

Analytical Mechanics: Link Mechanisms

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Agenda

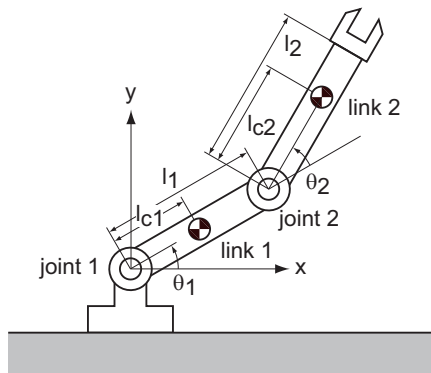
1 Open Link Mechanism

- Kinematics of Open Link Mechanism
- Dynamics of 2DOF open link mechanism

2 Closed Link Mechanism

- Kinematics of Closed Link Mechanism
- Dynamics of 2DOF closed link mechanism

Kinematics of 2DOF open link mechanism



two link open link mechanism

l_i length of link i

l_{ci} distance btw. joint i and the center of mass of link i

m_i mass of link i

J_i inertia of moment of link i around its center of mass

θ_1 rotation angle of joint 1

θ_2 rotation angle of joint 2

Kinematics of 2DOF open link mechanism

position of the center of mass of link 1:

$$\mathbf{x}_{c1} \triangleq \begin{bmatrix} x_{c1} \\ y_{c1} \end{bmatrix} = l_{c1} \begin{bmatrix} C_1 \\ S_1 \end{bmatrix}$$

position of the center of mass of link 2:

$$\mathbf{x}_{c2} \triangleq \begin{bmatrix} x_{c2} \\ y_{c2} \end{bmatrix} = l_1 \begin{bmatrix} C_1 \\ S_1 \end{bmatrix} + l_{c2} \begin{bmatrix} C_{1+2} \\ S_{1+2} \end{bmatrix}$$

orientation angle of link 1:

$$\theta_1$$

orientation angle of link 2:

$$\theta_1 + \theta_2$$

Kinetic energy

velocity of the center of mass of link 1:

$$\dot{\mathbf{x}}_{c1} = l_{c1} \dot{\theta}_1 \begin{bmatrix} -S_1 \\ C_1 \end{bmatrix}$$

angular velocity of link 1:

$$\dot{\theta}_1$$

kinetic energy of link 1:

$$\begin{aligned} T_1 &= \frac{1}{2} m_1 \dot{\mathbf{x}}_{c1}^T \dot{\mathbf{x}}_{c1} + \frac{1}{2} J_1 \dot{\theta}_1^2 \\ &= \frac{1}{2} (m_1 l_{c1}^2 + J_1) \dot{\theta}_1^2 \end{aligned}$$

Kinetic energy

velocity of the center of mass of link 2:

$$\dot{\mathbf{x}}_{c2} = l_1 \dot{\theta}_1 \begin{bmatrix} -S_1 \\ C_1 \end{bmatrix} + l_{c2}(\dot{\theta}_1 + \dot{\theta}_2) \begin{bmatrix} -S_{1+2} \\ C_{1+2} \end{bmatrix}$$

angular velocity of link 2:

$$\dot{\theta}_1 + \dot{\theta}_2$$

kinetic energy of link 2:

$$\begin{aligned} T_2 &= \frac{1}{2} m_2 \dot{\mathbf{x}}_{c2}^T \dot{\mathbf{x}}_{c2} + \frac{1}{2} J_2 (\dot{\theta}_1 + \dot{\theta}_2)^2 \\ &= \frac{1}{2} m_2 \{ l_1^2 \dot{\theta}_1^2 + l_{c2}^2 (\dot{\theta}_1 + \dot{\theta}_2)^2 + 2 l_1 l_{c2} C_2 \dot{\theta}_1 (\dot{\theta}_1 + \dot{\theta}_2) \} + \\ &\quad \frac{1}{2} J_2 (\dot{\theta}_1 + \dot{\theta}_2)^2 \end{aligned}$$

Kinetic energy

total kinetic energy

$$T = T_1 + T_2 = \frac{1}{2} \begin{bmatrix} \dot{\theta}_1 & \dot{\theta}_2 \end{bmatrix} \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

where

$$H_{11} = J_1 + m_1 l_{c1}^2 + J_2 + m_2 (l_1^2 + l_{c2}^2 + 2l_1 l_{c2} C_2)$$

$$H_{22} = J_2 + m_2 l_{c2}^2$$

$$H_{12} = H_{21} = J_2 + m_2 (l_{c2}^2 + l_1 l_{c2} C_2)$$

inertia matrix

$$H \triangleq \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}$$

Partial derivatives

H_{11} and $H_{12} = H_{21}$ depend on θ_2 :

$$\frac{\partial H_{11}}{\partial \theta_2} = -2h_{12}, \quad \frac{\partial H_{12}}{\partial \theta_2} = \frac{\partial H_{21}}{\partial \theta_2} = -h_{12} \quad (h_{12} \triangleq m_2 l_1 l_{c2} S_2)$$

$$\dot{H}_{11} = -2h_{12}\dot{\theta}_2, \quad \dot{H}_{12} = \dot{H}_{21} = -h_{12}\dot{\theta}_2$$

$$\frac{\partial T}{\partial \dot{\theta}_1} = H_{11}\dot{\theta}_1 + H_{12}\dot{\theta}_2, \quad \frac{\partial T}{\partial \dot{\theta}_2} = H_{21}\dot{\theta}_1 + H_{22}\dot{\theta}_2$$

$$\begin{aligned} -\frac{d}{dt} \frac{\partial T}{\partial \dot{\theta}_1} &= -\dot{H}_{11}\dot{\theta}_1 - H_{11}\ddot{\theta}_1 - \dot{H}_{12}\dot{\theta}_2 - H_{12}\ddot{\theta}_2 \\ &= 2h_{12}\dot{\theta}_1\dot{\theta}_2 + h_{12}\dot{\theta}_2^2 - H_{11}\ddot{\theta}_1 - H_{12}\ddot{\theta}_2 \end{aligned}$$

$$\begin{aligned} -\frac{d}{dt} \frac{\partial T}{\partial \dot{\theta}_2} &= -\dot{H}_{21}\dot{\theta}_1 - H_{21}\ddot{\theta}_1 - \dot{H}_{22}\dot{\theta}_2 - H_{22}\ddot{\theta}_2 \\ &= h_{12}\dot{\theta}_1\dot{\theta}_2 - H_{21}\ddot{\theta}_1 - H_{22}\ddot{\theta}_2 \end{aligned}$$

