

Analytical Mechanics: Link Mechanisms

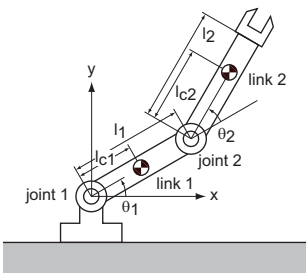
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Agenda

- 1 Open Link Mechanism
 - Kinematics of Open Link Mechanism
 - Dynamics of 2DOF open link mechanism
- 2 Closed Link Mechanism
 - Kinematics of Closed Link Mechanism
 - Dynamics of 2DOF closed link mechanism

Kinematics of 2DOF open link mechanism



two link open link mechanism

- l_i length of link i
- l_{ci} distance btw. joint i and the center of mass of link i
- m_i mass of link i
- J_i inertia of moment of link i around its center of mass
- θ_1 rotation angle of joint 1
- θ_2 rotation angle of joint 2

Kinematics of 2DOF open link mechanism

position of the center of mass of link 1:

$$\mathbf{x}_{c1} \triangleq \begin{bmatrix} x_{c1} \\ y_{c1} \end{bmatrix} = l_{c1} \begin{bmatrix} C_1 \\ S_1 \end{bmatrix}$$

position of the center of mass of link 2:

$$\mathbf{x}_{c2} \triangleq \begin{bmatrix} x_{c2} \\ y_{c2} \end{bmatrix} = l_1 \begin{bmatrix} C_1 \\ S_1 \end{bmatrix} + l_{c2} \begin{bmatrix} C_{1+2} \\ S_{1+2} \end{bmatrix}$$

orientation angle of link 1:

$$\theta_1$$

orientation angle of link 2:

$$\theta_1 + \theta_2$$

Kinetic energy

velocity of the center of mass of link 1:

$$\dot{\mathbf{x}}_{c1} = l_{c1} \dot{\theta}_1 \begin{bmatrix} -S_1 \\ C_1 \end{bmatrix}$$

angular velocity of link 1:

$$\dot{\theta}_1$$

kinetic energy of link 1:

$$\begin{aligned} T_1 &= \frac{1}{2} m_1 \dot{\mathbf{x}}_{c1}^T \dot{\mathbf{x}}_{c1} + \frac{1}{2} J_1 \dot{\theta}_1^2 \\ &= \frac{1}{2} (m_1 l_{c1}^2 + J_1) \dot{\theta}_1^2 \end{aligned}$$

Kinetic energy

velocity of the center of mass of link 2:

$$\dot{\mathbf{x}}_{c2} = l_1 \dot{\theta}_1 \begin{bmatrix} -S_1 \\ C_1 \end{bmatrix} + l_{c2} (\dot{\theta}_1 + \dot{\theta}_2) \begin{bmatrix} -S_{1+2} \\ C_{1+2} \end{bmatrix}$$

angular velocity of link 2:

$$\dot{\theta}_1 + \dot{\theta}_2$$

kinetic energy of link 2:

$$\begin{aligned} T_2 &= \frac{1}{2} m_2 \dot{\mathbf{x}}_{c2}^T \dot{\mathbf{x}}_{c2} + \frac{1}{2} J_2 (\dot{\theta}_1 + \dot{\theta}_2)^2 \\ &= \frac{1}{2} m_2 \{ l_1^2 \dot{\theta}_1^2 + l_{c2}^2 (\dot{\theta}_1 + \dot{\theta}_2)^2 + 2 l_1 l_{c2} C_2 \dot{\theta}_1 (\dot{\theta}_1 + \dot{\theta}_2) \} + \\ &\quad \frac{1}{2} J_2 (\dot{\theta}_1 + \dot{\theta}_2)^2 \end{aligned}$$

