Inelastic Deformation

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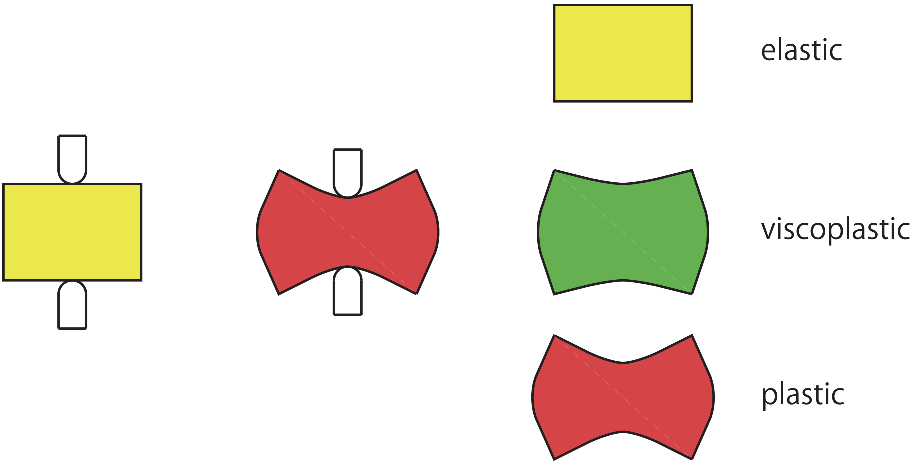
Agenda

1. One-dimensional Inelastic Deformation
2. Multi-dimensional Inelastic Deformation
3. Finite Element Method in Inelastic Deformation

Elastic/viscoplastic/plastic deformation



Maxwell model

E: Young's modulus
c: viscous modulus
$\varepsilon^{\rm ela}$: strain at elastic element
$\varepsilon^{\rm vis}$: strain at viscous element
ε: strain
σ: stress

$$\varepsilon = \varepsilon^{\rm ela} + \varepsilon^{\rm vis}$$

$$\sigma = E\varepsilon^{\rm ela}, \quad \sigma = c\dot{\varepsilon}^{\rm vis}$$

stress-strain relationship in Maxwell model:

$$\dot{\sigma} + \frac{E}{c}\sigma = E\dot{\varepsilon}$$

Maxwell model

ordinary differential equation of the first order:

$$\dot{\sigma} + \frac{E}{c}\sigma = E\dot{\varepsilon}$$

stress at time t:

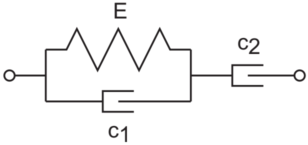
$$\sigma(t) = \int_0^t E e^{-\frac{E}{c}(t-t')} \dot{\varepsilon}(t') \, \mathrm{d}t'$$

In general,

$$\sigma(t) = \int_0^t r(t-t') \, \dot{\varepsilon}(t') \, \mathrm{d}t'$$

Function $r(t-t')$: *relaxation function*

Three-element model



E: Young's modulus
$c_1,\ c_2$: viscous moduli
$\varepsilon^{\rm voigt}$: strain at Voigt element
$\varepsilon^{\rm vis}$: strain at viscous element
ε: strain
σ: stress

$$\varepsilon = \varepsilon^{\rm voigt} + \varepsilon^{\rm vis}$$

$$\sigma = E\varepsilon^{\rm voigt} + c_1\dot{\varepsilon}^{\rm voigt}, \quad \sigma = c_2\dot{\varepsilon}^{\rm vis}$$

stress-strain relationship in three-element model:

$$\dot{\sigma} + \frac{E}{c_1 + c_2}\sigma = \frac{Ec_2}{c_1 + c_2}\dot{\varepsilon} + \frac{c_1 c_2}{c_1 + c_2}\ddot{\varepsilon}$$