Partial derivatives

H_{11} , H_{22} , and $H_{12} = H_{21}$ are independent of θ_1

$$\frac{\partial T}{\partial \theta_1} = \frac{1}{2} \begin{bmatrix} \dot{\theta}_1 & \dot{\theta}_2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = 0$$

H_{11} and $H_{12} = H_{21}$ depend on θ_2

$$\frac{\partial T}{\partial \theta_2} = \frac{1}{2} \begin{bmatrix} \dot{\theta}_1 & \dot{\theta}_2 \end{bmatrix} \begin{bmatrix} -2h_{12} & -h_{12} \\ -h_{12} & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = -h_{12}\dot{\theta}_1^2 - h_{12}\dot{\theta}_1\dot{\theta}_2$$

contribution of kinetic energy:

$$\frac{\partial T}{\partial \theta_1} - \frac{d}{dt} \frac{\partial T}{\partial \dot{\theta}_1} = 2h_{12}\dot{\theta}_1\dot{\theta}_2 + h_{12}\dot{\theta}_2^2 - H_{11}\ddot{\theta}_1 - H_{12}\ddot{\theta}_2$$

$$\frac{\partial T}{\partial \theta_2} - \frac{d}{dt} \frac{\partial T}{\partial \dot{\theta}_2} = -h_{12}\dot{\theta}_1^2 - H_{21}\ddot{\theta}_1 - H_{22}\ddot{\theta}_2$$

Gravitational potential energy

gravitational acceleration vector:

$$\mathbf{g} = \begin{bmatrix} 0 \\ -g \end{bmatrix}$$

potential energies of link 1 and 2:

$$U_1 = -m_1 \mathbf{g}^T \mathbf{x}_{c1}, \quad U_2 = -m_2 \mathbf{g}^T \mathbf{x}_{c2}$$

potential energy:

$$U = U_1 + U_2$$

contribution of potential energy:

$$-\frac{\partial U}{\partial \theta_1} = G_1 + G_2, \quad -\frac{\partial U}{\partial \theta_2} = G_2$$

where

$$G_1 = (m_1 l_{c1} + m_2 l_1) \mathbf{g}^T \begin{bmatrix} -S_1 \\ C_1 \end{bmatrix}, \quad G_2 = m_2 l_{c2} \mathbf{g}^T \begin{bmatrix} -S_{1+2} \\ C_{1+2} \end{bmatrix}$$

Work done by actuator torques

work done by τ_1 applied to rotational joint 1:

$$\tau_1\theta_1$$

work done by τ_2 applied to rotational joint 2:

$$\tau_2\theta_2$$

work done by the two actuator torques:

$$W = \tau_1\theta_1 + \tau_2\theta_2$$

contribution of work:

$$\frac{\partial W}{\partial \theta_1} = \tau_1, \quad \frac{\partial W}{\partial \theta_2} = \tau_2$$

Lagrange equations of motion

Lagrangian:

$$\mathcal{L} = T - U + W$$

Lagrange equations of motion

$$\frac{\partial \mathcal{L}}{\partial \theta_1} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\theta}_1} = 0$$

$$\frac{\partial \mathcal{L}}{\partial \theta_2} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\theta}_2} = 0$$

let $\omega_1 \triangleq \dot{\theta}_1$ and $\omega_2 \triangleq \dot{\theta}_2$:

$$-H_{11}\dot{\omega}_1 - H_{12}\dot{\omega}_2 + h_{12}\omega_2^2 + 2h_{12}\omega_1\omega_2 + G_1 + G_2 + \tau_1 = 0$$

$$-H_{22}\dot{\omega}_2 - H_{12}\dot{\omega}_1 - h_{12}\omega_1^2 + G_2 + \tau_2 = 0$$

Lagrange equations of motion

canonical form of ordinary differential equations:

$$\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}$$
$$\begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \end{bmatrix} = \begin{bmatrix} h_{12}\omega_2^2 + 2h_{12}\omega_1\omega_2 + G_1 + G_2 + \tau_1 \\ -h_{12}\omega_1^2 + G_2 + \tau_2 \end{bmatrix}$$

state variables: joint angles θ_1, θ_2 and angular velocities ω_1, ω_2

the inertia matrix is regular \longrightarrow 2nd eq. is solvable

\longrightarrow we can compute $\dot{\omega}_1$ and $\dot{\omega}_2$

$\dot{\theta}_1, \dot{\theta}_2, \dot{\omega}_1, \dot{\omega}_2$ are functions of $\theta_1, \theta_2, \omega_1, \omega_2$



we can sketch $\theta_1, \theta_2, \omega_1, \omega_2$ using an ODE solver.

Sample Programs

- class **Link**
- class **Link_Cylinder**
- class **Open_Mechanism_Two_DOF**
- class **Closed_Mechanism_Two_DOF**

class **Link_Cylinder** is a subclass of class **Link**

Sample Programs

file [Link.m](#)

```
classdef Link
    properties
        length;
        length_center;
        mass;
        inertia_of_moment_center;
        inertia_of_moment;
    end
    methods
        function obj = Link (l, lc, m, Jc, J)
            obj.length = l;
            obj.length_center = lc;
            obj.mass = m;
            obj.inertia_of_moment_center = Jc;
            obj.inertia_of_moment = J;
        end
    end
end
```

Sample Programs

Sentence

```
>> link1 = Link(2, 1, 0.0157, 0.0052, 0.0210)
```

builds a link with $l = 2$, $l_c = 1$, $m = 0.0157$, $J_c = 0.0052$, and $J = 0.0210$.

```
>> link1
```

```
link1 =
```

```
Link properties:
```

```
length: 2
```

```
length_center: 1
```

```
mass: 0.0157
```

```
inertia_of_moment_center: 0.0052
```

```
inertia_of_moment: 0.0210
```

```
>>
```


Sample Programs

building two cylindrical links of length 2, radius 0.05, and density 1

```
len = 2.00; radius = 0.05; density = 1;
```

```
len_c = len/2;
```

```
m = density * len * (pi*(radius)^2);
```

```
Jc = (1/12) * m * (3*radius^2 + len^2);
```

```
J = Jc + m * (len - len_c)^2;
```

```
link1 = Link (len, len_c, m, Jc, J);
```

```
link2 = Link (len, len_c, m, Jc, J);
```

```
>> link1
```

```
link1 =
```

```
Link properties:
```

```
length: 2
```

```
length_center: 1
```

```
mass: 0.0157
```