Kinetic energy

total kinetic energy

$$T = T_1 + T_2 = \frac{1}{2} \begin{bmatrix} \dot{\theta}_1 & \dot{\theta}_2 \end{bmatrix} \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

where

$$\begin{aligned} H_{11} &= J_1 + m_1 l_{c1}^2 + J_2 + m_2 (l_1^2 + l_{c2}^2 + 2 l_1 l_{c2} C_2) \\ H_{22} &= J_2 + m_2 l_{c2}^2 \\ H_{12} &= H_{21} = J_2 + m_2 (l_{c2}^2 + l_1 l_{c2} C_2) \end{aligned}$$

inertia matrix

$$H \triangleq \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}$$

Partial derivatives

H_{11} and $H_{12} = H_{21}$ depend on θ_2 :

$$\frac{\partial H_{11}}{\partial \theta_2} = -2h_{12}, \quad \frac{\partial H_{12}}{\partial \theta_2} = \frac{\partial H_{21}}{\partial \theta_2} = -h_{12} \quad (h_{12} \triangleq m_2 l_1 l_{c2} S_2)$$

$$\dot{H}_{11} = -2h_{12} \dot{\theta}_2, \quad \dot{H}_{12} = \dot{H}_{21} = -h_{12} \dot{\theta}_2$$

$$\frac{\partial T}{\partial \dot{\theta}_1} = H_{11} \dot{\theta}_1 + H_{12} \dot{\theta}_2, \quad \frac{\partial T}{\partial \dot{\theta}_2} = H_{21} \dot{\theta}_1 + H_{22} \dot{\theta}_2$$

$$-\frac{d}{dt} \frac{\partial T}{\partial \dot{\theta}_1} = -\dot{H}_{11} \dot{\theta}_1 - H_{11} \ddot{\theta}_1 - \dot{H}_{12} \dot{\theta}_2 - H_{12} \ddot{\theta}_2$$

$$= 2h_{12} \dot{\theta}_1 \dot{\theta}_2 + h_{12} \dot{\theta}_2^2 - H_{11} \ddot{\theta}_1 - H_{12} \ddot{\theta}_2$$

$$-\frac{d}{dt} \frac{\partial T}{\partial \dot{\theta}_2} = -\dot{H}_{21} \dot{\theta}_1 - H_{21} \ddot{\theta}_1 - \dot{H}_{22} \dot{\theta}_2 - H_{22} \ddot{\theta}_2$$

$$= h_{12} \dot{\theta}_1 \dot{\theta}_2 - H_{21} \ddot{\theta}_1 - H_{22} \ddot{\theta}_2$$

Partial derivatives

H_{11} , H_{22} , and $H_{12} = H_{21}$ are independent of θ_1

$$\frac{\partial T}{\partial \theta_1} = \frac{1}{2} \begin{bmatrix} \dot{\theta}_1 & \dot{\theta}_2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = 0$$

H_{11} and $H_{12} = H_{21}$ depend on θ_2

$$\frac{\partial T}{\partial \theta_2} = \frac{1}{2} \begin{bmatrix} \dot{\theta}_1 & \dot{\theta}_2 \end{bmatrix} \begin{bmatrix} -2h_{12} & -h_{12} \\ -h_{12} & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = -h_{12}\dot{\theta}_1^2 - h_{12}\dot{\theta}_1\dot{\theta}_2$$

contribution of kinetic energy:

$$\begin{aligned} \frac{\partial T}{\partial \theta_1} - \frac{d}{dt} \frac{\partial T}{\partial \dot{\theta}_1} &= 2h_{12}\dot{\theta}_1\dot{\theta}_2 + h_{12}\dot{\theta}_2^2 - H_{11}\ddot{\theta}_1 - H_{12}\ddot{\theta}_2 \\ \frac{\partial T}{\partial \theta_2} - \frac{d}{dt} \frac{\partial T}{\partial \dot{\theta}_2} &= -h_{12}\dot{\theta}_1^2 - H_{21}\ddot{\theta}_1 - H_{22}\ddot{\theta}_2 \end{aligned}$$

Gravitational potential energy

potential energy of link 1:

$$U_1 = m_1 g y_{c1} = m_1 g l_{c1} S_1$$

potential energy of link 2:

$$U_2 = m_2 g y_{c2} = m_2 g (l_1 S_1 + l_{c2} S_{1+2})$$

potential energy:

$$U = U_1 + U_2$$

partial derivatives w.r.t. joint angles:

$$\frac{\partial U}{\partial \theta_1} = G_1 + G_{12}, \quad \frac{\partial U}{\partial \theta_2} = G_{12}$$

where

$$G_1 = (m_1 l_{c1} + m_2 l_1) g C_1, \quad G_{12} = m_2 l_{c2} g C_{1+2}$$

Work done by actuator torques

work done by τ_1 applied to rotational joint 1:

$$\tau_1 \theta_1$$

work done by τ_2 applied to rotational joint 2:

$$\tau_2 \theta_2$$

work done by the two actuator torques:

$$W = \tau_1 \theta_1 + \tau_2 \theta_2$$

partial derivatives w.r.t. joint angles:

$$\frac{\partial W}{\partial \theta_1} = \tau_1, \quad \frac{\partial W}{\partial \theta_2} = \tau_2$$

