Elastic Deformation

Shinichi Hirai

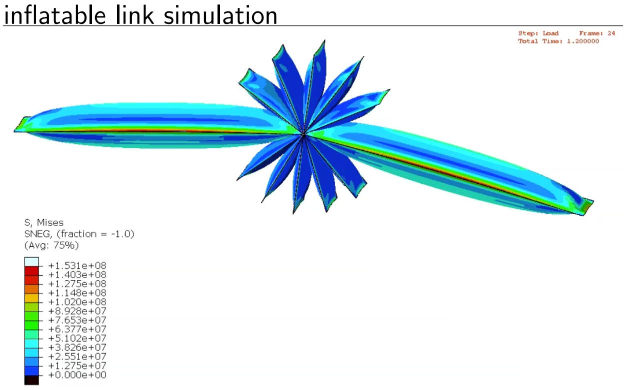
Dept. Robotics, Ritsumeikan Univ.

Agenda

1. Soft Body Models
2. Strain and Stress
3. One-dimensional Finite Element Method
4. Two/Three-dimensional Deformation
5. Two-dimensional Finite Element Method
6. Computing Static Deformation
7. Computing Dynamic Deformation
8. Summary
9. Green Strain

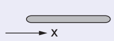
Finite Element Method (FEM)

inflatable link simulation

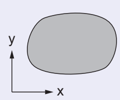


Soft Body Models

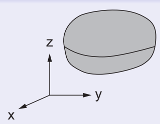
Soft-material Robots



1D model

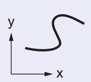


2D model

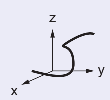


3D model

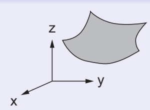
Geometrically Deformable Robots



linear in 2D

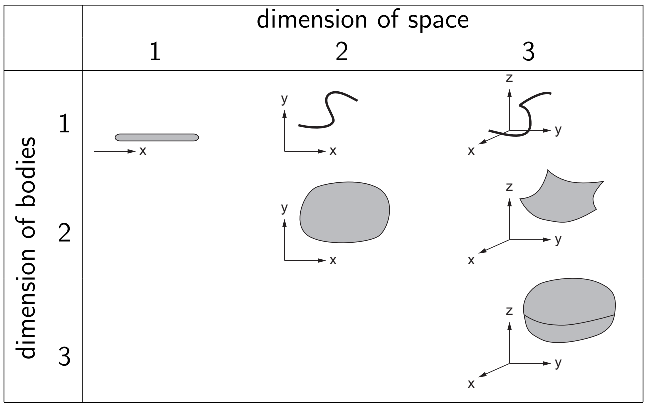


linear in 3D



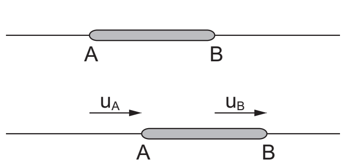
planar in 3D

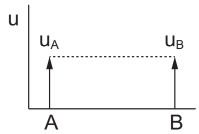
Soft Body Models



One-dimensional Soft Body Model

one-dimensional soft robot AB acts as

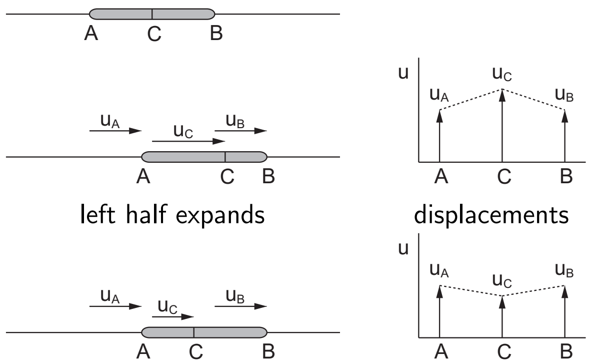




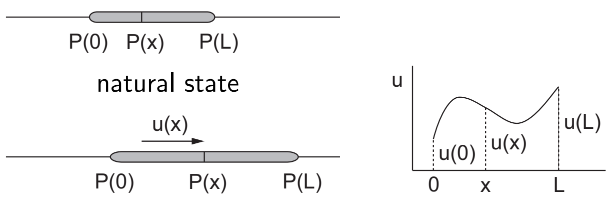
displacements

Can we conclude that AB moves but does not deform?

One-dimensional Soft Body Model



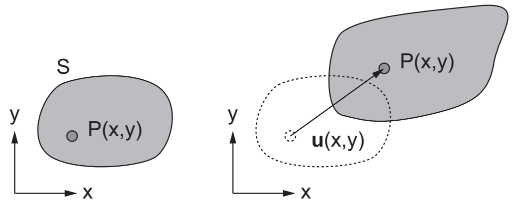
One-dimensional Soft Body Model



the motion and deformation: specified by function $u(x)$, where $x \in [0, L]$

Two-dimensional Soft Body Model

two-dimensional soft robot S acts as

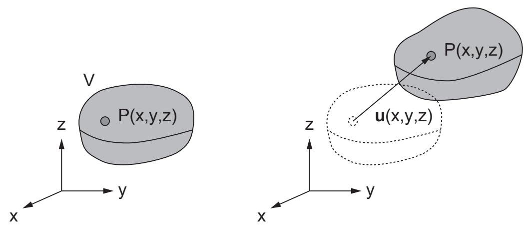


natural state   moved and deformed state

The motion and deformation: specified by a vector function $\boldsymbol{u}(x, y)$, that is, by its two components $u(x, y)$ and $v(x, y)$

Three-dimensional Soft Body Model

three-dimensional soft robot V acts as



natural state   moved and deformed state

The motion and deformation: specified by a vector function $\boldsymbol{u}(x, y, z)$, that is, by its three components $u(x, y, z)$, $v(x, y, z)$, and $w(x, y, z)$

Approach

Energies

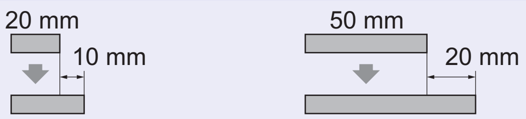
motion   kinetic energy T
deformation   strain potential energy U
strain and stress

Calculation

finite element approximation
divide-and-conquer approach
piecewise linear approximation

Strain and Stress

Which deforms more?



Strain

$$\text{strain} = \frac{\text{deformation}}{\text{size}}$$

$$\varepsilon = \frac{10\,\text{mm}}{20\,\text{mm}} = 0.50 \qquad \varepsilon = \frac{20\,\text{mm}}{50\,\text{mm}} = 0.40$$

Strain and Stress

Which pushes stronger?

Stress

$$\text{stress} = \frac{\text{force}}{\text{area}}$$

$$\sigma = \frac{0.4\,\text{N}}{(1\,\text{mm})^2} = 0.40\,\text{MPa} \qquad \sigma = \frac{0.8\,\text{N}}{(2\,\text{mm})^2} = 0.20\,\text{MPa}$$

Strain and Stress (Units)

Strain

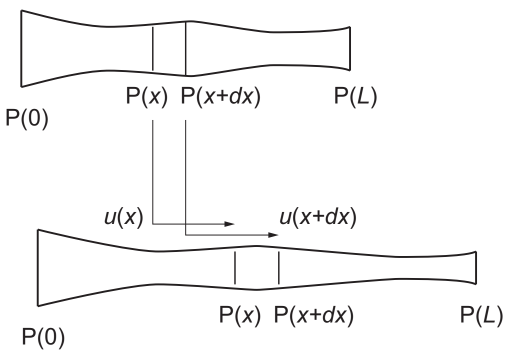
$$\frac{\text{deformation}}{\text{size}} \quad \frac{\text{m}}{\text{m}} = 1$$

Stress

$$\frac{\text{force}}{\text{area}} = \frac{\text{N}}{\text{m}^2} = \text{Pa}$$

$$\frac{\text{N}}{\text{mm}^2} = \frac{\text{N}}{(10^{-3}\,\text{m})^2} = \frac{\text{N}}{10^{-6}\,\text{m}^2} = 10^6\frac{\text{N}}{\text{m}^2} = 10^6\,\text{Pa} = \text{MPa}$$

One-dimensional Deformation



strain of small region $\mathrm{P}(x)\mathrm{P}(x + \mathrm{d}x)$

One-dimensional Deformation

$$\text{extension} = u(x + \mathrm{d}x) - u(x)$$

$$\text{strain} = \frac{\text{extension}}{\text{length}} = \frac{u(x + \mathrm{d}x) - u(x)}{\mathrm{d}x} \approx \frac{\partial u}{\partial x}$$

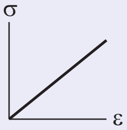
Strain

$$\varepsilon = \frac{\partial u}{\partial x}$$

Elasticity

relationship between stress σ and strain ε

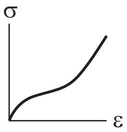
Linear elasticity



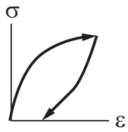
$$\sigma = E\varepsilon$$

E: Young's modulus (elastic modulus)
specific to materials

in reality

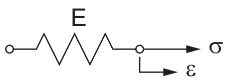


nonlinear

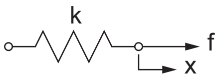


hysteresis

Elasticity

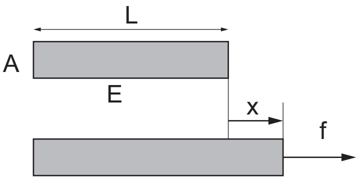


$$\sigma = E\varepsilon$$



$$f = kx$$

extending uniform cylinder

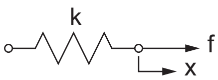


$$f = kx$$

$$k = E\,\frac{A}{L}$$

material geometry

Energy Density



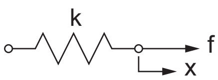
$$U = \frac{1}{2}fx = \frac{1}{2}kx^2$$

energy

N m

Energy Density

$$\frac{1}{2}\sigma\varepsilon = \frac{1}{2}E\varepsilon^2$$

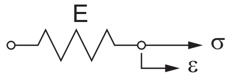


$$U = \frac{1}{2}fx = \frac{1}{2}kx^2$$

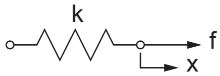
energy

N m

Energy Density



$$\frac{1}{2}\sigma\varepsilon = \frac{1}{2}E\varepsilon^2$$



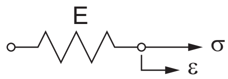
$$U = \frac{1}{2}fx = \frac{1}{2}kx^2$$

energy

$$\frac{\mathrm{N}}{\mathrm{m}^2} = \frac{\mathrm{N\,m}}{\mathrm{m}^3} = \frac{\text{energy}}{\text{volume}}$$

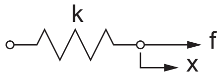
N m

Energy Density



$$\frac{1}{2}\sigma\varepsilon = \frac{1}{2}E\varepsilon^2$$

energy density



$$U = \frac{1}{2}fx = \frac{1}{2}kx^2$$

energy

$$\frac{\mathrm{N}}{\mathrm{m}^2} = \frac{\mathrm{N\,m}}{\mathrm{m}^3} = \frac{\text{energy}}{\text{volume}}$$

N m

Strain Potential Energy

energy density of one-dimensional deformation

$$\frac{1}{2}E\varepsilon^2 = \frac{1}{2}E\left(\frac{\partial u}{\partial x}\right)^2$$

A(x) cross-sectional area at point $\mathrm{P}(x)$

volume = (area) · (height) = $A\,\mathrm{d}x$

strain potential energy

$$U = \int_0^L \text{(energy density)} \cdot \text{(volume)}$$

$$= \int_0^L \frac{1}{2}E\left(\frac{\partial u}{\partial x}\right)^2 A\,\mathrm{d}x = \int_0^L \frac{1}{2}EA\left(\frac{\partial u}{\partial x}\right)^2 \mathrm{d}x$$

Kinetic Energy

velocity of point $\mathrm{P}(x)$

$$\dot{u} = \frac{\partial u}{\partial t}$$

mass of small region $\mathrm{P}(x)\mathrm{P}(x + \mathrm{d}x)$

$$\text{(density)} \cdot \text{(volume)} = \rho \cdot A\,\mathrm{d}x$$

kinetic energy

$$T = \int_0^L \frac{1}{2}\text{(mass)(velocity)}^2$$

$$= \int_0^L \frac{1}{2}\rho A\left(\frac{\partial u}{\partial t}\right)^2 \mathrm{d}x$$

One-dimensional Finite Element Method

energies

strain potential energy

$$U = \int_0^L \frac{1}{2} EA \left(\frac{\partial u}{\partial x} \right)^2 \mathrm{d}x$$

kinetic energy

$$T = \int_0^L \frac{1}{2} \rho A \left(\frac{\partial u}{\partial t} \right)^2 \mathrm{d}x$$

How calculate energies in integral forms?