Three-element model

ordinary differential equation of the first order:

$$\dot{\sigma} + \frac{E}{c_1 + c_2}\sigma = \frac{Ec_2}{c_1 + c_2}\dot{\varepsilon} + \frac{c_1 c_2}{c_1 + c_2}\ddot{\varepsilon}$$

stress at time t:

$$\sigma(t) = \int_0^t r(t - t')\,\dot{\varepsilon}(t')\,\mathrm{d}t'$$

where

$$r(t - t') = \frac{Ec_2}{c_1 + c_2} e^{-\frac{E}{c_1+c_2}(t-t')} \left(1 + \frac{c_1}{E}\frac{\mathrm{d}}{\mathrm{d}t}\right)$$

Isotropic deformation models

elastic deformation

specified by a constant E:

$$\sigma = E\varepsilon$$

2D isotropic elastic deformation

specified by two constants λ and μ (Lamé's constants):

$$\boldsymbol{\sigma} = (\lambda I_\lambda + \mu I_\mu)\boldsymbol{\varepsilon}$$

$$I_\lambda = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad I_\mu = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Isotropic deformation models

viscoelastic deformation

specified by an operator $E + c\,\mathrm{d}/\mathrm{d}t$:

$$\sigma = \left(E + c\,\frac{\mathrm{d}}{\mathrm{d}t} \right) \varepsilon$$

2D isotropic viscoelastic deformation

specified by two operators λ and μ:

$$\boldsymbol{\sigma} = (\lambda I_\lambda + \mu I_\mu)\boldsymbol{\varepsilon}$$

$$\lambda = \lambda^{\mathrm{ela}} + \lambda^{\mathrm{vis}}\frac{\mathrm{d}}{\mathrm{d}t}, \quad \mu = \mu^{\mathrm{ela}} + \mu^{\mathrm{vis}}\frac{\mathrm{d}}{\mathrm{d}t}$$

Isotropic deformation models

viscoplastic deformation

specified by a convolution with a relaxation function:

$$\sigma(t) = \int_0^t r(t - t')\,\dot{\varepsilon}(t')\,\mathrm{d}t'$$

2D isotropic viscoplastic deformation

specified by two relaxation functions:

$$\boldsymbol{\sigma}(t) = \int_0^t R(t - t')\,\dot{\boldsymbol{\varepsilon}}(t')\,\mathrm{d}t'$$

$$R(t - t') = r_\lambda(t - t') I_\lambda + r_\mu(t - t') I_\mu$$

Maxwell model

relaxation function

$$r(t - t') = E \exp\left\{-\frac{E}{c}(t - t')\right\}$$

2D isotropic viscoplastic deformation

two relaxation functions:

$$r_\lambda(t - t') = \lambda^{\rm ela} \exp\left\{-\frac{\lambda^{\rm ela}}{\lambda^{\rm vis}}(t - t')\right\}$$

$$r_\mu(t - t') = \mu^{\rm ela} \exp\left\{-\frac{\mu^{\rm ela}}{\mu^{\rm vis}}(t - t')\right\}$$

Three-element model

relaxation function

$$r(t-t') = \frac{Ec_2}{c_1+c_2} e^{-\frac{E}{c_1+c_2}(t-t')} \left(1 + \frac{c_1}{E}\frac{\mathrm{d}}{\mathrm{d}t}\right)$$

2D isotropic viscoplastic deformation

two relaxation functions:

$$r_\lambda(t-t') = \frac{\lambda^{\mathrm{ela}}\lambda_2^{\mathrm{vis}}}{\lambda_1^{\mathrm{vis}}+\lambda_2^{\mathrm{vis}}} \exp\left\{-\frac{\lambda^{\mathrm{ela}}}{\lambda_1^{\mathrm{vis}}+\lambda_2^{\mathrm{vis}}}(t-t')\right\} \left(1 + \frac{\lambda_1^{\mathrm{vis}}}{\lambda^{\mathrm{ela}}}\frac{\mathrm{d}}{\mathrm{d}t}\right)$$

$$r_\mu(t-t') = \frac{\mu^{\mathrm{ela}}\mu_2^{\mathrm{vis}}}{\mu_1^{\mathrm{vis}}+\mu_2^{\mathrm{vis}}} \exp\left\{-\frac{\mu^{\mathrm{ela}}}{\mu_1^{\mathrm{vis}}+\mu_2^{\mathrm{vis}}}(t-t')\right\} \left(1 + \frac{\mu_1^{\mathrm{vis}}}{\mu^{\mathrm{ela}}}\frac{\mathrm{d}}{\mathrm{d}t}\right)$$