Lagrange equations of motion

Lagrangian:

$$\mathcal{L} = T - U + W$$

Lagrange equations of motion

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \theta_1} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\theta}_1} &= 0 \\ \frac{\partial \mathcal{L}}{\partial \theta_2} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\theta}_2} &= 0 \end{aligned}$$

let $\omega_1 \triangleq \dot{\theta}_1$ and $\omega_2 \triangleq \dot{\theta}_2$:

$$\begin{aligned} -H_{11}\dot{\omega}_1 - H_{12}\dot{\omega}_2 + h_{12}\omega_2^2 + 2h_{12}\omega_1\omega_2 - G_1 - G_{12} + \tau_1 &= 0 \\ -H_{22}\dot{\omega}_2 - H_{12}\dot{\omega}_1 - h_{12}\omega_1^2 - G_{12} + \tau_2 &= 0 \end{aligned}$$

Lagrange equations of motion

canonical form of ordinary differential equations:

$$\begin{aligned} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} &= \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} \\ \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \end{bmatrix} &= \begin{bmatrix} h_{12}\omega_2^2 + 2h_{12}\omega_1\omega_2 - G_1 - G_{12} + \tau_1 \\ -h_{12}\omega_1^2 - G_{12} + \tau_2 \end{bmatrix} \end{aligned}$$

state variables: joint angles θ_1, θ_2 and angular velocities ω_1, ω_2

the inertia matrix is regular \rightarrow 2nd eq. is solvable

\rightarrow we can compute $\dot{\omega}_1$ and $\dot{\omega}_2$

$\dot{\theta}_1, \dot{\theta}_2, \dot{\omega}_1, \dot{\omega}_2$ are functions of $\theta_1, \theta_2, \omega_1, \omega_2$

\Downarrow

we can sketch $\theta_1, \theta_2, \omega_1, \omega_2$ using an ODE solver.

PD control

$$\tau_1 = K_{P1}(\theta_1^d - \theta_1) - K_{D1}\dot{\theta}_1$$

$$\tau_2 = K_{P2}(\theta_2^d - \theta_2) - K_{D2}\dot{\theta}_2$$

\Downarrow

$$\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}$$

$$\begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \end{bmatrix} = \begin{bmatrix} \dots + K_{P1}(\theta_1^d - \theta_1) - K_{D1}\omega_1 \\ \dots + K_{P2}(\theta_2^d - \theta_2) - K_{D2}\omega_2 \end{bmatrix}$$

current values of $\theta_1, \theta_2, \omega_1, \omega_2$

\Downarrow

their time derivatives $\dot{\theta}_1, \dot{\theta}_2, \dot{\omega}_1, \dot{\omega}_2$

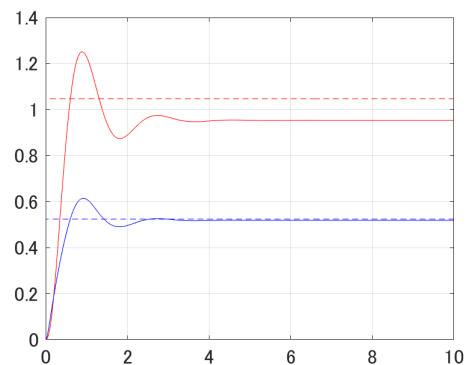
PD control of 2DOF open link mechanism

Sample Programs

- PD control of 2DOF open link mechanism
- equation of motion of 2DOF open link mechanism under PD control
- definition of 2DOF open link mechanism
- tip point coordinates with gradient vectors and Hessian matrices
- calculating inertia matrix and generated torques
- drawing 2DOF open link mechanism