Sample Programs

building two cylindrical links of length 2, radius 0.05, and density 1

```
len = 2.00; radius = 0.05; density = 1;
```

```
link1 = Link_Cylinder (len, radius, density);
```

```
link2 = Link_Cylinder (len, radius, density);
```

```
>> link1
```

```
link1 =
```

```
Link_Cylinder properties:
```

```
radius: 0.0500
```

```
density: 1
```

```
length: 2
```

```
length_center: 1
```

```
mass: 0.0157
```

```
inertia_of_moment_center: 0.0052
```

```
inertia_of_moment: 0.0210
```

```
>>
```

Sample Programs

building an open mechanism consisting of two links

```
base = [0; 0];
```

```
grav = [0; -9.8];
```

```
robot = Open_Mechanism_Two_DOF (link1, link2, base, grav)
```

```
>> robot
```

```
robot =
```

```
Open_Mechanism_Two_DOF properties:
```

```
    link1: [1 × 1 Link_Cylinder]
```

```
    link2: [1 × 1 Link_Cylinder]
```

```
base_position: [2 × 1 double]
```

```
gravity: [2 × 1 double]
```

```
theta1: []
```

```
theta2: []
```

```
omega1: []
```

```
omega2: []
```

```
    C1: []
```

Sample Programs

setting joint angles and angular velocities

```
theta = [ pi/3; pi/6 ];
```

```
omega = [ 0; 0 ];
```

```
robot = robot.joint_angles (theta, omega);
```

```
>> robot
```

```
robot =
```

```
Open_Mechanism_Two_DOF properties:
```

```
    link1: [1 × 1 Link_Cylinder]
```

```
    link2: [1 × 1 Link_Cylinder]
```

```
base_position: [2 × 1 double]
```

```
    gravity: [2 × 1 double]
```

```
    theta1: 1.0472
```

```
    theta2: 0.5236
```

```
    omega1: 0
```

```
    omega2: 0
```

```
    C1: 0.5000
```

Sample Programs

calculating inertia matrix and torque vector

```
[ mat, vec ] = robot.inertia_matrix_and_torque_vector
```

$$\text{mat} = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}, \quad \text{vec} = \begin{bmatrix} h_{12}\omega_2^2 + 2h_{12}\omega_1\omega_2 + G_1 + G_2 \\ -h_{12}\omega_1^2 + G_2 \end{bmatrix}$$

Note `vec` does not include τ_1 or τ_2 .

Solving

$$\text{mat} \dot{\omega} = \text{vec} + \tau$$

where $\tau = [\tau_1, \tau_2]^T$, yields angular acceleration $\dot{\omega}$.

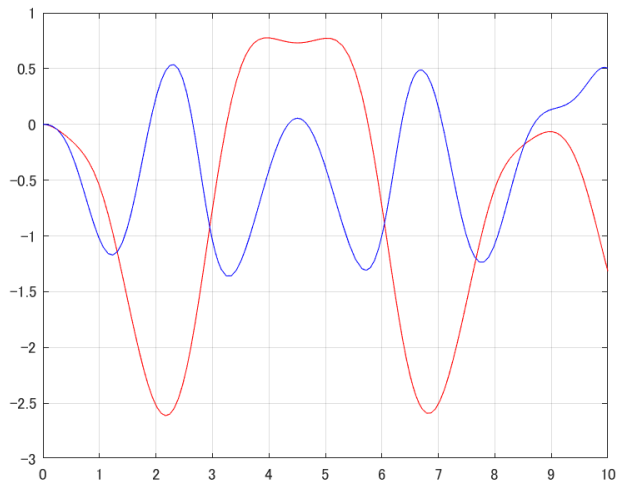
Driving by external torques

Sample Programs

- [open_mechanism_2DOF_external_torques.m](#)
2DOF open mechanism driven by external torques
- [open_mechanism_2DOF_external_torques_params.m](#)
equation of motion

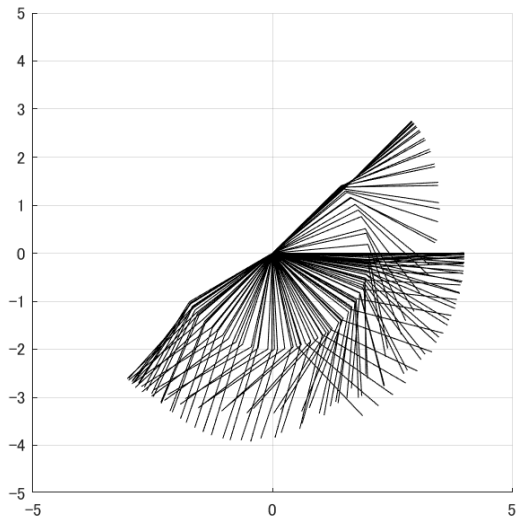
Driving by external torques

Result



Driving by external torques

Result



PD control

$$\tau_1 = -K_{P1}(\theta_1 - \theta_1^d) - K_{D1}\dot{\theta}_1$$

$$\tau_2 = -K_{P2}(\theta_2 - \theta_2^d) - K_{D2}\dot{\theta}_2$$

⇓

$$\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}$$

$$\begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \end{bmatrix} = \begin{bmatrix} \cdots - K_{P1}(\theta_1 - \theta_1^d) - K_{D1}\omega_1 \\ \cdots - K_{P2}(\theta_2 - \theta_2^d) - K_{D2}\omega_2 \end{bmatrix}$$

current values of $\theta_1, \theta_2, \omega_1, \omega_2$

⇓

their time derivatives $\dot{\theta}_1, \dot{\theta}_2, \dot{\omega}_1, \dot{\omega}_2$

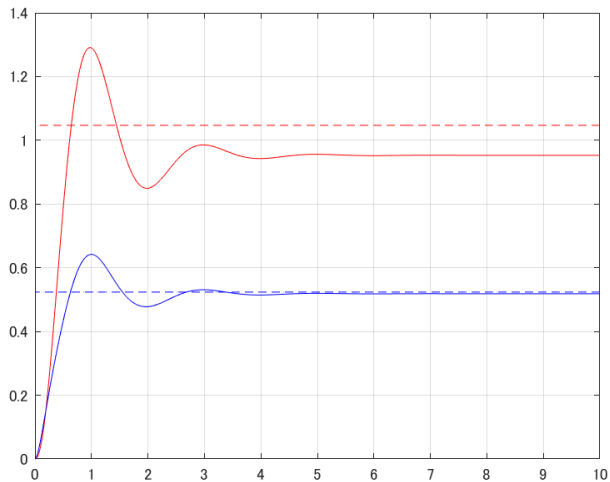
PD control

Sample Programs

- [open_mechanism_2DOF_PD.m](#)
PD control of 2DOF open mechanism
- [open_mechanism_2DOF_PD_params.m](#)
equation of motion

