Method of Variation

Example curve of fastest descent

Find a curve along that the mass falls fastest.

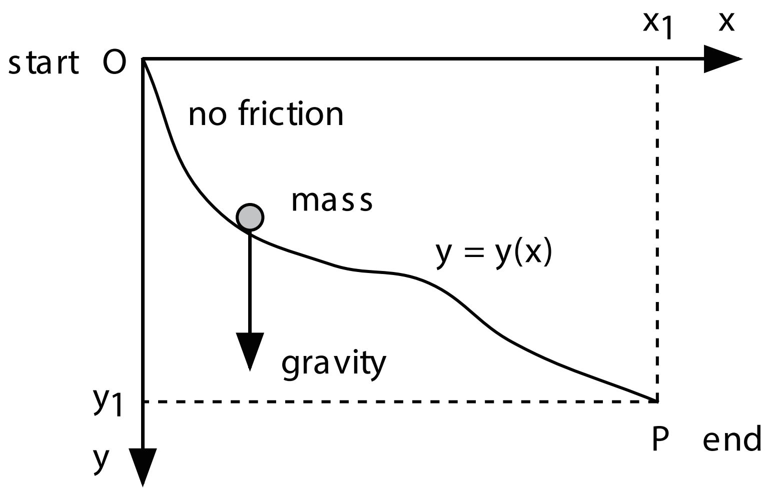

Figure 1: Mass falling along curve

Shape of the curve : $y(x)$
Time when the mass moves from O to P : $T = T[y]$
Scalar T depends on function y

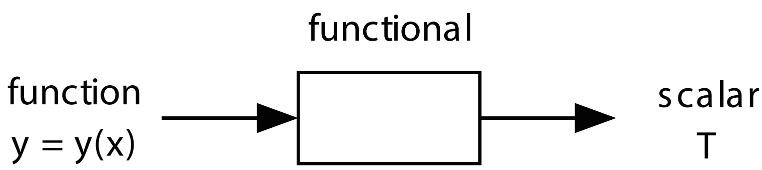

Figure 2: Functional $T[y]$

	kinetic energy	gravitational energy
point (x, y)	$\frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$	$-mgy$
point O	0	0

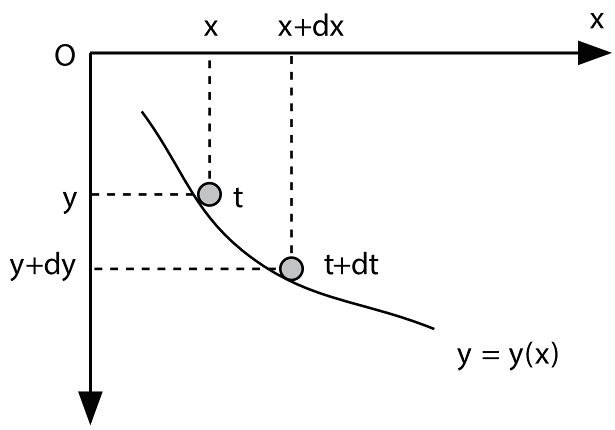

Figure 3: Neighboring points on curve

$$\frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy = 0$$
$$\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 = 2gy$$
$$(\mathrm{d}t)^2 = \frac{(\mathrm{d}x)^2 + (\mathrm{d}y)^2}{2gy} = \frac{1 + (\mathrm{d}y/\mathrm{d}x)^2}{2gy}(\mathrm{d}x)^2$$
$$\mathrm{d}t = \sqrt{\frac{1 + (y')^2}{2gy}}\mathrm{d}x$$

where

$$y' = \frac{\mathrm{d}y}{\mathrm{d}x}$$

Consequently,

$$T[y] = \int_{\text{point O}}^{\text{point P}} \mathrm{d}t = \int_0^{x_1} \sqrt{\frac{1 + (y')^2}{2gy}}\, \mathrm{d}x$$

Find a function $y(x)$ that minimizes (maximizes) a functional

$$T[y] = \int_{x_0}^{x_1} F(x, y, y')\, \mathrm{d}x$$

This problem is referred to as a **variational problem**.