Divide-and-Conquer Approach

divide

$$\int_0^L = \int_{x_1}^{x_2} + \int_{x_2}^{x_3} + \int_{x_3}^{x_4} + \int_{x_4}^{x_5}$$

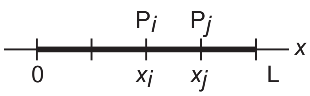
apply piecewise linear approximation

$$\int_{x_i}^{x_j} = \frac{1}{2} \begin{bmatrix} u_i & u_j \end{bmatrix} \begin{bmatrix} & \\ & \end{bmatrix} \begin{bmatrix} u_i \\ u_j \end{bmatrix}$$

synthesize

$$\int_0^L = \frac{1}{2} \begin{bmatrix} u_1 & u_2 & \cdots & u_5 \end{bmatrix} \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_5 \end{bmatrix}$$

Dividing Region

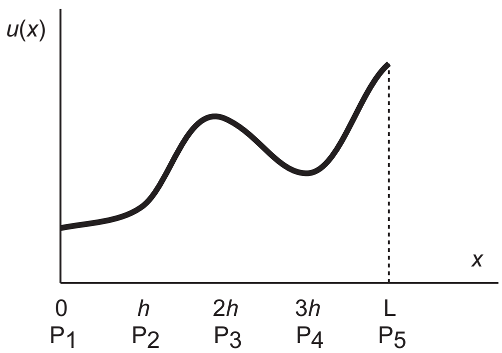


nodal points

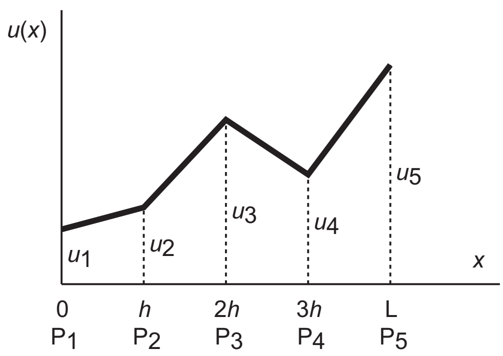
divide $[0, L]$ into four small regions
small region size $h = L/4$

$$x_1 = 0,\ x_2 = h,\ x_3 = 2h,\ x_4 = 3h,\ x_5 = L$$

Piecewise Linear Approximation



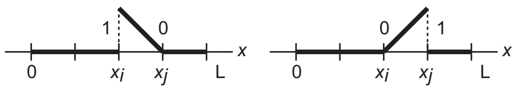
Piecewise Linear Approximation



Piecewise Linear Approximation

function $u(x)$ in small region $[x_i, x_j]$

$$u(x) = u_i\, N_{i,j}(x) + u_j\, N_{j,i}(x)$$



$$N_{i,j}(x) = \frac{x_j - x}{h} = \begin{cases} 1 & (x = x_i) \\ 0 & (x = x_j) \end{cases} \qquad N_{j,i}(x) = \frac{x - x_i}{h} = \begin{cases} 1 & (x = x_j) \\ 0 & (x = x_i) \end{cases}$$

$$u(x_i) = u_i N_{i,j}(x_i) + u_j N_{j,i}(x_i) = u_i \cdot 1 + u_j \cdot 0 = u_i$$
$$u(x_j) = u_i N_{i,j}(x_j) + u_j N_{j,i}(x_j) = u_i \cdot 0 + u_j \cdot 1 = u_j$$

Piecewise Linear Approximation

in small region $[x_i, x_j]$

$$N_{i,j}(x) = \frac{x_j - x}{h}, \qquad N_{j,i}(x) = \frac{x - x_i}{h}$$
$$N'_{i,j}(x) = \frac{-1}{h}, \qquad N'_{j,i}(x) = \frac{1}{h}$$

derivative $\partial u / \partial x$ in small region $[x_i, x_j]$

$$\frac{\mathrm{d}u}{\mathrm{d}x} = u_i N'_{i,j}(x) + u_j N'_{j,i}(x) = u_i \frac{-1}{h} + u_j \frac{1}{h} = \frac{-u_i + u_j}{h}$$

Piecewise Linear Approximation

assume Young's modulus E is constant

$$\int_{x_i}^{x_j} \frac{1}{2} EA \left(\frac{\mathrm{d}u}{\mathrm{d}x} \right)^2 \mathrm{d}x$$
$$= \int_{x_i}^{x_j} \frac{1}{2} EA \left(\frac{-u_i + u_j}{h} \right)^2 \mathrm{d}x$$
$$= \frac{1}{2} \frac{E}{h^2} (-u_i + u_j)^2 \int_{x_i}^{x_j} A \,\mathrm{d}x$$
$$= \frac{1}{2} \begin{bmatrix} u_i & u_j \end{bmatrix} \frac{E}{h^2} \begin{bmatrix} V_{i,j} & -V_{i,j} \\ -V_{i,j} & V_{i,j} \end{bmatrix} \begin{bmatrix} u_i \\ u_j \end{bmatrix}$$

Piecewise Linear Approximation

note

$$V_{i,j} = \int_{x_i}^{x_j} A \, \mathrm{d}x$$

represents volume in small region $[\, x_i, \, x_j \,]$

assume Young's modulus E and cross-sectional area A are constant

$$\int_{x_i}^{x_j} \frac{1}{2} EA \left(\frac{\mathrm{d}u}{\mathrm{d}x} \right)^2 \mathrm{d}x = \frac{1}{2} \begin{bmatrix} u_i & u_j \end{bmatrix} \frac{EA}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_i \\ u_j \end{bmatrix}$$

Synthesizing

nodal displacement vector

$$\boldsymbol{u}_{\mathrm{N}} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_5 \end{bmatrix}$$

describes soft robot motion and deformation

Synthesizing

assume E and A are constant

$$\begin{aligned} U = & \frac{1}{2} \begin{bmatrix} u_1 & u_2 \end{bmatrix} \frac{EA}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\ & + \frac{1}{2} \begin{bmatrix} u_2 & u_3 \end{bmatrix} \frac{EA}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_2 \\ u_3 \end{bmatrix} \\ & + \frac{1}{2} \begin{bmatrix} u_3 & u_4 \end{bmatrix} \frac{EA}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_3 \\ u_4 \end{bmatrix} \\ & + \frac{1}{2} \begin{bmatrix} u_4 & u_5 \end{bmatrix} \frac{EA}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_4 \\ u_5 \end{bmatrix} \end{aligned}$$

Synthesizing

strain potential energy

$$U = \frac{1}{2} \boldsymbol{u}_{\mathrm{N}}^{\mathrm{T}} K \boldsymbol{u}_{\mathrm{N}}$$

stiffness matrix

$$K = \frac{EA}{h} \begin{bmatrix} 1 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & -1 & 2 & -1 \\ & & & -1 & 1 \end{bmatrix}$$

Synthesizing

strain potential energy

$$U = \frac{1}{2} \boldsymbol{u}_{\mathrm{N}}^{\mathrm{T}} K \boldsymbol{u}_{\mathrm{N}}$$

stiffness matrix

$$K = \frac{EA}{h} \begin{bmatrix} 1 & -1 & & & \\ -1 & 1+1 & -1 & & \\ & -1 & 1+1 & -1 & \\ & & -1 & 1+1 & -1 \\ & & & -1 & 1 \end{bmatrix}$$

Synthesizing

strain potential energy

$$U = \frac{1}{2} \boldsymbol{u}_{\mathrm{N}}^{\mathrm{T}} K \boldsymbol{u}_{\mathrm{N}}$$

stiffness matrix

$$K = \frac{EA}{h} \begin{bmatrix} 1 & -1 & & & \\ -1 & 1+1 & -1 & & \\ & -1 & 1+1 & -1 & \\ & & -1 & 1+1 & -1 \\ & & & -1 & 1 \end{bmatrix}$$

Piecewise Linear Approximation

in small region $[\, x_i, \, x_j \,]$

$$\begin{aligned} u &= u_i \, N_{i,j} + u_j \, N_{j,i} \\ \dot{u} &= \dot{u}_i \, N_{i,j} + \dot{u}_j \, N_{j,i} \end{aligned}$$

assume density ρ and cross-sectional area A are constant

$$\begin{aligned} \int_{x_i}^{x_j} \frac{1}{2} \rho A \dot{u}^2 \, \mathrm{d}x &= \frac{1}{2} \rho A \begin{bmatrix} \dot{u}_i & \dot{u}_j \end{bmatrix} \begin{bmatrix} h/3 & h/6 \\ h/6 & h/3 \end{bmatrix} \begin{bmatrix} \dot{u}_i \\ \dot{u}_j \end{bmatrix} \\ &= \frac{1}{2} \frac{\rho A h}{6} \begin{bmatrix} \dot{u}_i & \dot{u}_j \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \dot{u}_i \\ \dot{u}_j \end{bmatrix} \end{aligned}$$

Synthesizing

kinetic energy

$$T = \frac{1}{2} \dot{\boldsymbol{u}}_{\mathrm{N}}^{\mathrm{T}} M \dot{\boldsymbol{u}}_{\mathrm{N}}$$

inertia matrix

$$M = \frac{\rho A h}{6} \begin{bmatrix} 2 & 1 & & & \\ 1 & 4 & 1 & & \\ & 1 & 4 & 1 & \\ & & 1 & 4 & 1 \\ & & & 1 & 2 \end{bmatrix}$$

Dynamic Equation

energies

$$U = \frac{1}{2}\boldsymbol{u}_{\mathrm{N}}^{\mathrm{T}} K \boldsymbol{u}_{\mathrm{N}}$$
$$T = \frac{1}{2}\dot{\boldsymbol{u}}_{\mathrm{N}}^{\mathrm{T}} M \dot{\boldsymbol{u}}_{\mathrm{N}}$$

work done by external forces

$$W = \boldsymbol{f}^{\mathrm{T}} \boldsymbol{u}_{\mathrm{N}}$$

constraint

$$R \triangleq \boldsymbol{a}^{\mathrm{T}} \boldsymbol{u}_{\mathrm{N}} = 0$$

Dynamic Equation

Lagrangian

$$\mathcal{L} = T - U + W + \lambda \boldsymbol{a}^{\mathrm{T}} \boldsymbol{u}_{\mathrm{N}}$$

λ: Lagrange multiplier

Lagrange equation of motion and deformation

$$\frac{\partial \mathcal{L}}{\partial \boldsymbol{u}_{\mathrm{N}}} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{u}}_{\mathrm{N}}} = \boldsymbol{0}$$
$$-K\boldsymbol{u}_{\mathrm{N}} + \boldsymbol{f} + \lambda \boldsymbol{a} - M\ddot{\boldsymbol{u}}_{\mathrm{N}} = \boldsymbol{0}$$