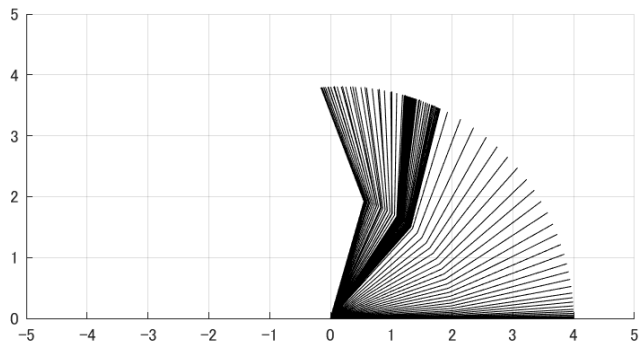
PD control

Result



PD control

Result



PI control

$$\tau_1 = -K_{P1}(\theta_1 - \theta_1^d) - K_{I1} \int_0^t \{(\theta_1 - \theta_1^d(\tau))\} d\tau$$

$$\tau_2 = -K_{P2}(\theta_2 - \theta_2^d) - K_{I2} \int_0^t \{(\theta_2 - \theta_2^d(\tau))\} d\tau$$

Introduce additional variables:

$$\xi_1 \triangleq \int_0^t \{(\theta_1 - \theta_1^d(\tau))\} d\tau$$

$$\xi_2 \triangleq \int_0^t \{(\theta_2 - \theta_2^d(\tau))\} d\tau$$

$$\dot{\xi}_1 = \theta_1 - \theta_1^d, \quad \tau_1 = -K_{P1}(\theta_1 - \theta_1^d) - K_{I1}\xi_1$$

$$\dot{\xi}_2 = \theta_2 - \theta_2^d, \quad \tau_2 = -K_{P2}(\theta_2 - \theta_2^d) - K_{I2}\xi_2$$

PI control

⇓

$$\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}$$

$$\begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \end{bmatrix} = \begin{bmatrix} \cdots - K_{P1}(\theta_1 - \theta_1^d) - K_{I1}\xi_1 \\ \cdots - K_{P2}(\theta_2 - \theta_2^d) - K_{I2}\xi_2 \end{bmatrix}$$

$$\dot{\xi}_1 = \theta_1 - \theta_1^d$$

$$\dot{\xi}_2 = \theta_2 - \theta_2^d$$

current values of $\theta_1, \theta_2, \omega_1, \omega_2, \xi_1, \xi_2$

⇓

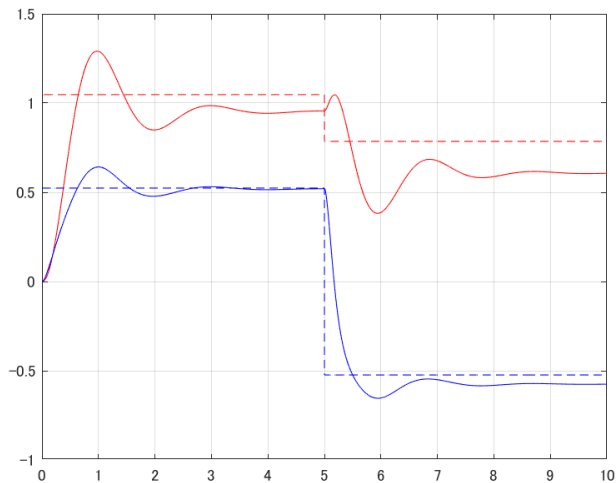
their time derivatives $\dot{\theta}_1, \dot{\theta}_2, \dot{\omega}_1, \dot{\omega}_2, \dot{\xi}_1, \dot{\xi}_2$

PD control (multiple desired values)

```
interval = [ 0, 5 ];  
qinit = [0;0; 0;0];  
thetad = [ pi/3; pi/6 ];  
open_mechanism_2DOF_PD_ode = @(t,q) open_mechanism_2DOF_PD_ode;  
[time1, q1] = ode45(open_mechanism_2DOF_PD_ode, interval,  
  
interval = [ 5, 10 ];  
qinit = q1(end,:);  
thetad = [ pi/4; -pi/6 ];  
open_mechanism_2DOF_PD_ode = @(t,q) open_mechanism_2DOF_PD_ode;  
[time2, q2] = ode45(open_mechanism_2DOF_PD_ode, interval,  
  
time = [time1;time2];  
q = [q1;q2];
```

PD control (multiple desired values)

Result

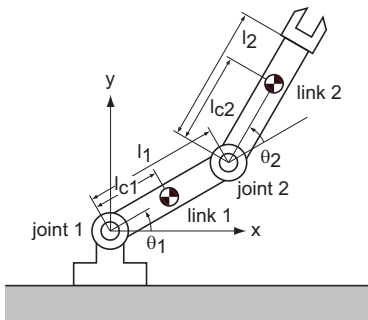


movie

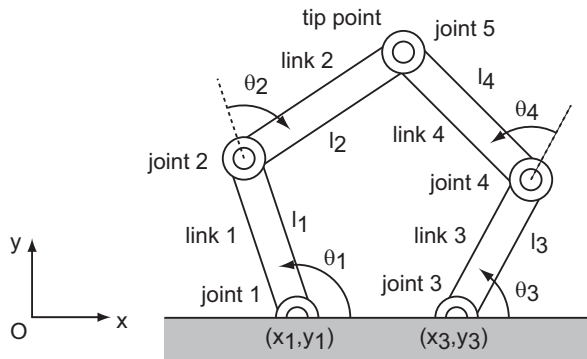
Report

Report #3 due date : Nov. 20 (Mon) 1:00 AM

Simulate the motion of a 2DOF open link mechanism under PID control. PID control is applied to active joints 1 and 2. Use appropriate values of geometrical and physical parameters of the manipulator.



