Modeling of Linear and Belt Object Deformation Based on Differential Geometry

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Agenda

- Introduction
- Modeling of Linear Object Deformation
- Application to Linear Object Structure
- Modeling of Belt Object Deformation
- Conclusions
Manipulation of Flexible Linear/Belt Objects

A modeling of linear/belt object deformation is required for planning of manipulative operations and their execution by a mechanical system.
Frenet-Serret Formulas

Frenet-Serret formulas

Angular velocities of rigid body
Assumption:

Deformation in any direction perpendicular to the central axis of a linear object is negligible.
Rotation Matrix

Rotation matrix:

\[
A = \begin{bmatrix}
\cos \theta \cos \phi \cos \psi - \sin \phi \sin \psi & -\cos \theta \cos \phi \sin \psi - \sin \phi \cos \psi & \sin \theta \cos \phi \\
\cos \theta \sin \phi \cos \psi + \cos \phi \sin \psi & -\cos \theta \sin \phi \sin \psi + \cos \phi \cos \psi & \sin \theta \sin \phi \\
-\sin \theta \cos \psi & \sin \theta \sin \psi & \cos \theta
\end{bmatrix}
\]
Rotation of Object Coordinate System

\[
\begin{bmatrix}
\xi(s) \\
\eta(s) \\
\zeta(s)
\end{bmatrix} =
\begin{bmatrix}
0 & \omega_\zeta & -\omega_\eta \\
-\omega_\zeta & 0 & \omega_\xi \\
\omega_\eta & -\omega_\xi & 0
\end{bmatrix}
\begin{bmatrix}
\xi(s) \\
\eta(s) \\
\zeta(s)
\end{bmatrix}
\]

Infinitesimal rotational angles:

\[
\begin{align*}
\omega_\xi &= \frac{d\theta}{ds} \sin \psi - \frac{d\phi}{ds} \sin \theta \cos \psi \\
\omega_\eta &= \frac{d\theta}{ds} \cos \psi + \frac{d\phi}{ds} \sin \theta \sin \psi \\
\omega_\zeta &= \frac{d\phi}{ds} \cos \theta + \frac{d\psi}{ds}
\end{align*}
\]
Infinitesimal Rotational Angles

Horizontal bend

Vertical bend

Twist
Curvature, Torsional Ratio, and Normal Strain

Curvature:

\[ \kappa^2 = \omega_\xi^2 + \omega_\eta^2 = \left( \frac{d\theta}{ds} \right)^2 + \left( \frac{d\phi}{ds} \right)^2 \sin^2 \theta \]

Torsional ratio:

\[ \omega^2 = \omega_\xi^2 = \left( \frac{d\psi}{ds} + \frac{d\phi}{ds} \cos \theta \right)^2 \]

Normal strain: \( \varepsilon(s) \)
The geometrical shape of a deformed linear object can be represented by four functions: $\phi(s), \theta(s), \psi(s), \varepsilon(s)$.

Mathematically, this can be expressed as:

$$x(s) = x(0) + \int_0^s \left\{ 1 - \varepsilon(s) \right\} \begin{bmatrix} \sin \theta(s) \cos \phi(s) \\ \sin \theta(s) \sin \phi(s) \\ \cos \theta(s) \end{bmatrix} ds$$
Potential Energy

Variational principle in statics:

The potential energy of a linear object attains its minimum value in its stable deformed state under the imposed constraints.

Potential energy: \[ U = U_{\text{flex}} + U_{\text{tor}} + U_{\text{ext}} + U_{\text{grav}} \]

Flexural energy: \[ U_{\text{flex}} = \int_{0}^{L} \frac{1}{2} R_f \kappa^2 ds \quad R_f : \text{Flexural rigidity} \]

Torsional energy: \[ U_{\text{tor}} = \int_{0}^{L} \frac{1}{2} R_t \omega^2 ds \quad R_t : \text{Torsional rigidity} \]

Extensional energy: \[ U_{\text{ext}} = \int_{0}^{L} \frac{1}{2} R_e \varepsilon^2 ds \quad R_e : \text{Extensional rigidity} \]

