Analytical Mechanics: Link Mechanisms

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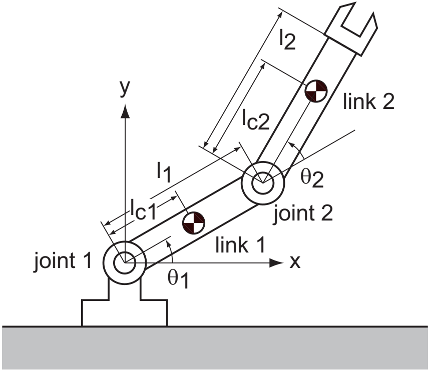
Agenda

1. Open Link Mechanism
   - Kinematics of Open Link Mechanism
   - Dynamics of 2DOF open link mechanism
2. Closed Link Mechanism
   - Kinematics of Closed Link Mechanism
   - Dynamics of 2DOF closed link mechanism

Kinematics of 2DOF open link mechanism



two link open link mechanism

l_i	length of link i
l_{ci}	distance btw. joint i and the center of mass of link i
m_i	mass of link i
J_i	inertia of moment of link i around its center of mass
θ_1	rotation angle of joint 1
θ_2	rotation angle of joint 2

Kinematics of 2DOF open link mechanism

position of the center of mass of link 1:

$$\boldsymbol{x}_{c1} \triangleq \begin{bmatrix} x_{c1} \\ y_{c1} \end{bmatrix} = l_{c1} \begin{bmatrix} C_1 \\ S_1 \end{bmatrix}$$

position of the center of mass of link 2:

$$\boldsymbol{x}_{c2} \triangleq \begin{bmatrix} x_{c2} \\ y_{c2} \end{bmatrix} = l_1 \begin{bmatrix} C_1 \\ S_1 \end{bmatrix} + l_{c2} \begin{bmatrix} C_{1+2} \\ S_{1+2} \end{bmatrix}$$

orientation angle of link 1:

$$\theta_1$$

orientation angle of link 2:

$$\theta_1 + \theta_2$$

Kinetic energy

velocity of the center of mass of link 1:

$$\dot{\boldsymbol{x}}_{c1} = l_{c1}\dot{\theta}_1 \begin{bmatrix} -S_1 \\ C_1 \end{bmatrix}$$

angular velocity of link 1:

$$\dot{\theta}_1$$

kinetic energy of link 1:

$$\begin{aligned} T_1 &= \frac{1}{2} m_1 \dot{\boldsymbol{x}}_{c1}^{\mathrm{T}} \dot{\boldsymbol{x}}_{c1} + \frac{1}{2} J_1 \dot{\theta}_1^2 \\ &= \frac{1}{2} (m_1 l_{c1}^2 + J_1) \dot{\theta}_1^2 \end{aligned}$$

Kinetic energy

velocity of the center of mass of link 2:

$$\dot{x}_{c2} = l_1\dot{\theta}_1 \begin{bmatrix} -S_1 \\ C_1 \end{bmatrix} + l_{c2}(\dot{\theta}_1 + \dot{\theta}_2) \begin{bmatrix} -S_{1+2} \\ C_{1+2} \end{bmatrix}$$

angular velocity of link 2:

$$\dot{\theta}_1 + \dot{\theta}_2$$

kinetic energy of link 2:

$$\begin{aligned} T_2 =& \frac{1}{2} m_2 \dot{x}_{c2}^{\mathrm{T}} \dot{x}_{c2} + \frac{1}{2} J_2 (\dot{\theta}_1 + \dot{\theta}_2)^2 \\ =& \frac{1}{2} m_2 \{ l_1^2 \dot{\theta}_1^2 + l_{c2}^2 (\dot{\theta}_1 + \dot{\theta}_2)^2 + 2 l_1 l_{c2} C_2 \dot{\theta}_1 (\dot{\theta}_1 + \dot{\theta}_2) \} + \\ & \frac{1}{2} J_2 (\dot{\theta}_1 + \dot{\theta}_2)^2 \end{aligned}$$

Kinetic energy

total kinetic energy

$$T = T_1 + T_2 = \frac{1}{2} \begin{bmatrix} \dot{\theta}_1 & \dot{\theta}_2 \end{bmatrix} \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

where

$$H_{11} = J_1 + m_1 l_{c1}^2 + J_2 + m_2(l_1^2 + l_{c2}^2 + 2 l_1 l_{c2} C_2)$$
$$H_{22} = J_2 + m_2 l_{c2}^2$$
$$H_{12} = H_{21} = J_2 + m_2(l_{c2}^2 + l_1 l_{c2} C_2)$$

inertia matrix

$$H \triangleq \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}$$

Partial derivatives

H_{11} and $H_{12} = H_{21}$ depend on θ_2:

$$\frac{\partial H_{11}}{\partial \theta_2} = -2h_{12}, \quad \frac{\partial H_{12}}{\partial \theta_2} = \frac{\partial H_{21}}{\partial \theta_2} = -h_{12} \quad (h_{12} \triangleq m_2 l_1 l_{c2} S_2)$$