Nodal elastic forces

stress-strain relationship

$$\boldsymbol{\sigma} = (\lambda I_\lambda + \mu I_\mu)\boldsymbol{\varepsilon}$$

a set of elastic forces applied to nodal points:

$$\text{elastic force} = -(\lambda J_\lambda + \mu J_\mu)\boldsymbol{u}_\mathrm{N}$$

from stress-strain relationship to nodal force set

replacing I_λ by J_λ, I_μ by J_μ, and $\boldsymbol{\varepsilon}$ by $\boldsymbol{u}_\mathrm{N}$ in the stress-strain relationship yields the elastic force set

Nodal viscoelastic forces

stress-strain relationship

$$\boldsymbol{\sigma} = I_{\lambda}(\lambda^{\mathrm{ela}}\boldsymbol{\varepsilon} + \lambda^{\mathrm{vis}}\dot{\boldsymbol{\varepsilon}}) + I_{\mu}(\mu^{\mathrm{ela}}\boldsymbol{\varepsilon} + \mu^{\mathrm{vis}}\dot{\boldsymbol{\varepsilon}})$$

replacing I_{λ} by J_{λ}, I_{μ} by J_{μ}, and $\boldsymbol{\varepsilon}$ by $\boldsymbol{u}_{\mathrm{N}}$ in the stress-strain relationship yields a viscoelastic force set

$$\Downarrow$$

a set of viscoelastic forces applied to nodal points:

$$\begin{aligned}\text{viscoelastic force} = & - J_{\lambda}(\lambda^{\mathrm{ela}}\boldsymbol{u}_{\mathrm{N}} + \lambda^{\mathrm{vis}}\dot{\boldsymbol{u}}_{\mathrm{N}}) \\ & - J_{\mu}(\mu^{\mathrm{ela}}\boldsymbol{u}_{\mathrm{N}} + \mu^{\mathrm{vis}}\dot{\boldsymbol{u}}_{\mathrm{N}})\end{aligned}$$

Nodal viscoplastic forces

stress-strain relationship

$$\boldsymbol{\sigma}(t) = I_\lambda \int_0^t r_\lambda(t-t')\,\dot{\boldsymbol{\varepsilon}}(t')\,\mathrm{d}t' + I_\mu \int_0^t r_\mu(t-t')\,\dot{\boldsymbol{\varepsilon}}(t')\,\mathrm{d}t'$$

replacing I_λ by J_λ, I_μ by J_μ, and $\boldsymbol{\varepsilon}$ by $\boldsymbol{u}_\mathrm{N}$ in the stress-strain relationship yields a viscoplastic force set

$$\Downarrow$$

a set of viscoplastic forces applied to nodal points

$$\text{viscoplastic force} = -J_\lambda \int_0^t r_\lambda(t-t')\,\dot{\boldsymbol{u}}_\mathrm{N}(t')\,\mathrm{d}t' - J_\mu \int_0^t r_\mu(t-t')\,\dot{\boldsymbol{u}}_\mathrm{N}(t')\,\mathrm{d}t'$$

Nodal viscoplastic forces

introduce

$$\boldsymbol{f}_\lambda = \int_0^t r_\lambda(t - t')\, \dot{\boldsymbol{u}}_{\rm N}(t')\, {\rm d}t'$$
$$\boldsymbol{f}_\mu = \int_0^t r_\mu(t - t')\, \dot{\boldsymbol{u}}_{\rm N}(t')\, {\rm d}t'$$

Vectors $\boldsymbol{f}_\lambda$ and $\boldsymbol{f}_\mu$ have dimension of force/length

Nodal vescoplastic forces

$$\text{viscoplastic force} = -J_\lambda \boldsymbol{f}_\lambda - J_\mu \boldsymbol{f}_\mu$$