Dynamic Equation

constraint stabilization method

$$\ddot{R} + 2\alpha\dot{R} + \alpha^2 R = 0$$

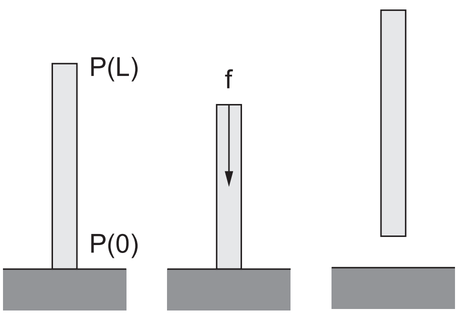
$$-\boldsymbol{a}^{\mathrm{T}}\ddot{\boldsymbol{u}}_{\mathrm{N}} = 2\alpha\boldsymbol{a}^{\mathrm{T}}\dot{\boldsymbol{u}}_{\mathrm{N}} + \alpha^2\boldsymbol{a}^{\mathrm{T}}\boldsymbol{u}_{\mathrm{N}}$$

canonical form of ODE

$$\dot{\boldsymbol{u}}_{\mathrm{N}} = \boldsymbol{v}_{\mathrm{N}}$$
$$M\dot{\boldsymbol{v}}_{\mathrm{N}} - \lambda\boldsymbol{a} = -K\boldsymbol{u}_{\mathrm{N}} + \boldsymbol{f}$$
$$-\boldsymbol{a}^{\mathrm{T}}\dot{\boldsymbol{v}}_{\mathrm{N}} = 2\alpha\boldsymbol{a}^{\mathrm{T}}\boldsymbol{v}_{\mathrm{N}} + \alpha^2\boldsymbol{a}^{\mathrm{T}}\boldsymbol{u}_{\mathrm{N}}$$

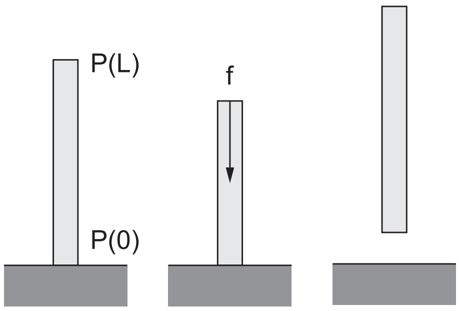
Example

one-dimensional soft body of length L and area A
Young's modulus E, viscous modulus c, density ρ



Example

$[0,\ t_{push}]$ fix the bottom & force f to the top
$[t_{push},\ t_{end}]$ free motion



Example

nodal point number $n = 6$
dividing $[0,\ L]$ into $(n-1)$ small regions:

$$h = \frac{L}{n-1}$$

nodal displacement vector

$$\boldsymbol{u}_{\mathrm{N}} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_6 \end{bmatrix}$$

$u_1 = u(0) \quad u_6 = u(L)$

Example

$[0,\ t_{push}]$
constraint and work done by pushing force

$$R = u_1 = \boldsymbol{a}^{\mathrm{T}}\boldsymbol{u}_{\mathrm{N}}$$
$$W = f_{push} \cdot u_6 = \boldsymbol{f}^{\mathrm{T}}\boldsymbol{u}_{\mathrm{N}}$$

where

$$\boldsymbol{a} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \boldsymbol{f} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ f_{push} \end{bmatrix}$$

Example

$[0,\ t_{push}]$
canonical form of ODE

$$\dot{\boldsymbol{u}}_{\mathrm{N}} = \boldsymbol{v}_{\mathrm{N}}$$
$$\begin{bmatrix} M & -\boldsymbol{a} \\ -\boldsymbol{a}^{\mathrm{T}} & \end{bmatrix}\begin{bmatrix} \dot{\boldsymbol{v}}_{\mathrm{N}} \\ \lambda \end{bmatrix} = \begin{bmatrix} -K\boldsymbol{u}_{\mathrm{N}} - B\boldsymbol{v}_{\mathrm{N}} + \boldsymbol{f} \\ 2\alpha\boldsymbol{a}^{\mathrm{T}}\boldsymbol{v}_{\mathrm{N}} + \alpha^2\boldsymbol{a}^{\mathrm{T}}\boldsymbol{u}_{\mathrm{N}} \end{bmatrix}$$

where

$$K = \frac{EA}{h}\begin{bmatrix} \quad \end{bmatrix} \qquad B = \frac{cA}{h}\begin{bmatrix} \quad \end{bmatrix}$$

$-K\boldsymbol{u}_{\mathrm{N}}$: elastic force $\qquad -B\boldsymbol{v}_{\mathrm{N}}$: viscous force

Example

$[\, t_{push},\ t_{end} \,]$
canonical form of ODE

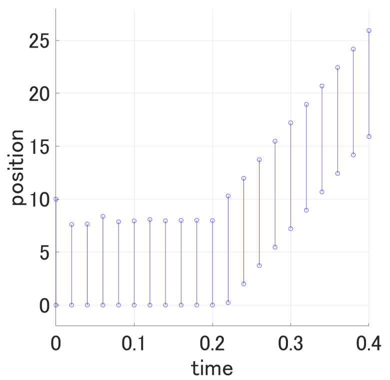
$$\dot{\boldsymbol{u}}_{\rm N} = \boldsymbol{v}_{\rm N}$$
$$M\dot{\boldsymbol{v}}_{\rm N} = -K\boldsymbol{u}_{\rm N} - B\boldsymbol{v}_{\rm N} + \boldsymbol{f}$$

where $\boldsymbol{f} = [\, f_{floor},\ 0,\ \cdots\ ,\ 0\,]^{\rm T}$ and

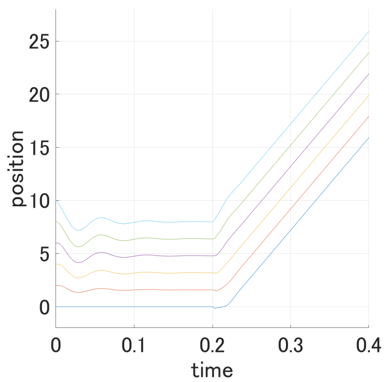
$$f_{floor} = p_{floor} A$$
$$p_{floor} = \begin{cases} -E'_{floor} u_1 - c'_{floor} v_1 & u_1 < 0 \\ 0 & \text{otherwise} \end{cases}$$

f_{floor}: reaction force from floor (penalty method)

Example (body)



Example (nodal point position)



Two/Three-dimensional Deformation

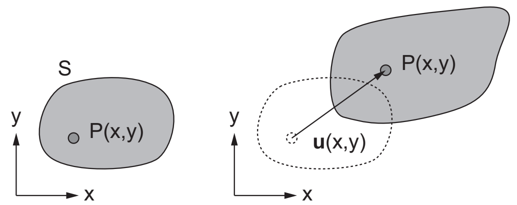
one-dimensional deformation

extensional strain ε
Young's modulus E
strain potential energy density $\dfrac{1}{2}E\varepsilon^2$

two/three-dimensional deformation

extensional & shear strains $\rightarrow$ strain vector $\boldsymbol{\varepsilon}$
Lamé's constants λ, μ $\rightarrow$ elasticity matrix $\lambda I_\lambda + \mu I_\mu$
strain potential energy density $\dfrac{1}{2}\boldsymbol{\varepsilon}^{\rm T}(\lambda I_\lambda + \mu I_\mu)\boldsymbol{\varepsilon}$

Two-dimensional Deformation

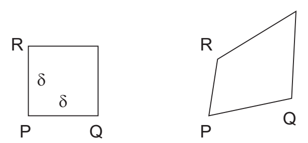


natural state   moved and deformed state

displacement vector

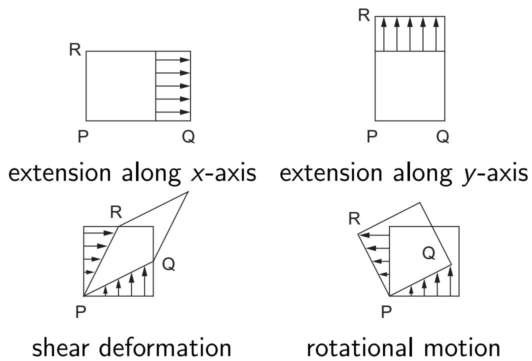
$$\boldsymbol{u}(x,y) = \begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix}$$

Two-dimensional Deformation



natural   deformed and rotated

Two-dimensional Deformation



Two-dimensional Deformation

$\dfrac{\partial u}{\partial x}$ = extension along x-axis $\dfrac{\partial v}{\partial y}$ = extension along y-axis

$\dfrac{\partial v}{\partial x}$ = shear + rotation $\dfrac{\partial u}{\partial y}$ = shear − rotation

$$\Downarrow$$

Cauchy strain

$$\varepsilon_{xx} = \frac{\partial u}{\partial x}, \qquad \varepsilon_{yy} = \frac{\partial v}{\partial y}, \qquad 2\varepsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

Two-dimensional Deformation

strain vector

$$\boldsymbol{\varepsilon} \stackrel{\triangle}{=} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ 2\varepsilon_{xy} \end{bmatrix}$$

Two-dimensional Deformation

Strain potential energy density

linear isotropic elastic material

$$\frac{1}{2}\boldsymbol{\varepsilon}^{\mathrm{T}}(\lambda I_{\lambda} + \mu I_{\mu})\boldsymbol{\varepsilon}$$

where λ and μ are Lamé's constants and

$$I_{\lambda} = \begin{bmatrix} 1 & 1 & \\ 1 & 1 & \\ & & \end{bmatrix}, \quad I_{\mu} = \begin{bmatrix} 2 & & \\ & 2 & \\ & & 1 \end{bmatrix}$$

Two-dimensional Deformation

Volume element

$$h\,\mathrm{d}S = h\,\mathrm{d}x\,\mathrm{d}y$$

Strain potential energy

$$U = \int_S \frac{1}{2}\boldsymbol{\varepsilon}^{\mathrm{T}}(\lambda I_{\lambda} + \mu I_{\mu})\boldsymbol{\varepsilon}\, h\,\mathrm{d}S$$

Two-dimensional Deformation

Volume element

$$h\,\mathrm{d}S = h\,\mathrm{d}x\,\mathrm{d}y$$

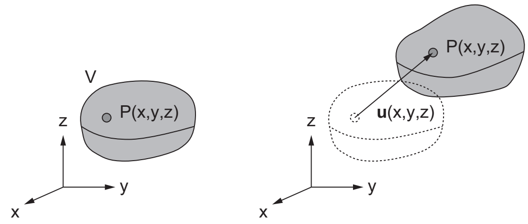
Strain potential energy

$$U = \int_S \frac{1}{2}\boldsymbol{\varepsilon}^{\mathrm{T}}(\lambda I_{\lambda} + \mu I_{\mu})\boldsymbol{\varepsilon}\, h\,\mathrm{d}S$$

Kinetic energy

$$T = \int_S \frac{1}{2}\rho\, \dot{\boldsymbol{u}}^{\mathrm{T}}\dot{\boldsymbol{u}}\, h\,\mathrm{d}S$$

Three-dimensional Deformation

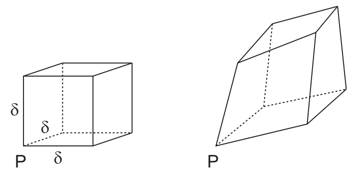


natural state   moved and deformed state

displacement vector

$$\boldsymbol{u}(x,y,z) = \begin{bmatrix} u(x,y,z) \\ v(x,y,z) \\ w(x,y,z) \end{bmatrix}$$

Three-dimensional Deformation



natural   deformed and rotated

Three-dimensional Deformation

	u	v	w
$\partial/\partial x$	ext. along x	shr − rot in xy	shr + rot in zx
$\partial/\partial y$	shr + rot in xy	ext. along y	shr − rot in yz
$\partial/\partial z$	shr − rot in zx	shr + rot in yz	ext. along z

$$2 \cdot \text{shear in } yz\text{-plane} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}$$

$$2 \cdot \text{shear in } zx\text{-plane} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

$$2 \cdot \text{shear in } xy\text{-plane} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

Three-dimensional Deformation

Cauchy strain

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} \quad \varepsilon_{yy} = \frac{\partial v}{\partial y} \quad \varepsilon_{zz} = \frac{\partial w}{\partial z}$$

$$2\varepsilon_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}$$

$$2\varepsilon_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

$$2\varepsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

Three-dimensional Deformation

strain vector

$$\varepsilon = \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ 2\varepsilon_{yz} \\ 2\varepsilon_{zx} \\ 2\varepsilon_{xy} \end{bmatrix}$$

