

Analytical Mechanics: Rigid Body Rotation

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Agenda

1 Planar Rotation

- Description of Planar Rotation
- Lagrange Equation of Planar Rotation
- Rotation Matrix

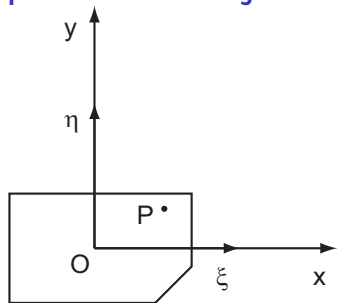
2 Spatial Rotation

- Description of Spatial Rotation
- Lagrange Equation of Spatial Rotation
- Forced Spatial Rotation

3 Quaternion

- Describing Spatial Rotation using Quaternions
- Rotation Dynamics using Quaternion
- Description of Forced Rotation

Spatial and object coordinate systems



$O - xy$ spatial coordinate system

$O - \xi\eta$ object coordinate system

\mathbf{a} unit vector along ξ -axis

\mathbf{b} unit vector along η -axis

$$\|\mathbf{a}\| = 1 \rightarrow \mathbf{a}^T \mathbf{a} = 1$$

$$\|\mathbf{b}\| = 1 \rightarrow \mathbf{b}^T \mathbf{b} = 1$$

$$\mathbf{a} \perp \mathbf{b} \rightarrow \mathbf{a}^T \mathbf{b} = \mathbf{b}^T \mathbf{a} = 0$$

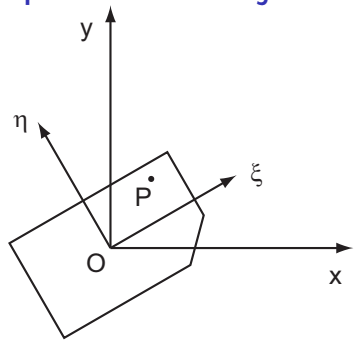
ξ, η : object coordinates of point P spatial coordinates of point P

$$\mathbf{x} = \xi \mathbf{a} + \eta \mathbf{b} = \begin{bmatrix} \mathbf{a} & \mathbf{b} \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix}$$

$R \quad \xi$

Note: \mathbf{a} and \mathbf{b} depend on time. ξ and η are independent of time.

Spatial and object coordinate systems



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$R \quad \xi$

Note: \mathbf{a} and \mathbf{b} depend on time. ξ and η are independent of time.

Angular velocity in planar rotation

differentiating relationships between \mathbf{a} and \mathbf{b} w.r.t time:

$$\mathbf{a}^T \mathbf{a} = 1 \rightarrow \mathbf{a}^T \dot{\mathbf{a}} = 0$$

$$\mathbf{b}^T \mathbf{b} = 1 \rightarrow \mathbf{b}^T \dot{\mathbf{b}} = 0$$

$$\mathbf{a}^T \mathbf{b} = 0 \rightarrow \underbrace{\mathbf{a}^T \dot{\mathbf{b}}}_{-\omega} + \underbrace{\mathbf{b}^T \dot{\mathbf{a}}}_{\omega} = 0$$

describing $\dot{\mathbf{a}}$ and $\dot{\mathbf{b}}$ in object coordinate system:

$$\dot{\mathbf{a}} = (\mathbf{a}^T \dot{\mathbf{a}})\mathbf{a} + (\mathbf{b}^T \dot{\mathbf{a}})\mathbf{b} = \omega \mathbf{b}$$

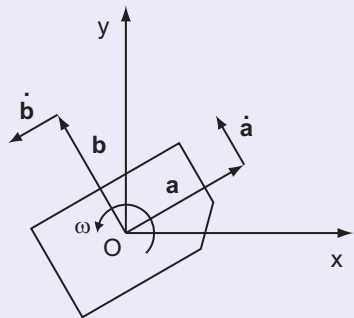
$$\dot{\mathbf{b}} = (\mathbf{a}^T \dot{\mathbf{b}})\mathbf{a} + (\mathbf{b}^T \dot{\mathbf{b}})\mathbf{b} = -\omega \mathbf{a}$$

velocity of point $P(\xi, \eta)$

$$\dot{\mathbf{x}} = \xi \dot{\mathbf{a}} + \eta \dot{\mathbf{b}} = \omega(\xi \mathbf{b} - \eta \mathbf{a})$$