Gravitational energy: \[ U_{\text{grav}} = \int_{0}^{L} Dgxds \quad D : \text{Linear density} \]
Positional/Orientational Constraints

\[
\mathbf{x}(s_b) - \mathbf{x}(s_a) = \begin{bmatrix}
  l_x \\
  l_y \\
  l_z
\end{bmatrix}
\]

\[
\begin{bmatrix}
  \phi(s_c) \\
  \theta(s_c) \\
  \psi(s_c)
\end{bmatrix} = \begin{bmatrix}
  \phi_c \\
  \theta_c \\
  \psi_c
\end{bmatrix}
\]
Consideration of Contact with Obstacles

Surface: \( f(x, y, z) = 0 \)

Inside: \( f(x, y, z) > 0 \)

Outside: \( f(x, y, z) < 0 \)

\( f(x(s), y(s), z(s)) \leq 0, \quad \forall s \in [0, L] \)
Consideration of Self-interaction

The geometrical constraints imposed on a linear object are given by not only equational conditions but also inequality conditions.

\[ |x(s_i) - x(s_j)| \geq 2r, \quad \forall s_i, s_j \in [0, L], \text{ s.t. } |s_i - s_j| \geq 2r \]
Minimization Problem

\[
\phi(s) = \sum_{i=1}^{n} a_i^\phi e_i(s), \quad \theta(s) = \sum_{i=1}^{n} a_i^\theta e_i(s), \\
\psi(s) = \sum_{i=1}^{n} a_i^\psi e_i(s), \quad \varepsilon(s) = \sum_{i=1}^{n} a_i^\varepsilon e_i(s)
\]

\[
\phi(s) = a^\phi \cdot e(s), \quad \theta(s) = a^\theta \cdot e(s), \quad \psi(s) = a^\psi \cdot e(s), \quad \varepsilon(s) = a^\varepsilon \cdot e(s)
\]

\[
a = \begin{bmatrix} a^\phi & a^\theta & a^\psi & a^\varepsilon \end{bmatrix}
\]

Minimize potential energy \( U(a) \)
Subject to \( f_j(a) = 0 \quad (j = 1, \ldots, J) \)
Positional/orientational constraints
\( g_k(a) \leq 0 \quad (k = 1, \ldots, K) \)
Avoidance of (self-)interference
Computational Results

Basis functions:

\[ e_1 = 1, \quad e_2 = s, \]

\[ e_{2i+1} = \sin \frac{\pi is}{L}, \]

\[ e_{2i+2} = \cos \frac{\pi is}{L} \quad (i = 1, 2, 3, 4) \]
Experimental Verification (1)

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length</td>
<td>$8.7 \times 10^2$ [mm]</td>
</tr>
<tr>
<td>Flexural rigidity</td>
<td>$6.6 \times 10^{-4}$ [Nm$^2$]</td>
</tr>
<tr>
<td>Torsional rigidity</td>
<td>$2.3 \times 10^{-4}$ [Nm$^2$]</td>
</tr>
<tr>
<td>Weight per unit length</td>
<td>$1.0 \times 10^{-2}$ [N/m]</td>
</tr>
</tbody>
</table>

\[ l_z = 800 \text{ [mm]} \]

\[ l_z = 600 \text{ [mm]} \]
Experimental Verification (2)

$l_z = 400 \, [mm]$  \hspace{1cm}  l_z = 200 \, [mm]$
Knotted Shape of Linear Object

\[
\begin{align*}
  z(s_1^u) - z(s_1^l) &= 0, \\
  z(s_2^u) - z(s_2^l) &= 0, \\
  z(s_3^u) - z(s_3^l) &= 0, \\
  x(s_1^u) - x(s_1^l) &= 0, \\
  x(s_2^u) - x(s_2^l) &= 0, \\
  x(s_3^u) - x(s_3^l) &= 0, \\
  y(s_1^u) - y(s_1^l) &= 2r, \\
  y(s_2^u) - y(s_2^l) &= 2r, \\
  y(s_3^u) - y(s_3^l) &= 2r,
\end{align*}
\]

\[
0 \leq s_1^l < s_2^u < s_3^l < s_1^u < s_2^l < s_3^u \leq L
\]

\[
a = \begin{bmatrix}
a^\phi & a^\theta & a^\psi & s_1^u & s_1^l & s_2^u & s_2^l & s_3^u & s_3^l
\end{bmatrix}
\]
Computational Result of Overhand Knot
Knitted Shape of Linear Objects