Three-dimensional Deformation

Strain potential energy density

linear isotropic elastic material

$$\frac{1}{2}\varepsilon^{\mathrm{T}}(\lambda I_\lambda + \mu I_\mu)\varepsilon$$

$$I_\lambda = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & & & \\ 1 & 1 & 1 & & & \\ 1 & 1 & 1 & & & \\ \hline & & & & & \\ & & & & & \\ & & & & & \end{array}\right], \quad I_\mu = \left[\begin{array}{ccc|ccc} 2 & & & & & \\ & 2 & & & & \\ & & 2 & & & \\ \hline & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{array}\right]$$

Three-dimensional Deformation

Volume element

$$\mathrm{d}V = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

Strain potential energy

$$U = \int_V \frac{1}{2}\varepsilon^{\mathrm{T}}(\lambda I_\lambda + \mu I_\mu)\varepsilon\,\mathrm{d}V$$

Three-dimensional Deformation

Volume element

$$\mathrm{d}V = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

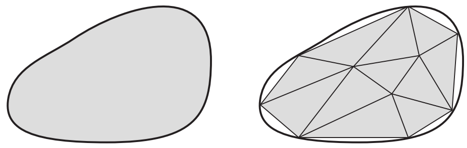
Strain potential energy

$$U = \int_V \frac{1}{2}\varepsilon^{\mathrm{T}}(\lambda I_\lambda + \mu I_\mu)\varepsilon\,\mathrm{d}V$$

Kinetic energy

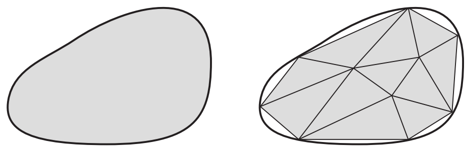
$$T = \int_V \frac{1}{2}\rho\,\dot{\boldsymbol{u}}^{\mathrm{T}}\dot{\boldsymbol{u}}\,\mathrm{d}V$$

Two-dimensional FEM



region S cover by triangles

Two-dimensional FEM



region S cover by triangles

$$\int_S \mathrm{d}S \approx \sum_{\text{triangles}} \int_{\triangle \mathrm{P}_i\mathrm{P}_j\mathrm{P}_k} \mathrm{d}S$$

Two-dimensional FEM

assume density ρ and thickness h are constants
kinetic energy of $\triangle = \triangle \mathrm{P}_i\mathrm{P}_j\mathrm{P}_k$

$$T_{i,j,k} = \int_\triangle \frac{1}{2}\rho\,\dot{\boldsymbol{u}}^{\mathrm{T}}\dot{\boldsymbol{u}}\,h\,\mathrm{d}S$$

$$= \frac{1}{2}\begin{bmatrix} \dot{\boldsymbol{u}}_i^{\mathrm{T}} & \dot{\boldsymbol{u}}_j^{\mathrm{T}} & \dot{\boldsymbol{u}}_k^{\mathrm{T}} \end{bmatrix} \frac{\rho h \triangle}{12} \begin{bmatrix} 2I_{2\times2} & I_{2\times2} & I_{2\times2} \\ I_{2\times2} & 2I_{2\times2} & I_{2\times2} \\ I_{2\times2} & I_{2\times2} & 2I_{2\times2} \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{u}}_i \\ \dot{\boldsymbol{u}}_j \\ \dot{\boldsymbol{u}}_k \end{bmatrix}$$

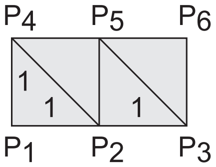
(see Finite_Element_Approximation.pdf for details)

Two-dimensional FEM

Partial inertia matrix

$$M_{i,j,k} = \frac{\rho h \triangle}{12} \begin{bmatrix} 2I_{2\times2} & I_{2\times2} & I_{2\times2} \\ I_{2\times2} & 2I_{2\times2} & I_{2\times2} \\ I_{2\times2} & I_{2\times2} & 2I_{2\times2} \end{bmatrix}$$

Example (inertia matrix)



assume $\rho h \triangle / 12$ is constantly equal to 1
partial inertia matrices

$$M_{1,2,4} = M_{2,3,5} = M_{5,4,2} = M_{6,5,3} = \begin{bmatrix} 2I_{2\times 2} & I_{2\times 2} & I_{2\times 2} \\ I_{2\times 2} & 2I_{2\times 2} & I_{2\times 2} \\ I_{2\times 2} & I_{2\times 2} & 2I_{2\times 2} \end{bmatrix}.$$

Example (inertia matrix)

total kinetic energy

$$T = \frac{1}{2} \dot{\boldsymbol{u}}_{\mathrm{N}}^{\mathrm{T}} \, M \, \dot{\boldsymbol{u}}_{\mathrm{N}}$$

$$= \frac{1}{2} \left[\; \dot{\boldsymbol{u}}_1^{\mathrm{T}} \;\; \dot{\boldsymbol{u}}_2^{\mathrm{T}} \;\; \cdots \;\; \dot{\boldsymbol{u}}_6^{\mathrm{T}} \; \right] \begin{bmatrix} & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{u}}_1 \\ \dot{\boldsymbol{u}}_2 \\ \vdots \\ \dot{\boldsymbol{u}}_6 \end{bmatrix}$$

M: inertia matrix (6×6 block matrix)

Example (inertia matrix)

$$M_{1,2,4} = \left[\begin{array}{c|c|c} (1,1) \text{ block} & (1,2) \text{ block} & (1,4) \text{ block} \\ \hline (2,1) \text{ block} & (2,2) \text{ block} & (2,4) \text{ block} \\ \hline (4,1) \text{ block} & (4,2) \text{ block} & (4,4) \text{ block} \end{array} \right]$$

contribution of $M_{1,2,4}$ to M

$$\left[\begin{array}{c|c|c|c|c|c} 2I_{2\times 2} & I_{2\times 2} & & I_{2\times 2} & & \\ \hline I_{2\times 2} & 2I_{2\times 2} & & I_{2\times 2} & & \\ \hline & & & & & \\ \hline I_{2\times 2} & I_{2\times 2} & & 2I_{2\times 2} & & \\ \hline & & & & & \\ \hline & & & & & \end{array} \right]$$

Example (inertia matrix)

$$M_{2,3,5} = \left[\begin{array}{c|c|c} (2,2) \text{ block} & (2,3) \text{ block} & (2,5) \text{ block} \\ \hline (3,2) \text{ block} & (3,3) \text{ block} & (3,5) \text{ block} \\ \hline (5,2) \text{ block} & (5,3) \text{ block} & (5,5) \text{ block} \end{array} \right]$$

contribution of $M_{2,3,5}$ to M

$$\left[\begin{array}{c|c|c|c|c|c} & & & & & \\ \hline & 2I_{2\times 2} & I_{2\times 2} & & I_{2\times 2} & \\ \hline & I_{2\times 2} & 2I_{2\times 2} & & I_{2\times 2} & \\ \hline & & & & & \\ \hline & I_{2\times 2} & I_{2\times 2} & & 2I_{2\times 2} & \\ \hline & & & & & \end{array} \right]$$

Example (inertia matrix)

$$M_{5,4,2} = \left[\begin{array}{c|c|c} (5,5) \text{ block} & (5,4) \text{ block} & (5,2) \text{ block} \\ \hline (4,5) \text{ block} & (4,4) \text{ block} & (4,2) \text{ block} \\ \hline (2,5) \text{ block} & (2,4) \text{ block} & (2,2) \text{ block} \end{array} \right]$$

contribution of $M_{5,4,2}$ to M

$$\left[\begin{array}{c|c|c|c|c|c} & & & & & \\ \hline & 2I_{2\times 2} & & I_{2\times 2} & I_{2\times 2} & \\ \hline & & & & & \\ \hline & I_{2\times 2} & & 2I_{2\times 2} & I_{2\times 2} & \\ \hline & I_{2\times 2} & & I_{2\times 2} & 2I_{2\times 2} & \\ \hline & & & & & \end{array} \right]$$

Example (inertia matrix)

$$M_{6,5,3} = \left[\begin{array}{c|c|c} (6,6) \text{ block} & (6,5) \text{ block} & (6,3) \text{ block} \\ \hline (5,6) \text{ block} & (5,5) \text{ block} & (5,3) \text{ block} \\ \hline (3,6) \text{ block} & (3,5) \text{ block} & (3,3) \text{ block} \end{array} \right]$$

contribution of $M_{6,5,3}$ to M

$$\left[\begin{array}{c|c|c|c|c|c} & & & & & \\ \hline & & & & & \\ \hline & & 2I_{2\times 2} & & I_{2\times 2} & I_{2\times 2} \\ \hline & & & & & \\ \hline & & I_{2\times 2} & & 2I_{2\times 2} & I_{2\times 2} \\ \hline & & I_{2\times 2} & & I_{2\times 2} & 2I_{2\times 2} \end{array} \right]$$

Example (inertia matrix)

inertia matrix

$$M = M_{1,2,4} \oplus M_{2,3,5} \oplus M_{5,4,2} \oplus M_{6,5,3}$$

$$= \begin{bmatrix} 2I_{2\times 2} & I_{2\times 2} & & I_{2\times 2} & & \\ I_{2\times 2} & 6I_{2\times 2} & I_{2\times 2} & 2I_{2\times 2} & 2I_{2\times 2} & \\ & I_{2\times 2} & 4I_{2\times 2} & & 2I_{2\times 2} & I_{2\times 2} \\ I_{2\times 2} & 2I_{2\times 2} & & 4I_{2\times 2} & I_{2\times 2} & \\ & 2I_{2\times 2} & 2I_{2\times 2} & I_{2\times 2} & 6I_{2\times 2} & I_{2\times 2} \\ & & I_{2\times 2} & & I_{2\times 2} & 2I_{2\times 2} \end{bmatrix}$$

Two-dimensional FEM

assume λ, μ and h are constants
strain potential energy stored in $\triangle = \triangle \mathrm{P}_i \mathrm{P}_j \mathrm{P}_k$

$$U_{i,j,k} = \int_{\triangle} \frac{1}{2} \boldsymbol{\varepsilon}^{\mathrm{T}} (\lambda I_{\lambda} + \mu I_{\mu}) \boldsymbol{\varepsilon} \, h \, \mathrm{d}S$$

$$= \frac{1}{2} \left[\; \boldsymbol{u}_i^{\mathrm{T}} \;\; \boldsymbol{u}_j^{\mathrm{T}} \;\; \boldsymbol{u}_k^{\mathrm{T}} \; \right] K_{i,j,k} \begin{bmatrix} \boldsymbol{u}_i \\ \boldsymbol{u}_j \\ \boldsymbol{u}_k \end{bmatrix}$$

where

$$K_{i,j,k} = \lambda J_{\lambda}^{i,j,k} + \mu J_{\mu}^{i,j,k}$$

(see Finite_Element_Approximation.pdf for details)

Two-dimensional FEM

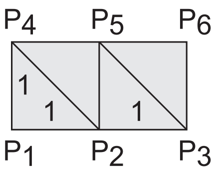
$$\boldsymbol{a} = \frac{1}{2\triangle}\begin{bmatrix} y_j - y_k \\ y_k - y_i \\ y_i - y_j \end{bmatrix}, \qquad \boldsymbol{b} = \frac{-1}{2\triangle}\begin{bmatrix} x_j - x_k \\ x_k - x_i \\ x_i - x_j \end{bmatrix}$$

$$H_\lambda = \begin{bmatrix} \boldsymbol{a}\boldsymbol{a}^{\mathrm{T}} & \boldsymbol{a}\boldsymbol{b}^{\mathrm{T}} \\ \boldsymbol{b}\boldsymbol{a}^{\mathrm{T}} & \boldsymbol{b}\boldsymbol{b}^{\mathrm{T}} \end{bmatrix} h\triangle$$

$$H_\mu = \begin{bmatrix} 2\boldsymbol{a}\boldsymbol{a}^{\mathrm{T}} + \boldsymbol{b}\boldsymbol{b}^{\mathrm{T}} & \boldsymbol{b}\boldsymbol{a}^{\mathrm{T}} \\ \boldsymbol{a}\boldsymbol{b}^{\mathrm{T}} & 2\boldsymbol{b}\boldsymbol{b}^{\mathrm{T}} + \boldsymbol{a}\boldsymbol{a}^{\mathrm{T}} \end{bmatrix} h\triangle$$