Angular velocity in planar rotation

interpretation



$$\omega = \mathbf{b}^T \dot{\mathbf{a}} \quad \text{angular velocity}$$

$$\|\mathbf{a}\| = 1 \rightarrow \dot{\mathbf{a}} \perp \mathbf{a} \rightarrow \dot{\mathbf{a}} \parallel \mathbf{b}$$

$$\|\mathbf{b}\| = 1 \rightarrow \dot{\mathbf{b}} \perp \mathbf{b} \rightarrow \dot{\mathbf{b}} \parallel \mathbf{a}$$

$$\|\mathbf{a}\| = 1 \rightarrow \|\dot{\mathbf{a}}\| = \omega$$

$$\|\mathbf{b}\| = 1 \rightarrow \|\dot{\mathbf{b}}\| = \omega$$

Kinetic energy of rigid body

Divide a rigid body into a finite number of masses.

m_i the i -th mass
 (ξ_i, η_i) object coordinates of the i -th mass

position of mass m_i $\mathbf{x}_i = \xi_i \mathbf{a} + \eta_i \mathbf{b}$

velocity of mass m_i $\dot{\mathbf{x}}_i = \xi_i \dot{\mathbf{a}} + \eta_i \dot{\mathbf{b}} = \omega(\xi_i \mathbf{b} - \eta_i \mathbf{a})$

kinetic energy of mass m_i

$$\begin{aligned} \frac{1}{2} m_i \dot{\mathbf{x}}_i^T \dot{\mathbf{x}}_i &= \frac{1}{2} m_i \omega^2 (\xi_i \mathbf{b} - \eta_i \mathbf{a})^T (\xi_i \mathbf{b} - \eta_i \mathbf{a}) \\ &= \frac{1}{2} m_i \omega^2 (\xi_i^2 + \eta_i^2) \end{aligned}$$

Kinetic energy of rigid body

kinetic energy of rigid body rotating on plane

$$\begin{aligned}\sum_i \frac{1}{2} m_i \dot{\mathbf{x}}_i^T \dot{\mathbf{x}}_i &= \sum_i \frac{1}{2} m_i \omega^2 (\xi_i^2 + \eta_i^2) \\ &= \frac{1}{2} J \omega^2\end{aligned}$$

where

$$J = \sum_i m_i (\xi_i^2 + \eta_i^2) \quad \text{inertia of moment}$$

Note: J is constant (independent of time)

Computing Lagrange equation of planar rotation

description that satisfies relationships between \mathbf{a} and \mathbf{b}

$$\mathbf{a} = \begin{bmatrix} C_\theta \\ S_\theta \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -S_\theta \\ C_\theta \end{bmatrix}$$

angular velocity

$$\omega = \mathbf{b}^T \dot{\mathbf{a}} = \begin{bmatrix} -S_\theta & C_\theta \end{bmatrix} \begin{bmatrix} -S_\theta \\ C_\theta \end{bmatrix} \dot{\theta} = \dot{\theta}$$

kinetic energy

$$T = \frac{1}{2} J \dot{\theta}^2$$

work done by external torque τ around point O

$$W = \tau \theta$$

Computing Lagrange equation of planar rotation

Lagrangian

$$L = \frac{1}{2}J\dot{\theta}^2 + \tau\theta$$

Lagrange equation of motion

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} = 0$$

$$\tau - J\ddot{\theta} = 0$$

$$J\ddot{\theta} = \tau$$

equation of planar rotation

Inertia of moment in rigid continuum

Let ρ be planar density of a rigid body.

$$m_i \rightarrow \rho \, d\xi d\eta$$

$$\sum_i \rightarrow \int_S$$

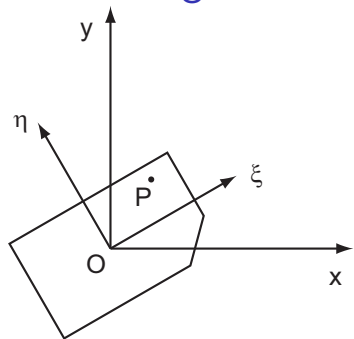
inertia of moment:

$$J = \sum_i m_i (\xi_i^2 + \eta_i^2)$$

↓

$$J = \int_S \rho (\xi^2 + \eta^2) \, d\xi d\eta$$

Introducing rotation matrix



spatial coordinates of point $P(\xi, \eta)$

$$\begin{aligned} \mathbf{x} &= \xi \mathbf{a} + \eta \mathbf{b} \\ &= \left[\begin{array}{c|c} \mathbf{a} & \mathbf{b} \end{array} \right] \begin{bmatrix} \xi \\ \eta \end{bmatrix} \\ &= R \boldsymbol{\xi} \end{aligned}$$