Maxwell model

introduce

$$\boldsymbol{f}_\lambda = \int_0^t \lambda^{\rm ela} \exp\left\{ -\frac{\lambda^{\rm ela}}{\lambda^{\rm vis}}(t - t') \right\} \dot{\boldsymbol{u}}_{\rm N}(t')\,{\rm d}t'$$

$$\boldsymbol{f}_\mu = \int_0^t \mu^{\rm ela} \exp\left\{ -\frac{\mu^{\rm ela}}{\mu^{\rm vis}}(t - t') \right\} \dot{\boldsymbol{u}}_{\rm N}(t')\,{\rm d}t'$$

Nodal vescoplastic forces

$$\text{viscoplastic force} = -J_\lambda \boldsymbol{f}_\lambda - J_\mu \boldsymbol{f}_\mu$$

$$\dot{\boldsymbol{f}}_\lambda = -\frac{\lambda^{\rm ela}}{\lambda^{\rm vis}}\boldsymbol{f}_\lambda + \lambda^{\rm ela}\dot{\boldsymbol{u}}_{\rm N} = -\frac{\lambda^{\rm ela}}{\lambda^{\rm vis}}\boldsymbol{f}_\lambda + \lambda^{\rm ela}\boldsymbol{v}_{\rm N}$$

$$\dot{\boldsymbol{f}}_\mu = -\frac{\mu^{\rm ela}}{\mu^{\rm vis}}\boldsymbol{f}_\mu + \mu^{\rm ela}\dot{\boldsymbol{u}}_{\rm N} = -\frac{\mu^{\rm ela}}{\mu^{\rm vis}}\boldsymbol{f}_\mu + \mu^{\rm ela}\boldsymbol{v}_{\rm N}$$

Maxwell model

ordinary differential equation of the first order:

$$\dot{\sigma} + \frac{E}{c}\sigma = E\dot{\varepsilon}$$

$$\Downarrow$$

$$\dot{\boldsymbol{f}}_{\lambda} = -\frac{\lambda^{\mathrm{ela}}}{\lambda^{\mathrm{vis}}}\boldsymbol{f}_{\lambda} + \lambda^{\mathrm{ela}}\dot{\boldsymbol{u}}_{\mathrm{N}} = -\frac{\lambda^{\mathrm{ela}}}{\lambda^{\mathrm{vis}}}\boldsymbol{f}_{\lambda} + \lambda^{\mathrm{ela}}\boldsymbol{v}_{\mathrm{N}}$$

$$\dot{\boldsymbol{f}}_{\mu} = -\frac{\mu^{\mathrm{ela}}}{\mu^{\mathrm{vis}}}\boldsymbol{f}_{\mu} + \mu^{\mathrm{ela}}\dot{\boldsymbol{u}}_{\mathrm{N}} = -\frac{\mu^{\mathrm{ela}}}{\mu^{\mathrm{vis}}}\boldsymbol{f}_{\mu} + \mu^{\mathrm{ela}}\boldsymbol{v}_{\mathrm{N}}$$

Three-element model

introduce

$$\boldsymbol{f}_{\lambda} = \int_0^t \frac{\lambda^{\rm ela}\lambda_2^{\rm vis}}{\lambda_1^{\rm vis} + \lambda_2^{\rm vis}} e^{-\frac{\lambda^{\rm ela}}{\lambda_1^{\rm vis}+\lambda_2^{\rm vis}}(t-t')} \left(\dot{\boldsymbol{u}}_{\rm N} + \frac{\lambda_1^{\rm vis}}{\lambda^{\rm ela}} \ddot{\boldsymbol{u}}_{\rm N} \right) (t') \, {\rm d}t'$$

$$\boldsymbol{f}_{\mu} = \int_0^t \frac{\mu^{\rm ela}\mu_2^{\rm vis}}{\mu_1^{\rm vis} + \mu_2^{\rm vis}} e^{-\frac{\mu^{\rm ela}}{\mu_1^{\rm vis}+\mu_2^{\rm vis}}(t-t')} \left(\dot{\boldsymbol{u}}_{\rm N} + \frac{\mu_1^{\rm vis}}{\mu^{\rm ela}} \ddot{\boldsymbol{u}}_{\rm N} \right) (t') \, {\rm d}t'$$