Kinematics of 2DOF closed link mechanism



joint 1, 3: active
joint 2, 4, 5: passive

$\theta_1, \theta_2, \theta_3, \theta_4$:
rotation angles

τ_1, τ_3 :
actuator torques

Kinematics of 2DOF closed link mechanism

decomposition of closed link mechanism into open link mechanisms:

left arm link 1 and 2

right arm link 3 and 4

end point of the left arm:

$$\mathbf{x}_{1,2} = \begin{bmatrix} x_{1,2} \\ y_{1,2} \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + l_1 \begin{bmatrix} C_1 \\ S_1 \end{bmatrix} + l_2 \begin{bmatrix} C_{1+2} \\ S_{1+2} \end{bmatrix}$$

end point of the right arm:

$$\mathbf{x}_{3,4} = \begin{bmatrix} x_{3,4} \\ y_{3,4} \end{bmatrix} = \begin{bmatrix} x_3 \\ y_3 \end{bmatrix} + l_3 \begin{bmatrix} C_3 \\ S_3 \end{bmatrix} + l_4 \begin{bmatrix} C_{3+4} \\ S_{3+4} \end{bmatrix}$$

Kinematics of 2DOF closed link mechanism

constraint vector:

$$\mathbf{R} \triangleq \mathbf{x}_{1,2} - \mathbf{x}_{3,4} = \mathbf{0}$$

components of vector \mathbf{R} :

$$X \triangleq x_{1,2} - x_{3,4} = l_1 C_1 + l_2 C_{1+2} - l_3 C_3 - l_4 C_{3+4} + x_1 - x_3$$

$$Y \triangleq y_{1,2} - y_{3,4} = l_1 S_1 + l_2 S_{1+2} - l_3 S_3 - l_4 S_{3+4} + y_1 - y_3$$

Kinematics of 2DOF closed link mechanism

Jacobian of left arm:

$$\begin{aligned} J_{1,2} &= \begin{bmatrix} \frac{\partial \mathbf{x}_{1,2}}{\partial \theta_1} & \frac{\partial \mathbf{x}_{1,2}}{\partial \theta_2} \end{bmatrix} = \begin{bmatrix} \partial x_{1,2}/\partial \theta_1 & \partial x_{1,2}/\partial \theta_2 \\ \partial y_{1,2}/\partial \theta_1 & \partial y_{1,2}/\partial \theta_2 \end{bmatrix} \\ &= \begin{bmatrix} -l_1 S_1 - l_2 S_{1+2} & -l_2 S_{1+2} \\ l_1 C_1 + l_2 C_{1+2} & l_2 C_{1+2} \end{bmatrix} \end{aligned}$$

Jacobian of right arm:

$$\begin{aligned} J_{3,4} &= \begin{bmatrix} \frac{\partial \mathbf{x}_{3,4}}{\partial \theta_3} & \frac{\partial \mathbf{x}_{3,4}}{\partial \theta_4} \end{bmatrix} = \begin{bmatrix} \partial x_{3,4}/\partial \theta_3 & \partial x_{3,4}/\partial \theta_4 \\ \partial y_{3,4}/\partial \theta_3 & \partial y_{3,4}/\partial \theta_4 \end{bmatrix} \\ &= \begin{bmatrix} -l_3 S_3 - l_4 S_{3+4} & -l_4 S_{3+4} \\ l_3 C_3 + l_4 C_{3+4} & l_4 C_{3+4} \end{bmatrix} \end{aligned}$$

Lagrangian

Lagrangian of the closed link mechanism:

$$\mathcal{L} = \mathcal{L}_{1,2} + \mathcal{L}_{3,4} + \boldsymbol{\lambda}^T \mathbf{R}$$

$\mathcal{L}_{1,2}, \mathcal{L}_{3,4}$ Lagrangians of the left and right arms

$\boldsymbol{\lambda} = [\lambda_x, \lambda_y]^T$ Lagrange multiplier vector

Lagrange equations of motion:

$$\frac{\partial \mathcal{L}}{\partial \boldsymbol{\theta}_{1,2}} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \boldsymbol{\omega}_{1,2}} = \mathbf{0}$$

$$\frac{\partial \mathcal{L}}{\partial \boldsymbol{\theta}_{3,4}} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \boldsymbol{\omega}_{3,4}} = \mathbf{0}$$

where

$$\boldsymbol{\theta}_{1,2} = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}, \boldsymbol{\omega}_{1,2} = \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}, \boldsymbol{\theta}_{3,4} = \begin{bmatrix} \theta_3 \\ \theta_4 \end{bmatrix}, \boldsymbol{\omega}_{3,4} = \begin{bmatrix} \omega_3 \\ \omega_4 \end{bmatrix}$$

Contributions of $\mathcal{L}_{1,2}$

contributions of Lagrangian $\mathcal{L}_{1,2}$ to the Lagrange eqs:

$$-H_{1,2} \dot{\omega}_{1,2} + \tau_{1,2} + \tau_{left}$$
$$\mathbf{0}$$

where

$$H_{1,2} = \begin{bmatrix} *** & J_2 + m_2(l_{c2}^2 + l_1 l_{c2} C_2) \\ J_2 + m_2(l_{c2}^2 + l_1 l_{c2} C_2) & J_2 + m_2 l_{c2}^2 \end{bmatrix}$$
$$\tau_{1,2} = \begin{bmatrix} +h_{12}\omega_2^2 + 2h_{12}\omega_1\omega_2 + G_1 + G_2 \\ -h_{12}\omega_1^2 + G_2 \end{bmatrix}$$
$$\tau_{left} = \begin{bmatrix} \tau_1 \\ 0 \end{bmatrix}$$

$$*** = J_1 + m_1 l_{c1}^2 + J_2 + m_2(l_1^2 + l_{c2}^2 + 2l_1 l_{c2} C_2)$$

Contributions of $\mathcal{L}_{3,4}$

contributions of Lagrangian $\mathcal{L}_{3,4}$ to the Lagrange eqs:

0

$$- H_{3,4} \dot{\omega}_{3,4} + \tau_{3,4} + \tau_{right}$$

where

$$H_{3,4} = \begin{bmatrix} *** & J_4 + m_4(l_{c4}^2 + l_3 l_{c4} C_4) \\ J_4 + m_4(l_{c4}^2 + l_3 l_{c4} C_4) & J_4 + m_4 l_{c4}^2 \end{bmatrix}$$
$$\tau_{3,4} = \begin{bmatrix} +h_{34}\omega_4^2 + 2h_{34}\omega_3\omega_4 + G_3 + G_4 \\ -h_{34}\omega_3^2 + G_4 \end{bmatrix}$$
$$\tau_{right} = \begin{bmatrix} \tau_3 \\ 0 \end{bmatrix}$$

$$*** = J_3 + m_3 l_{c3}^2 + J_4 + m_4(l_3^2 + l_{c4}^2 + 2l_3 l_{c4} C_4)$$

Contributions of $\lambda^T R$

since $\mathbf{x}_{3,4}$ is independent of θ_1 and θ_2

$$\frac{\partial R}{\partial \theta_1} = \frac{\partial \mathbf{x}_{1,2}}{\partial \theta_1}, \quad \frac{\partial R}{\partial \theta_2} = \frac{\partial \mathbf{x}_{1,2}}{\partial \theta_2}$$