$$\dot{H}_{11} = -2h_{12}\dot{\theta}_2, \quad \dot{H}_{12} = \dot{H}_{21} = -h_{12}\dot{\theta}_2$$

$$\frac{\partial T}{\partial \dot{\theta}_1} = H_{11}\dot{\theta}_1 + H_{12}\dot{\theta}_2, \quad \frac{\partial T}{\partial \dot{\theta}_2} = H_{21}\dot{\theta}_1 + H_{22}\dot{\theta}_2$$

$$\begin{aligned}
-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{\theta}_1} &= -\dot{H}_{11}\dot{\theta}_1 - H_{11}\ddot{\theta}_1 - \dot{H}_{12}\dot{\theta}_2 - H_{12}\ddot{\theta}_2 \\
&= 2h_{12}\dot{\theta}_1\dot{\theta}_2 + h_{12}\dot{\theta}_2^2 - H_{11}\ddot{\theta}_1 - H_{12}\ddot{\theta}_2 \\
-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{\theta}_2} &= -\dot{H}_{21}\dot{\theta}_1 - H_{21}\ddot{\theta}_1 - \dot{H}_{22}\dot{\theta}_2 - H_{22}\ddot{\theta}_2 \\
&= h_{12}\dot{\theta}_1\dot{\theta}_2 - H_{21}\ddot{\theta}_1 - H_{22}\ddot{\theta}_2
\end{aligned}$$

Partial derivatives

H_{11}, H_{22}, and $H_{12} = H_{21}$ are independent of θ_1

$$\frac{\partial T}{\partial \theta_1} = \frac{1}{2} \begin{bmatrix} \dot{\theta}_1 & \dot{\theta}_2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = 0$$

H_{11} and $H_{12} = H_{21}$ depend on θ_2

$$\frac{\partial T}{\partial \theta_2} = \frac{1}{2} \begin{bmatrix} \dot{\theta}_1 & \dot{\theta}_2 \end{bmatrix} \begin{bmatrix} -2h_{12} & -h_{12} \\ -h_{12} & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = -h_{12}\dot{\theta}_1^2 - h_{12}\dot{\theta}_1\dot{\theta}_2$$

contribution of kinetic energy:

$$\begin{aligned} \frac{\partial T}{\partial \theta_1} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{\theta}_1} &= 2h_{12}\dot{\theta}_1\dot{\theta}_2 + h_{12}\dot{\theta}_2^2 - H_{11}\ddot{\theta}_1 - H_{12}\ddot{\theta}_2 \\ \frac{\partial T}{\partial \theta_2} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{\theta}_2} &= -h_{12}\dot{\theta}_1^2 - H_{21}\ddot{\theta}_1 - H_{22}\ddot{\theta}_2 \end{aligned}$$

Gravitational potential energy

potential energy of link 1:

$$U_1 = m_1 g \, y_{c1} = m_1 g \, l_{c1} S_1$$

potential energy of link 2:

$$U_2 = m_2 g \, y_{c2} = m_2 g \, (l_1 S_1 + l_{c2} S_{1+2})$$

potential energy:

$$U = U_1 + U_2$$

partial derivatives w.r.t. joint angles:

$$\frac{\partial U}{\partial \theta_1} = G_1 + G_{12}, \qquad \frac{\partial U}{\partial \theta_2} = G_{12}$$

where

$$G_1 = (m_1 l_{c1} + m_2 l_1) \, g C_1, \qquad G_{12} = m_2 l_{c2} \, g C_{1+2}$$

Work done by actuator torques

work done by τ_1 applied to rotational joint 1:

$$\tau_1 \theta_1$$

work done by τ_2 applied to rotational joint 2:

$$\tau_2 \theta_2$$

work done by the two actuator torques:

$$W = \tau_1 \theta_1 + \tau_2 \theta_2$$

partial derivatives w.r.t. joint angles:

$$\frac{\partial W}{\partial \theta_1} = \tau_1, \qquad \frac{\partial W}{\partial \theta_2} = \tau_2$$

Lagrange equations of motion

Lagrangian:

$$\mathcal{L} = T - U + W$$

Lagrange equations of motion

$$\frac{\partial \mathcal{L}}{\partial \theta_1} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1} = 0$$
$$\frac{\partial \mathcal{L}}{\partial \theta_2} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{\theta}_2} = 0$$

let $\omega_1 \triangleq \dot{\theta}_1$ and $\omega_2 \triangleq \dot{\theta}_2$:

$$- H_{11}\dot{\omega}_1 - H_{12}\dot{\omega}_2 + h_{12}\omega_2^2 + 2h_{12}\omega_1\omega_2 - G_1 - G_{12} + \tau_1 = 0$$
$$- H_{22}\dot{\omega}_2 - H_{12}\dot{\omega}_1 - h_{12}\omega_1^2 - G_{12} + \tau_2 = 0$$

Lagrange equations of motion

canonical form of ordinary differential equations:

$$\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}$$

$$\begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \end{bmatrix} = \begin{bmatrix} h_{12}\omega_2^2 + 2h_{12}\omega_1\omega_2 - G_1 - G_{12} + \tau_1 \\ -h_{12}\omega_1^2 - G_{12} + \tau_2 \end{bmatrix}$$

state variables: joint angles θ_1, θ_2 and angular velocities ω_1, ω_2

the inertia matrix is regular $\longrightarrow$ 2nd eq. is solvable
$\longrightarrow$ we can compute $\dot{\omega}_1$ and $\dot{\omega}_2$

$\dot{\theta}_1$, $\dot{\theta}_2$, $\dot{\omega}_1$, $\dot{\omega}_2$ are functions of θ_1, θ_2, ω_1, ω_2

$$\Downarrow$$

we can sketch θ_1, θ_2, ω_1, ω_2 using an ODE solver.

