Analytical expression of confinement loss in Bragg fibers and its relationship with generalized quarter-wave stack condition

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Analytical expression of confinement loss (CL) is presented for the TE mode of Bragg fibers by applying the perturbation theory to electromagnetic fields under an asymptotic expansion approximation. The expression can be used to evaluate CL for general cases as well as the quarter-wave stack (QWS) condition. Under the generalized QWS condition, the expression gives an explicit dependence of CL on the fiber structural parameters, the number of periodic layer pairs, and so on. We also present numerical results for dependence of the CL on various parameters, such as the core radius, the cladding layer thicknesses, the refractive indices, the wavelengths, the number of periodic layer pairs, and the mode number. The relation between the CL and the generalized QWS condition is scrutinized. The position of the minimum CL corresponds well to the condition satisfying a generalized QWS condition. © 2011 Optical Society of America

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1. INTRODUCTION

The periodicity of cladding in photonic crystal fibers (PCFs) helps in optical confinement due to the photonic bandgap. However, the finite thickness of the periodic cladding causes an optical leakage, and this phenomenon is called the confinement loss (CL) [1]. Reduction in CL is important for appropriate design of PCFs because CL can arise even without material and scattering losses.

CL is usually evaluated by numerical means. An index-guiding holey fiber, which includes air holes in the cladding, makes use of the multipole method [1,2]. A hollow-core photonic bandgap fiber (PBF) with air holes in the cladding utilizes the vectorial finite element method [3]. In an all-solid PBF based on the antiresonant reflection, the adjustable boundary condition method is used [4].

CL of a Bragg fiber [5,6], which consists of air core and cylindrically symmetric cladding with alternating high and low refractive indices, has been studied by the transfer matrix method [7,8], Chew’s method [9,10], a mixed method of exact and asymptotic analyses [11], a multilayer division (MLD) method [12], and the finite element method [13]. The combined effect of confinement and material losses in the Bragg fiber has been investigated numerically [11,14]. Although numerical means give detailed numerical results, they not only require much computing time but also are disadvantageous to wholly grasping the characteristics.

An analytical approach to characterizing the Bragg fiber is an asymptotic expansion approximation [15–18], where electromagnetic fields are exactly treated for the core and are treated in an asymptotic form for the cladding. Use of approximate Bloch theorem in the cylindrical coordinates improves the accuracy of eigenvalue equations especially for hybrid modes [19].

Enhancement of the optical confinement to the core is realized by the introduction of quarter-wave stack (QWS) condition [5]. Its employment vastly simplifies the eigenvalue equation from the viewpoint of wave [18] and geometrical optics [20]. Relation of QWS condition with the optimal index profile has been discussed [21].

The antiresonant reflecting optical waveguide (ARROW) model was presented to confine optical waves in the waveguide with high-index single cladding [22]. The model has been generalized to optical waveguides and fibers with many cladding layers by counting the phase at each round trip within individual layers [23]. The ARROW model has been extended to a stratified planar ARROW (SPARROW) model [13]. A phase theory has shown the equivalence between the in-phase condition at the core–cladding boundary and the antiresonant reflection condition at each round trip in the one-dimensional (1D) and cylindrically symmetric two-dimensional (2D) structures with periodic cladding [24]. It also shows that a generalized QWS condition developed from the QWS condition is equivalent to the central gap point in the SPARROW model.

Evaluation of the CL requires the imaginary part of the complex propagation constant. It is generally cumbersome to numerically solve the eigenvalue equation, including the complex propagation constant in the Bragg fiber. The first purpose of the present paper is to present and verify an analytical expression of CL for the TE mode, the lowest CL mode [7,12], of Bragg fibers with finite periodic region to give physical insight into the PBFs. The CL is evaluated by treating the finite periodic cladding as a perturbation from the Bragg fiber with infinite periodic cladding [18], which is analyzed using an asymptotic expansion approximation. The second purpose is to relate the CL characteristics with the generalized QWS condition and to find the operation condition showing the low CL.
This paper proceeds as follows. Section 2 shows how to derive the propagation constant in the perturbed system, especially for a Bragg fiber. Section 3 derives perturbed fields in the Bragg fiber. Section 4 provides the relationship between amplitude coefficients of perturbed fields in restricted regions. Amplitude coefficients of perturbed fields are determined in Section 5. In Section 6, we present an analytical expression of the propagation constant in the perturbed system. Section 7 gives an explicit expression of the propagation constant under the WQS condition. Section 8 compares the present results with those of the MLD method to verify the present perturbation theory. Section 9 clarifies its relationship with the generalized WQS condition. In addition, we show the design method of a low CL. Sections 10 and 11 are devoted to discussion and conclusion.

2. BASIC EQUATIONS BASED ON PERTURBATION THEORY

A. General Expressions of Propagation Constant

We use cylindrical coordinates \((r, \theta, z)\) with the optical propagation axis \(z\). Assume that the waveguide structure is independent of \(z\). Electromagnetic field components are assumed to have a spatiotemporal factor of \(U_{r,z} = \exp[i(\omega t - \beta z)]\) with the angular frequency \(\omega\) and the complex propagation constant \(\beta\), and the factor is dropped off hereafter. The CL is calculated from the imaginary part \(\text{Im}(\beta)\) of propagation constant \(\beta\) and it is required that \(\text{Im}(\beta) < 0\) in the present framework.

The electromagnetic component \(\Psi\) satisfies the wave equation

\[
[\nabla^2 + \epsilon(r)k_0^2 - \beta^2]\Psi = 0, \tag{1}
\]

where \(\epsilon(r)\) denotes the dielectric constant consisting of a step function, \(k_0 = 2\pi/\lambda_0\) denotes the wavenumber of vacuum, and \(\lambda_0\) denotes the wavelength of vacuum. CL is calculated here by treating the present system as a perturbation from the Bragg fiber with infinite periodic cladding, which has the same fiber parameters as those of the perturbed system, except that the periodic cladding extends to infinity. In this case, the change in dielectric constant can be represented by

\[
\epsilon(r) = \epsilon_\infty(r) + \delta\epsilon_p(r). \tag{2}
\]

Here, \(\delta\) is a parameter representing the order of perturbation, and subscripts \(\infty\) and \(p\) indicate parameters of unperturbed and perturbed systems.

To solve Eq. (1) in the perturbed system, fields and propagation constant are set as

\[
\Psi = \Psi_\infty + \delta\Psi_{p1} + \delta^2\Psi_{p2}, \tag{3a}
\]

\[
\beta = \beta_\infty + \delta\beta_{p1} + \delta^2\beta_{p2}. \tag{3b}
\]

Substitution of these equations into Eq. (1) yields

\[
[\nabla^2 + \epsilon_\infty(r)k_0^2 - \beta_\infty^2]\Psi_{p1} = [-\epsilon_p(r)k_0^2 + 2\beta_\infty\beta_{p1}]\Psi_\infty \tag{4}
\]

for the first-order perturbation and

\[
[\nabla^2 + \epsilon_\infty(r)k_0^2 - \beta_\infty^2]\Psi_{p2} = [-\epsilon_p(r)k_0^2 + 2\beta_\infty\beta_{p1}]\Psi_{p1} + (2\beta_\infty\beta_{p2} + \beta_{p1}^2)\Psi_\infty \tag{5}
\]

for the second-order perturbation. Equations (4) and (5) hold even when \(\Psi\) is replaced with a scalar function.

We first multiply both sides of Eq. (5) by \(\Psi_\ast\) and integrate the entire fiber cross section. Moreover, we apply the Green theorem to the result and employ \(\Psi_\ast = \nabla \cdot \Psi = 0\) at \(r = \infty\). Then we have

\[
\oint \Psi_\ast \cdot \nabla^2 \Psi_{p1} dS = \oint \Psi_{p1} \cdot \nabla^2 \Psi_\ast dS \tag{6}
\]

through the standard procedure. In the case of real \(\beta_\ast\), which can be applied to the Bragg fiber with infinite periodic cladding, we finally have the propagation constant \(\beta_{p1}\) of the first-order perturbation as

\[
\beta_{p1} = \frac{\oint \epsilon_p(r)k_0^2 |\Psi_\ast|^2 dS}{2\beta_\infty \oint |\Psi_\ast|^2 dS}. \tag{7}
\]

It can be seen from Eq. (7) that \(\beta_{p1}\) is real for a real value of \(\epsilon_p(r)\). We need a higher-order perturbation term to evaluate the CL.

In the same way as with \(\beta_{p1}\), we can obtain the second-order perturbation of the propagation constant from Eq. (5) as

\[
\beta_{p2} = \frac{-\beta_{p1}^2 \oint |\Psi_\ast|^2 dS + \oint (\epsilon_p(r)k_0^2 - 2\beta_\infty\beta_{p1})|\Psi_\ast|^2 \Psi_{p1} dS}{2\beta_\infty \oint |\Psi_\ast|^2 dS}. \tag{8}
\]

Equation (8) implies that \(\beta_{p2}\) can be evaluated using the first-order perturbation functions, \(\Psi_{p1}\) and \(\beta_{p1}\), and unperturbed functions, \(\Psi_\infty\) and \(\beta_\infty\).

B. Formal Expressions of CL for Bragg Fibers

Consider a Bragg fiber with finite periodic cladding and a cylindrically symmetric index (see Fig. 1). The fiber consists of a core, periodic cladding, and infinite external region. The core index is \(n_c\) and its radius is \(r_c\). The periodic cladding has a finite number \(N\) of layer pairs that consist of high \(n_m\) and low \(n_a\) indices.
low indices \( n_b \) \((n_a > n_b > n_c)\). The corresponding layer thicknesses are \( a \) and \( b \), and the cladding period is \( \Lambda = a + b \). The index of the external region is denoted by \( n_{ex} \) and is set to \( n_{ex} = n_b \) in the present treatment.