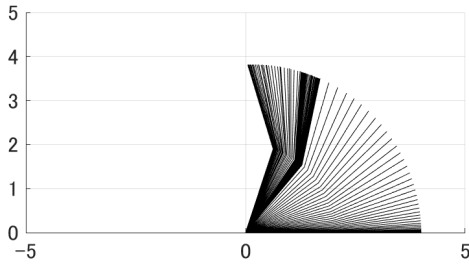
PD control of 2DOF open link mechanism

Result

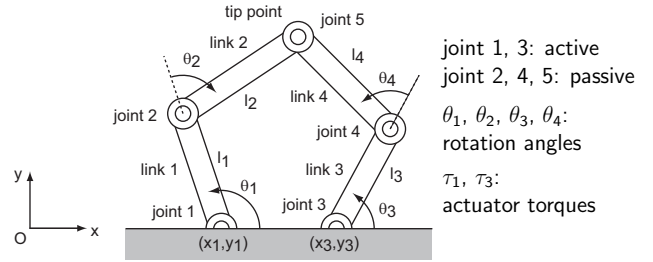


PD control of 2DOF open link mechanism

Result



Kinematics of 2DOF closed link mechanism



joint 1, 3: active
joint 2, 4, 5: passive
 $\theta_1, \theta_2, \theta_3, \theta_4$:
rotation angles
 τ_1, τ_3 :
actuator torques

PI control

$$\tau_1 = K_{P1}(\theta_1^d - \theta_1) + K_{I1} \int_0^t \{(\theta_1^d - \theta_1(\tau))\} d\tau$$

$$\tau_2 = K_{P2}(\theta_2^d - \theta_2) + K_{I2} \int_0^t \{(\theta_2^d - \theta_2(\tau))\} d\tau$$

Introduce additional variables:

$$\xi_1 \triangleq \int_0^t \{(\theta_1^d - \theta_1(\tau))\} d\tau$$

$$\xi_2 \triangleq \int_0^t \{(\theta_2^d - \theta_2(\tau))\} d\tau$$

$$\dot{\xi}_1 = \theta_1^d - \theta_1, \quad \tau_1 = K_{P1}(\theta_1^d - \theta_1) + K_{I1}\xi_1$$

$$\dot{\xi}_2 = \theta_2^d - \theta_2, \quad \tau_2 = K_{P2}(\theta_2^d - \theta_2) + K_{I2}\xi_2$$

PI control

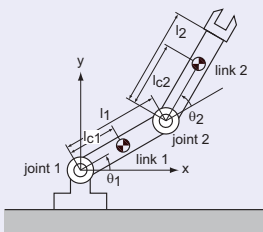
$$\begin{aligned} &\downarrow \\ &\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} \\ \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \end{bmatrix} &= \begin{bmatrix} \cdots + K_{P1}(\theta_1^d - \theta_1) + K_{I1}\xi_1 \\ \cdots + K_{P2}(\theta_2^d - \theta_2) + K_{I2}\xi_2 \end{bmatrix} \\ &\dot{\xi}_1 = \theta_1^d - \theta_1 \\ &\dot{\xi}_2 = \theta_2^d - \theta_2 \end{aligned}$$

current values of $\theta_1, \theta_2, \omega_1, \omega_2, \xi_1, \xi_2$
 \downarrow
their time derivatives $\dot{\theta}_1, \dot{\theta}_2, \dot{\omega}_1, \dot{\omega}_2, \dot{\xi}_1, \dot{\xi}_2$

Report

Report #2 due date : Nov. 16 (Mon.)

Simulate the motion of a 2DOF open link mechanism under PID control. PID control is applied to active joints 1 and 2. Use appropriate values of geometrical and physical parameters of the manipulator.



Kinematics of 2DOF closed link mechanism

decomposition of closed link mechanism into open link mechanisms:

left arm link 1 and 2
right arm link 3 and 4

end point of the left arm:

$$\begin{bmatrix} x_{1,2} \\ y_{1,2} \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + l_1 \begin{bmatrix} C_1 \\ S_1 \end{bmatrix} + l_2 \begin{bmatrix} C_{1+2} \\ S_{1+2} \end{bmatrix}$$

end point of the right arm:

$$\begin{bmatrix} x_{3,4} \\ y_{3,4} \end{bmatrix} = \begin{bmatrix} x_3 \\ y_3 \end{bmatrix} + l_3 \begin{bmatrix} C_3 \\ S_3 \end{bmatrix} + l_4 \begin{bmatrix} C_{3+4} \\ S_{3+4} \end{bmatrix}$$

two constraints:

$$X \triangleq x_{1,2} - x_{3,4} = l_1 C_1 + l_2 C_{1+2} - l_3 C_3 - l_4 C_{3+4} + x_1 - x_3 = 0$$

$$Y \triangleq y_{1,2} - y_{3,4} = l_1 S_1 + l_2 S_{1+2} - l_3 S_3 - l_4 S_{3+4} + y_1 - y_3 = 0$$