PD control

$$\tau_1 = K_{\mathrm{P1}}(\theta_1^d - \theta_1) - K_{\mathrm{D1}}\dot{\theta}_1$$
$$\tau_2 = K_{\mathrm{P2}}(\theta_2^d - \theta_2) - K_{\mathrm{D2}}\dot{\theta}_2$$

$$\Downarrow$$

$$\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}$$
$$\begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \end{bmatrix} = \begin{bmatrix} \cdots + K_{\mathrm{P1}}(\theta_1^d - \theta_1) - K_{\mathrm{D1}}\omega_1 \\ \cdots + K_{\mathrm{P2}}(\theta_2^d - \theta_2) - K_{\mathrm{D2}}\omega_2 \end{bmatrix}$$

current values of θ_1, θ_2, ω_1, ω_2

$$\Downarrow$$

their time derivatives $\dot{\theta}_1$, $\dot{\theta}_2$, $\dot{\omega}_1$, $\dot{\omega}_2$

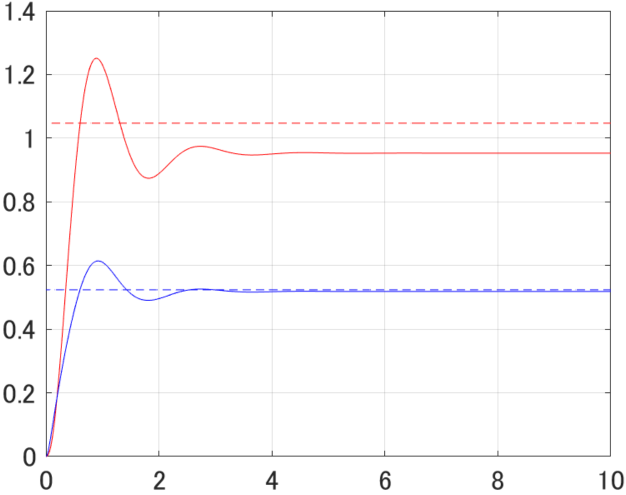
PD control of 2DOF open link mechanism

Sample Programs

- PD control of 2DOF open link mechanism
- equation of motion of 2DOF open link mechanism under PD control
- definition of 2DOF open link mechanism
- tip point coordinates with gradient vectors and Hessian matrices
- calculating inertia matrix and generated torques
- drawing 2DOF open link mechanism

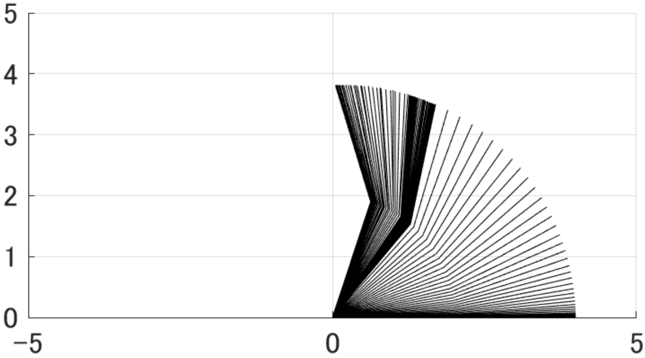
PD control of 2DOF open link mechanism

Result



PD control of 2DOF open link mechanism

Result



PI control

$$\tau_1 = K_{\mathrm{P1}}(\theta_1^d - \theta_1) + K_{\mathrm{I1}} \int_0^t \{(\theta_1^d - \theta_1(\tau))\} \, \mathrm{d}\tau$$

$$\tau_2 = K_{\mathrm{P2}}(\theta_2^d - \theta_2) + K_{\mathrm{I2}} \int_0^t \{(\theta_2^d - \theta_2(\tau))\} \, \mathrm{d}\tau$$

Introduce additional variables:

$$\xi_1 \triangleq \int_0^t \{(\theta_1^d - \theta_1(\tau))\} \, \mathrm{d}\tau$$

$$\xi_2 \triangleq \int_0^t \{(\theta_2^d - \theta_2(\tau))\} \, \mathrm{d}\tau$$

$$\dot{\xi}_1 = \theta_1^d - \theta_1, \quad \tau_1 = K_{\mathrm{P1}}(\theta_1^d - \theta_1) + K_{\mathrm{I1}}\xi_1$$

$$\dot{\xi}_2 = \theta_2^d - \theta_2, \quad \tau_2 = K_{\mathrm{P2}}(\theta_2^d - \theta_2) + K_{\mathrm{I2}}\xi_2$$