CL is calculated here by treating the present system as a perturbation from the Bragg fiber with infinite periodic cladding [18], which has the core and periodic cladding parameters identical with those of the perturbed system. Namely, the fiber with infinite periodic cladding is regarded as an unperturbed system. Then, the perturbed dielectric constant can be represented by

\[
epsilon_p(r) = \begin{cases} (n_a^2 - n_b^2); & r_c + m\Lambda \leq r \leq r_c + m\Lambda + a \quad (m = N - \infty). \\ 0, & \text{otherwise} \end{cases}
\]

\[ \beta_{p1} = -\frac{(n_a^2 - n_b^2)k_0^2I_{ex,a}}{2\rho_a I_{tot}}, \]

where \( I_{tot} \) denotes the total power and is defined by

\[ I_{tot} \equiv \int |\Psi_a|^2 dS = I_{core} + I_{oa} + I_{ob}. \]

Here, parameters \( I_{core}, I_{oa}, \) and \( I_{ob} \) indicate integrals over the core, entire periodic cladding layer \( a \), and entire periodic cladding layer \( b \), respectively, in the infinite periodic cladding case. \( I_{ex,a} \) is an integral value of \( \int |\Psi_a|^2 dS \) over the entire external layer \( a \).

For the second term of the numerator of Eq. (8), we have

\[
I_{p2} \equiv \int |\epsilon_p(r)k_0^2 - 2\beta_a \rho_{p1}\Psi_p^* \cdot \Psi_{p1}| dS \\
= -2\beta_a \rho_{p1}(I_{p,core} + I_{p,pe,a} + I_{p,pe,b} + I_{p,ex,a}) \\
- (n_a^2 - n_b^2)k_0^2 + 2\beta_a \rho_{p1})I_{p,ex,a}.
\]

where integral parameters, \( I_{p,core}, I_{p,pe,a}, I_{p,pe,b}, I_{p,ex,a}, \) and \( I_{p,ex,b} \) indicate the contribution of the core, periodic cladding layers \( a \) and \( b \) and external layers \( a \) and \( b \), respectively.

3. DERIVATION OF PERTURBED FIELDS IN BRAGG FIBER

Fields of unperturbed Bragg fiber that has an infinite periodic cladding are represented by exact fields for the core and by fields analyzed using an asymptotic expansion approximation for the cladding [18]. A Bragg fiber with the finite periodic region is treated as the perturbation from the unperturbed Bragg fiber. Nonzero electromagnetic fields to be continuous at the interfaces are \( H_z \) and \( E_{\theta} \) for the TE mode.

A. Unperturbed Fields

For a fiber with infinite periodic cladding, nonzero electromagnetic fields for \( \Psi_a \) are expressed in a matrix form as [18] \( \mathbf{D}_a(r) \Psi_a(r) = 0 \) and \( \mathbf{D}_b(r) \Psi_b(r) = 0 \) for the core and cladding, respectively.

\[
(\begin{bmatrix} H_z \nabla E_{\theta} 
\end{bmatrix})_{\text{ex}} = \begin{bmatrix} D_c(r) & 0 \\
0 & D_c(r) \end{bmatrix} \begin{bmatrix} A_c \\
B_c \end{bmatrix} \]

\[ D_c(r) \equiv \begin{bmatrix} d_{11} & d_{12} \\
d_{21} & d_{22} \end{bmatrix}, \]

\[ D_c(r) \equiv \begin{bmatrix} (\frac{2}{\pi r})^{1/2} G_i(r)Q_i(r) \end{bmatrix} \]

Here, \( A_c \) and \( B_c \) denote the amplitude coefficients of the core, and \( a_{m} \) and \( b_{m} \) of the \( m \)th cladding layer \( a \) (Fig. 1). Those for cladding layer \( b \) are expressed by adding the prime. These amplitude coefficients apply to the unperturbed fiber. We set \( B_b = A_b \) from a requirement that fields must be finite at the core center.

Elements of representation matrix \( D_c(r) \) in the core are given by

\[ d_{11} = H_{\nu}^{(1)}(k_{c}r), \quad d_{12} = H_{\nu}^{(1)}(k_{c}r), \]

\[ d_{21} = -\frac{\alpha_{10}}{\mu_{\kappa}} H_{\nu}^{(2)}(k_{c}r), \quad d_{22} = -\frac{\alpha_{10}}{\mu_{\kappa}} H_{\nu}^{(2)}(k_{c}r), \]

with the lateral propagation constant

\[ \kappa_{i} = \left[(n_{i}k_{0})_{1}^{2} - \beta_{i}^{2}\right]^{1/2} \quad (i = a, b, c). \]

\[ H_{\nu}^{(1)} \equiv J_{\nu} + iN_{\nu} \quad \text{and} \quad H_{\nu}^{(2)} \equiv J_{\nu} - iN_{\nu} \]

indicate the Hankel functions of the first and second kinds, respectively, \( \nu \) denotes the azimuthal mode number, and \( \mu_{\kappa} \) denotes the magnetic permeability of vacuum. Although \( \nu = 0 \) for the TE mode, it is retained for reference. The prime of cylinder functions indicates differentiation with respect to their argument throughout this paper.

The boundary matrix \( G_i(r) \) and the displacement matrix \( Q_i(r) \) included in the cladding representation are expressed as

\[ G_i(r) = \begin{cases} \frac{1}{\sqrt{\kappa_i}} \left[ \frac{1}{\ii \alpha_{0}/\kappa_i} \frac{1}{\ii \alpha_{0}/\kappa_i} \right] \quad (i = a, b). \end{cases} \]

\[ Q_i(r) = \begin{cases} \exp(-\ii \kappa_{i}r) \quad \text{for } \kappa_{i}r \gg 1, \end{cases} \]

under the asymptotic expansion approximation, which is valid only for \( \kappa_{i}r \gg 1 \). Functions \( \exp(-\ii \kappa_{i}r) \) and \( \exp(\ii \kappa_{i}r) \) correspond to outward and inward waves, respectively. Relative radial coordinates, \( r_{a,m} \) and \( r_{b,m} \), are introduced to calculate the phase factor of the cladding layers \( a \) and \( b \), respectively, where \( r_{a,m} \equiv r - [r_c + (m - 1)\Lambda] \), \( r_{b,m} \equiv r - [r_c + (m - 1)\Lambda + a] \), \( 0 \leq r_{a,m} \leq a \), and \( 0 \leq r_{b,m} \leq b \).

After the propagation constant \( \beta_{i} \) is determined through solving eigenvalue equations, amplitude coefficients, \( a_{m} \), \( b_{m}, a'_{m} \), and \( b'_{m} \), of cladding layers are represented in terms of amplitude coefficients \( a_{1} \) and \( b_{1} \) or \( A_{c} \) [17,18]. Some important properties in deriving equations of the periodic structure are summarized in Appendix A.
B. Perturbed Fields

We use a parameter variation method to derive \( H_{z,p1}(r) \) from a differential equation, where \( \Psi \) is set as \( H_z(r) \) in Eq. (4). \( iE_{\theta,p1}(r) \) can be derived from \( H_{z,n}(r) \) and \( H_{z,p1}(r) \). Perturbed fields are summarized in terms of matrix form. For the core, we have

\[
\begin{align*}
H_{z,p1}(r) & = D_z(r) \left( A_{p1,c} B_{p1,c} - \frac{\beta_n}{k_z} r M_c(r) A_c \right), \\
\begin{math}
iE_{\theta,p1}(r) \end{math} & = -D_z(r) \left( A_{p1,c} B_{p1,c} - \frac{\beta_n}{k_z} r M_c(r) A_c \right) .
\end{align*}
\]

(17a)

Here, \( A_{p1,c} \) and \( B_{p1,c} \) stand for the amplitude coefficients of the perturbed system. Matrix elements \( m_{ij} \) of \( M_z(r) \) are given by

\[
\begin{align*}
\begin{array}{l}
m_{11} = H_I^{(i)}(\kappa_x r), \\
m_{12} = \frac{\eta_0}{k_x} \left( \frac{1 - \frac{\nu^2}{k_x^2 r^2}}{H_I^{(0)}(\kappa_x r) + \frac{2 \kappa_x}{k_x r} H_I^{(2)}(\kappa_x r)} \right), \\
m_{21} = \frac{\eta_0}{k_x} \left( \frac{1 - \frac{\nu^2}{k_x^2 r^2}}{H_I^{(0)}(\kappa_x r) + \frac{2 \kappa_x}{k_x r} H_I^{(2)}(\kappa_x r)} \right), \\
m_{22} = \frac{\eta_0}{k_x} \left( \frac{1 - \frac{\nu^2}{k_x^2 r^2}}{H_I^{(0)}(\kappa_x r) + \frac{2 \kappa_x}{k_x r} H_I^{(2)}(\kappa_x r)} \right) .
\end{array}
\end{align*}
\]

(17b)

Similarly, one obtains perturbed fields of \( m \)th periodic and external layers \( a \) as

\[
\begin{align*}
\begin{array}{l}
H_{z,p1}(r) \\
\begin{math}
iE_{\theta,p1}(r) \end{math}
\end{array} = \begin{cases} 
D_i(r) a_{pe,m}(r) & \text{for periodic layers} \\
D_i(r) a_{ex,m}(r) & \text{for external layers}
\end{cases},
\end{align*}
\]

(18)

where two column vectors are defined by

\[
\begin{align*}
a_{pe,m}(r) & = \begin{pmatrix} a_{p1,m} \\ b_{p1,m} \end{pmatrix} - i \beta_\nu b_{p1} r \begin{pmatrix} -a_m \\ b_m \end{pmatrix}, \\
(1 \leq m \leq N, i = a, b),
\end{align*}
\]

(19)

\[
\begin{align*}
a_{ex,m}(r) & = \left( \begin{pmatrix} a_{p1,m} \\ b_{p1,m} \end{pmatrix} - i \eta_{pi} r \begin{pmatrix} -a_m \\ b_m \end{pmatrix} \right), \\
(m \geq N + 1, i = a, b),
\end{align*}
\]

(20)

\[
\eta_{pi} = \frac{1}{\kappa_i} \left[ (n_i^2 - n_b^2) h_i^2 + 2 \beta_m \beta_{p1} \right] (i = a, b).
\]

(21)

Here, \( a_{p1,m} \) and \( b_{p1,m} \) stand for the amplitude coefficients of the \( m \)th layer \( a \) (see Fig. 1). The \( \beta_\nu \beta_{p1}/\kappa_i \) and \( \eta_{pi} \) in Eqs. (19) and (20) indicate effective layers of the propagation constant and refractive index changes due to the finite thickness of the periodic layer, respectively. Actually, \( \eta_{pi} \) is equal to \( \beta_\nu \beta_{p1}/\kappa_i \). In \( \eta_{pi} \) of Eq. (21), the first is much larger than the second, as shown in the last paragraph of Subsection 7.B. The amplitude coefficients for \( m \)th layers \( b \) are expressed by adding the prime to column vectors \( a_{pe,m}(r) \) and \( a_{ex,m}(r) \).