Recall : minimization of function $f(x)$

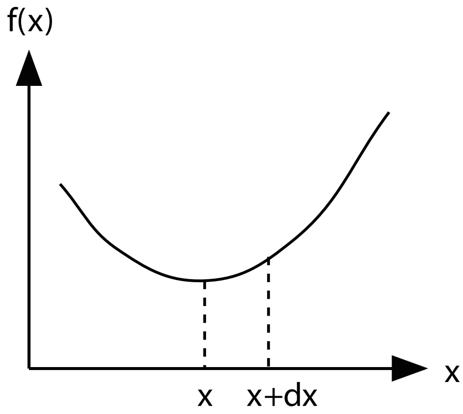

Figure 4: Minimum point

The following condition must be satisfied for any deviation $\mathrm{d}x$ at the minimum point:

$$f(x + \mathrm{d}x) - f(x) = f'(x)\mathrm{d}x = 0.$$

This condition is satisfied if, and only if,

$$f'(x) = 0.$$

Thus, solving $f'(x) = 0$ yields minimum points.
(exactly speaking, local minimum points)

Let us introduce the deviation of function $y(x)$ into a variational problem:

$$\delta y(x) \quad \text{where} \quad \delta y(x_0) = 0, \quad \delta y(x_1) = 0$$

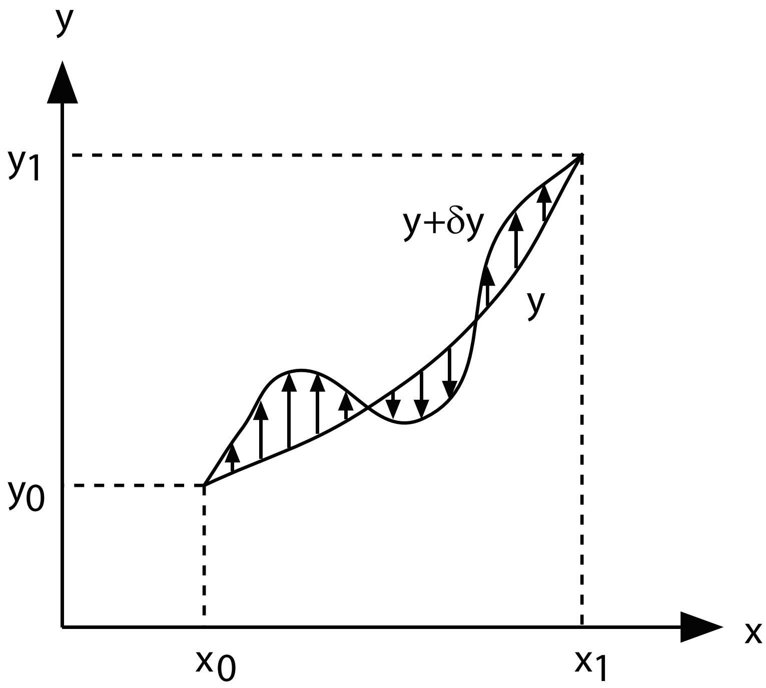

Figure 5: Deviation of function

Evaluating functional $T[y]$, we have

$$T[y] = \int_{x_0}^{x_1} F(x, y, y')\,\mathrm{d}x$$
$$T[y + \delta y] = \int_{x_0}^{x_1} F(x, y + \delta y, y' + \delta y')\,\mathrm{d}x$$

Note that

$$F(x, y + \delta y, y' + \delta y') = F(x, y, y') + \frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta y'$$

$$T[y+\delta y]-T[y]=\int_{x_0}^{x_1}\left(\frac{\partial F}{\partial y}\delta y+\frac{\partial F}{\partial y'}\delta y'\right)\,\mathrm{d}x,$$

$$\int_{x_0}^{x_1}\frac{\partial F}{\partial y'}\delta y'\mathrm{d}x=\underbrace{\left[\frac{\partial F}{\partial y'}\delta y\right]_{x=x_0}^{x=x_1}}_{\substack{\uparrow\\ 0}}-\int_{x_0}^{x_1}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'}\right)\delta y\,\mathrm{d}x$$

Thus,

$$T[y+\delta y]-T[y]=\int_{x_0}^{x_1}\left[\frac{\partial F}{\partial y}-\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'}\right)\right]\delta y\,\mathrm{d}x$$

$$y:\ \text{optimal}\quad\Leftrightarrow\quad T[y+\delta y]-T[y]=0\quad\text{for any}\quad\delta y$$

Consequently,

$$\frac{\partial F}{\partial y}-\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'}\right)=0 \tag{1}$$

Euler-Lagrange's equation

The above equation is simply described as

$$F_y-\frac{\mathrm{d}}{\mathrm{d}x}F_{y'}=0$$

where

$$F_y=\frac{\partial F}{\partial y},\quad F_{y'}=\frac{\partial F}{\partial y'}$$