PI control

$$\Downarrow$$

$$\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}$$

$$\begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \end{bmatrix} = \begin{bmatrix} \cdots + K_{\mathrm{P1}}(\theta_1^d - \theta_1) + K_{\mathrm{I1}}\xi_1 \\ \cdots + K_{\mathrm{P2}}(\theta_2^d - \theta_2) + K_{\mathrm{I2}}\xi_2 \end{bmatrix}$$

$$\dot{\xi}_1 = \theta_1^d - \theta_1$$

$$\dot{\xi}_2 = \theta_2^d - \theta_2$$

current values of θ_1, θ_2, ω_1, ω_2, ξ_1, ξ_2

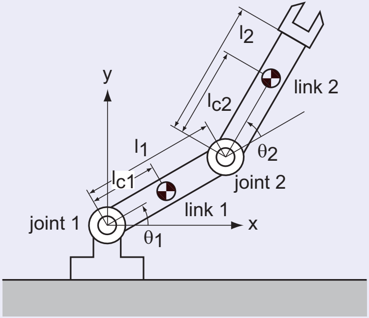
$$\Downarrow$$

their time derivatives $\dot{\theta}_1$, $\dot{\theta}_2$, $\dot{\omega}_1$, $\dot{\omega}_2$, $\dot{\xi}_1$, $\dot{\xi}_2$

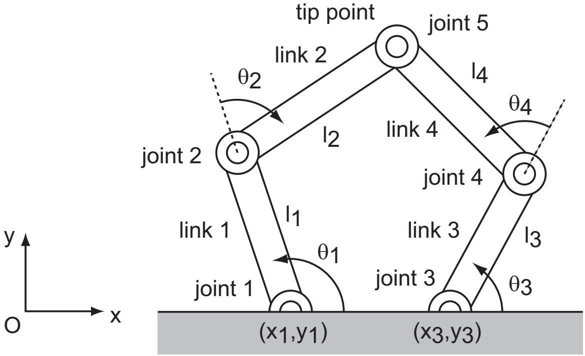
Report

Report #2 due date : Nov. 16 (Mon.)

Simulate the motion of a 2DOF open link mechanism under PID control. PID control is applied to active joints 1 and 2. Use appropriate values of geometrical and physical parameters of the manipulator.



Kinematics of 2DOF closed link mechanism



joint 1, 3: active
joint 2, 4, 5: passive

θ_1, θ_2, θ_3, θ_4:
rotation angles

τ_1, τ_3:
actuator torques

Kinematics of 2DOF closed link mechanism

decomposition of closed link mechanism into open link mechanisms:

left arm	link 1 and 2
right arm	link 3 and 4

end point of the left arm:

$$\begin{bmatrix} x_{1,2} \\ y_{1,2} \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + l_1 \begin{bmatrix} C_1 \\ S_1 \end{bmatrix} + l_2 \begin{bmatrix} C_{1+2} \\ S_{1+2} \end{bmatrix}$$

end point of the right arm:

$$\begin{bmatrix} x_{3,4} \\ y_{3,4} \end{bmatrix} = \begin{bmatrix} x_3 \\ y_3 \end{bmatrix} + l_3 \begin{bmatrix} C_3 \\ C_3 \end{bmatrix} + l_4 \begin{bmatrix} C_{3+4} \\ S_{3+4} \end{bmatrix}$$

two constraints:

$$X \triangleq x_{1,2} - x_{3,4} = l_1 C_1 + l_2 C_{1+2} - l_3 C_3 - l_4 C_{3+4} + x_1 - x_3 = 0$$

$$Y \triangleq y_{1,2} - y_{3,4} = l_1 S_1 + l_2 S_{1+2} - l_3 S_3 - l_4 S_{3+4} + y_1 - y_3 = 0$$

Lagrangian

Lagrangian of the closed link mechanism:

$$\mathcal{L} = \mathcal{L}_{1,2} + \mathcal{L}_{3,4} + \lambda_x X + \lambda_y Y$$

$\mathcal{L}_{1,2}$, $\mathcal{L}_{3,4}$	Lagrangians of the left and right arms
λ_x, λ_y	Lagrange multipliers

Lagrange equations of motion:

$$\frac{\partial \mathcal{L}}{\partial \boldsymbol{\theta}_{1,2}} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \boldsymbol{\omega}_{1,2}} = \mathbf{0}$$

$$\frac{\partial \mathcal{L}}{\partial \boldsymbol{\theta}_{3,4}} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \boldsymbol{\omega}_{3,4}} = \mathbf{0}$$

where

$$\boldsymbol{\theta}_{1,2} = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}, \; \boldsymbol{\omega}_{1,2} = \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}, \; \boldsymbol{\theta}_{3,4} = \begin{bmatrix} \theta_3 \\ \theta_4 \end{bmatrix}, \; \boldsymbol{\omega}_{3,4} = \begin{bmatrix} \omega_3 \\ \omega_4 \end{bmatrix}$$