Equation (18) indicates that the relationship between amplitude coefficients in the perturbed system is identical to that in the unperturbed system if perturbed fields are defined by expressions in Eqs. (19) and (20). This property is peculiar to the asymptotic expansion approximation and useful in that if \( a_{pe,m}(r) \) or \( a_{pe,m}(r) \) corresponds to the column vector given in Eq. (A1), then properties similar to those of the unperturbed system can be used in the periodic region.

4. RELATIONSHIP BETWEEN AMPLITUDE COEFFICIENTS OF PERTURBED FIELDS

There are many amplitude coefficients to be related with each other in the perturbed fields. The relationship between the amplitude coefficients must be treated systematically. Conditions of amplitude coefficients are summarized in the following way: (i) fields must be finite at the core center; (ii) tangential components must be continuous at the interface of the core, periodic layers, and external layers; and (iii) inward field components must disappear in the external layers. The last condition is peculiar to the present problem.

A. Amplitude Coefficients in Core and Periodic Region

From a requirement that fields must be finite at the core center, one obtains \( A_{p1,c} = B_{p1,c} \) because the unperturbed fields have been set to be finite at the core center.

For perturbed fields between the \( m \)th periodic layers \( a \) and \( b \), we require

\[
G_a(r_{m,a})Q_a(r_{a,m} = a)a_{pe,m}(r_{m,a}) = G_b(r_{m,a})Q_b(r_{b,m} = 0)a_{pe,m}(r_{m,a})
\]

(22)

with \( r_{m,a} \equiv r_c + (m - 1)\Lambda + a \) (see Fig. 1). Here, \( Q_b(r_{b,m} = 0) \) is a unit matrix. Multiplication of the above equation by \( G_b^{-1}(r_{m,a}) \) from the left-hand side produces the relation

\[
a_{pe,m}(r_{m,a}) = H_a a_{pe,m}(r_{m,a}) \quad (1 \leq m \leq N),
\]

(23)

where

\[
H_a = G_b^{-1}(r_{m,a})G_a(r_{m,a})Q_a(r_{a,m} = a) = \frac{1}{2} \left[ \frac{\kappa_a}{\kappa_f} \right]^{1/2} \left[ h_0^2 \right].
\]

(24)

Matrix elements of \( H_a \) are the same as those of the unperturbed system:

\[
h_n^{(0)} = h_n^{(2)} = (1 + \kappa_b/\kappa_a) \exp(-i\kappa_a) \quad \text{and} \quad h_{n+1}^{(2)} = (1 - \kappa_b/\kappa_a) \exp(i\kappa_a).
\]

Equation (23) indicates the amplitude coefficients of cladding layer \( b \) in terms of those of cladding layer \( a \), and the left-hand side is symmetrical to the right-hand side regarding matrix \( H_a \).

Amplitude coefficients of \( (m + 1) \)th cladding layer \( a \) are similarly related with those of the \( m \)th cladding layer \( b \) at \( r = r_{m,a} \equiv r_c + m\Lambda \); that is, \( a_{pe,m+1}(r_{m+1,b}) = H_b a_{pe,m}(r_{m,a}) \). Here, we define \( H_b = (1/2)(\kappa_f/\kappa_b)^{1/2}[h_0^{(2)}] \), the parameters of which are obtained by interchanging \( a \) and \( b \) in \( h_0^{(2)} \).

The relationship between amplitude coefficients in adjacent layers \( a \) of the periodic region can be written by

\[
\begin{align*}
\left( \begin{array}{c}
a_{p1,m+1} \\ b_{p1,m+1}
\end{array} \right) & = \exp(-iK_j\Lambda) \left( \begin{array}{c}
a_{p1,m} \\ b_{p1,m}
\end{array} \right) \\
& + i\beta_\nu b_{p1} \exp(-iK_j\Lambda) \left( \begin{array}{c}
-\frac{a_1}{b_1} \\ \frac{1}{K_b} \end{array} \right) \\
& - \frac{1}{K_b} \exp(iK_j\Lambda) H_b \left( \begin{array}{c}
-\frac{a_1'}{b_1'}
\end{array} \right)
\end{align*}
\]

(25)

using the above two relations and a property concerning \( H_jH_a \) (see Appendix A) with \( K_j \) being the Bloch wavenumber \( j = 1, 2 \).

Summation of both sides of Eq. (25) multiplied by \( \exp(-iK_j(m - 1 - m'))\Lambda \) over \( m = 1 \) to \( m - 1 \) yields amplitude coefficients of the \( m \)th periodic layer \( a \) as
In the external region can be represented by

\[ a_{\Psi,1,N+q} = a_{p1,N+1} C_{ex}^{q-1} - a_{N+1} \exp[-iK_j(q-1)\Lambda] \]

\[ + 1 + in_{pa}r_{N+q-1}\Lambda - \frac{in_{pa}aC_{ex} \exp(iK_j\Lambda)}{1 - C_{ex} \exp(iK_j\Lambda)} \]

\[ + a_{N+1}C_{ex}^{q-1} \left[ 1 + \frac{in_{pa}r_{N+1}\Lambda - \frac{in_{pa}a}{1 - C_{ex} \exp(iK_j\Lambda)} \right]. \]

(32)

In a similar way, we have the amplitude coefficient of the qth external layer b as

\[ a'_{p1,N+q} = a_{p1,N+1} C_{ex}^{q-1} \left( \frac{\kappa_b}{\kappa_a} \right)^{1/2} \left( 1 + \frac{\kappa_b}{\kappa_a} \right) \exp(-i\kappa_a\alpha) \]

\[ + a_{N+1}C_{ex}^{q-1} \left( 1 + \frac{\kappa_b}{\kappa_a} \right)^{1/2} \left( 1 + \frac{\kappa_b}{\kappa_a} \right) \exp(-i\kappa_a\alpha) \]

\[ + a'_{N+1} \{ \exp[-iK_j(q-1)\Lambda] - C_{ex}^{q-1} \frac{in_{pb}bC_{ex} \exp(iK_j\Lambda)}{1 - C_{ex} \exp(iK_j\Lambda)} \}

- a'_{N+1} \{ \exp[-iK_j(q-1)\Lambda] - C_{ex}^{q-1} \frac{in_{pb}bC_{ex} \exp(iK_j\Lambda)}{1 - C_{ex} \exp(iK_j\Lambda)} \}. \]

(33)

Equations (32) and (33) mean that amplitude coefficients of the (N + q)th layers a and b in the perturbed system are expressible as a function of amplitude coefficients of the first external layer in the perturbed and unperturbed systems.

Perturbed fields of the qth external layer a are expressed as

\[ H_{2,p1}(i\epsilon_{\theta,p1}) = D_{\alpha}(r) \left[ \begin{array}{c} a_{N+q} + a_{p1,N+q} + in_{pa}r_{N+q} \\ 0 \end{array} \right] \]

\[ - \left[ \begin{array}{c} 0 \\ b_{N+1} \end{array} \right] + \left( \frac{a_{N+q}}{0} \right) + C_{ex}C_{ex}^{q-1} D_{\alpha}(r) \left( \frac{a_{p1,N+1}}{0} \right). \]

(34)

where we used Eq. (32) and set

\[ a_{N+q} \equiv -a_{N+1} \left[ 1 + in_{pa}r_{N+q-1}\Lambda - \frac{in_{pa}aC_{ex} \exp(iK_j\Lambda)}{1 - C_{ex} \exp(iK_j\Lambda)} \right]. \]

(35a)

\[ a'_{p1,N+1} \equiv a_{p1,N+1} \]

\[ + a_{N+1} \left[ 1 + in_{pa}r_{N+1}\Lambda - \frac{in_{pa}a}{1 - C_{ex} \exp(iK_j\Lambda)} \right]. \]

(35b)

In Eq. (34), the second term takes place to cancel out the inward wave in unperturbed fields, and the remaining terms consist of only the outward wave component. The first term includes the radial coordinate r. Here, a_{N+q} depends on q, but a'_{p1,N+1} is independent of q.

5. DETERMINATION OF AMPLITUDE COEFFICIENTS IN PERTURBED FIELDS

Amplitude coefficients between the core and innermost external layer in the perturbed system can be related with each other with the help of the relationship among amplitude coefficients in the periodic region. Through this process, all the amplitude coefficients in the perturbed system can be
explicitly expressed in terms of known parameters of the unperturbed system.