contributions of $\lambda^T R$ to the first Lagrange eq:

$$\begin{aligned} \begin{bmatrix} \lambda^T \partial R / \partial \theta_1 \\ \lambda^T \partial R / \partial \theta_2 \end{bmatrix} &= \begin{bmatrix} \lambda^T \partial \mathbf{x}_{1,2} / \partial \theta_1 \\ \lambda^T \partial \mathbf{x}_{1,2} / \partial \theta_2 \end{bmatrix} = \begin{bmatrix} (\partial \mathbf{x}_{1,2} / \partial \theta_1)^T \lambda \\ (\partial \mathbf{x}_{1,2} / \partial \theta_2)^T \lambda \end{bmatrix} \\ &= \begin{bmatrix} (\partial \mathbf{x}_{1,2} / \partial \theta_1)^T \\ (\partial \mathbf{x}_{1,2} / \partial \theta_2)^T \end{bmatrix} \lambda \\ &= \begin{bmatrix} \frac{\partial \mathbf{x}_{1,2}}{\partial \theta_1} & \frac{\partial \mathbf{x}_{1,2}}{\partial \theta_2} \end{bmatrix}^T \lambda \\ &= J_{1,2}^T \lambda \end{aligned}$$

Contributions of $\lambda^T R$

since $\mathbf{x}_{1,2}$ is independent of θ_3 and θ_4

$$\frac{\partial R}{\partial \theta_3} = -\frac{\partial \mathbf{x}_{3,4}}{\partial \theta_3}, \quad \frac{\partial R}{\partial \theta_4} = -\frac{\partial \mathbf{x}_{3,4}}{\partial \theta_4}$$

contributions of $\lambda^T R$ to the second Lagrange eq:

$$\begin{aligned} \begin{bmatrix} \lambda^T \partial R / \partial \theta_3 \\ \lambda^T \partial R / \partial \theta_4 \end{bmatrix} &= \begin{bmatrix} -\lambda^T \partial \mathbf{x}_{3,4} / \partial \theta_3 \\ -\lambda^T \partial \mathbf{x}_{3,4} / \partial \theta_4 \end{bmatrix} = \begin{bmatrix} -(\partial \mathbf{x}_{3,4} / \partial \theta_3)^T \lambda \\ -(\partial \mathbf{x}_{3,4} / \partial \theta_4)^T \lambda \end{bmatrix} \\ &= \begin{bmatrix} -(\partial \mathbf{x}_{3,4} / \partial \theta_3)^T \\ -(\partial \mathbf{x}_{3,4} / \partial \theta_4)^T \end{bmatrix} \lambda \\ &= - \begin{bmatrix} \frac{\partial \mathbf{x}_{3,4}}{\partial \theta_3} & \frac{\partial \mathbf{x}_{3,4}}{\partial \theta_4} \end{bmatrix}^T \lambda \\ &= -J_{3,4}^T \lambda \end{aligned}$$

Contributions of $\lambda^T R$

contributions of constraint term $\lambda^T R$ to the Lagrange eqs:

$$J_{1,2}^T \lambda \\ - J_{3,4}^T \lambda$$

where $J_{1,2}$ and $J_{3,4}$ are Jacobians:

$$J_{1,2} = \begin{bmatrix} -l_1 S_1 - l_2 S_{1+2} & -l_2 S_{1+2} \\ l_1 C_1 + l_2 C_{1+2} & l_2 C_{1+2} \end{bmatrix}$$
$$J_{3,4} = \begin{bmatrix} -l_3 S_3 - l_4 S_{3+4} & -l_4 S_{3+4} \\ l_3 C_3 + l_4 C_{3+4} & l_4 C_{3+4} \end{bmatrix}$$

Lagrange equations of motion

$$-H_{1,2} \dot{\omega}_{1,2} + \tau_{1,2} + \tau_{left} + J_{1,2}^T \lambda = 0$$

$$-H_{3,4} \dot{\omega}_{3,4} + \tau_{3,4} + \tau_{right} - J_{3,4}^T \lambda = 0$$

⇓

$$\begin{bmatrix} H_{1,2} & O_{2 \times 2} & -J_{1,2}^T \\ O_{2 \times 2} & H_{3,4} & J_{3,4}^T \end{bmatrix} \begin{bmatrix} \dot{\omega}_{1,2} \\ \dot{\omega}_{3,4} \\ \lambda \end{bmatrix} = \begin{bmatrix} \tau_{1,2} + \tau_{left} \\ \tau_{3,4} + \tau_{right} \end{bmatrix}$$

Equation stabilizing constraint

constraint vector

$$\mathbf{R} = \mathbf{x}_{1,2}(\theta_1, \theta_2) - \mathbf{x}_{3,4}(\theta_3, \theta_4)$$

time-derivative

$$\begin{aligned}\dot{\mathbf{R}} &= \frac{\partial \mathbf{x}_{1,2}}{\partial \theta_1} \omega_1 + \frac{\partial \mathbf{x}_{1,2}}{\partial \theta_2} \omega_2 - \frac{\partial \mathbf{x}_{3,4}}{\partial \theta_3} \omega_3 - \frac{\partial \mathbf{x}_{3,4}}{\partial \theta_4} \omega_4 \\ &= \begin{bmatrix} \frac{\partial \mathbf{x}_{1,2}}{\partial \theta_1} & \frac{\partial \mathbf{x}_{1,2}}{\partial \theta_2} \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} - \begin{bmatrix} \frac{\partial \mathbf{x}_{3,4}}{\partial \theta_3} & \frac{\partial \mathbf{x}_{3,4}}{\partial \theta_4} \end{bmatrix} \begin{bmatrix} \omega_3 \\ \omega_4 \end{bmatrix} \\ &= \mathbf{J}_{1,2} \boldsymbol{\omega}_{1,2} - \mathbf{J}_{3,4} \boldsymbol{\omega}_{3,4}\end{aligned}$$

second-order time-derivative

$$\ddot{\mathbf{R}} = \dot{\mathbf{J}}_{1,2} \boldsymbol{\omega}_{1,2} + \mathbf{J}_{1,2} \dot{\boldsymbol{\omega}}_{1,2} - \dot{\mathbf{J}}_{3,4} \boldsymbol{\omega}_{3,4} - \mathbf{J}_{3,4} \dot{\boldsymbol{\omega}}_{3,4}$$