Contributions of $\mathcal{L}_{1,2}$

contributions of Lagrangian $\mathcal{L}_{1,2}$ to the Lagrange eqs:

$$- H_{1,2}\, \dot{\boldsymbol{\omega}}_{1,2} + \boldsymbol{\tau}_{1,2}$$

$$\mathbf{0}$$

where

$$H_{1,2} = \begin{bmatrix} *** & J_2 + m_2(l_{c2}^2 + l_1 l_{c2} C_2) \\ J_2 + m_2(l_{c2}^2 + l_1 l_{c2} C_2) & J_2 + m_2 l_{c2}^2 \end{bmatrix}$$

$$\boldsymbol{\tau}_{1,2} = \begin{bmatrix} +h_{12}\omega_2^2 + 2h_{12}\omega_1\omega_2 - G_1 - G_{12} + \tau_1 \\ -h_{12}\omega_1^2 - G_{12} \end{bmatrix}$$

$$*** = J_1 + m_1 l_{c1}^2 + J_2 + m_2(l_1^2 + l_{c2}^2 + 2 l_1 l_{c2} C_2),$$

$$h_{12} = m_2 l_1 l_{c2} S_2,$$

$$G_1 = (m_1 l_{c1} + m_2 l_1)\, g C_1, \quad G_{12} = m_2 l_{c2}\, g C_{1+2}$$

Contributions of $\mathcal{L}_{3,4}$

contributions of Lagrangian $\mathcal{L}_{3,4}$ to the Lagrange eqs:

$$\mathbf{0}$$

$$- H_{3,4}\,\dot{\boldsymbol{\omega}}_{3,4} + \boldsymbol{\tau}_{3,4}$$

where

$$H_{3,4} = \begin{bmatrix} *** & J_4 + m_4(l_{c4}^2 + l_3 l_{c4} C_4) \\ J_4 + m_4(l_{c4}^2 + l_3 l_{c4} C_4) & J_4 + m_4 l_{c4}^2 \end{bmatrix}$$

$$\boldsymbol{\tau}_{3,4} = \begin{bmatrix} +h_{34}\omega_4^2 + 2h_{34}\omega_3\omega_4 - G_3 - G_{34} + \tau_3 \\ -h_{34}\omega_3^2 - G_{34} \end{bmatrix}$$

$$*** = J_3 + m_3 l_{c3}^2 + J_4 + m_4(l_3^2 + l_{c4}^2 + 2 l_3 l_{c4} C_4),$$

$$h_{34} = m_4 l_3 l_{c4} S_4,$$

$$G_3 = (m_3 l_{c3} + m_4 l_3)\, g C_3, \quad G_{34} = m_4 l_{c4}\, g C_{3+4}$$

Contributions of $\lambda_x X$

contributions of $\lambda_x X$ to the Lagrange eqs:

$$\begin{aligned} &\lambda_x \boldsymbol{g}_{x;1,2} \\ -&\lambda_x \boldsymbol{g}_{x;3,4} \end{aligned}$$

where

$$\boldsymbol{g}_{x;1,2} = \frac{\partial x_{1,2}}{\partial \boldsymbol{\theta}_{1,2}} = \begin{bmatrix} -l_1 S_1 - l_2 S_{1+2} \\ -l_2 S_{1+2} \end{bmatrix}$$

$$\boldsymbol{g}_{x;3,4} = \frac{\partial x_{3,4}}{\partial \boldsymbol{\theta}_{3,4}} = \begin{bmatrix} -l_3 S_3 - l_4 S_{3+4} \\ -l_4 S_{3+4} \end{bmatrix}$$

Contributions of $\lambda_y Y$

contributions of $\lambda_y Y$ to the Lagrange eqs:

$$\begin{aligned} &\lambda_y \boldsymbol{g}_{y;1,2} \\ -&\lambda_y \boldsymbol{g}_{y;3,4} \end{aligned}$$

where

$$\boldsymbol{g}_{y;1,2} = \frac{\partial y_{1,2}}{\partial \boldsymbol{\theta}_{1,2}} = \begin{bmatrix} l_1 C_1 + l_2 C_{1+2} \\ l_2 C_{1+2} \end{bmatrix}$$

$$\boldsymbol{g}_{y;3,4} = \frac{\partial y_{3,4}}{\partial \boldsymbol{\theta}_{3,4}} = \begin{bmatrix} l_3 C_3 + l_4 C_{3+4} \\ l_4 C_{3+4} \end{bmatrix}$$