Lagrangian

Lagrangian of the closed link mechanism:

$$\mathcal{L} = \mathcal{L}_{1,2} + \mathcal{L}_{3,4} + \lambda_x X + \lambda_y Y$$

$\mathcal{L}_{1,2}, \mathcal{L}_{3,4}$ Lagrangians of the left and right arms
 λ_x, λ_y Lagrange multipliers

Lagrange equations of motion:

$$\frac{\partial \mathcal{L}}{\partial \theta_{1,2}} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \omega_{1,2}} = 0$$

$$\frac{\partial \mathcal{L}}{\partial \theta_{3,4}} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \omega_{3,4}} = 0$$

where

$$\theta_{1,2} = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}, \quad \omega_{1,2} = \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}, \quad \theta_{3,4} = \begin{bmatrix} \theta_3 \\ \theta_4 \end{bmatrix}, \quad \omega_{3,4} = \begin{bmatrix} \omega_3 \\ \omega_4 \end{bmatrix}$$

Contributions of $\mathcal{L}_{1,2}$

contributions of Lagrangian $\mathcal{L}_{1,2}$ to the Lagrange eqs:

$$-H_{1,2} \dot{\omega}_{1,2} + \tau_{1,2}$$

where

$$H_{1,2} = \begin{bmatrix} *** & J_2 + m_2(l_{c2}^2 + l_1 l_{c2} C_2) \\ J_2 + m_2(l_{c2}^2 + l_1 l_{c2} C_2) & J_2 + m_2 l_{c2}^2 \end{bmatrix}$$

$$\tau_{1,2} = \begin{bmatrix} +h_{12}\omega_2^2 + 2h_{12}\omega_1\omega_2 - G_1 - G_{12} + \tau_1 \\ -h_{12}\omega_1^2 - G_{12} \end{bmatrix}$$

$$*** = J_1 + m_1 l_{c1}^2 + J_2 + m_2(l_1^2 + l_{c2}^2 + 2l_1 l_{c2} C_2),$$

$$h_{12} = m_2 l_1 l_{c2} S_2,$$

$$G_1 = (m_1 l_{c1} + m_2 l_1) g C_1, \quad G_{12} = m_2 l_{c2} g C_{1+2}$$

Contributions of $\mathcal{L}_{3,4}$

contributions of Lagrangian $\mathcal{L}_{3,4}$ to the Lagrange eqs:

$$\mathbf{0} \\ -H_{3,4} \dot{\omega}_{3,4} + \tau_{3,4}$$

where

$$H_{3,4} = \begin{bmatrix} *** & J_4 + m_4(l_{c4}^2 + l_3 l_{c4} C_4) \\ J_4 + m_4(l_{c4}^2 + l_3 l_{c4} C_4) & J_4 + m_4 l_{c4}^2 \end{bmatrix} \\ \tau_{3,4} = \begin{bmatrix} +h_{34} \omega_4^2 + 2h_{34} \omega_3 \omega_4 - G_3 - G_{34} + \tau_3 \\ -h_{34} \omega_3^2 - G_{34} \end{bmatrix}$$

$$*** = J_3 + m_3 l_{c3}^2 + J_4 + m_4(l_3^2 + l_{c4}^2 + 2l_3 l_{c4} C_4),$$

$$h_{34} = m_4 l_3 l_{c4} S_4,$$

$$G_3 = (m_3 l_{c3} + m_4 l_3) g C_3, \quad G_{34} = m_4 l_{c4} g C_{3+4}$$

Contributions of $\lambda_x X$

contributions of $\lambda_x X$ to the Lagrange eqs:

$$\lambda_x \mathbf{g}_{x;1,2} \\ -\lambda_x \mathbf{g}_{x;3,4}$$

where

$$\mathbf{g}_{x;1,2} = \frac{\partial x_{1,2}}{\partial \theta_{1,2}} = \begin{bmatrix} -l_1 S_1 - l_2 S_{1+2} \\ -l_2 S_{1+2} \end{bmatrix} \\ \mathbf{g}_{x;3,4} = \frac{\partial x_{3,4}}{\partial \theta_{3,4}} = \begin{bmatrix} -l_3 S_3 - l_4 S_{3+4} \\ -l_4 S_{3+4} \end{bmatrix}$$