1, 4, 2, 5, 3, 6 rows and columns of H_λ, H_μ $\rightarrow$
1, 2, 3, 4, 5, 6 rows and columns of $J_\lambda^{i,j,k}$, $J_\mu^{i,j,k}$

Example (stiffness matrix)



assume $h = 2$

stiffness matrix

$$K = K_{1,2,4} \oplus K_{2,3,5} \oplus K_{5,4,2} \oplus K_{6,5,3}$$

Example (stiffness matrix)

assume λ and μ are constants over region

$$\begin{aligned} K &= K_{1,2,4} \oplus K_{2,3,5} \oplus K_{5,4,2} \oplus K_{6,5,3} \\ &= (\lambda J_\lambda^{1,2,4} + \mu J_\mu^{1,2,4}) \oplus (\lambda J_\lambda^{2,3,5} + \mu J_\mu^{2,3,5}) \oplus \cdots \\ &= \lambda (J_\lambda^{1,2,4} \oplus J_\lambda^{2,3,5} \oplus \cdots) + \mu (J_\mu^{1,2,4} \oplus J_\mu^{2,3,5} \oplus \cdots) \\ &= \lambda J_\lambda + \mu J_\mu \end{aligned}$$

where

$$J_\lambda = J_\lambda^{1,2,4} \oplus J_\lambda^{2,3,5} \oplus J_\lambda^{5,4,2} \oplus J_\lambda^{6,5,3}$$
$$J_\mu = J_\mu^{1,2,4} \oplus J_\mu^{2,3,5} \oplus J_\mu^{5,4,2} \oplus J_\mu^{6,5,3}$$

Example (stiffness matrix)

$\mathrm{P}_1\mathrm{P}_2\mathrm{P}_4$: $\boldsymbol{a} = [-1,\ 1,\ 0]^{\mathrm{T}}$ and $\boldsymbol{b} = [-1,\ 0,\ 1]^{\mathrm{T}}$

$$H_\lambda = \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & -1 \\ -1 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 1 & -1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & -1 & 0 & 1 \end{array}\right]$$

$$J_\lambda^{1,2,4} = \left[\begin{array}{cc|cc|cc} 1 & 1 & -1 & 0 & 0 & -1 \\ 1 & 1 & -1 & 0 & 0 & -1 \\ \hline -1 & -1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 & 0 & 1 \end{array}\right]$$

Example (stiffness matrix)

$\mathrm{P}_1\mathrm{P}_2\mathrm{P}_4$: $\boldsymbol{a} = [-1,\ 1,\ 0]^{\mathrm{T}}$ and $\boldsymbol{b} = [-1,\ 0,\ 1]^{\mathrm{T}}$

$$H_\mu = \left[\begin{array}{ccc|ccc} 3 & -2 & -1 & 1 & -1 & 0 \\ -2 & 2 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & -1 & 1 & 0 \\ \hline 1 & 0 & -1 & 3 & -1 & -2 \\ -1 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & -2 & 0 & 2 \end{array}\right]$$

$$J_\mu^{1,2,4} = \left[\begin{array}{cc|cc|cc} 3 & 1 & -2 & -1 & -1 & 0 \\ 1 & 3 & 0 & -1 & -1 & -2 \\ \hline -2 & 0 & 2 & 0 & 0 & 0 \\ -1 & -1 & 0 & 1 & 1 & 0 \\ \hline -1 & -1 & 0 & 1 & 1 & 0 \\ 0 & -2 & 0 & 0 & 0 & 2 \end{array}\right]$$

Example (stiffness matrix)

$$J_\lambda^{1,2,4} = J_\lambda^{2,3,5} = J_\lambda^{5,4,2} = J_\lambda^{6,5,3}$$

$$J_\mu^{1,2,4} = J_\mu^{2,3,5} = J_\mu^{5,4,2} = J_\mu^{6,5,3}$$

Example (stiffness matrix)

contribution of $J_\lambda^{1,2,4}$ to J_λ

$$\left[\begin{array}{cc|cc|cc|cc|cc|cc} 1 & 1 & -1 & 0 & & & 0 & -1 & & & & \\ 1 & 1 & -1 & 0 & & & 0 & -1 & & & & \\ \hline -1 & -1 & 1 & 0 & & & 0 & 1 & & & & \\ 0 & 0 & 0 & 0 & & & 0 & 0 & & & & \\ \hline & & & & & & & & & & & \\ & & & & & & & & & & & \\ \hline 0 & 0 & 0 & 0 & & & 0 & 0 & & & & \\ -1 & -1 & 1 & 0 & & & 0 & 1 & & & & \\ \hline & & & & & & & & & & & \\ & & & & & & & & & & & \\ \hline & & & & & & & & & & & \\ & & & & & & & & & & & \end{array}\right]$$

Example (stiffness matrix)

contribution of $J_\lambda^{2,3,5}$ to J_λ

$$\left[\begin{array}{cc|cc|cc|cc|cc|cc} & & & & & & & & & & & \\ & & & & & & & & & & & \\ \hline & & 1 & 1 & -1 & 0 & & & 0 & -1 & & \\ & & 1 & 1 & -1 & 0 & & & 0 & -1 & & \\ \hline & & -1 & -1 & 1 & 0 & & & 0 & 1 & & \\ & & 0 & 0 & 0 & 0 & & & 0 & 0 & & \\ \hline & & & & & & & & & & & \\ & & & & & & & & & & & \\ \hline & & 0 & 0 & 0 & 0 & & & 0 & 0 & & \\ & & -1 & -1 & 1 & 0 & & & 0 & 1 & & \\ \hline & & & & & & & & & & & \\ & & & & & & & & & & & \end{array}\right]$$

Example (stiffness matrix)

contribution of $J_\lambda^{5,4,2}$ to J_λ

$$
\left[\begin{array}{cc|cc|cc|cc|cc|cc}
 & & & & & & & & & & & \\
 & & & & & & & & & & & \\
\hline
 & & 0 & 0 & & & 0 & 0 & 0 & 0 & & \\
 & & 0 & 1 & & & 1 & 0 & -1 & -1 & & \\
\hline
 & & & & & & & & & & & \\
 & & & & & & & & & & & \\
\hline
 & & 0 & 1 & & & 1 & 0 & -1 & -1 & & \\
 & & 0 & 0 & & & 0 & 0 & 0 & 0 & & \\
\hline
 & & 0 & -1 & & & -1 & 0 & 1 & 1 & & \\
 & & 0 & -1 & & & -1 & 0 & 1 & 1 & & \\
\hline
 & & & & & & & & & & & \\
 & & & & & & & & & & &
\end{array}\right]
$$

Example (stiffness matrix)

contribution of $J_\lambda^{6,5,3}$ to J_λ

$$
\left[\begin{array}{cc|cc|cc|cc|cc|cc}
 & & & & & & & & & & & \\
 & & & & & & & & & & & \\
\hline
 & & & & & & & & & & & \\
 & & & & & & & & & & & \\
\hline
 & & & & 0 & 0 & & & 0 & 0 & 0 & 0 \\
 & & & & 0 & 1 & & & 1 & 0 & -1 & -1 \\
\hline
 & & & & & & & & & & & \\
 & & & & & & & & & & & \\
\hline
 & & & & 0 & 1 & & & 1 & 0 & -1 & -1 \\
 & & & & 0 & 0 & & & 0 & 0 & 0 & 0 \\
\hline
 & & & & 0 & -1 & & & -1 & 0 & 1 & 1 \\
 & & & & 0 & -1 & & & -1 & 0 & 1 & 1
\end{array}\right]
$$

Example (stiffness matrix)

$$J_\lambda = J_\lambda^{1,2,4} \oplus J_\lambda^{2,3,5} \oplus J_\lambda^{5,4,2} \oplus J_\lambda^{6,5,3}$$

$$
= \left[\begin{array}{cc|cc|cc|cc|cc|cc}
1 & 1 & -1 & 0 & & & 0 & -1 & & & & \\
1 & 1 & -1 & 0 & & & 0 & -1 & & & & \\
\hline
-1 & -1 & 2 & 1 & -1 & 0 & 0 & 1 & 0 & -1 & & \\
0 & 0 & 1 & 2 & -1 & 0 & 1 & 0 & -1 & -2 & & \\
\hline
 & & -1 & -1 & 1 & 0 & & & 0 & 1 & 0 & 0 \\
 & & 0 & 0 & 0 & 1 & & & 1 & 0 & -1 & -1 \\
\hline
0 & 0 & 0 & 1 & & & 1 & 0 & -1 & -1 & & \\
-1 & -1 & 1 & 0 & & & 0 & 1 & 0 & 0 & & \\
\hline
 & & 0 & -1 & 0 & 1 & -1 & 0 & 2 & 1 & -1 & -1 \\
 & & -1 & -2 & 1 & 0 & -1 & 0 & 1 & 2 & 0 & 0 \\
\hline
 & & & & 0 & -1 & & & -1 & 0 & 1 & 1 \\
 & & & & 0 & -1 & & & -1 & 0 & 1 & 1
\end{array}\right]
$$

Example (stiffness matrix)

$$J_\mu = J_\mu^{1,2,4} \oplus J_\mu^{2,3,5} \oplus J_\mu^{5,4,2} \oplus J_\mu^{6,5,3}$$

$$
= \left[\begin{array}{cc|cc|cc|cc|cc|cc}
3 & 1 & -2 & -1 & & & -1 & 0 & & & & \\
1 & 3 & 0 & -1 & & & -1 & -2 & & & & \\
\hline
-2 & 0 & 6 & 1 & -2 & -1 & 0 & 1 & -2 & -1 & & \\
-1 & -1 & 1 & 6 & 0 & -1 & 1 & 0 & -1 & -4 & & \\
\hline
 & & -2 & 0 & 3 & 0 & & & 0 & 1 & -1 & -1 \\
 & & -1 & -1 & 0 & 3 & & & 1 & 0 & 0 & -2 \\
\hline
-1 & -1 & 0 & 1 & & & 3 & 0 & -2 & 0 & & \\
0 & -2 & 1 & 0 & & & 0 & 3 & -1 & -1 & & \\
\hline
 & & -2 & -1 & 0 & 1 & -2 & -1 & 6 & 1 & -2 & 0 \\
 & & -1 & -4 & 1 & 0 & 0 & -1 & 1 & 6 & -1 & -1 \\
\hline
 & & & & -1 & 0 & & & -2 & -1 & 3 & 1 \\
 & & & & -1 & -2 & & & 0 & -1 & 1 & 3
\end{array}\right]
$$

Example (stiffness matrix)

stiffness matrix

$$K = \lambda J_\lambda + \mu J_\mu$$

λ, μ	material-specific
J_λ, J_μ	geometric

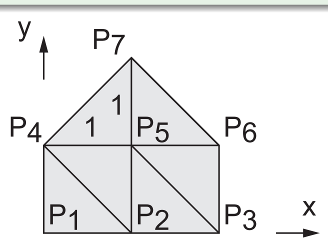
strain potential energy

$$U = \frac{1}{2} \boldsymbol{u}_{\rm N}^{\rm T} \, K \, \boldsymbol{u}_{\rm N}$$

Inertia and Connection Matrices

Report #7 due date : Jan. 9 (Mon) 1:00 AM

Caculate inertia matrix M and connection matrices J_λ, J_μ for a two-dimensional object shown in the figure. Length of the orthogonal sides of isosceles right triangles is 1 and thickness h is equal to 2.