Lagrange equations of motion

$$-H_{1,2}\,\dot{\boldsymbol{\omega}}_{1,2} + \boldsymbol{\tau}_{1,2} + \lambda_x \boldsymbol{g}_{x;1,2} + \lambda_y \boldsymbol{g}_{y;1,2} = \mathbf{0}$$
$$-H_{3,4}\,\dot{\boldsymbol{\omega}}_{3,4} + \boldsymbol{\tau}_{3,4} - \lambda_x \boldsymbol{g}_{x;3,4} - \lambda_y \boldsymbol{g}_{y;3,4} = \mathbf{0}$$
$$\Downarrow$$
$$\begin{bmatrix} H_{1,2} & O_{2\times 2} & -J_{1,2}^{\mathrm{T}} \\ O_{2\times 2} & H_{3,4} & J_{3,4}^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{\omega}}_{1,2} \\ \dot{\boldsymbol{\omega}}_{3,4} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\tau}_{1,2} \\ \boldsymbol{\tau}_{3,4} \end{bmatrix}$$

where

$$J_{1,2}^{\mathrm{T}} = \begin{bmatrix} \boldsymbol{g}_{x;1,2} & \boldsymbol{g}_{y;1,2} \end{bmatrix}$$
$$J_{3,4}^{\mathrm{T}} = \begin{bmatrix} \boldsymbol{g}_{x;3,4} & \boldsymbol{g}_{y;3,4} \end{bmatrix}$$
$$\boldsymbol{\lambda} = \begin{bmatrix} \lambda_x \\ \lambda_y \end{bmatrix}$$

Equation stabilizing constraint X

Recall that $X = x_{1,2} - x_{3,4}$

$$\dot{X} = \left(\frac{\partial x_{1,2}}{\partial \boldsymbol{\theta}_{1,2}}\right)^{\mathrm{T}} \dot{\boldsymbol{\theta}}_{1,2} - \left(\frac{\partial x_{3,4}}{\partial \boldsymbol{\theta}_{3,4}}\right)^{\mathrm{T}} \dot{\boldsymbol{\theta}}_{3,4} = \boldsymbol{g}_{x;1,2}^{\mathrm{T}}\,\boldsymbol{\omega}_{1,2} - \boldsymbol{g}_{x;3,4}^{\mathrm{T}}\,\boldsymbol{\omega}_{3,4}$$

$$\ddot{X} = \dot{\boldsymbol{g}}_{x;1,2}^{\mathrm{T}}\,\boldsymbol{\omega}_{1,2} + \boldsymbol{g}_{x;1,2}^{\mathrm{T}}\,\dot{\boldsymbol{\omega}}_{1,2} - \dot{\boldsymbol{g}}_{x;3,4}^{\mathrm{T}}\,\boldsymbol{\omega}_{3,4} - \boldsymbol{g}_{x;3,4}^{\mathrm{T}}\,\dot{\boldsymbol{\omega}}_{3,4}$$

Since

$$\dot{\boldsymbol{g}}_{x;1,2} = \frac{\partial \boldsymbol{g}_{x;1,2}}{\partial \boldsymbol{\theta}_{1,2}^{\mathrm{T}}}\dot{\boldsymbol{\theta}}_{1,2} = \frac{\partial \boldsymbol{g}_{x;1,2}}{\partial \boldsymbol{\theta}_{1,2}^{\mathrm{T}}}\boldsymbol{\omega}_{1,2}$$

we have $\dot{\boldsymbol{g}}_{x;1,2}^{\mathrm{T}}\,\boldsymbol{\omega}_{1,2} = \boldsymbol{\omega}_{1,2}^{\mathrm{T}}\,Q_{x;1,2}\,\boldsymbol{\omega}_{1,2}$, where

$$Q_{x;1,2} = \frac{\partial \boldsymbol{g}_{x;1,2}^{\mathrm{T}}}{\partial \boldsymbol{\theta}_{1,2}} = \begin{bmatrix} \partial^2 x_{1,2}/\partial\theta_1^2 & \partial^2 x_{1,2}/\partial\theta_1\partial\theta_2 \\ \partial^2 x_{1,2}/\partial\theta_2\partial\theta_1 & \partial^2 x_{1,2}/\partial\theta_2^2 \end{bmatrix}$$

Hessian matrix

Equation stabilizing constraint X

$$\ddot{X} + 2\alpha\dot{X} + \alpha^2 X = 0$$

$$\Downarrow$$

$$\begin{aligned} -\boldsymbol{g}_{x;1,2}^{\mathrm{T}}\,\dot{\boldsymbol{\omega}}_{1,2} + \boldsymbol{g}_{x;3,4}^{\mathrm{T}}\,\dot{\boldsymbol{\omega}}_{3,4} &= \boldsymbol{\omega}_{1,2}^{\mathrm{T}}\,Q_{x;1,2}\,\boldsymbol{\omega}_{1,2} - \boldsymbol{\omega}_{3,4}^{\mathrm{T}}\,Q_{x;3,4}\,\boldsymbol{\omega}_{3,4} \\ &\quad + 2\alpha(\boldsymbol{g}_{x;1,2}^{\mathrm{T}}\,\boldsymbol{\omega}_{1,2} - \boldsymbol{g}_{x;3,4}^{\mathrm{T}}\,\boldsymbol{\omega}_{3,4}) + \alpha^2 X \end{aligned}$$

where

$$\boldsymbol{g}_{x;1,2} = \begin{bmatrix} -l_1 S_1 - l_2 S_{1+2} \\ -l_2 S_{1+2} \end{bmatrix},\; Q_{x;1,2} = \begin{bmatrix} -l_1 C_1 - l_2 C_{1+2} & -l_2 C_{1+2} \\ -l_2 C_{1+2} & -l_2 C_{1+2} \end{bmatrix}$$

$$\boldsymbol{g}_{x;3,4} = \begin{bmatrix} -l_3 S_3 - l_4 S_{3+4} \\ -l_4 S_{3+4} \end{bmatrix},\; Q_{x;3,4} = \begin{bmatrix} -l_3 C_3 - l_4 C_{3+4} & -l_4 C_{3+4} \\ -l_4 C_{3+4} & -l_4 C_{3+4} \end{bmatrix}$$