Contributions of $\lambda_y Y$

contributions of $\lambda_y Y$ to the Lagrange eqs:

$$\lambda_y \mathbf{g}_{y;1,2} \\ -\lambda_y \mathbf{g}_{y;3,4}$$

where

$$\mathbf{g}_{y;1,2} = \frac{\partial y_{1,2}}{\partial \theta_{1,2}} = \begin{bmatrix} l_1 C_1 + l_2 C_{1+2} \\ l_2 C_{1+2} \end{bmatrix} \\ \mathbf{g}_{y;3,4} = \frac{\partial y_{3,4}}{\partial \theta_{3,4}} = \begin{bmatrix} l_3 C_3 + l_4 C_{3+4} \\ l_4 C_{3+4} \end{bmatrix}$$

Lagrange equations of motion

$$-H_{1,2} \dot{\omega}_{1,2} + \tau_{1,2} + \lambda_x \mathbf{g}_{x;1,2} + \lambda_y \mathbf{g}_{y;1,2} = \mathbf{0} \\ -H_{3,4} \dot{\omega}_{3,4} + \tau_{3,4} - \lambda_x \mathbf{g}_{x;3,4} - \lambda_y \mathbf{g}_{y;3,4} = \mathbf{0}$$

↓

$$\begin{bmatrix} H_{1,2} & O_{2 \times 2} & -J_{1,2}^T \\ O_{2 \times 2} & H_{3,4} & J_{3,4}^T \end{bmatrix} \begin{bmatrix} \dot{\omega}_{1,2} \\ \dot{\omega}_{3,4} \\ \lambda \end{bmatrix} = \begin{bmatrix} \tau_{1,2} \\ \tau_{3,4} \end{bmatrix}$$

where

$$J_{1,2}^T = \begin{bmatrix} \mathbf{g}_{x;1,2} & \mathbf{g}_{y;1,2} \end{bmatrix} \\ J_{3,4}^T = \begin{bmatrix} \mathbf{g}_{x;3,4} & \mathbf{g}_{y;3,4} \end{bmatrix} \\ \lambda = \begin{bmatrix} \lambda_x \\ \lambda_y \end{bmatrix}$$

Equation stabilizing constraint X

Recall that $X = x_{1,2} - x_{3,4}$

$$\dot{X} = \left(\frac{\partial x_{1,2}}{\partial \theta_{1,2}} \right)^T \dot{\theta}_{1,2} - \left(\frac{\partial x_{3,4}}{\partial \theta_{3,4}} \right)^T \dot{\theta}_{3,4} = \mathbf{g}_{x;1,2}^T \omega_{1,2} - \mathbf{g}_{x;3,4}^T \omega_{3,4}$$

$$\ddot{X} = \dot{\mathbf{g}}_{x;1,2}^T \omega_{1,2} + \mathbf{g}_{x;1,2}^T \dot{\omega}_{1,2} - \dot{\mathbf{g}}_{x;3,4}^T \omega_{3,4} - \mathbf{g}_{x;3,4}^T \dot{\omega}_{3,4}$$

Since

$$\dot{\mathbf{g}}_{x;1,2} = \frac{\partial \mathbf{g}_{x;1,2}}{\partial \theta_{1,2}} \dot{\theta}_{1,2} = \frac{\partial \mathbf{g}_{x;1,2}}{\partial \theta_{1,2}^T} \omega_{1,2}$$

we have $\dot{\mathbf{g}}_{x;1,2}^T \omega_{1,2} = \omega_{1,2}^T Q_{x;1,2} \omega_{1,2}$, where

$$Q_{x;1,2} = \frac{\partial \mathbf{g}_{x;1,2}^T}{\partial \theta_{1,2}} = \begin{bmatrix} \partial^2 x_{1,2} / \partial \theta_1^2 & \partial^2 x_{1,2} / \partial \theta_1 \partial \theta_2 \\ \partial^2 x_{1,2} / \partial \theta_2 \partial \theta_1 & \partial^2 x_{1,2} / \partial \theta_2^2 \end{bmatrix}$$