Nodal vescoplastic forces

$$\text{vescoplastic force} = -J_{\lambda}\boldsymbol{f}_{\lambda} - J_{\mu}\boldsymbol{f}_{\mu}$$

$$\dot{\boldsymbol{f}}_{\lambda} = -\frac{\lambda^{\rm ela}}{\lambda_1^{\rm vis} + \lambda_2^{\rm vis}} \boldsymbol{f}_{\lambda} + \frac{\lambda^{\rm ela}\lambda_2^{\rm vis}}{\lambda_1^{\rm vis} + \lambda_2^{\rm vis}} \boldsymbol{v}_{\rm N} + \frac{\lambda_1^{\rm vis}\lambda_2^{\rm vis}}{\lambda_1^{\rm vis} + \lambda_2^{\rm vis}} \dot{\boldsymbol{v}}_{\rm N}$$

$$\dot{\boldsymbol{f}}_{\mu} = -\frac{\mu^{\rm ela}}{\mu_1^{\rm vis} + \mu_2^{\rm vis}} \boldsymbol{f}_{\mu} + \frac{\mu^{\rm ela}\mu_2^{\rm vis}}{\mu_1^{\rm vis} + \mu_2^{\rm vis}} \boldsymbol{v}_{\rm N} + \frac{\mu_1^{\rm vis}\mu_2^{\rm vis}}{\mu_1^{\rm vis} + \mu_2^{\rm vis}} \dot{\boldsymbol{v}}_{\rm N}$$

Three-element model

ordinary differential equation of the first order:

$$\dot{\sigma} + \frac{E}{c_1 + c_2}\sigma = \frac{Ec_2}{c_1 + c_2}\dot{\varepsilon} + \frac{c_1 c_2}{c_1 + c_2}\ddot{\varepsilon}$$

$$\Downarrow$$

$$\dot{\boldsymbol{f}}_\lambda = -\frac{\lambda^{\rm ela}}{\lambda_1^{\rm vis} + \lambda_2^{\rm vis}}\boldsymbol{f}_\lambda + \frac{\lambda^{\rm ela}\lambda_2^{\rm vis}}{\lambda_1^{\rm vis} + \lambda_2^{\rm vis}}\boldsymbol{v}_{\rm N} + \frac{\lambda_1^{\rm vis}\lambda_2^{\rm vis}}{\lambda_1^{\rm vis} + \lambda_2^{\rm vis}}\dot{\boldsymbol{v}}_{\rm N}$$

$$\dot{\boldsymbol{f}}_\mu = -\frac{\mu^{\rm ela}}{\mu_1^{\rm vis} + \mu_2^{\rm vis}}\boldsymbol{f}_\mu + \frac{\mu^{\rm ela}\mu_2^{\rm vis}}{\mu_1^{\rm vis} + \mu_2^{\rm vis}}\boldsymbol{v}_{\rm N} + \frac{\mu_1^{\rm vis}\mu_2^{\rm vis}}{\mu_1^{\rm vis} + \mu_2^{\rm vis}}\dot{\boldsymbol{v}}_{\rm N}$$

FE formulation of viscoplastic deformation

a set of equations of elastic deformation:

$$\underbrace{-K\boldsymbol{u}_{\rm N}}_{\text{elastic force}} + \boldsymbol{f}_{\rm ext} + A\boldsymbol{\lambda}_A - M\ddot{\boldsymbol{u}}_{\rm N} = \boldsymbol{0}$$

$$\Downarrow$$

a set of equations of viscoplastic deformation:

$$\underbrace{-J_\lambda \boldsymbol{f}_\lambda - J_\mu \boldsymbol{f}_\mu}_{\text{viscoplastic force}} + \boldsymbol{f}_{\rm ext} + A\boldsymbol{\lambda}_A - M\ddot{\boldsymbol{u}}_{\rm N} = \boldsymbol{0}$$