In the case that $F = F(y, y')$, say, no explicit x is involved in a functional F,

$$\frac{\mathrm{d}}{\mathrm{d}x}(y'F_{y'} - F) = y''F_{y'} + y'\underbrace{\frac{\mathrm{d}}{\mathrm{d}x}F_{y'}}_{\substack{\uparrow \\ F_y}} - \underbrace{\frac{\mathrm{d}}{\mathrm{d}x}F}_{\substack{\uparrow \\ \frac{\partial F}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\partial F}{\partial y'}\frac{\mathrm{d}y'}{\mathrm{d}x} \\ = F_y y' + F_{y'} y''}}$$

$$= 0$$

Thus, Euler-Lagrange's equation can be simplified to

$$y'F_{y'} - F = c \ \ \text{(const)}.$$

Solving curve of fastest descent problem

$$F(y, y') = \sqrt{\frac{1 + (y')^2}{2gy}}$$
$$T[y] = \int_0^{x_1} F(y, y')\,\mathrm{d}x$$

$$\frac{\partial F}{\partial y'} = \frac{y'}{\sqrt{(1 + y'^2)2gy}}$$

Euler-Lagrange's equation

$$y'F_{y'} - F = \frac{y'^2}{\sqrt{(1 + y'^2)2gy}} - \sqrt{\frac{1 + (y')^2}{2gy}} = c_1 \quad \text{(const)} \tag{2}$$

Then,

$$\frac{y'^2 - (1 + y'^2)}{\sqrt{(1 + y'^2)2gy}} = c_1$$
$$-1 = c_1\sqrt{(1 + y'^2)2gy}$$
$$1 = c_1^2(1 + y'^2)2gy$$
$$y'^2 = \frac{c_2 - y}{y} \quad \text{where } \; c_2 = \frac{1}{2gc_1^2} \quad \text{(const)}$$

Let

$$y = \frac{c_2}{2}(1 - \cos u)$$

Then,

$$y' = \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{c_2}{2}u'\sin u$$
$$\left(\frac{c_2}{2}u'\sin u\right)^2 = \frac{c_2 - c_2/2(1 - \cos u)}{c_2/2(1 - \cos u)} = \frac{1 + \cos u}{1 - \cos u} = \frac{\sin^2 u}{(1 - \cos u)^2}$$

The above differential equation is converted into:

$$\frac{c_2}{2}u' = \frac{1}{1-\cos u}$$

$$\frac{c_2}{2}(1-\cos u)\,\mathrm{d}u = \mathrm{d}x$$

$$x = \int \frac{c_2}{2}(1-\cos u)\,\mathrm{d}u = \frac{c_2}{2}(u-\sin u) + c_3$$

Noting that $x = c_3$ and $y = 0$ at $u = 0$, we find that $c_3 = 0$. Finally, we have

$$x = \frac{c_2}{2}(u - \sin u) \tag{3a}$$

$$y = \frac{c_2}{2}(1 - \cos u) \qquad \text{cycloid} \tag{3b}$$

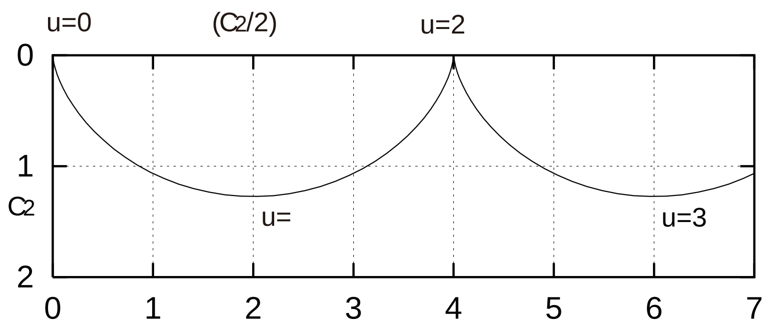

Figure 6: Cycloid

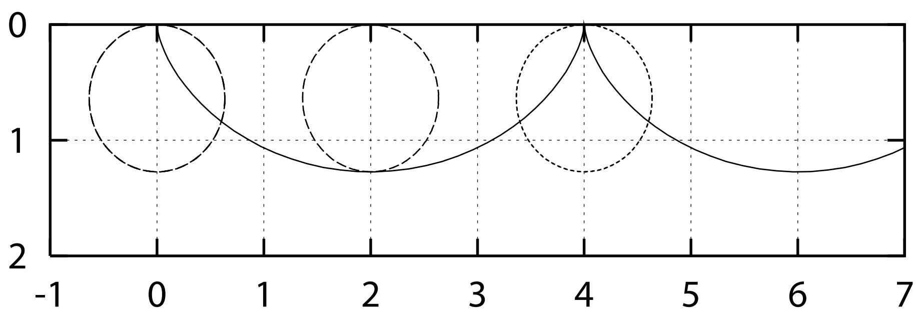

Figure 7: Path of point on rolling circle