Hessian matrix

Equation stabilizing constraint X

$$\ddot{X} + 2\alpha \dot{X} + \alpha^2 X = 0$$

↓

$$-\mathbf{g}_{x;1,2}^T \dot{\omega}_{1,2} + \mathbf{g}_{x;3,4}^T \dot{\omega}_{3,4} = \omega_{1,2}^T Q_{x;1,2} \omega_{1,2} - \omega_{3,4}^T Q_{x;3,4} \omega_{3,4} \\ + 2\alpha(\mathbf{g}_{x;1,2}^T \omega_{1,2} - \mathbf{g}_{x;3,4}^T \omega_{3,4}) + \alpha^2 X$$

where

$$\mathbf{g}_{x;1,2} = \begin{bmatrix} -l_1 S_1 - l_2 S_{1+2} \\ -l_2 S_{1+2} \end{bmatrix}, \quad Q_{x;1,2} = \begin{bmatrix} -l_1 C_1 - l_2 C_{1+2} & -l_2 C_{1+2} \\ -l_2 C_{1+2} & -l_2 C_{1+2} \end{bmatrix} \\ \mathbf{g}_{x;3,4} = \begin{bmatrix} -l_3 S_3 - l_4 S_{3+4} \\ -l_4 S_{3+4} \end{bmatrix}, \quad Q_{x;3,4} = \begin{bmatrix} -l_3 C_3 - l_4 C_{3+4} & -l_4 C_{3+4} \\ -l_4 C_{3+4} & -l_4 C_{3+4} \end{bmatrix}$$

Equation stabilizing constraint Y

$$\ddot{Y} + 2\alpha \dot{Y} + \alpha^2 Y = 0$$

↓

$$-\mathbf{g}_{y;1,2}^T \dot{\omega}_{1,2} + \mathbf{g}_{y;3,4}^T \dot{\omega}_{3,4} = \omega_{1,2}^T Q_{y;1,2} \omega_{1,2} - \omega_{3,4}^T Q_{y;3,4} \omega_{3,4} \\ + 2\alpha(\mathbf{g}_{y;1,2}^T \omega_{1,2} - \mathbf{g}_{y;3,4}^T \omega_{3,4}) + \alpha^2 Y$$

where

$$\mathbf{g}_{y;1,2} = \begin{bmatrix} l_1 C_1 + l_2 C_{1+2} \\ l_2 C_{1+2} \end{bmatrix}, \quad Q_{y;1,2} = \begin{bmatrix} -l_1 S_1 - l_2 S_{1+2} & -l_2 S_{1+2} \\ -l_2 S_{1+2} & -l_2 S_{1+2} \end{bmatrix} \\ \mathbf{g}_{y;3,4} = \begin{bmatrix} l_3 C_3 + l_4 C_{3+4} \\ l_4 C_{3+4} \end{bmatrix}, \quad Q_{y;3,4} = \begin{bmatrix} -l_3 S_3 - l_4 S_{3+4} & -l_4 S_{3+4} \\ -l_4 S_{3+4} & -l_4 S_{3+4} \end{bmatrix}$$

Equations stabilizing constraint X and Y

$$-\mathbf{g}_{x;1,2}^T \dot{\omega}_{1,2} + \mathbf{g}_{x;3,4}^T \dot{\omega}_{3,4} = C_x(\theta_{1,2}, \theta_{3,4}, \omega_{1,2}, \omega_{3,4}) \\ -\mathbf{g}_{y;1,2}^T \dot{\omega}_{1,2} + \mathbf{g}_{y;3,4}^T \dot{\omega}_{3,4} = C_y(\theta_{1,2}, \theta_{3,4}, \omega_{1,2}, \omega_{3,4})$$

where

$$C_x = \omega_{1,2}^T Q_{x;1,2} \omega_{1,2} - \omega_{3,4}^T Q_{x;3,4} \omega_{3,4} + 2\alpha(\mathbf{g}_{x;1,2}^T \omega_{1,2} - \mathbf{g}_{x;3,4}^T \omega_{3,4}) + \alpha^2 X \\ C_y = \omega_{1,2}^T Q_{y;1,2} \omega_{1,2} - \omega_{3,4}^T Q_{y;3,4} \omega_{3,4} + 2\alpha(\mathbf{g}_{y;1,2}^T \omega_{1,2} - \mathbf{g}_{y;3,4}^T \omega_{3,4}) + \alpha^2 Y$$