Equation stabilizing constraint

$$\frac{d}{dt} \frac{\partial \mathbf{x}_{1,2}}{\partial \theta_1} = \frac{\partial^2 \mathbf{x}_{1,2}}{\partial \theta_1 \partial \theta_1} \omega_1 + \frac{\partial^2 \mathbf{x}_{1,2}}{\partial \theta_1 \partial \theta_2} \omega_2$$

$$\frac{d}{dt} \frac{\partial \mathbf{x}_{1,2}}{\partial \theta_2} = \frac{\partial^2 \mathbf{x}_{1,2}}{\partial \theta_2 \partial \theta_1} \omega_1 + \frac{\partial^2 \mathbf{x}_{1,2}}{\partial \theta_2 \partial \theta_2} \omega_2$$

introduce Hessian matrices

$$Q_{1,2;x} = \begin{bmatrix} \frac{\partial^2 x_{1,2}}{\partial \theta_1 \partial \theta_1} & \frac{\partial^2 x_{1,2}}{\partial \theta_1 \partial \theta_2} \\ \frac{\partial^2 x_{1,2}}{\partial \theta_2 \partial \theta_1} & \frac{\partial^2 x_{1,2}}{\partial \theta_2 \partial \theta_2} \end{bmatrix} = \begin{bmatrix} -l_1 C_1 - l_2 C_{1+2} & -l_2 C_{1+2} \\ -l_2 C_{1+2} & -l_2 C_{1+2} \end{bmatrix}$$

$$Q_{1,2;y} = \begin{bmatrix} \frac{\partial^2 y_{1,2}}{\partial \theta_1 \partial \theta_1} & \frac{\partial^2 y_{1,2}}{\partial \theta_1 \partial \theta_2} \\ \frac{\partial^2 y_{1,2}}{\partial \theta_2 \partial \theta_1} & \frac{\partial^2 y_{1,2}}{\partial \theta_2 \partial \theta_2} \end{bmatrix} = \begin{bmatrix} -l_1 S_1 - l_2 S_{1+2} & -l_2 S_{1+2} \\ -l_2 S_{1+2} & -l_2 S_{1+2} \end{bmatrix}$$

Equation stabilizing constraint

$$\begin{aligned}
 j_{1,2}\omega_{1,2} &= \begin{bmatrix} \frac{d}{dt} \frac{\partial \mathbf{x}_{1,2}}{\partial \theta_1} & \frac{d}{dt} \frac{\partial \mathbf{x}_{1,2}}{\partial \theta_2} \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} \\
 &= \begin{bmatrix} \frac{\partial^2 \mathbf{x}_{1,2}}{\partial \theta_1 \partial \theta_1} \omega_1 + \frac{\partial^2 \mathbf{x}_{1,2}}{\partial \theta_1 \partial \theta_2} \omega_2 & \frac{\partial^2 \mathbf{x}_{1,2}}{\partial \theta_2 \partial \theta_1} \omega_1 + \frac{\partial^2 \mathbf{x}_{1,2}}{\partial \theta_2 \partial \theta_2} \omega_2 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} \\
 &= \begin{bmatrix} \frac{\partial^2 x_{1,2}}{\partial \theta_1 \partial \theta_1} \omega_1^2 + \frac{\partial^2 x_{1,2}}{\partial \theta_1 \partial \theta_2} \omega_1 \omega_2 + \frac{\partial^2 x_{1,2}}{\partial \theta_2 \partial \theta_1} \omega_2 \omega_1 + \frac{\partial^2 x_{1,2}}{\partial \theta_2 \partial \theta_2} \omega_2^2 \\ \frac{\partial^2 y_{1,2}}{\partial \theta_1 \partial \theta_1} \omega_1^2 + \frac{\partial^2 y_{1,2}}{\partial \theta_1 \partial \theta_2} \omega_1 \omega_2 + \frac{\partial^2 y_{1,2}}{\partial \theta_2 \partial \theta_1} \omega_2 \omega_1 + \frac{\partial^2 y_{1,2}}{\partial \theta_2 \partial \theta_2} \omega_2^2 \end{bmatrix} \\
 &= \begin{bmatrix} \begin{bmatrix} \omega_1 & \omega_2 \end{bmatrix} Q_{1,2;x} \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} \\ \begin{bmatrix} \omega_1 & \omega_2 \end{bmatrix} Q_{1,2;y} \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \omega_{1,2}^T Q_{1,2;x} \omega_{1,2} \\ \omega_{1,2}^T Q_{1,2;y} \omega_{1,2} \end{bmatrix}
 \end{aligned}$$

Equation stabilizing constraint

similarly

$$j_{3,4}\omega_{3,4} = \begin{bmatrix} \omega_{3,4}^T Q_{3,4;x} \omega_{3,4} \\ \omega_{3,4}^T Q_{3,4;y} \omega_{3,4} \end{bmatrix}$$

where Hessian matrices are

$$Q_{3,4;x} = \begin{bmatrix} \frac{\partial^2 x_{3,4}}{\partial \theta_3 \partial \theta_3} & \frac{\partial^2 x_{3,4}}{\partial \theta_3 \partial \theta_4} \\ \frac{\partial^2 x_{3,4}}{\partial \theta_4 \partial \theta_3} & \frac{\partial^2 x_{3,4}}{\partial \theta_4 \partial \theta_4} \end{bmatrix} = \begin{bmatrix} -l_3 C_3 - l_4 C_{3+4} & -l_4 C_{3+4} \\ -l_4 C_{3+4} & -l_4 C_{3+4} \end{bmatrix}$$
$$Q_{3,4;y} = \begin{bmatrix} \frac{\partial^2 y_{3,4}}{\partial \theta_3 \partial \theta_3} & \frac{\partial^2 y_{3,4}}{\partial \theta_3 \partial \theta_4} \\ \frac{\partial^2 y_{3,4}}{\partial \theta_4 \partial \theta_3} & \frac{\partial^2 y_{3,4}}{\partial \theta_4 \partial \theta_4} \end{bmatrix} = \begin{bmatrix} -l_3 S_3 - l_4 S_{3+4} & -l_4 S_{3+4} \\ -l_4 S_{3+4} & -l_4 S_{3+4} \end{bmatrix}$$

Equation stabilizing constraint

$$\ddot{\mathbf{R}} + 2\alpha\dot{\mathbf{R}} + \alpha^2\mathbf{R} = \mathbf{0}$$