Only the perturbed fields must be continuous at the interface of the core and innermost periodic layer since the unperturbed fields have been set to be continuous there. This is accomplished by applying the boundary condition to Eqs. (17) and (18) at \( r = r_c \), and by utilizing the finite field requirement at the core center. In the external region, only the outward fields exist in the form of \( \Psi_\infty + \Psi_{\psi} \), as shown in Eq. (29). From this condition, fields are related at the interface of the outermost periodic and innermost external layers, namely \( r = r_{NB} \). Amplitude coefficients, \( A_{p1,x} \) and \( a_{p1,N+1} \), are treated here as independent parameters. Thus, we have a relationship of amplitude coefficients between the core and innermost external layer as

\[
\begin{bmatrix}
J_{+,a} - \exp(iK_NA) \\
J_{-,a}
\end{bmatrix}
= \begin{bmatrix}
A_{p1,1}(\pi \kappa_c r_c/2)^{1/2} \\
0
\end{bmatrix}
\begin{bmatrix}
P_a \\
P_b
\end{bmatrix},
\]

(36a)

where

\[
\begin{aligned}
P_a &\equiv i\eta_{\text{ex}}r_{NB} \left[ \frac{a_1}{0} - \beta_c \beta_{p1} \frac{r_c}{\kappa_c} F_a \left( \frac{\pi \kappa_c r_c}{2} \right)^{1/2} - \frac{0}{b_1} \right] \\
+ i\beta_c \beta_{p1} \frac{N_a}{\kappa_a} \left[ -a_1 + i\beta_c \beta_{p1} \frac{N_b}{\kappa_b} \exp(iK_NA) H_b \left( -a_1' b_1 \right) \right],
\end{aligned}
\]

(36b)

\[
J_{+,a} \equiv J_a \pm \frac{\kappa_c}{\kappa_a} J'_a \quad (i = a, b),
\]

(36c)

\[
F_c \equiv
\begin{cases}
-J_c + \frac{\kappa_c}{\kappa_a} \left( 1 - \frac{i^2}{\kappa_c^2} \right) J_c + 2J'_c \\
-J_c + \frac{\kappa_c}{\kappa_a} \left( 1 - \frac{i^2}{\kappa_c^2} \right) J_c + 2J'_c
\end{cases}
\]

(36d)

In the double sign notation of Eq. (36c), upper and lower signs correspond to each other. The argument \( \kappa_c r_c \) of \( J_c \) and \( J'_c \) is dropped off here. The product \( H_b H_a \) was replaced by \( \exp(-iK_NA) \) in the periodic region (see Appendix A).

Equation (36a) enables us to solve the amplitude coefficients, \( A_{p1,e} \), and \( a_{p1,N+1} \), of the perturbed system in terms of structural parameters and amplitude coefficients of the unperturbed system. Note that \( \beta_{p1} \) can be evaluated using parameters of the unperturbed system, as shown in Eq. (7) or Eq. (10). In Eq. (36b), the first term indicates the effect of index change from \( n_a \) to \( n_b \) in the external layer, the second indicates an influence on the core, the third indicates the effect of no inward wave in the external layer, the fourth indicates the phase change arising from all periodic layers \( a \), and the last term indicates the phase change caused by all periodic layers \( b \).

Solving Eq. (36a), we obtain the amplitude coefficients of the perturbed system (see Appendix B). We take into consideration that terms being proportional to \( \beta_c \beta_{p1} \) are small compared to other terms. Then, we can find that amplitude coefficient \( b_1 \) has a significant influence on amplitude coefficients, \( A_{p1,e} \), \( a_{p1,1} \), and \( b_{p1,1} \), for the core and periodic region.

This is because amplitude coefficient \( b_0 \) corresponding to the inward wave acts on the suppression of inward waves in the external region. In amplitude coefficients \( a_{p1,N+1} \) and \( a_{p1,N+1} \) of the external region, \( \eta_{\text{ex}} \) corresponding to the index change is added to \( b_0 \).

Remaining amplitude coefficients of the perturbed system can be calculated from the above amplitude coefficients with the help of Eqs. (26), (27), (32), and (33). We are now in a position that all the amplitude coefficients of the perturbed system are represented in terms of those of the unperturbed system. The amplitude coefficients of the perturbed system as well as the unperturbed system are indispensable for the evaluation of \( \beta_{p2} \) for general cases.

6. ANALYTICAL EXPRESSION OF PROPAGATION CONSTANT IN PERTURBED SYSTEM

This section gives analytic expressions that are available for general cases. \( E_\theta \) component is used as \( \Psi \) in calculating integral parameters.

A. Propagation Constant of First-Order Perturbation

When \( E_\theta \) is used as \( \Psi \) in Eq. (10), the propagation constant of the first-order perturbation can be expressed with the help of

\[
I_{\text{core}} = 4\pi |\lambda_1|^2 (a_{\psi_1})^2 \left( J_0^2 + J_1^2 - \frac{2}{\kappa_c r_c} I_{\text{core}} J_1 \right),
\]

(37)

\[
I_{\text{ex},a} = 4 \frac{(a_{\psi_1})^2}{\kappa_a^2} \left[ \frac{R_{\text{TE}}}{1 - R_{\text{TE}}} \left\{ |(a_1|^2 + |b_1|^2)(\kappa_a a) \right. \right.

- \left. \left. |(a_1^2 + b_1^2) \cos(\kappa_a a) - i(a_1^2 - b_1^2) \sin(\kappa_a a) \sin(\kappa_a a) \right) \right],
\]

(38)

with \( R_{\text{TE}} \equiv \text{Re}(X_{\text{TE}}) \pm \{ \text{Re}(X_{\text{TE}}) \}^{1/2} \) (see Appendix A). The argument of Bessel function \( J_1 \) is all \( \kappa_c r_c \) in this section, and it is abbreviated for brevity. An expression of \( I_{\text{ex},b} \) can be given where the symbol and subscript \( a \) are replaced by \( b \), and the prime is added to amplitude coefficients in Eq. (38). \( I_{\text{ex},a} \) and \( I_{\text{ex},b} \) are obtained by setting \( N = 0 \), in \( I_{\text{ex},a} \) and \( I_{\text{ex},b} \), respectively. \( I_{\text{core}} \) is obtained using Eq. (11).

B. Propagation Constant of Second-Order Perturbation

We set \( \Psi_{\psi} = E_{\theta} \) and \( \Psi_{p1} = E_{\psi_1} \) in Eq. (12) to evaluate integral parameters. Then, we can use Eq. (18) for the periodic region and Eq. (34) for the external region, and employ amplitude coefficients of the perturbed system derived in Section 5. Integral parameters are shown by neglecting terms in proportion to \( \beta_c \beta_{p1} \) in the following way. For the core, we have

\[
I_{p,\text{core}} = 4\pi A_{p1,e}(a_{\psi_1})^2 \left( J_0^2 + J_1^2 - \frac{2}{\kappa_c r_c} I_{\text{core}} J_1 \right).
\]

(39)

For the periodic region \( a \), we obtain
\[ I_{p,pe,a,b} = \frac{4}{\kappa_a} \left( \frac{\omega_{0b}}{\kappa_a} \right)^2 \left[ 2 - \frac{R_{2b}^C}{R_{TE}^C} \right] \left( -a_1^* b_1 + b_1^* a_1 \right) a \]
\[ + \frac{\sin(\kappa_a a)}{\kappa_a} \left[ a_1^* b_{p11} \exp(-i\kappa_a a) + b_1^* a_{p11} \exp(-i\kappa_a a) \right] \]
\[ \text{after summing up } I_{p,pe,a,b} \text{ for the } m \text{th periodic layer } a \text{ from } m = 1 \text{ to } N. \]
For the periodic region \( b \), we have
\[ I_{p,pe,b} = \frac{4}{\kappa_b} \left( \frac{\omega_{0b}}{\kappa_b} \right)^2 \left[ 2 - \frac{R_{2b}^C}{R_{TE}^C} \right] \left( -1 - \frac{\kappa_b^2}{2 R_{2b}^C} \right)^{1/2} \]
\[ \times \left\{ \left( h_{11}^a a_{p11} + h_{12}^b b_{p11} \right) \left[ -a_1^* b_1 + b_1^* a_1 \sin(\kappa_b b) \exp(-i\kappa_b b) \right] \right. \]
\[ + \left. \left( h_{21}^a a_{p11} + h_{22}^b b_{p11} \right) \left[ -b_1^* b_1 + a_1^* \sin(\kappa_b b) \exp(-i\kappa_b b) \right] \right\} \]
\[ \text{in the similar way.} \]
For the external region \( a \), one obtains
\[ I_{p,ex,a} = \frac{4}{\kappa_a} \left( \frac{\omega_{0a}}{\kappa_a} \right)^2 \frac{R_{2a}^C}{R_{TE}^C} \left[ -1 - R_{2a}^C \right]^{-1} \]
\[ \times \left\{ \left[ 1 - \frac{i}{2} \eta_{pe} a + C_{ex} \exp(iK_{a}A) \right] a |a_1| \right. \]
\[ - \frac{\sin(\kappa_a a)}{\kappa_a} a_1^* b_1 \exp(-i\kappa_a a) + a |b_1| \right\} \]
\[ + \left. \frac{1}{1 - C_{ex} \exp(-iK_{a}A)} \left[ a a_1^* - \sin(\kappa_a a) b_1 \exp(-i\kappa_a a) \right] \right\} \]
\[ \times \left\{ a_1 \left[ 1 - i \eta_{pe} a + C_{ex} \exp(iK_{a}A) \right] b_1 b_1^* \right\} \]
\[ \text{using a relation, } \exp(iK_{a}A) = \exp(-iK_{a}A), \text{ derived from } [17]. \]
\[ \text{The term } \eta_{pe} \text{ represents the effect of index change in the external layer } a, \text{ and } R_{2a}^C \text{ reflects the dependence of number } N \text{ of cladding layer pairs in Eq. (22). Similarly, we have the integral parameter for the external region } b \text{ as} \]
\[ I_{p,ex,b} = \frac{4}{\kappa_b} \left( \frac{\omega_{0b}}{\kappa_b} \right)^2 \frac{R_{2b}^C}{R_{TE}^C} \left[ -1 - R_{2b}^C \right]^{-1} \]
\[ \times \left\{ \left[ 1 - \frac{i}{2} \eta_{pe} b + C_{ex} \exp(iK_{b}A) \right] b |a_1| \right. \]
\[ - \frac{\sin(\kappa_b b)}{\kappa_b} b_1^* a_1 \exp(-i\kappa_b b) + b |b_1| \right\} \]
\[ + \left. \frac{1}{1 - C_{ex} \exp(-iK_{b}A)} \left[ b b_1^* - \sin(\kappa_b b) a_1 \exp(-i\kappa_b b) \right] \right\} \]
\[ \times \left\{ a_1 \left[ 1 - i \eta_{pe} a + C_{ex} \exp(iK_{b}A) \right] b_1 b_1^* \right\} \]
\[ \text{Both } I_{p,ex,a} \text{ and } I_{p,ex,b} \text{ include } C_{ex}, \text{ which reflects on the existence of only the outward waves in the external region, unlike other integral parameters.} \]
All integral parameters in Eqs. (39)–(43) are represented in terms of parameters of the unperturbed system and \( \eta_{pe} \). The propagation constant \( \beta_{pe} \) of second-order perturbation is obtainable by substituting these integral parameters into Eq. (5). Thus, the CL for general cases can be computed.