Maxwell model

$$\dot{\boldsymbol{u}}_{\rm N} = \boldsymbol{v}_{\rm N}$$

$$\begin{bmatrix} M & -A \\ -A^{\rm T} & \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{v}}_{\rm N} \\ \boldsymbol{\lambda}_A \end{bmatrix} = \begin{bmatrix} -J_\lambda \boldsymbol{f}_\lambda - J_\mu \boldsymbol{f}_\mu + \boldsymbol{f}_{\rm ext} \\ A^{\rm T}(2\alpha \boldsymbol{v}_{\rm N} + \alpha^2 \boldsymbol{u}_{\rm N}) \end{bmatrix}$$

$$\dot{\boldsymbol{f}}_\lambda = -\frac{\lambda^{\rm ela}}{\lambda^{\rm vis}} \boldsymbol{f}_\lambda + \lambda^{\rm ela} \boldsymbol{v}_{\rm N}$$

$$\dot{\boldsymbol{f}}_\mu = -\frac{\mu^{\rm ela}}{\mu^{\rm vis}} \boldsymbol{f}_\mu + \mu^{\rm ela} \boldsymbol{v}_{\rm N}$$

Three-element model

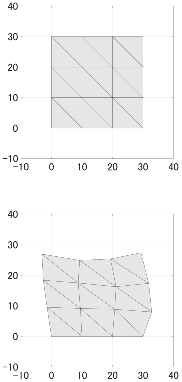
$$\dot{\boldsymbol{u}}_{\rm N} = \boldsymbol{v}_{\rm N}$$

$$\left[\begin{array}{cc} M & -A \\ -A^{\rm T} & \end{array} \right] \left[\begin{array}{c} \dot{\boldsymbol{v}}_{\rm N} \\ \boldsymbol{\lambda}_A \end{array} \right] = \left[\begin{array}{c} -J_\lambda \boldsymbol{f}_\lambda - J_\mu \boldsymbol{f}_\mu + \boldsymbol{f}_{\rm ext} \\ A^{\rm T}(2\alpha \boldsymbol{v}_{\rm N} + \alpha^2 \boldsymbol{u}_{\rm N}) \end{array} \right]$$

$$\dot{\boldsymbol{f}}_\lambda = -\frac{\lambda^{\rm ela}}{\lambda_1^{\rm vis} + \lambda_2^{\rm vis}} \boldsymbol{f}_\lambda + \frac{\lambda^{\rm ela} \lambda_2^{\rm vis}}{\lambda_1^{\rm vis} + \lambda_2^{\rm vis}} \boldsymbol{v}_{\rm N} + \frac{\lambda_1^{\rm vis} \lambda_2^{\rm vis}}{\lambda_1^{\rm vis} + \lambda_2^{\rm vis}} \dot{\boldsymbol{v}}_{\rm N}$$

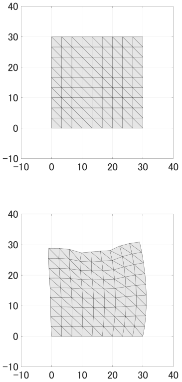
$$\dot{\boldsymbol{f}}_\mu = -\frac{\mu^{\rm ela}}{\mu_1^{\rm vis} + \mu_2^{\rm vis}} \boldsymbol{f}_\mu + \frac{\mu^{\rm ela} \mu_2^{\rm vis}}{\mu_1^{\rm vis} + \mu_2^{\rm vis}} \boldsymbol{v}_{\rm N} + \frac{\mu_1^{\rm vis} \mu_2^{\rm vis}}{\mu_1^{\rm vis} + \mu_2^{\rm vis}} \dot{\boldsymbol{v}}_{\rm N}$$

Example (Sample Program)



simulation movie

Example (Sample Program)



simulation movie

Summary

one-dimensional inelastic deformation

- Maxwell model
- three-element model

2D/3D inelastic deformation

- isotropic deformation models
- formulating nodel force sets
- finite element formulation

Handouts

Sample programs (MATLAB) are available at:

http://www.ritsumei.ac.jp/~hirai/edu/common/soft_robotics/Physics_Soft_Bodies.html