Equation stabilizing constraint Y

$$\ddot{Y} + 2\alpha\dot{Y} + \alpha^2 Y = 0$$

$$\Downarrow$$

$$-\boldsymbol{g}_{y;1,2}^{\mathrm{T}}\,\dot{\boldsymbol{\omega}}_{1,2} + \boldsymbol{g}_{y;3,4}^{\mathrm{T}}\,\dot{\boldsymbol{\omega}}_{3,4} = \boldsymbol{\omega}_{1,2}^{\mathrm{T}}\,Q_{y;1,2}\,\boldsymbol{\omega}_{1,2} - \boldsymbol{\omega}_{3,4}^{\mathrm{T}}\,Q_{y;3,4}\,\boldsymbol{\omega}_{3,4} + 2\alpha(\boldsymbol{g}_{y;1,2}^{\mathrm{T}}\,\boldsymbol{\omega}_{1,2} - \boldsymbol{g}_{y;3,4}^{\mathrm{T}}\,\boldsymbol{\omega}_{3,4}) + \alpha^2 Y$$

where

$$\boldsymbol{g}_{y;1,2} = \begin{bmatrix} l_1 C_1 + l_2 C_{1+2} \\ l_2 C_{1+2} \end{bmatrix}, \; Q_{y;1,2} = \begin{bmatrix} -l_1 S_1 - l_2 S_{1+2} & -l_2 S_{1+2} \\ -l_2 S_{1+2} & -l_2 S_{1+2} \end{bmatrix}$$

$$\boldsymbol{g}_{y;3,4} = \begin{bmatrix} l_3 C_3 + l_4 C_{3+4} \\ l_4 C_{3+4} \end{bmatrix}, \; Q_{y;3,4} = \begin{bmatrix} -l_3 S_3 - l_4 S_{3+4} & -l_4 S_{3+4} \\ -l_4 S_{3+4} & -l_4 S_{3+4} \end{bmatrix}$$

Equations stabilizing constraint X and Y

$$-\boldsymbol{g}_{x;1,2}^{\mathrm{T}}\,\dot{\boldsymbol{\omega}}_{1,2} + \boldsymbol{g}_{x;3,4}^{\mathrm{T}}\,\dot{\boldsymbol{\omega}}_{3,4} = C_x(\boldsymbol{\theta}_{1,2}, \boldsymbol{\theta}_{3,4}, \boldsymbol{\omega}_{1,2}, \boldsymbol{\omega}_{3,4})$$
$$-\boldsymbol{g}_{y;1,2}^{\mathrm{T}}\,\dot{\boldsymbol{\omega}}_{1,2} + \boldsymbol{g}_{y;3,4}^{\mathrm{T}}\,\dot{\boldsymbol{\omega}}_{3,4} = C_y(\boldsymbol{\theta}_{1,2}, \boldsymbol{\theta}_{3,4}, \boldsymbol{\omega}_{1,2}, \boldsymbol{\omega}_{3,4})$$

where

$$C_x = \boldsymbol{\omega}_{1,2}^{\mathrm{T}}\,Q_{x;1,2}\,\boldsymbol{\omega}_{1,2} - \boldsymbol{\omega}_{3,4}^{\mathrm{T}}\,Q_{x;3,4}\,\boldsymbol{\omega}_{3,4} + 2\alpha(\boldsymbol{g}_{x;1,2}^{\mathrm{T}}\,\boldsymbol{\omega}_{1,2} - \boldsymbol{g}_{x;3,4}^{\mathrm{T}}\,\boldsymbol{\omega}_{3,4}) + \alpha^2 X$$
$$C_y = \boldsymbol{\omega}_{1,2}^{\mathrm{T}}\,Q_{y;1,2}\,\boldsymbol{\omega}_{1,2} - \boldsymbol{\omega}_{3,4}^{\mathrm{T}}\,Q_{y;3,4}\,\boldsymbol{\omega}_{3,4} + 2\alpha(\boldsymbol{g}_{y;1,2}^{\mathrm{T}}\,\boldsymbol{\omega}_{1,2} - \boldsymbol{g}_{y;3,4}^{\mathrm{T}}\,\boldsymbol{\omega}_{3,4}) + \alpha^2 Y$$

$$\Downarrow$$

$$\begin{bmatrix} -J_{1,2} & J_{3,4} & O_{2\times 2} \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{\omega}}_{1,2} \\ \dot{\boldsymbol{\omega}}_{3,4} \\ \boldsymbol{\lambda} \end{bmatrix} = \boldsymbol{C}$$

where $\boldsymbol{C} = [\, C_x,\ C_y \,]^{\mathrm{T}}$

Dynamic equations for closed link mechanism

$$\begin{bmatrix} H_{1,2} & O_{2\times 2} & -J_{1,2}^{\mathrm{T}} \\ O_{2\times 2} & H_{3,4} & J_{3,4}^{\mathrm{T}} \\ -J_{1,2} & J_{3,4} & O_{2\times 2} \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{\omega}}_{1,2} \\ \dot{\boldsymbol{\omega}}_{3,4} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\tau}_{1,2} \\ \boldsymbol{\tau}_{3,4} \\ \boldsymbol{C} \end{bmatrix}$$

the coefficient matrix is regular $\longrightarrow$ we can compute $\dot{\omega}_1$ through $\dot{\omega}_4$