7. EXPLICIT EXPRESSION OF PROPAGATION CONSTANT UNDER QWS CONDITION
The QWS condition, \( \kappa_a a = \pi / 2 \), has been introduced as a condition where an optical wave is efficiently confined to the core in the Bragg fiber [5]. The QWS condition has been extended to a generalized QWS condition [24]. Under the QWS and generalized QWS conditions, the eigenvalue equation of the Bragg fiber can be simplified [18,24]. Employment of simplified eigenvalue equations leads to an explicit expression of the propagation constant, as shown next. It provides several useful properties of CL.

A. Eigenvalue Equations under Generalized QWS Condition
The simplified eigenvalue equation of the TE mode in Bragg fiber is given by [18,24]
\[ \kappa_a r_c = 2 \frac{\rho_{\infty}}{\rho_0} \sqrt{\frac{\rho_{\infty}}{\rho_0}} \left[ \left( \left( n_a^2 - n_0^2 \right)^2 + \left( \frac{U_{0WSG} \rho_0}{2 \pi r_c} \right)^2 \right)^{1/2} \right] \]
\[ = U_{QWS} \] \[(44) \]
with \( U_{QWS} = j_{1,\varepsilon} = j_{0,\varepsilon+1} \) for the TE\(_{0b}\) mode. Here, \( j_{\varepsilon,\mu} \) and \( j_{\varepsilon,\mu} \) indicate the \( \varepsilon \)th zeros of Bessel function \( J_{\varepsilon} \) and its derivative \( J_{\varepsilon}^1 \), respectively. \( \varepsilon \) and \( \mu \) stand for the azimuthal and radial mode numbers. Then, cladding thicknesses \( a \) and \( b \) are determined by
\[ a / \lambda_0 = \frac{q_1 - 1/2}{2} \left[ \left( n_a^2 - n_0^2 \right)^2 + \left( \frac{U_{0WSG} \rho_0}{2 \pi r_c} \right)^2 \right]^{-1/2} \] \[(45a) \]
\[ b / \lambda_0 = \frac{q_2 - 1/2}{2} \left[ \left( n_b^2 - n_0^2 \right)^2 + \left( \frac{U_{0WSG} \rho_0}{2 \pi r_c} \right)^2 \right]^{-1/2} \] \[(45b) \]
where \( q_1 \) and \( q_2 \) are integers with \( q_2 \geq q_1 \geq 1 \) [24]. Equations (45a) and (45b) consist of core, cladding, and mode parameters. If we set \( q_1 = q_2 = 1 \) in Eqs. (45a) and (45b), then the above expressions are reduced to the prior results under the QWS condition [18]. It is found from Eq. (45a) that cladding thickness \( a \) can be increased by increasing the value of \( q_1 \) with the remaining parameters unchanged.

Guided modes are allowed for \( \left| \text{Re}(X_{TE}) \right| > 1 \) [17]. Which Bloch wavenumber \( K_1 \) or \( K_2 \) is chosen depends on the value of \( \text{Re}(X_{TE}) \). We make use of \( K_1 \) for \( \text{Re}(X_{TE}) < -1 \), including the fundamental photonic band, and we offer explicit expressions under the QWS condition in this section.

B. Propagation Constant of First-Order Perturbation
Integral parameters relating to unperturbed fields are obtained to be
\[ I_{tot} = 4 \pi |A_0|^2 \left( \omega_{0b} \right)^2 \left( \frac{\rho_{\infty}}{\rho_0} \right) \]
\[ \times \frac{\rho_c}{U_{QWS}} \left[ 1 + \frac{1}{n_b^2 - n_a^2} \frac{\Lambda}{\eta_{pe} r_c} \left( \frac{U_{QWS} \rho_0}{2 \pi r_c} \right)^2 \right] \] \[(46) \]
\[ I_{\text{core}} = 4\pi |A_1|^2 (a\mu_0)^2 J_0^2(U_{\text{QWS}}) \frac{U_{\text{QWS}}^2}{k_0^2}. \]  

(47)

\[ I_{\text{ex.a}} = \frac{1}{\pi} |A_1|^2 (a\mu_0)^2 J_0^2(U_{\text{QWS}}) \left( \frac{a}{b} \right)^{2N} \frac{r_2 a_0^2}{n_a^2 - n_b^2}. \]  

(48)

Expressions for \( I_{\text{ex.b}}, I_{\text{ex.a}}, \) and \( I_{\text{ex.b}} \) are obtained in a manner similar to that given just after Eq. (38). It is natural that the value shown in Eq. (46) is real and positive.

The propagation constant of the first-order perturbation can be given as

\[ \beta_{p1} = -\frac{(a/b)^{2N} a U_{\text{QWS}}^2}{2(b/a) r_c k_0^2} \left[ 1 + \frac{1}{n_a^2 - n_b^2} \frac{\Delta (U_{\text{QWS}}^2)}{\beta (U_{\text{QWS}})} \right]^{-1}. \]  

(49)

by substituting Eqs. (46) and (48) into Eq. (10). Equation (49) brings out several properties, which are described later because they are almost identical to \( \beta_{p2} \), namely, CL.

Let us estimate the magnitude of several important parameters under the QWS condition. For the TE01 mode \( (U_{\text{QWS}} = 3.83171) \) with \( n_a = 2.5, n_b = n_{\text{ex}} = 1.5, n_c = 1.0, r_2 = 2.0 \mu m, \) and \( \lambda_0 = 1.0 \mu m \), we have \( \beta_{a1}/k_0 = 0.9524 \) from Eq. (44). In addition, we obtain \( \beta_{p1}/k_0 = -2.63 \times 10^{-3} \) from Eq. (49). One obtains \( \beta_{p1}/k_0 = -2.64 \times 10^{-6} \) for \( N = 5 \) and \( 10 \), respectively. Value of the second term in square brackets of Eq. (49) is \( 3.76 \times 10^{-3} \), which is negligibly small compared to the unity for \( r_c/\lambda_0 \gg 1 \).

We compare the value of \( (n_a^2 - n_b^2)k_0^2 \) with \( 2\beta_{p1} \) in the \( \eta_{p2} \) of Eq. (21). The former is nearly of the same order as that of \( 2\beta_{p2} \) since \( (n_a^2 - n_b^2) \) is nearly of the same order as that of \( n_c \). Hence, the ratio of the former to the latter is roughly identical to the ratio of \( \beta_{p2} \) to \( \beta_{p1} \). The fact of \( \beta_{p2} \gg \beta_{p1} \) indicates that \( (n_a^2 - n_b^2)k_0^2 \) is markedly larger than \( 2\beta_{p1} \).

C. Expression and Properties of CL

The \( \beta_{p2} \) can be calculated by substituting integral parameters, Eqs. (30)-(49), into Eq. (8). CL is obtained from the imaginary part of the propagation constant. \( \text{Im}(I_{\text{p.ex.}}) \) mainly contributes to the CL of the TE mode because \( (n_a^2 - n_b^2)k_0^2 \gg |2\beta_{p1}|. \) We show here only the results that apply to \( \text{Re}(X_{\text{TE}}) < -1. \) An explicit expression of CL results in

\[ \text{Im}(\beta_{p2}) = -\frac{(n_a^2 - n_b^2)^2 (a/b)^{2N} U_{\text{QWS}}^2}{\pi^2 (b/a) k_0^2} \]  

\[ \times \left[ 1 + \frac{1}{n_a^2 - n_b^2} \frac{\Delta (U_{\text{QWS}}^2)}{\beta (U_{\text{QWS}})} \right]^{-1} \]  

\[ \times \left\{ \frac{b/a}{(4/4a)(2+b/a+a/b)} + \frac{1}{1-(a^2/b^2)(2+b/a+a/b)} \right\}. \]  

(50)

under the generalized QWS condition.

We can deduce several important properties from Eq. (50) as follows.

i. \( \text{Im}(\beta_{p2}) \) is always negative because the factor including \( a/b \) is positive for \( 0 < a/b < 1 \), namely, \( n_a > n_b \).

ii. \( \text{Im}(\beta_{p2}) \) includes the dependence of \( (a/b)^{2N} \) on the number \( N \) of finite periodic layer pairs. Its absolute value is reduced with an increase in \( N \) because \( 0 < a/b < 1 \).

iii. \( \text{Im}(\beta_{p2}) \) is proportional to \( r_c^3 \) and \( U_{\text{QWS}}^2 \). The last term is peculiar to the fact that only outward waves exist in the external region.

iv. Moreover, the dependence of \( (n_a^2 - n_b^2)^2 \) is added to the above result.

vi. If fiber structural parameters, such as \( r_c, a, \) and \( b, \) are changed so as to satisfy the QWS condition for every wavelength \( \lambda_0 \) at constant refractive indices, then the right-hand side of Eq. (50) is independent of \( \lambda_0 \). Hence, the CL value is in inverse proportion to \( \lambda_0 \). This means that the CL value is small for a long wavelength when the QWS condition is always maintained.

Items (i)-(iii) also hold for \( \beta_{p1} \), as can be seen from Eq. (49).