↓

$$\begin{bmatrix} \omega_{1,2}^T & Q_{1,2;x} & \omega_{1,2} \\ \omega_{1,2}^T & Q_{1,2;y} & \omega_{1,2} \end{bmatrix} + J_{1,2}\dot{\omega}_{1,2} - \begin{bmatrix} \omega_{3,4}^T & Q_{3,4;x} & \omega_{3,4} \\ \omega_{3,4}^T & Q_{3,4;y} & \omega_{3,4} \end{bmatrix} - J_{3,4}\dot{\omega}_{3,4} \\ + 2\alpha(J_{1,2}\omega_{1,2} - J_{3,4}\omega_{3,4}) + \alpha^2\mathbf{R} = \mathbf{0}$$

↓

Equation stabilizing constraint

$$\begin{bmatrix} -J_{1,2} & J_{3,4} \end{bmatrix} \begin{bmatrix} \dot{\omega}_{1,2} \\ \dot{\omega}_{3,4} \end{bmatrix} = \mathbf{C}$$

where

$$\mathbf{C} = \begin{bmatrix} \omega_{1,2}^T Q_{1,2;x} \omega_{1,2} \\ \omega_{1,2}^T Q_{1,2;y} \omega_{1,2} \end{bmatrix} - \begin{bmatrix} \omega_{3,4}^T Q_{3,4;x} \omega_{3,4} \\ \omega_{3,4}^T Q_{3,4;y} \omega_{3,4} \end{bmatrix} + 2\alpha(J_{1,2}\omega_{1,2} - J_{3,4}\omega_{3,4}) + \alpha^2 \mathbf{R}$$

Dynamic equations for closed link mechanism

Combining Lagrange equation of motion and equation stabilizing constraint yields

$$\begin{bmatrix} H_{1,2} & O_{2 \times 2} & -J_{1,2}^T \\ O_{2 \times 2} & H_{3,4} & J_{3,4}^T \\ -J_{1,2} & J_{3,4} & O_{2 \times 2} \end{bmatrix} \begin{bmatrix} \dot{\omega}_{1,2} \\ \dot{\omega}_{3,4} \\ \lambda \end{bmatrix} = \begin{bmatrix} \tau_{1,2} + \tau_{left} \\ \tau_{3,4} + \tau_{right} \\ C \end{bmatrix}$$

coefficient matrix is regular \longrightarrow we can compute $\dot{\omega}_1$ through $\dot{\omega}_4$

Physical Interpretation

$J_{1,2}$ and $J_{3,4}$: Jacobian matrices of the left and right arms

$\boldsymbol{\lambda} = [\lambda_x, \lambda_y]^T$: constraint force

equivalent torques around rotational joints 1 and 2:

$$J_{1,2}^T \boldsymbol{\lambda} = \begin{bmatrix} \lambda_x(-l_1 S_1 - l_2 S_{1+2}) + \lambda_y(l_1 C_1 + l_2 C_{1+2}) \\ \lambda_x(-l_2 S_{1+2}) + \lambda_y l_2 C_{1+2} \end{bmatrix}$$

reaction force $-\boldsymbol{\lambda}$

equivalent torques around rotational joint 3 and 4:

$$J_{3,4}^T(-\boldsymbol{\lambda}) = \begin{bmatrix} \lambda_x(l_3 S_3 + l_4 S_{3+4}) + \lambda_y(-l_3 C_3 - l_4 C_{3+4}) \\ \lambda_x l_4 S_{3+4} + \lambda_y(-l_4 C_{3+4}) \end{bmatrix}$$

PD control

$$\tau_1 = -K_{P1}(\theta_1 - \theta_1^d) - K_{D1}\dot{\theta}_1$$

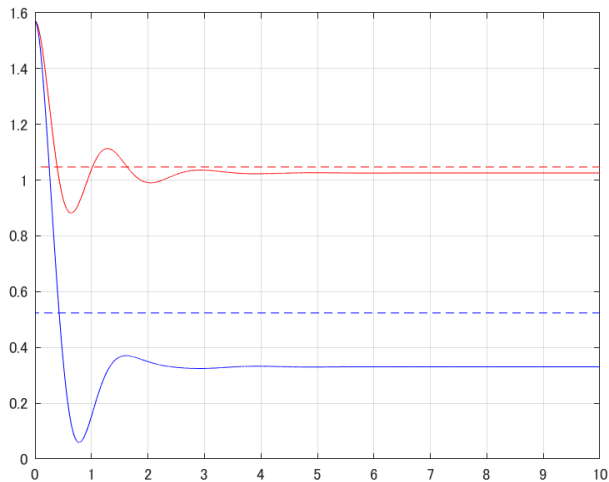
$$\tau_3 = -K_{P3}(\theta_3 - \theta_3^d) - K_{D3}\dot{\theta}_3$$

Sample Programs

- class **Closed_Mechanism_Two_DOF**
- [closed_mechanism_2DOF_PD.m](#)
PD control of 2DOF closed mechanism
- [closed_mechanism_2DOF_PD_params.m](#) equation of motion

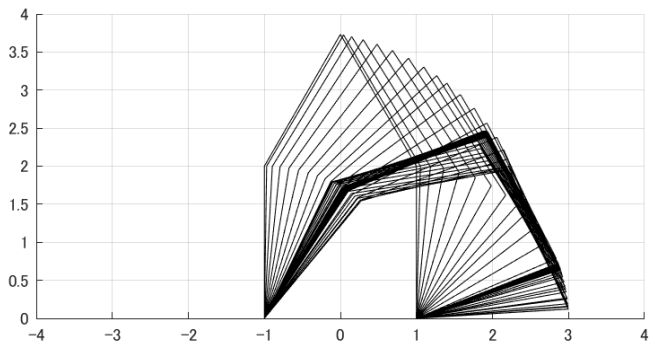
PD control

Result



PD control

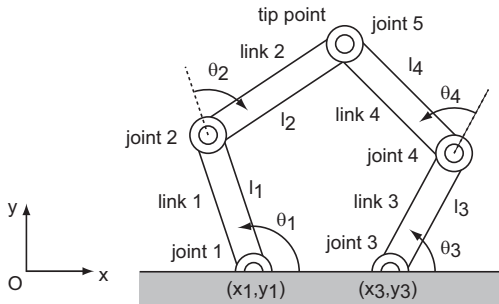
Result



Report

Report #4 due date : Nov. 27 (Mon) 1:00 AM

Simulate the motion of a 2DOF closed link mechanism under PID control. PID control is applied to active joints 1 and 3. Use appropriate values of geometrical and physical parameters of the manipulator.



Summary

Open link mechanism

- inertia matrix depends on joint angles
- Lagrange equations of motion of open link mechanism

Closed link mechanism

- two open link mechanisms with geometric constraints
- synthesized from Lagrange equations of open link mechanisms