Computing Static Deformation

- Step 1 formulate internal energy and constraints
- Step 2 derive linear equation (if possible)
- Step 3 solve the derived linear equation
- Step 4 visualize obtained numerical solution

or

- Step 1 formulate internal energy and constraints
- Step 2 apply (conditional) numerical optimization to the internal energy with constraints
- Step 3 visualize obtained numerical solution

Internal energy

Strain potential energy

$$U = \frac{1}{2} \boldsymbol{u}_{\rm N}^{\rm T} \, K \, \boldsymbol{u}_{\rm N}$$

Work done by external forces

$$W = \boldsymbol{f}^{\rm T} \boldsymbol{u}_{\rm N}$$

Constraints

$$\boldsymbol{R} = A^{\rm T} \boldsymbol{u}_{\rm N} - \boldsymbol{b} = \boldsymbol{0}$$

Variational principle in statics

$$\text{minimize} \quad I = U - W$$
$$\text{subject to} \quad \boldsymbol{R} = \boldsymbol{0}$$

Minimization

$$\text{minimize} \quad I = U - W$$
$$\text{subject to} \quad \boldsymbol{R} = \boldsymbol{0}$$
$$\Downarrow$$
$$J = U - W - \boldsymbol{\lambda}^{\mathrm{T}}\boldsymbol{R}$$
$$= \frac{1}{2}\boldsymbol{u}_{\mathrm{N}}^{\mathrm{T}}\, K\, \boldsymbol{u}_{\mathrm{N}} - \boldsymbol{f}^{\mathrm{T}}\boldsymbol{u}_{\mathrm{N}} - \boldsymbol{\lambda}^{\mathrm{T}}(A^{\mathrm{T}}\boldsymbol{u}_{\mathrm{N}} - \boldsymbol{b})$$
$$\Downarrow$$

Minimization

$$\frac{\partial J}{\partial \boldsymbol{u}_{\mathrm{N}}} = K\boldsymbol{u}_{\mathrm{N}} - \boldsymbol{f} - A\boldsymbol{\lambda} = \boldsymbol{0}$$
$$\frac{\partial J}{\partial \boldsymbol{\lambda}} = -(A^{\mathrm{T}}\boldsymbol{u}_{\mathrm{N}} - \boldsymbol{b}) = \boldsymbol{0}$$
$$\Downarrow$$
$$\begin{bmatrix} K & -A \\ -A^{\mathrm{T}} & \end{bmatrix} \begin{bmatrix} \boldsymbol{u}_{\mathrm{N}} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \boldsymbol{f} \\ -\boldsymbol{b} \end{bmatrix}$$

solving the above linear equation numerically yields displacement vector $\boldsymbol{u}_{\mathrm{N}}$

Implementation

two-dimentional finite element calculation on MATLAB

`https://www.hirailab.com/edu/common/soft_robotics/Physics_Soft_Bodies.html`

Classes : `NodalPoint`, `Triangle`, `Body`

Implementation

```
classdef NodalPoint
    properties
        Coordinates;
        Displacement;
        Velocity
    end
    methods
        function obj = NodalPoint(p)
            obj.Coordinates = p;
        end
    end
end
```


Implementation

```
classdef Triangle
    properties
        Vertices;
        Area;
        Thickness;
        Density; lambda; mu;
        vector_a; vector_b;
        u_x; u_y; v_x; v_y;
        Cauchy_strain;
        Green_strain;
        Partial_J_lambda; Partial_J_mu;
        Partial_Stiffness_Matrix;
        Partial_Inertia_Matrix;
        Partial_Gravitational_Vector;
    end
    methods
```


Implementation

```
classdef Body
    properties
        numNodalPoints; NodalPoints;
        numTriangles; Triangles;
        strain_potential_energy;
        gravitational_potential_energy;
        J_lambda; J_mu;
        Stiffness_Matrix;
        Inertia_Matrix;
        Gravitational_Vector;
    end
    methods
        function obj = Body(npoints, points, ntris, tris,
            obj.numNodalPoints = npoints;
            for k=1:npoints
                pt(k) = NodalPoint(points(:,k));
```


Implementation

methods of class `Triangle`

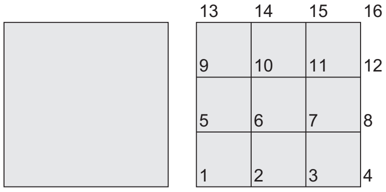
- partial_derivaties — calculating partial derivatives $\partial u/\partial x$, $\partial u/\partial y$, $\partial v/\partial x$, $\partial v/\partial y$
- calculate_Cauchy_strain — calculating Cauchy strain in the triangle
- partial_strain_potential_energy — strain potential energy stored in the triangle
- calculate_Green_strain — calculating Green strain in the triangle
- partial_strain_potential_energy_Green_strain — strain potential energy using Green strain
- partial_gravitational_potential_energy — gravitational potential energy stored in the triangle
- partial_stiffness_matrix — calculating partial stiffness matrix $K_{i,j,k}$
- partial_inertia_matrix — calculating partial inertia matrix $M_{i,j,k}$
- partial_gravitational_vector — calculating partial gravitational vector $\boldsymbol{g}_{i,j,k}$

Implementation

methods of class `Body`

- total_strain_potential_energy — calculating strain potential energy stored in the body
- total_strain_potential_energy_Green_strain — strain potential energy using Green strain
- total_gravitational_potential_energy — gravitational potential energy stored in the body
- calculate_stiffness_matrix — calculating stiffness matrix K
- calculate_inertia_matrix — calculating inertia matrix M
- calculate_gravitational_vector — calculating gravitational vector $\boldsymbol{g}$
- constraint_matrix — constraint matrix when specified nodal points are fixed
- draw — draw the shape of the body

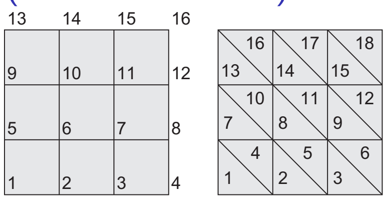
Example (static simulation)



Sample program 'get_started.m'.

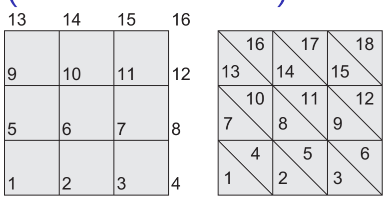
$$\text{points} = \begin{bmatrix} 0 & 1 & 2 & 3 & 0 & 1 & \cdots & 2 & 3 \\ 0 & 0 & 0 & 0 & 1 & 1 & \cdots & 3 & 3 \end{bmatrix}$$

Example (static simulation)



$$\text{triangles} = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 6 \\ 3 & 4 & 7 \\ 6 & 5 & 2 \\ & \vdots & \\ 15 & 14 & 11 \\ 16 & 15 & 12 \end{bmatrix}$$

Example (static simulation)




```
npoints = size(points,2);
ntriangles = size(triangles,1);
thickness = 1;
elastic = Body(npoints, points, ntriangles, tri
```

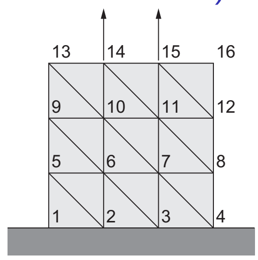
Variable 'elastic' represents the rectangle body.

Example (static simulation)

Defining elatic property to calculate stiffness matrix.

```
% E = 0.1 MPa; \nu = 0.48; rho = 1 g/cm^2
Young = 1.0*1e+6; nu = 0.48; density = 1.00;
[ lambda, mu ] = Lame_constants( Young, nu );
elastic = elastic.mechanical_parameters(density

% stiffness matrix
elastic = elastic.calculate_stiffness_matrix;
K = elastic.Stiffness_Matrix;
```

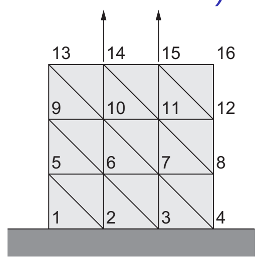

Example (static simulation)



Bottom face is fixed to floor.
Edge $P_{14}P_{15}$ is pulled up / pushed down.

$$A^{\mathrm{T}} \boldsymbol{u}_{\mathrm{N}} = \boldsymbol{b}$$

Example (static simulation)



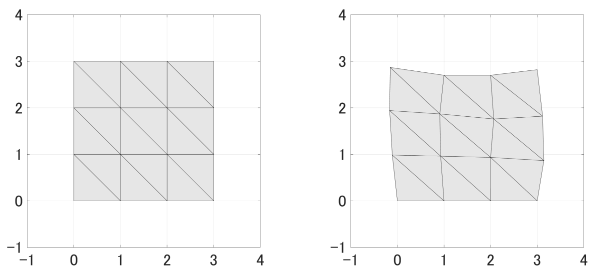

```
% constraints
nconstraints = 12;
A = elastic.constraint_matrix([1, 2, 3, 4, 14, 1
dy = -0.3;
b = [ 0;0; 0;0; 0;0; 0;0; 0;dy; 0;dy ];
```


Example (static simulation)

Building and solving linear equation

```
mat = [ K, -A; -A', zeros(nconstraints,nconstra
vec = [ zeros(2*npoints,1); -b ];
sol = mat \ vec;
un = sol(1:2*npoints);
```


Example (static simulation)



natural deformed

Computing Dynamic Deformation

Step 1 formulate Lagrangian
Step 2 derive Lagrange equations of motion and deformation
Step 3 derive equations for constraint stabilization (if necesary)
Step 4 formulate canonical form of ODEs
Step 5 solve the canonical form of ODEs
Step 6 visualize obtained numerical solution

Lagrange equation

Kinetic and strain potential energies

$$T = \frac{1}{2}\dot{\boldsymbol{u}}_{\rm N}^{\rm T} \, M \, \dot{\boldsymbol{u}}_{\rm N}, \qquad U = \frac{1}{2}\boldsymbol{u}_{\rm N}^{\rm T} \, K \, \boldsymbol{u}_{\rm N}$$

Work done by external forces

$$W = \boldsymbol{f}^{\rm T}\boldsymbol{u}_{\rm N}$$

Constraints

$$\boldsymbol{R} = A^{\rm T}\boldsymbol{u}_{\rm N} - \boldsymbol{b}(t) = \boldsymbol{0}$$

Lagrangian

$$\mathcal{L} = T - U + W + \boldsymbol{\lambda}^{\rm T}\boldsymbol{R}$$

Lagrange equation

Lagrange equation of motion and deformation

$$\frac{\partial \mathcal{L}}{\partial \boldsymbol{u}} - \frac{\rm d}{{\rm d}t}\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{u}}} = \boldsymbol{0}$$

$$\Downarrow$$

$$-K\boldsymbol{u}_{\rm N} + \boldsymbol{f} + A\boldsymbol{\lambda} - M\ddot{\boldsymbol{u}}_{\rm N} = \boldsymbol{0}$$

$$\Downarrow$$

$$\dot{\boldsymbol{u}}_{\rm N} = \boldsymbol{v}_{\rm N}$$
$$M\dot{\boldsymbol{v}}_{\rm N} - A\boldsymbol{\lambda} = -K\boldsymbol{u}_{\rm N} + \boldsymbol{f}$$

Constraint stabilization

Equations for constraint stabilization

$$\ddot{\boldsymbol{R}} + 2\alpha\dot{\boldsymbol{R}} + \alpha^2\boldsymbol{R} = \boldsymbol{0}$$

$$\Downarrow$$

$$(A^{\rm T}\ddot{\boldsymbol{u}}_{\rm N} - \ddot{\boldsymbol{b}}(t)) + 2\alpha(A^{\rm T}\dot{\boldsymbol{u}}_{\rm N} - \dot{\boldsymbol{b}}(t)) + \alpha^2(A^{\rm T}\boldsymbol{u}_{\rm N} - \boldsymbol{b}(t)) = \boldsymbol{0}$$

$$\Downarrow$$

$$-A^{\rm T}\dot{\boldsymbol{v}}_{\rm N} = -\ddot{\boldsymbol{b}}(t) + 2\alpha(A^{\rm T}\boldsymbol{v}_{\rm N} - \dot{\boldsymbol{b}}(t)) + \alpha^2(A^{\rm T}\boldsymbol{u}_{\rm N} - \boldsymbol{b}(t))$$