↓

$$\begin{bmatrix} -J_{1,2} & J_{3,4} & O_{2 \times 2} \end{bmatrix} \begin{bmatrix} \dot{\omega}_{1,2} \\ \dot{\omega}_{3,4} \\ \lambda \end{bmatrix} = \mathbf{C}$$

where $\mathbf{C} = [C_x, C_y]^T$

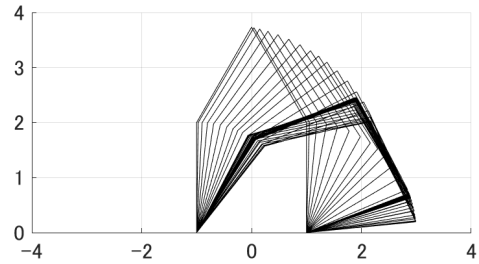
Dynamic equations for closed link mechanism

$$\begin{bmatrix} H_{1,2} & O_{2 \times 2} & -J_{1,2}^T \\ O_{2 \times 2} & H_{3,4} & J_{3,4}^T \\ -J_{1,2} & J_{3,4} & O_{2 \times 2} \end{bmatrix} \begin{bmatrix} \dot{\omega}_{1,2} \\ \dot{\omega}_{3,4} \\ \lambda \end{bmatrix} = \begin{bmatrix} \tau_{1,2} \\ \tau_{3,4} \\ C \end{bmatrix}$$

the coefficient matrix is regular \rightarrow we can compute $\dot{\omega}_1$ through $\dot{\omega}_4$

PD control of 2DOF closed link mechanism

Result



Physical Interpretation

$J_{1,2}$ and $J_{3,4}$: Jacobian matrices of the left and right arms

$\lambda = [\lambda_x, \lambda_y]^T$: constraint force

equivalent torques around rotational joints 1 and 2:

$$J_{1,2}^T \lambda = \begin{bmatrix} \lambda_x(-l_1 S_1 - l_2 S_{1+2}) + \lambda_y(l_1 C_1 + l_2 C_{1+2}) \\ \lambda_x(-l_2 S_{1+2}) + \lambda_y l_2 C_{1+2} \end{bmatrix}$$

reaction force $-\lambda$

equivalent torques around rotational joint 3 and 4:

$$J_{3,4}^T (-\lambda) = \begin{bmatrix} \lambda_x(l_3 S_3 + l_4 S_{3+4}) + \lambda_y(-l_3 C_3 - l_4 C_{3+4}) \\ \lambda_x l_4 S_{3+4} + \lambda_y(-l_4 C_{3+4}) \end{bmatrix}$$

PD control of 2DOF closed link mechanism

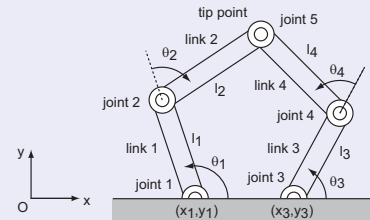
Sample Programs

- PD control of 2DOF closed link mechanism
- equation of motion of 2DOF closed link mechanism under PD control
- definition of 2DOF closed link mechanism
- drawing 2DOF closed link mechanism

Report

Report #3 due date: Nov. 30 (Mon.)

Simulate the motion of a 2DOF closed link mechanism under PID control. PID control is applied to active joints 1 and 3. Use appropriate values of geometrical and physical parameters of the manipulator.



Summary

Open link mechanism

- inertia matrix depends on joint angles
- Lagrange equations of motion of open link mechanism

Closed link mechanism

- two open link mechanisms with geometric constraints
- synthesized from Lagrange equations of open link mechanisms

PD control of 2DOF closed link mechanism

Result