The \( r_c^3 \) dependence of CL of the TE mode in Bragg fibers has been shown by a consideration of the field form \( \{7\} \) and by numerical examples for the large \( r_c/\lambda_0 \). The agreement is reasonable because the asymptotic expansion approximation is valid for \( r_c/\lambda_0 \gg 1 \). The \( (a/b)^{2N} \) dependence of the TE mode has been pointed out and confirmed by numerical examples \( \{12\} \). The \( U_{\text{QWS}}^2 \) dependence of CL is presented here for the first time. This means that higher-order modes exhibit a large CL for fixed structural parameters, as confirmed in Subsection 8.A. It has been stated that the TE01 mode shows the lowest CL among all the modes \( \{7,12\} \).

8. VERIFICATION OF CL BASED ON PRESENT THEORY

This section is devoted to the verification of CL derived from the present theory. The CL of the Bragg fiber has been calculated by various methods, as stated in Section 1. It has numerically been shown \( \{12\} \) that the MLD results are in excellent agreement with those of the transfer matrix and Chew’s methods. Therefore, we use the results of the MLD method as the touchstone of the CL values. Most fiber parameters treated here are set to meet the QWS condition for infinite cladding layer pairs, even though the present paper is targeted for Bragg fibers with finite cladding layer pairs.

In the following examples in this section, fiber parameters are always set to satisfy the QWS condition, namely, Eqs. (45a) and (45b) with \( q_1 = q_2 = 1 \) except for Fig. 6. Therefore, Eq. (50) is used to evaluate the CL for the present theory except for Fig. 2. The core index is assumed to be \( n_c = 1.0 \) throughout this paper.

A. Confirmation of Mode Dependence

Figure 2 shows the normalized CL divided by \( U_{\text{QWS}}^2 \) as a function of core radius, where the CL of the TE0y mode is redrawn from the previous result numerically evaluated by the MLD method \( \{12\} \). As the core radius \( r_c \) increases, the normalized CL with identical \( n_a \) converges to a certain value despite radial mode number \( \mu \). This supports the mode dependence described in item (iii) of Subsection 7.C, which is highly precise, especially for \( r_c \geq 7.0 \mu m \). This agreement for large \( r_c \) is due to the fact that the asymptotic expansion approximation secures a high accuracy for large \( r_c \). The converging value depends on
the refractive index because the second term in square brackets of Eqs. (45a) and (45b) is negligibly small compared to its first term, and hence both $a$ and $b$ are nearly independent of $r_c$ and the mode number.

B. Dependence on Number of Cladding Layer Pairs

The dependence of CL on number $N$ of cladding layer pairs is semilogarithmically shown in Fig. 3 for several $\text{TE}_{0\mu}$ modes to compare the results of the present theory with those of the MLD method. CL exhibits a nearly linear change with $N$. Slope $S$ for each mode is almost identical over the entire region, since CL is proportional to $(a/b)^{2\mu}$ for the TE mode [12]. For example, the ratios of the values by the present at $N = 10$ to those by the MLD method are 0.209, 0.226, and 0.250 for $\text{TE}_{01}$, $\text{TE}_{02}$, and $\text{TE}_{03}$ modes, respectively. This implies that the CL value of the present theory ranges from about one fifth to one fourth of that of the MLD method. CL is low for lower-order modes among the $\text{TE}_{0\mu}$ modes, as expected from the value of $U_{\text{QWS}}$ in Eq. (50).

C. Dependence on Core Radius

The core radius dependence of CL for the $\text{TE}_{0\mu}$ modes is logarithmically plotted in Fig. 4 to compare the results of both methods. CL gradually decreases against the core radius except for the small core radius, which is close to the guiding limit. CL is inversely proportional to $r_c^2$ for the large $r_c$ of the $\text{TE}_{0\mu}$ modes, as seen from the present theory. $r_c^{2\mu}$ dependence has been shown by a consideration of field form [2] and by numerical means [11]. The tendency of CL is nearly the same for both methods, although the values of the present theory are about one fifth of those of the MLD method. For example, the ratios are 0.200, 0.201, and 0.202 for the $\text{TE}_{01}$, $\text{TE}_{02}$, and $\text{TE}_{03}$ modes at $r_c = 10.0 \mu m$.

D. Dependence on Cladding Index Contrast

The cladding index contrast dependence of CL is shown in Fig. 5 for $\text{TE}_{0\mu}$ modes with $n_c = n_{\text{ex}} = 1.5$. CL is reduced with increasing $n_c$. The discrepancy in the results between the present theory and MLD method increases with an increase in $n_c$ and is nearly the same for a fixed value of $n_c - n_{\text{ex}}$, despite mode number $\mu$. For example, we have loss ratios of 0.209, 0.125, and 0.0860 at $n_c = 2.5$, 3.5, and 4.5 for the $\text{TE}_{01}$ mode. The difference in the results between the present theory and MLD method is noteworthy for the cladding index contrast compared to other fiber parameters. The ratios of the present theory to the MLD method are about one fifth, one eighth, and one twelfth for $n_c = 2.5$, 3.5, and 4.5. The present
theory is useful for evaluating CL, since it provides a simple calculation method.

E. Dependence on Wavelength

Figure 6 semilogarithmically indicates the wavelength dependence of CL for the TE\(r_o\) modes to compare the results of the present theory with those of MLD method. Fiber structure parameters are set to satisfy the QWS condition at \(\lambda_{QWS} = 1.0 \mu m\). After the fiber structure was fixed, the wavelength was varied. The loss is low for lower-order modes among the TE\(r_o\) modes and shows its minimum value in the neighborhood of the QWS condition. The loss discrepancy between both methods is maximum near the QWS condition. CL values of the present theory are about one fifth and one fourth of the MLD method for the TE\(r_{01}\) and TE\(r_{03}\) modes, respectively, near the QWS condition.

Summarizing these results, the present theory is inclined to underestimate the CL under the QWS condition. The difference in CL between both methods is noteworthy in the cladding high-index dependence. However, the present theory enables us to compare CL values relatively.

9. RELATIONSHIP OF CL WITH GENERALIZED QWS CONDITION

This section presents the numerical results of CL, mainly the cladding layer thickness dependence, and studies the relationship of CL characteristics with the generalized QWS condition. Hence, we exploit the CL expression for general cases. The wavelength is fixed at \(\lambda_0 = 1.0 \mu m\) in most cases.

A. Some Expressions under Generalized QWS Condition

The relation between the photonic bandgap and antiresonance was presented in the SPARROW model \[13\]. The in-phase condition is equivalent to a generalized QWS condition in the 1D and cylindrically symmetric 2D structures with periodic cladding, and, moreover, the generalized QWS condition is identical with the central gap point in the SPARROW model \[24\].

When the Bragg fiber exactly satisfies the generalized QWS condition, Eqs. (45a) and (45b), effective index \(n_{eff}/k_0\) is given by \[18, 24\].

\[
\tilde{n}_{co} = \left[ n_c^2 - \frac{U_{QWS}^2}{2\pi q_c} \right]^{1/2}. 
\]

Equation (51) includes the core and mode parameters.

The generalized QWS condition, namely, the central gap point, can also be represented using only cladding parameters in terms of the wavelength \[13, 24\]:

\[
\lambda_{QWS} = 2 \left\{ \frac{1}{n_c^2 - n_a^2} \left[ \frac{(q_1 - 1/2)^2}{a} - \frac{(q_2 - q_1 + 1/2)^2}{b} \right] \right\}^{1/2}
\]

and the effective index:

\[
\tilde{n}_{cl} = \left( \frac{n_c^2 - n_a^2 + b}{1 - n_a^2} \right)^{1/2}
\]

with \(n_a \equiv [(q_1 - 1/2)b]/[(q_2 - q_1 + 1/2)a] \). The expressions in Eqs. (52) and (53) agree exactly with those in \[13\] by setting \(q_1 = m_1\) and \(q_2 - q_1 = m_2\).

B. Cladding Layer Thickness Dependence of CL

Although cladding layer thickness \(b\) is set to satisfy the QWS condition in Eq. (45b) for \(n_0, n_a, \) and \(r_c\) for the moment, layer thickness \(a\) is arbitrarily changed. Figure 7 shows the \(a\) dependence of the CL of the TE\(r_{01}\) mode as a function of core radius \(r_c\). We show only the guided modes that satisfy \(\text{Re}(\mathbf{X}_{TE}) > 1\) \[17\]. Hence, the region without curves corresponds to the radiation mode. This peculiar property is attributed to the fact that the optical wave is confined to the core based on the photonic bandgap mechanism in the Bragg fiber. Regions with finite CL values are regarded as photonic bands. The CL for the guided mode decreases with increasing \(r_c\) at a fixed value of \(a\). We find, for any case, that CL exhibits its minimum value near the cross where cladding layer thickness \(a\) meets the generalized QWS condition. CL is the lowest at \(q_1 = q_2 = 1\), namely, the QWS condition. For example, the value of \(a\) is 0.1061 \(\mu m\) at the lowest CL and is 0.1082 \(\mu m\) under the QWS condition for \(r_c = 2.0 \mu m\). Their relative difference in \(a\) is about 2.0% for \(r_c = 2.0\) to 10.0 \(\mu m\). Their CL values have also nearly the same relative difference as in \(a\). The values of \(a\) showing the minimum CL hardly depend on \(r_c\) for each.
photonic band. As \( r_c \) changes from 2.0 to 5.0 \( \mu m \), CL is reduced by about 1.5 orders.

The dependence of CL on cladding layer thickness \( a \) is plotted in Fig. 8 for the \( TE_{01} \) mode as a function of cladding high-index \( n_a \). We can also see the regions where no guided modes exist, as in Fig. 7. As \( n_a \) increases, the CL for the guided mode greatly decreases and thickness \( a \) exhibiting the minimum loss shifts to a small value. There is a tendency that the minimum loss is achieved near the generalized QWS condition and the lowest loss is obtained near the QWS condition: at \( q_1 = q_2 = 1 \). In addition, the spacing between the photonic bands becomes narrow with increasing \( n_a \) because the generalized QWS condition depends much more on the cladding layer indices than on other parameters, as seen from Eqs. (45a) and (45b).