Ordinary differential equations

Canonical form

$$\dot{\boldsymbol{u}}_{\rm N} = \boldsymbol{v}_{\rm N}$$

$$\begin{bmatrix} M & -A \\ -A^{\rm T} & \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{v}}_{\rm N} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} -K\boldsymbol{u}_{\rm N} + \boldsymbol{f} \\ \boldsymbol{C}(\boldsymbol{u}_{\rm N}, \boldsymbol{v}_{\rm N}) \end{bmatrix}$$

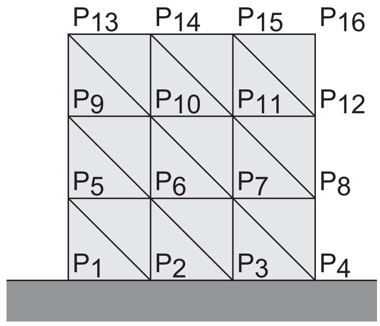
where

$$\boldsymbol{C}(\boldsymbol{u}_{\rm N}, \boldsymbol{v}_{\rm N}) = -\ddot{\boldsymbol{b}}(t) + 2\alpha(A^{\rm T}\boldsymbol{v}_{\rm N} - \dot{\boldsymbol{b}}(t)) + \alpha^2(A^{\rm T}\boldsymbol{u}_{\rm N} - \boldsymbol{b}(t))$$

any ODE solver can be applied to the canonical form

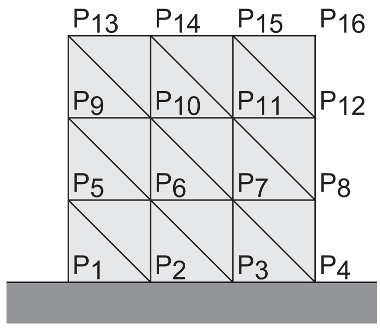
Example (dynamic simulation)

two-dimensional square soft body of width w
Young's modulus E, viscous modulus c, density ρ
divide square into $3 \times 3 \times 2$ triangles



Example (dynamic simulation)

$[\,0,\ t_{push}\,]$ fix the bottom & push $\mathrm{P}_{14}\mathrm{P}_{15}$ downward
$[\,t_{push},\ t_{hold}\,]$ fix the bottom & keep $\mathrm{P}_{14}\mathrm{P}_{15}$
$[\,t_{hold},\ t_{end}\,]$ fix the bottom & release $\mathrm{P}_{14}\mathrm{P}_{15}$



Example (dynamic simulation)

$[\,0,\ t_{push}\,]$ pushing velocity v_{push}

$$\boldsymbol{u}_1 = \boldsymbol{u}_2 = \boldsymbol{u}_3 = \boldsymbol{u}_4 = \boldsymbol{0}$$
$$\boldsymbol{u}_{14} = \boldsymbol{u}_{15} = \boldsymbol{0} + \boldsymbol{v}_{push}t$$

where $\boldsymbol{v}_{push} = [\,0,\ -v_{push}\,]^{\rm T}$

$$A^{\rm T} = \begin{bmatrix} I & & & & \cdots & & \\ & I & & & \cdots & & \\ & & I & & \cdots & & \\ & & & I & \cdots & & \\ & & & & \cdots & I & \\ & & & & \cdots & & I \end{bmatrix}$$

1 2 3 4 14 15-th block columns

Example (dynamic simulation)

$[\,0,\ t_{push}\,]$

note

$$A^{\mathrm{T}} \boldsymbol{u}_{\mathrm{N}} = \begin{bmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_2 \\ \boldsymbol{u}_3 \\ \boldsymbol{u}_4 \\ \boldsymbol{u}_{14} \\ \boldsymbol{u}_{15} \end{bmatrix}$$

speficies nodal points under constraints

Example (dynamic simulation)

$[\,0,\ t_{push}\,]$

$$\boldsymbol{b}(t) = \boldsymbol{b}_0 + \boldsymbol{b}_1 t$$

where

$$\boldsymbol{b}_0 = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{0} \\ \boldsymbol{0} \\ \boldsymbol{0} \\ \boldsymbol{0} \\ \boldsymbol{0} \end{bmatrix}, \qquad \boldsymbol{b}_1 = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{0} \\ \boldsymbol{0} \\ \boldsymbol{0} \\ \boldsymbol{v}_{push} \\ \boldsymbol{v}_{push} \end{bmatrix}$$

note $\dot{\boldsymbol{b}}(t) = \boldsymbol{b}_1$ and $\ddot{\boldsymbol{b}}(t) = \boldsymbol{0}$, yielding

$$\boldsymbol{C}(\boldsymbol{u}_{\mathrm{N}}, \boldsymbol{v}_{\mathrm{N}}) = 2\alpha(A^{\mathrm{T}}\boldsymbol{v}_{\mathrm{N}} - \boldsymbol{b}_1) + \alpha^2(A^{\mathrm{T}}\boldsymbol{u}_{\mathrm{N}} - (\boldsymbol{b}_0 + \boldsymbol{b}_1 t))$$

Example (dynamic simulation)

$[\,t_{push},\ t_{hold}\,]$

$$\boldsymbol{b}_0 = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{0} \\ \boldsymbol{0} \\ \boldsymbol{0} \\ \boldsymbol{v}_{push} t_{push} \\ \boldsymbol{v}_{push} t_{push} \end{bmatrix}, \qquad \boldsymbol{b}_1 = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{0} \\ \boldsymbol{0} \\ \boldsymbol{0} \\ \boldsymbol{0} \\ \boldsymbol{0} \end{bmatrix}$$

Example (dynamic simulation)

$[\,t_{hold},\ t_{end}\,]$

$$\boldsymbol{u}_1 = \boldsymbol{u}_2 = \boldsymbol{u}_3 = \boldsymbol{u}_4 = \boldsymbol{0}$$

$$A^{\mathrm{T}} = \begin{bmatrix} I & & & & \cdots \\ & I & & & \cdots \\ & & I & & \cdots \\ & & & I & \cdots \end{bmatrix}$$

$$\boldsymbol{b}_0 = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{0} \\ \boldsymbol{0} \\ \boldsymbol{0} \end{bmatrix}, \qquad \boldsymbol{b}_1 = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{0} \\ \boldsymbol{0} \\ \boldsymbol{0} \end{bmatrix}$$

Example (dynamic simulation)

```
% Dynamic deformation of an elastic square object (4&time
% g, cm, sec

addpath('../two_dim_fea');

width = 30; height = 30; thickness = 1;
m = 4; n = 4;
[points, triangles] = rectangular_object(m, n, width, hei

% E = 1 MPa; c = 0.04 kPa s; rho = 1 g/cm^2
Young = 10.0*1e+6; c = 0.4*1e+3; nu = 0.48; density = 1.0
[lambda, mu] = Lame_constants(Young, nu);
[lambda_vis, mu_vis] = Lame_constants(c, nu);

npoints = size(points,2);
ntriangles = size(triangles,1);
```

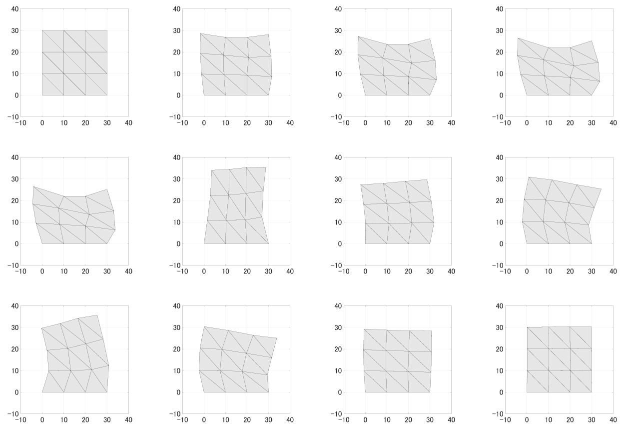

Example (dynamic simulation)

```
elastic = Body(npoints, points, ntriangles, triangles, th
elastic = elastic.mechanical_parameters(density, lambda,
elastic = elastic.viscous_parameters(lambda_vis, mu_vis);
elastic = elastic.calculate_stiffness_matrix;
elastic = elastic.calculate_damping_matrix;
elastic = elastic.calculate_inertia_matrix;

tp = 0.5; vpush = 0.8*(height/3)/tp;
th = 0.5;
tf = 2.0;

alpha = 1e+6;
```

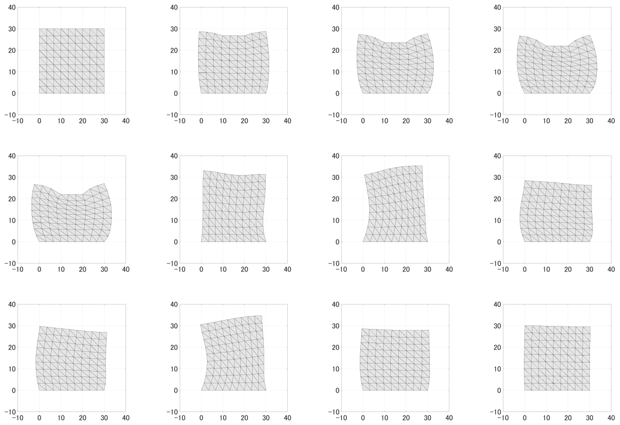

Example (dynamic simulation)

```
% pushing top region
A = elastic.constraint_matrix([1,2,3,4,14,15]);
b0 = zeros(2*6,1);
b1 = [ zeros(2*4,1); 0; -vpush; 0; -vpush ];
interval = [0, tp];
qinit = zeros(4*npoints,1);
square_object_push = @(t,q) square_object_constraint_para
[time_push, q_push] = ode15s(square_object_push, interval

% holding top region
b0 = [ zeros(2*4,1); 0; -vpush*tp; 0; -vpush*tp ];
b1 = zeros(2*6,1);
interval = [tp, tp+th];
qinit = q_push(end,:);
square_object_hold = @(t,q) square_object_constraint_para
[time_hold, q_hold] = ode15s(square_object_hold, interval
```


Example (dynamic simulation)

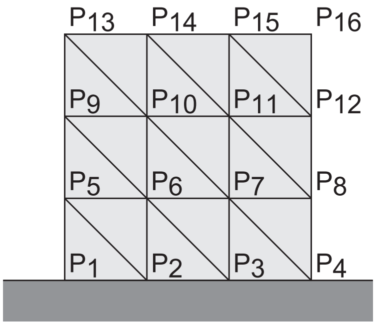


Example (dynamic simulation)



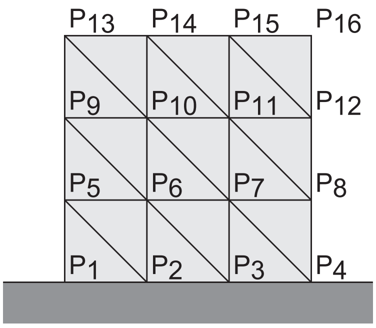
Example (dynamic simulation)

two-dimensional square soft body of width w
Young's modulus E, viscous modulus c, density ρ
divide square into $3 \times 3 \times 2$ triangles



Example (dynamic simulation)

$[\,0,\ t_{push}\,]$ fix the bottom & push $\mathrm{P}_{14}\mathrm{P}_{15}$ downward
$[\,t_{push},\ t_{hold}\,]$ fix the bottom & keep $\mathrm{P}_{14}\mathrm{P}_{15}$
$[\,t_{hold},\ t_{end}\,]$ free (reaction force by penalty method)