Assumption:
The shape of the fabric can be represented by repetitions of the same shape of one loop.

\[

d(z(s_i) - z(s_{i+2})) = 0, \quad (i = 1, 2, 5, 6)
\]

\[
x(s_i) - x(s_{i-2}) = l_w, \quad (i = 3, 4)
\]

\[
x(s_i) - x(s_{i+2}) = l_w, \quad (i = 5, 6)
\]

\[
y(s_i) - y(s_{i-2}) = 2r, \quad (i = 3, 7)
\]

\[
y(s_i) - y(s_{i+2}) = 2r, \quad (i = 2, 6)
\]

\[
0 \leq s_i < s_{i+1} \leq L, \quad (i = 1, \ldots, 7)
\]
Computational Result of Plain Knitted Fabric
Modeling of Belt Object Deformation

Assumptions:
- A belt object is rectangular.
- The width of the object is sufficiently small compared to its length.
- The object is inextensible. Namely, it can be bent and twisted but cannot be expanded or contracted.
- Its both ends cannot be deformed because connectors are attached to the ends.
Infinitesimal Rotational Angles

Shape in $uv$-space

Bend

Twist

Assumption:
A belt object is inextensible.

In case of rectangular object:

$\omega_{\xi} \equiv 0$
Assumption:
A belt object is inextensible. Its surface is developable.

Developable surface:
- It can be generated by sweeping a straight line in 3D space.
- It includes straight lines.

Cylindrical surface

Conic surface
Fishbone Model

The shape of a belt object:
- Shape of the bent and twisted spine line: $\phi(u), \theta(u), \psi(u)$
- Direction of straight rib lines: $\alpha(u)$
Potential Energy of Belt Object

Potential energy: \[ I = \int_0^U \frac{R_f}{2} \kappa_{\text{max}}^2 \, du = \int_0^U \frac{R_f}{2} \left( \frac{\omega_\zeta^2 + \omega_\eta^2}{\omega_\zeta^2} \right)^2 \, du \]

\[ \kappa_{\text{max}} = -\frac{\omega_\zeta^2 + \omega_\eta^2}{\omega_\zeta} \]

\[ R_f \]: flexural rigidity along the spine line

Cylindrical surface
Constraints

- **Necessary constraints for developability**
  - To maintain initial shape in uv-space:
    \[
    \omega_{u\xi} = 0, \forall u \in [0, U]
    \]
  - To prevent rib lines from intersecting with themselves:
    \[
    - \frac{2 \cos^2 \alpha}{V} \leq \frac{d\alpha}{du} \leq \frac{2 \cos^2 \alpha}{V}, \forall u \in [0, U]
    \]
  - Relationship between the rib angle and infinitesimal rotational angles:
    \[
    \alpha = -\tan^{-1}\frac{\omega_\eta}{\omega_\zeta}, \forall u \in [0, U]
    \]

- **Geometric constraints**
Experimental Verification

(a) Computational result
(b) Experimental result

Obverse side
Reverse side

Polystyrene
200[mm] long
20[mm] wide
140[μm] thick

200[mm] long
20[mm] wide
140[μm] thick

OSAKA University, Department of Manufacturing Science
Advanced Manufacturing Systems Lab.
Application to Curved/Bent Belt Object

Assumption:
The rib line at the bent point coincides with the connecting line.

Shape in $uv$-space:

$$\omega_\xi \equiv \kappa_c$$
Deformed Shape of Curved Belt Object

(a) Initial shape
(b) Computational result
(c) Experimental result
Deformed Shape of Bent Belt Object

(a) Initial shape

(b) Computational result

(c) Experimental result
Conclusions

A modeling method of linear/belt object deformation based on differential geometry was proposed.

- Differential geometry was extended to describe linear object deformation including flexure, torsion, and extension.

- The shape of a linear object can be described by four independent variables if it is extensible and by three otherwise.

- It was shown that more complex shapes such as knots and knitted fabrics also can be computed using our proposed approach.

- This approach was applied to deformation of an inextensible belt object.

- It was found that the belt object shape can be described by two independent variables.