The \( a \) dependence of CL is shown in Fig. 9 for several \( TE_{n0} \) modes. CL is low for the \( TE_{01} \) mode and increases by about one order every radial mode number \( \mu \). The minimum loss can be found at thickness \( a \), where the generalized QWS condition is satisfied. The lowest loss is also obtained near the QWS condition despite mode number \( \mu \).

Contrary to the above, cladding layer thickness \( b \) is arbitrarily changed on the condition that thickness \( a \) is kept at the QWS condition. Figure 10 shows the CL versus \( b \) relation of the \( TE_{01} \) mode as a function of core radius \( r_c \). As \( r_c \) increases, CL decreases and thickness \( b \) exhibiting the minimum loss is nearly unchanged. The minimum loss tends to be obtained near the generalized QWS condition and the lowest loss is obtained near the QWS condition, as in the case of the \( a \) change. For example, the value of \( b = 0.2154 \mu m \) at the lowest CL and is 0.2157 \( \mu m \) under the QWS condition for \( r_c = 2.0 \mu m \). Their relative differences in \( b \) are 0.15 to 0.37% for \( r_c = 2.0 \) to 10.0 \( \mu m \). The difference in cladding thickness for the QWS condition and the minimum CL is smaller in \( b \) than in \( a \).

From Figs. 7–10, CL exhibits its minimum loss in the vicinity of the generalized QWS condition and the lowest loss is obtained almost at the QWS condition if cladding layer thickness \( b \) or \( a \) satisfies the QWS condition. Cladding thickness \( a \) showing the minimum CL is nearly independent of the core radius and the radial mode number, but it strongly depends on the cladding index contrast.

The border between the guided and radiation modes is determined from the value of \( \text{Re}(X_{TE}) \): guided and radiation modes satisfy \( |\text{Re}(X_{TE})| > 1 \) and \( |\text{Re}(X_{TE})| \leq 1 \), respectively. The contour curve of \( \text{Re}(X_{TE}) = \pm 1 \) is illustrated in Fig. 11 for \( \lambda_0 = 1.0 \mu m \), \( r_c = 2.0 \mu m \), \( n_a = 2.5 \), and \( n_b = n_{ex} = 1.5 \) and \( N = 10 \). Values of cladding layer thickness \( b \) are 0.2157, 0.2001, and 0.1811 \( \mu m \) for \( TE_{01} \), \( TE_{02} \), and \( TE_{03} \) modes that meet Eq. (45b) with \( q_1 = q_2 = 1 \), namely, the QWS condition. Crosses indicate positions at which \( a \) satisfies the generalized QWS condition. Three crosses for each mode correspond to \( q_1 = 1, 2, \) and 3 from the left.

**C. How to Find Operation Condition Exhibiting Low CL**

We plot the wavelength dependence of CL of the \( TE_{01} \) mode in Fig. 12 for point A in Fig. 11. The fiber parameters are set to satisfy the QWS condition at \( \lambda_{QWS} = 1.0 \mu m \). The results of the MLD method are redrawn from a previous paper [24]. We can see good agreement of the results between the two methods. The loss values near \( 10^6 \text{dB/km} \) for \( N = 20 \) are attributed to the computation limit of the MLD method. The present theory shows that no CL values appear near \( \lambda_{QWS}/t \) with even \( t \).
Figure 12 shows that the fundamental photonic band appears near $\lambda_{\text{QWS}} = 1.0 \, \mu m$. From Eqs. (52) and (53), $\lambda_{\text{QWS}} = 1.0, 0.933$, and $0.2 \, \mu m$ for three combinations of $q_1 = 1$ and $q_2 - q_1 = 0$, $q_2 - q_1 = 2$, and $q_2 - q_1 = 1$, and $q_1 = 3$ and $q_2 - q_1 = 2$, respectively, and $\tilde{n}_c = 0.9524$ for three combinations. On the other hand, $\tilde{n}_{co} = 0.9524, 0.9948$, and 0.9981 for the three corresponding combinations despite $N$ from Eq. (51). For the three cases, both the present theory and MLD method provide identical effective indices of 0.9524, 0.9948, and 0.9981 because the values of $|p_{n1}/k_0|$ are of the order of $10^{-8}$ and $10^{-9}$ for $N = 5$ and 10. It is natural that Eq. (51) gives a precise effective index under the exact generalized QWS condition. This means that the core parameters are indispensable for the precise estimation of the effective index.

Point B in Fig. 11 indicates where both $q_1$ and $q_2 - q_1$ are half integers, namely, $q_1 = 1.5$ and $q_2 - q_1 = 0.5$ in Eqs. (45a) and (45b), at $\lambda_0 = 1.0 \, \mu m$. After the fiber structural parameters are fixed at point B, only the wavelength is varied (Fig. 13). The fiber has a guided region near $\lambda_0 = 2.0 \, \mu m$. This can be explained as follows. After substituting the cladding parameters into Eq. (52), we have $\lambda_{\text{QWS}} = 2.000 \, \mu m$ for $q_1 = q_2 = 1$, $\lambda_{\text{QWS}} = 0.667 \, \mu m$ for $q_1 = 2$ and $q_2 = 3$, and $\lambda_{\text{QWS}} = 0.400 \, \mu m$ for $q_1 = 3$ and $q_2 = 5$. Equation (53) gives an effective index of $\tilde{n}_c = 0.9524$ for these three combinations, and Eq. (51) gives $\tilde{n}_{co} = 0.9925$ at $\lambda_0 = 0.4 \, \mu m$, $\tilde{n}_{co} = 0.9791$ at $\lambda_0 = 0.667 \, \mu m$, and $\tilde{n}_{co} = 0.7925$ at $\lambda_0 = 2.0 \, \mu m$. Actually, since we have $\beta_n/k_0 = 0.9664, 0.9790$, and 0.9790 for $\lambda_0 = 0.4, 0.667$, and $2.0 \, \mu m$, respectively, we can estimate $\lambda_{\text{QWS}}$ but not the effective index from only the cladding parameters, as in Fig. 12. Point B satisfies the QWS condition at $\lambda_{\text{QWS}} = 2.0 \, \mu m$.

CL also shows its minimum loss at $\lambda_0 = 2.0 / \ell \, \mu m$ with odd $\ell$. The phase theory explained that CL shows a minimum loss at $\lambda_{\text{QWS}} / \ell$ with odd $\ell$ under the generalized QWS condition [24]. This condition is equivalent to that in which waves reflected from each cladding interface become in phase just inside the core–cladding boundary of the Bragg fiber. Note from Fig. 13 that no CL values appear near $\lambda_0 = 1.0$ and $0.5 \, \mu m$, where guided modes exist. These wavelengths correspond to $\lambda_{\text{QWS}} / \ell$ with even $\ell$. The present theory shows that the CL is too high to exhibit in the neighborhood of $\lambda_{\text{QWS}} / \ell$ with even $\ell$.

Point C in Fig. 11 indicates the position where $q_1$ is a half integer and $q_2 - q_1$ is an integer: $q_1 = 1.5$ and $q_2 - q_1 = 0$ at $\lambda_0 = 1.0 \, \mu m$. If we use the same procedure as in Fig. 12 to find $\lambda_{\text{QWS}}$ under a restriction of integers $q_i$, we find them for a combination of large $q_i$s; that is, $q_1 = 5$ and 7, for $0 < n_{cl} < 1$.

How do we estimate the wavelength of the photonic band for general cladding thicknesses? The width of the fundamental photonic band is not so narrow. We seek a combination of real $q_1$ and $q_2$ near $q_1 = q_2 = 1$ such that the real solutions of $\lambda_{\text{QWS}}, \tilde{n}_{co}$, and $\tilde{n}_c$ exist simultaneously and the value of $|\tilde{n}_{co} - \tilde{n}_{cl}|$ is as small as possible. For cladding indices and thicknesses at point C, we seek solutions by changing $q_1$ and $q_2$ around solution candidates every 0.01 steps. Then, we have $\lambda_{\text{QWS}} = 1.486 \, \mu m, \tilde{n}_{co} = 0.8915$, and $\tilde{n}_c = 0.8931$ for $q_1 = 1.18$ and $q_2 = 1.03$.

Figure 14 indicates the CL of the TE$_{01}$ mode with fiber structural parameters at point C. The fundamental photonic

![Fig. 12.](image-url) Color online) Wavelength dependence of CL of the TE$_{01}$ mode for two kinds of N. Cladding thicknesses, $a = 0.1082 \, \mu m$ and $b = 0.2175 \, \mu m$, are set at point A in Fig. 11. $r = 2.0 \, \mu m$, $n_c = 2.5$, and $n_g = n_{ex} = 1.5$. Solid curves, perturbation theory; dotted-dashed curves, MLD method ($N = 10$); dashed curves, MLD method ($N = 20$). Crosses indicate data at wavelengths satisfying the generalized QWS condition.

![Fig. 13.](image-url) (Color online) Wavelength dependence of CL for the TE$_{01}$ mode with cladding thicknesses at point B in Fig. 11. $r = 2.0 \, \mu m$, $n_c = 2.5$, $n_g = n_{ex} = 1.5$, $a = 0.2163 \, \mu m$, $b = 0.7315 \, \mu m$, and $N = 10$. Solid circles indicate data at wavelengths satisfying the generalized QWS condition.

![Fig. 14.](image-url)
band is situated near $\lambda_0 = 1.55 \mu m$, which is shifted from the above $\lambda_{QWS}$ by about 4.1%. This slight discrepancy implies that the above estimation is reasonable. We have $\beta_0/k_0 = 0.8966$ at $\lambda_0 = 1.486 \mu m$ from the numerical data of the present theory. We cannot estimate the low CL regions using $\lambda_{QWS}/\ell$ with odd $\ell$ because there is no $\lambda_{QWS}/\ell$ in the present case. The second and third photonic bands cannot be found in a manner similar to the above because their width is narrow. In the present case, far from the exact generalized QWS condition, we can neither find a high loss near $\lambda_0 = 0.5 \mu m$ nor in the wavelength between the 0.75 and 1.55 $\mu m$ bands.