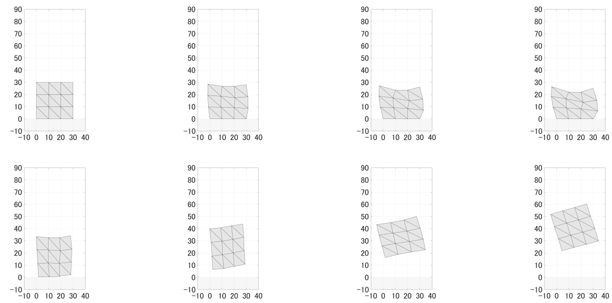
Example (dynamic simulation)

```
% Jumping of an elastic square object (4&times;4)
% g, cm, sec

addpath('../two_dim_fea');

width = 30; height = 30; thickness = 1;
m = 4; n = 4;
[points, triangles] = rectangular_object(m, n, width, hei

% E = 1 MPa; c = 0.04 kPa s; rho = 1 g/cm^2
Young = 10.0*1e+6; c = 0.4*1e+3; nu = 0.48; density = 1.0
% Kfloor = 0.002 MPa/m = 2 KPa/cm
Epfloor = 0.02*1e+6;
[lambda, mu] = Lame_constants(Young, nu);
[lambda_vis, mu_vis] = Lame_constants(c, nu);
```


Example (dynamic simulation)

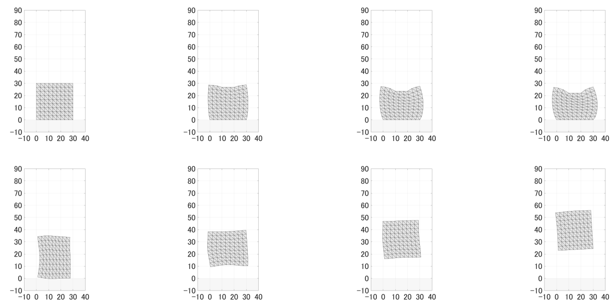
```
% holding top region
b0 = [ zeros(2*4,1); 0; -vpush*tp; 0; -vpush*tp ];
b1 = zeros(2*6,1);
interval = [tp, tp+th];
qinit = q_push(end,:);
square_object_hold = @(t,q) square_object_constraint_para
[time_hold, q_hold] = ode15s(square_object_hold, interval

% releasing all constraints
floor_force = @(t,npoints,un,vn) floor_force_param(t,npoi
interval = [tp+th, tp+th+tf];
qinit = q_hold(end,:);
square_object_free = @(t,q) square_object_free_param(t,q,
[time_free, q_free] = ode15s(square_object_free, interval

time = [time_push; time_hold; time_free];
```


Example (dynamic simulation)



jump simulation movie

Example (dynamic simulation)



jump simulation movie

Example (dynamic simulation)

- motion and deformation can be simulated properly
- results depend on mesh and include artifacts
- finer mesh yields better result but needs more computation time

Summary

energies in integral forms

potential energy

$$U = \int (\text{potential energy density}) \cdot (\text{volume element})$$

kinetic energy

$$T = \int (\text{kinetic energy density}) \cdot (\text{volume element})$$

Summary

integrals

$$\int_{\text{region}} = \sum_{\text{small regions}} \int_{\text{small region}}$$

1D line segments
2D triangles / rectangles / $\cdots$
3D tetrahedra / cubes / $\cdots$

Summary

one-dimensional deformation

extensional strain	ε
Young's modulus	E
strain potential energy density	$\frac{1}{2}E\varepsilon^2$
kinetic energy density	$\frac{1}{2}\rho\dot{\varepsilon}^2$
volume element	$A\,\mathrm{d}x$

Summary

two/three-dimensional deformation

strain vector	$\boldsymbol{\varepsilon}$ (extensional & shear strains)
elasticity matrix	$\lambda I_\lambda + \mu I_\mu$ (Lamé's constants λ, μ)
strain potential energy density	$\frac{1}{2}\boldsymbol{\varepsilon}^{\mathrm{T}}(\lambda I_\lambda + \mu I_\mu)\boldsymbol{\varepsilon}$
kinetic energy density	$\frac{1}{2}\rho\dot{\boldsymbol{\varepsilon}}^{\mathrm{T}}\dot{\boldsymbol{\varepsilon}}$
volume element	$h\,\mathrm{d}S$ or $\mathrm{d}V$

Summary

strain potential energy

quadratic form with respect to $\boldsymbol{u}_{\mathrm{N}}$

$$U = \frac{1}{2}\boldsymbol{u}_{\mathrm{N}}^{\mathrm{T}}\,K\,\boldsymbol{u}_{\mathrm{N}} \quad (K\text{: stiffness matrix})$$

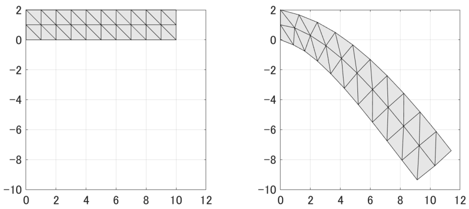
kinetic energy

quadratic form with respect to $\dot{\boldsymbol{u}}_{\mathrm{N}}$

$$T = \frac{1}{2}\dot{\boldsymbol{u}}_{\mathrm{N}}^{\mathrm{T}}\,M\,\dot{\boldsymbol{u}}_{\mathrm{N}} \quad (M\text{: inertia matrix})$$

Calculating based on Cauchy Strain

elastic beam
one end of the beam is fixed to a wall
force is applied to the center of the other end



natural deformed

Calculating based on Cauchy Strain

Cauchy strain

$$\varepsilon_{xx} = \frac{\partial u}{\partial x}, \qquad \varepsilon_{yy} = \frac{\partial v}{\partial y}, \qquad 2\varepsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

Displacements caused by pure rotation

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} C_\theta & -S_\theta \\ S_\theta & C_\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} C_\theta - 1 & -S_\theta \\ S_\theta & C_\theta - 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\Downarrow$$

Calculating based on Cauchy Strain

Cauchy strain

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = C_\theta - 1$$
$$\varepsilon_{yy} = \frac{\partial v}{\partial y} = C_\theta - 1$$
$$2\varepsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = (-S_\theta) + S_\theta = 0$$

Pure rotation (no deformation) yields
non-zero Cauchy strain components

Green Strain

Green strain

$$\boldsymbol{E} = \begin{bmatrix} E_{xx} \\ E_{yy} \\ 2E_{xy} \end{bmatrix}$$

Green strain components

$$E_{xx} = u_x + \frac{1}{2}(u_x^2 + v_x^2)$$
$$E_{yy} = v_y + \frac{1}{2}(u_y^2 + v_y^2)$$
$$2E_{xy} = u_y + v_x + (u_x u_y + v_x v_y)$$

Green Strain

under pure rotation

$$\begin{aligned} E_{xx} &= u_x + \frac{1}{2}(u_x^2 + v_x^2) \\ &= (C_\theta - 1) + \frac{1}{2}\left\{(C_\theta - 1)^2 + (S_\theta)^2\right\} \\ &= 0 \\ E_{yy} &= u_x + \frac{1}{2}(u_x^2 + v_x^2) \\ &= (C_\theta - 1) + \frac{1}{2}\left\{(-S_\theta)^2 + (C_\theta - 1)^2\right\} \\ &= 0 \end{aligned}$$

Green Strain

under pure rotation

$$\begin{aligned} 2E_{xy} &= u_y + v_x + (u_x u_y + v_x v_y) \\ &= (-S_\theta) + (S_\theta) + \{(C_\theta - 1)(-S_\theta) + (S_\theta)(C_\theta - 1)\} \\ &= 0 \end{aligned}$$

Pure rotation (no deformation) yields
Green strain components be zero

$$\Downarrow$$

able to eliminate effect of rotation
rotation-invariant strain

Calculating based on Green Strain

Calculating Green strain energy stored in $\triangle \mathrm{P}_i \mathrm{P}_j \mathrm{P}_k$

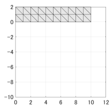
- calculate vector $\boldsymbol{a}$ and $\boldsymbol{b}$
- calculate partial derivatives $\partial u/\partial x$, $\partial u/\partial y$, $\partial v/\partial x$, and $\partial v/\partial y$
- calculate Green strain components
- calculate $U_{i,j,k} = (1/2)\, \boldsymbol{E}^{\mathrm{T}} K_{i,j,k} \boldsymbol{E}$

$$\begin{aligned} &\text{minimize} \quad I(\boldsymbol{u}_{\mathrm{N}}) = U - W \\ &\text{subject to} \quad A^{\mathrm{T}} \boldsymbol{u}_{\mathrm{N}} = \boldsymbol{0} \end{aligned}$$

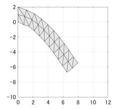
Applying `fmincon` results in static deformation

Calculating based on Green Strain

elastic beam
one end of the beam is fixed to a wall
force is applied to the center of the other end

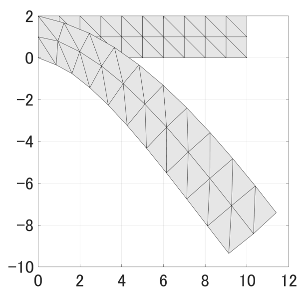


natural

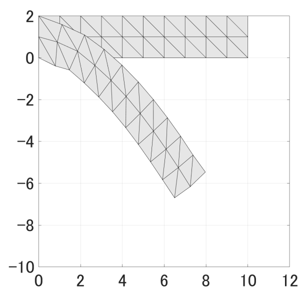


deformed

Calculating based on Green Strain



Cauchy strain



Green strain

Green Strain

two neighboring points $\mathrm{P}(x, y)$ and $\mathrm{Q}(x + \delta x, y + \delta y)$
square of distance betweeen P and Q in natural shape

$$(\delta s)^2 = \delta x^2 + \delta y^2$$

vector from P to Q in deformed shape

$$\begin{aligned} &\begin{bmatrix} \delta x \\ \delta y \end{bmatrix} + \begin{bmatrix} u(x + \delta x, y + \delta y) \\ v(x + \delta x, y + \delta y) \end{bmatrix} - \begin{bmatrix} u(x, y) \\ v(x, y) \end{bmatrix} \\ = &\begin{bmatrix} \delta x \\ \delta y \end{bmatrix} + \begin{bmatrix} u_x \delta x + u_y \delta y \\ v_x \delta x + v_y \delta y \end{bmatrix} \end{aligned}$$

Green Strain

square of distance betweeen P and Q in deformed shape

$$(\delta s')^2 = (\delta x + u_x \delta x + u_y \delta y)^2 + (\delta y + v_x \delta x + v_y \delta y)^2$$

difference

$$\begin{aligned} (\delta s')^2 - (\delta s)^2 &= 2\delta x(u_x \delta x + u_y \delta y) + 2\delta y(v_x \delta x + v_y \delta y) \\ &\quad + (u_x \delta x + u_y \delta y)^2 + (v_x \delta x + v_y \delta y)^2 \\ &= 2 \begin{bmatrix} \delta x & \delta y \end{bmatrix} \begin{bmatrix} E_{xx} & E_{xy} \\ E_{xy} & E_{yy} \end{bmatrix} \begin{bmatrix} \delta x \\ \delta y \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \text{pure rotation} &\Rightarrow (\delta s')^2 - (\delta s)^2 = 0, \ \forall \delta x, \delta y \\ &\Rightarrow E_{xx} = 0, \ E_{yy} = 0, \ E_{xy} = 0 \end{aligned}$$

Handouts

Sample programs (MATLAB) are available at:

`http://www.ritsumei.ac.jp/~hirai/edu/common/soft_robotics/Physics_Soft_Bodies.html`

Simulating Viscoelastic Deformation

Report #8 due date : Jan. 30 (Mon) 1:00 AM

Simulate the deformation of a rectangular viscoelastic object shown in the figure. The bottom surface is fixed to the ground. A pair of forces with the same magnitude f are applied to the both sides for a while, then the forces are released. Action lines of the forces do not coincide. Use appropriate values of geometrical and physical parameters of the object.

Simulating Viscoelastic Deformation