Physical Interpretation

$J_{1,2}$ and $J_{3,4}$: Jacobian matrices of the left and right arms
$\boldsymbol{\lambda} = [\, \lambda_x, \, \lambda_y \,]^{\rm T}$: constraint force
equivalent torques around rotational joints 1 and 2:

$$J_{1,2}^{\rm T}\boldsymbol{\lambda} = \left[\begin{array}{c} \lambda_x(-l_1 S_1 - l_2 S_{1+2}) + \lambda_y(l_1 C_1 + l_2 C_{1+2}) \\ \lambda_x(-l_2 S_{1+2}) + \lambda_y l_2 C_{1+2} \end{array} \right]$$

reaction force $-\boldsymbol{\lambda}$
equivalent torques around rotational joint 3 and 4:

$$J_{3,4}^{\rm T}(-\boldsymbol{\lambda}) = \left[\begin{array}{c} \lambda_x(l_3 S_3 + l_4 S_{3+4}) + \lambda_y(-l_3 C_3 - l_4 C_{3+4}) \\ \lambda_x l_4 S_{3+4} + \lambda_y(-l_4 C_{3+4}) \end{array} \right]$$

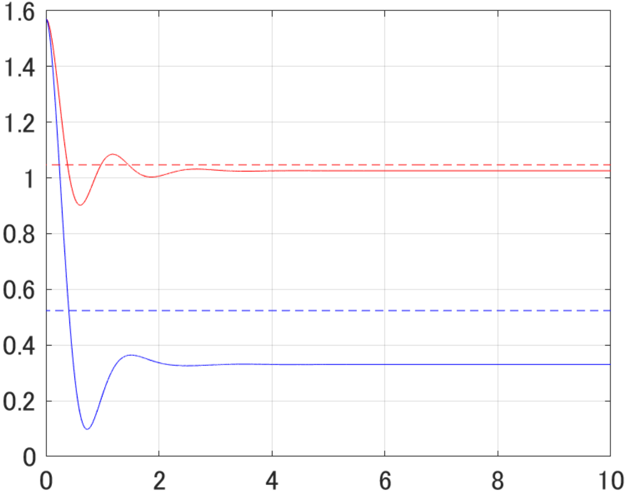
PD control of 2DOF closed link mechanism

Sample Programs

- PD control of 2DOF closed link mechanism
- equation of motion of 2DOF closed link mechanism under PD control
- definition of 2DOF closed link mechanism
- drawing 2DOF closed link mechanism

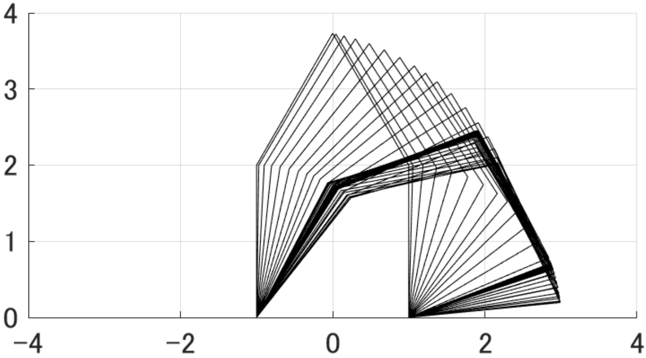
PD control of 2DOF closed link mechanism

Result



PD control of 2DOF closed link mechanism

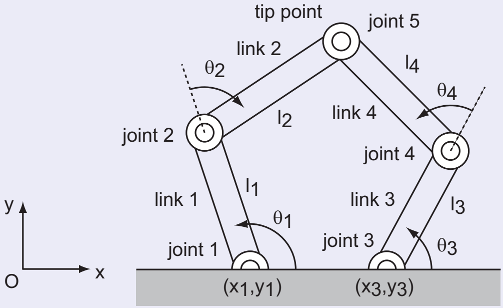
Result



Report

Report #3 due date : Nov. 30 (Mon.)

Simulate the motion of a 2DOF closed link mechanism under PID control. PID control is applied to active joints 1 and 3. Use appropriate values of geometrical and physical parameters of the manipulator.



Summary

Open link mechanism

- inertia matrix depends on joint angles
- Lagrange equations of motion of open link mechanism

Closed link mechanism

- two open link mechanisms with geometric constraints
- synthesized from Lagrange equations of open link mechanisms