It is possible to estimate the photonic band wavelength and the effective index using Eqs. (52) and (51) when the Bragg fiber approximately satisfies the generalized QWS condition. If the photonic band is set at a certain wavelength, we can employ Eqs. (45a) and (45b) to design the fiber structural parameters.

10. DISCUSSION

Although the present theory can predict the dependence of CL on fiber structural parameters, the number of finite periodic layer pairs, wavelength, and mode number in an explicit form, it usually underestimates CL, as shown in Section 8. The reason is explored next.

CL is evaluated from the imaginary part $\text{Im}(\beta)$ of the propagation constant. $\text{Im}(\beta)$ is extremely small compared to its real part $\text{Re}(\beta)$. The asymptotic expansion approximation and perturbation theory are used here to formulate CL. According to the present theory, fields in external layer $a$ mainly contribute to $\text{Im}(\beta)$, as shown in Section 7. The perturbation is given in the external region, where electromagnetic fields are markedly small in the unperturbed system. Amplitude coefficients of the perturbed system have a strong dependence on amplitude coefficient $b_1$, corresponding to the inward wave of the unperturbed system, as shown in Section 5. It is required for CL to calculate perturbed fields as well as unperturbed fields with high precision. The asymptotic expansion approximation secures good accuracy for large $r_e/\lambda_0$, namely, $\kappa_2 r_e \gg 1$. However, since perturbed fields are roughly obtained by differentiating unperturbed fields, their accuracy is degraded by the differentiation. The accuracy is insufficient to evaluate the subtle $\text{Im}(\beta)$, although it gives satisfactory accuracy for $\text{Re}(\beta)$. Therefore, the discrepancy between the present theory and previous numerical results is attributed to the use of asymptotic expansion approximation.

11. CONCLUSION

An analytical expression of CL was presented for the TE mode of Bragg fibers using the perturbation theory. Under the generalized QWS condition, the present theory leads to a formula of CL that is proportional to $(a/b)^{N r_e^2 \mu} U_{QWS}$ with the core radius $r_e$, the number $N$ of finite periodic layer pairs, the zero $U_{QWS}$ of the Bessel function peculiar to each mode, and cladding layer thicknesses $a$ and $b$.

The present theory is useful for characterizations of CL, as several dependencies on the fiber structural parameters and the wavelength. However, the present theory has a tendency to underestimate CL. A significant difference can be seen in the dependence of CL on cladding high-index $n_a$. Although the present theory usually underestimates CL, it provides deep insight into the PBF.

We scrutinized the relation between CL and the generalized QWS condition. CL exhibits its minimum loss near the generalized QWS condition. How to find the photonic band was shown for general cases that do not satisfy the generalized QWS condition.

APPENDIX A: SEVERAL PROPERTIES OF PERIODIC STRUCTURE UNDER ASYMPTOTIC EXPANSION APPROXIMATION

For the Bragg fiber with infinite periodic cladding, amplitude coefficients of cladding layers $a$ and $b$ are related with each other as $17,18$

\[ a_{a,m} = H_a a_{b,m}, \quad a_{a,m+1} = H_b a_{a,m} \]  

under the asymptotic expansion approximation. Here, matrix $H_a$ is identical to that in Eq. (24), and matrix elements $H_a$ and $H_b$ are independent of $m$. Usage of the Bloch theorem gives an eigenvalue of the product of matrices $H_a H_b$ as

\[ R_{TE} \equiv \exp(-iK/L) = \text{Re}(X_{TE}) \pm \{\text{Re}(X_{TE})^2 - 1\}^{1/2} \]

with

\[ X_{TE} = \left[ \cos(k_0 b) - i \left( \frac{\kappa_a}{\kappa_e} + \frac{\kappa_b}{\kappa_e} \right) \sin(k_0 b) \right] \exp(-i\kappa_e a). \]

Here, upper and lower signs correspond to $j = 1, 2$, and Bloch wavenumbers $K_1$ and $K_2$ apply to $\text{Re}(X_{TE}) < -1$ and $\text{Re}(X_{TE}) > 1$, respectively. $H_a H_b$ can be transformed into a scalar value $\exp(-iK/L)$ in the periodic region but not in the external region. In addition, amplitude coefficients between plural cladding layers $a$ are related to each other as

\[ a_{a,m} = \exp(-iK_j/L)a_{a,m-1} = \exp[-iK_j(m-1)\Lambda]a_{a,1}. \]

Amplitude coefficients of cladding layers $b$ are obtained by adding the prime to each $a_{a,m}$.
APPENDIX B: AMPLITUDE COEFFICIENTS OF PERTURBED SYSTEM

From Eq. (36a), we have the amplitude coefficient

\[
A_{p1,1}(\frac{\nu p a r}{2} J_{a}^{1/2}) = \frac{1}{J_{a}} \left( \begin{array}{c} b_{1} + \beta_{a} \frac{p_{1} \nu a r}{2} b_{1} A_{\nu p a r} \left( \frac{\nu p a r}{2} \right)^{1/2} \\
J_{0} + i \frac{\nu a}{\nu b} \left( J_{0} + 2J_{a} \right) \\
+ i \beta_{a} \frac{p_{1} \nu a r}{2} b_{1} \frac{J_{0}}{J_{a}} + \frac{1}{J_{a}} \exp(iK_{\lambda}) \frac{\nu a}{\nu b} \left( J_{0} + 2J_{a} \right) \right] \\
\times \left( \frac{\nu a}{\nu b} \right)^{1/2} \left( -a_{j} h_{21}^{0} + b_{j} h_{22}^{0} \right) \right) \quad (B1)
\]

for the core and

\[
\frac{a_{p1,N+1}}{\exp(-iK_{\lambda} N \nu b)} = -b_{1} J_{a}^{1/2} - i a_{1} \eta_{p b} \nu b b_{1} \nu a r \\
+ i \beta_{a} \frac{p_{1} \nu a r}{2} \left( 2A_{\nu p a r} \left( \frac{\nu p a r}{2} \right)^{1/2} J_{0} \left( J_{0} + J_{a} \right)^{2} \\
+ \frac{Na}{\nu a} \frac{J_{a}}{J_{a}} \left( a_{1} + b_{1} \right) \\
+ \frac{Nb}{\nu b} \exp(iK_{\lambda} \nu a r) \left( \frac{\nu a}{\nu b} \right)^{3/2} \left[ a_{1} \exp(-iK_{\lambda} \nu b \nu a) J_{a}^{1/2} \\
+ b_{1} \exp(iK_{\lambda} \nu b \nu a) J_{a}^{1/2} \right] \right) \quad (B2)
\]

for the innermost external layer \( a \), in a normalized form. Here, the argument \( \nu a r \) of \( J_{a} \) and \( J_{a} \) is suppressed.

Use of Eq. (B1) leads to amplitude coefficients of innermost periodic layer \( a \) as

\[
a_{p1,1} = -b_{1} J_{a}^{1/2} + i \beta_{a} \frac{p_{1} \nu a r}{2} \left[ -a_{j} \frac{\nu a r}{\nu a} \left( \frac{J_{0}}{J_{a}} + \frac{J_{a}}{J_{a}} \right)^{2} \\
+ 2A_{\nu p a r} \left( \frac{\nu p a r}{2} \right)^{1/2} J_{0} \left( J_{0} + J_{a} \right)^{2} \\
+ \frac{1}{J_{a}} \exp(iK_{\lambda} \nu a r) \left( \frac{\nu a}{\nu b} \right)^{1/2} \left( -a_{j} \frac{\nu a r}{\nu a} \left( \frac{J_{0}}{J_{a}} + \frac{J_{a}}{J_{a}} \right) J_{a}^{1/2} \right) \right) \quad (B3)
\]

\[
b_{p1,1} = -b_{1} + i \beta_{a} \frac{p_{1} \nu a r}{2} \left[ b_{j} \frac{\nu a r}{\nu a} \left( \frac{J_{0}}{J_{a}} + \frac{J_{a}}{J_{a}} \right) \\
+ \frac{1}{J_{a}} \exp(iK_{\lambda} \nu a r) \left( \frac{\nu a}{\nu b} \right)^{1/2} \left( -a_{j} \frac{\nu a r}{\nu a} \left( \frac{J_{0}}{J_{a}} + \frac{J_{a}}{J_{a}} \right) J_{a}^{1/2} \right) \right]. \quad (B4)
\]

Substituting Eq. (B2) into Eq. (33) with \( q = 1 \), we have the amplitude coefficient of innermost external layer \( b \) as

\[
\frac{d_{p1,N+1}}{\exp(-iK_{\lambda} N \nu a r)} = -a_{j} \left( 1 + i \eta_{p b} \nu a r \right) \left( \frac{\nu a}{\nu b} \right)^{1/2} \left( 1 + \frac{\nu a}{\nu b} \right) \exp(-iK_{\lambda} \nu a r) \left( \frac{\nu a}{\nu b} \right)^{1/2} \\
\times \left( a_{1} - b_{1} \right) J_{a}^{1/2} - i a_{1} \eta_{p b} \nu a r \\
+ i \beta_{a} \frac{p_{1} \nu a r}{2} \left( 2A_{\nu p a r} \left( \frac{\nu p a r}{2} \right)^{1/2} J_{a}^{1/2} \left( J_{0} + J_{a} \right)^{2} \\
+ \frac{Na}{\nu a} \left( a_{1} + b_{1} \right) J_{a}^{1/2} + \frac{Nb}{\nu b} \exp(iK_{\lambda} \nu a r) \left( \frac{\nu a}{\nu b} \right)^{3/2} \left[ a_{1} \exp(-iK_{\lambda} \nu b \nu a) \\
+ b_{1} \exp(iK_{\lambda} \nu b \nu a) \right] \right) \quad (B5)
\]

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