mu \text{m} \). Solid curves, the double correction method; dashed curves, asymptotic expansion method; open circles, multilayer division method that is used to check the accuracy of double correction method.](image2)

![Fig. 10. Cladding high index dependence of the group velocity dispersion of the TE mode with \( n_t = 0.8 \). Parameters are the same as in Fig. 9 except for the \( n_a \).](image3)
D. Wavelength Dependence of Group Velocity Dispersion for the TE₀₁ Mode

The TE₀₁ mode is expected as a candidate for the single correction method in the Bragg fiber [18]. The TE₀₁ mode exhibits the highest optical power confinement factor among the modes [16]. This subsection treats the group velocity dispersion of the TE₀₁ mode.

Wavelength dependence of the group velocity dispersion is illustrated in Fig. 11 for the TE₀₁ mode with several combinations of core radius and \( n_c \) under the quasi-QWS condition. Parameters are \( n_c = 1.5 \), and \( \lambda_{QWS} = 1.0 \) μm. The long wavelength side in corrected curves is restricted to a wavelength at which \( n_{cl, p} \) value reaches zero in Eq. (15) and it can no longer be calculated. The short wavelength side is determined by the photonic band edge. A range where the two methods show a good agreement becomes wide, as the core radius or the cladding index contrast increases. In particular, agreement is good for a case where the core radius dependence of group velocity dispersion can be evaluated only near the \( \lambda_{QWS} \) for relatively small cladding index contrast and small core radius. The present correction methods give a good accuracy for the group velocity dispersion of TE and HEₘₙ modes (\( \mu \neq 1 \)), especially the TE₀₁ mode.

5. SUMMARY

Several correction methods were presented for the dispersion relation and the group velocity dispersion of TE, TM, and hybrid modes in the Bragg fiber. Their errors were also estimated. Since they can readily be calculated from the simplified eigenvalue equation of the non-QWS condition without solving the transcendental eigenvalue equations, their programming is very simple and they need little computational time.

The correction methods of the dispersion relation can be used for any modes with a satisfactory accuracy. Relative errors are less than 1.0% in most cases of all the modes except for the TM₀₁ and HE₁₁ modes. The core radius dependence of group velocity dispersion can be evaluated with an accuracy less than about 5.0%, although there exists an application range. Application range of the correction method is restricted to the vicinity of the \( \lambda_{QWS} \). Applicability of these results to Eq. (B4) of [13] finally produces Eq. (10a).

APPENDIX A: DERIVATION OF \( \delta_\varepsilon \) FOR TE, TM, AND HYBRID MODES

Inserting Eq. (5) into Bessel functions appearing in Eqs. (B2) to (B6) of [13] and applying the Taylor expansion to them around the QWS condition, we have

\[
J_\varepsilon^r(\kappa, r_c) \equiv J_\varepsilon^r(U_{QWS}) + \delta U_{QWS} J_\varepsilon^r(U_{QWS}), \quad (A1)
\]

\[
J'\varepsilon^r(\kappa, r_c) \equiv J'\varepsilon^r(U_{QWS}) + \delta U_{QWS} J'\varepsilon^r(U_{QWS}), \quad (A2)
\]

up to the first order in \( \delta \). Here, the prime indicates differentiation with respect to its argument.

For the TE₀₁ mode, we have \( U_{QWS} = J_{0, \mu} = J_{0, \mu+1} \) from Eq. (4b). Then we obtain

\[
J_\varepsilon^r(\kappa, r_c) \equiv J_{0, \mu}^r + J_{0, \mu+1}^r \quad \text{and} \quad J'\varepsilon^r(\kappa, r_c) \equiv -\delta J_{0, \mu}^r + \delta J_{0, \mu+1}^r. \]

Substitution of these results into Eq. (B2) of [13] leads to a very simple form given in Eq. (8).

For the TM₀₁ mode, one obtains \( U_{QWS} = J_{0, \mu} \) from Eq. (4b). Then we have

\[
J_\varepsilon^r(\kappa, r_c) \equiv \delta J_{0, \mu}^r J_{0, \mu}^r + J_{0, \mu}^r \quad \text{and} \quad J'\varepsilon^r(\kappa, r_c) \equiv (1 - \delta J_{0, \mu}^r) J_{0, \mu}^r. \]

Application of these results to Eq. (B4) of [13] finally produces Eq. (9).

For the HEₘₙ mode, we have

\[
J_\varepsilon^r(\kappa, r_c) \equiv J_{\mu}^r J_{\mu}^r \quad \text{and} \quad J'\varepsilon^r(\kappa, r_c) = \delta J_{\mu}^r J_{\mu}^r \left[ 1 - \left( \frac{\mu}{J_{\mu}^r} \right)^2 \right] J_{\mu}^r J_{\mu}^r. \quad (A3)
\]

In deriving the above results, we made use of Eq. (4a) and some formulas concerning Bessel functions [19]. Substituting Eqs. (A1), (A3), and (7) into Eq. (B6) of [13], and neglecting minute quantities equal to and higher than second order in \( \delta_\varepsilon, \delta_n, \) and \( \delta_\varepsilon \), we obtain Eq. (10a).
Similar procedures can be used to derive the eigenvalue equation of the EH_{\mu} mode. Application of the Taylor expansions to Bessel functions appearing in Eq. (B6) of [13] leads to \( J_{\nu}(k_r r_0) \equiv \delta_{\nu,\mu} J_{\nu}(j_{\mu} r_0) \) and \( J_{\nu}'(k_r r_0) \equiv (1 - \delta_{\nu,\mu}) J_{\nu}'(j_{\mu} r_0) \) with the help of Eq. (4a) and some formulas concerning Bessel functions [19]. Inserting these results into Eq. (B6) of [13] and neglecting minute quantities equal to and higher than second order in \( \delta_a, \delta_b, \) and \( \delta_c, \) one obtains Eq. (11).

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