R rotation matrix

$$R^T R = \begin{bmatrix} \mathbf{a}^T \\ \mathbf{b}^T \end{bmatrix} \left[\begin{array}{c|c} \mathbf{a} & \mathbf{b} \end{array} \right] = \begin{bmatrix} \mathbf{a}^T \mathbf{a} & \mathbf{a}^T \mathbf{b} \\ \mathbf{b}^T \mathbf{a} & \mathbf{b}^T \mathbf{b} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

R is an orthogonal matrix

$$R^T R = R R^T = I_{2 \times 2} \quad (\text{unit matrix})$$

Computing kinetic energy using rotation matrix

differentiating $R^T R = I_{2 \times 2}$ w.r.t time:

$$\dot{R}^T R + R^T \dot{R} = O_{2 \times 2} \quad (\text{zero matrix})$$

$$(R^T \dot{R})^T + (R^T \dot{R}) = O_{2 \times 2}$$

$R^T \dot{R}$ is a skew-symmetric matrix:

$$R^T \dot{R} = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix}$$

$$\begin{aligned} (R^T \dot{R})^T (R^T \dot{R}) &= \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} = \begin{bmatrix} \omega^2 & 0 \\ 0 & \omega^2 \end{bmatrix} \\ &= \omega^2 I_{2 \times 2} \end{aligned}$$

Computing kinetic energy using rotation matrix

differentiating $\mathbf{x} = R\boldsymbol{\xi}$ with respect to time:

$$\dot{\mathbf{x}} = \dot{R}\boldsymbol{\xi}$$

quadratic form:

$$\begin{aligned}\dot{\mathbf{x}}^T \dot{\mathbf{x}} &= \boldsymbol{\xi}^T \dot{R}^T \dot{R} \boldsymbol{\xi} = \boldsymbol{\xi}^T \dot{R}^T R R^T \dot{R} \boldsymbol{\xi} \\ &= \boldsymbol{\xi}^T (R^T \dot{R})^T (R^T \dot{R}) \boldsymbol{\xi} = \boldsymbol{\xi}^T \omega^2 I_{2 \times 2} \boldsymbol{\xi} \\ &= \omega^2 \boldsymbol{\xi}^T \boldsymbol{\xi}\end{aligned}$$

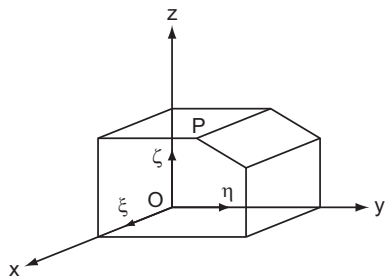
kinetic energy of a rigid body:

$$T = \sum_i \frac{1}{2} m_i \dot{\mathbf{x}}_i^T \dot{\mathbf{x}}_i = \sum_i \frac{1}{2} m_i \omega^2 \boldsymbol{\xi}_i^T \boldsymbol{\xi}_i = \frac{1}{2} J \omega^2$$

where

$$J = \sum_i m_i \boldsymbol{\xi}_i^T \boldsymbol{\xi}_i = \sum_i m_i (\xi_i^2 + \eta_i^2)$$

Spatial and object coordinate systems



O – xyz spatial coordinate system

O – $\xi\eta\zeta$ object coordinate system

\mathbf{a} , \mathbf{b} , \mathbf{c} unit vectors along
 ξ -, η -, and ζ axes

$$\mathbf{a}^T \mathbf{a} = \mathbf{b}^T \mathbf{b} = \mathbf{c}^T \mathbf{c} = 1$$

$$\mathbf{a}^T \mathbf{b} = \mathbf{b}^T \mathbf{c} = \mathbf{c}^T \mathbf{a} = 0$$

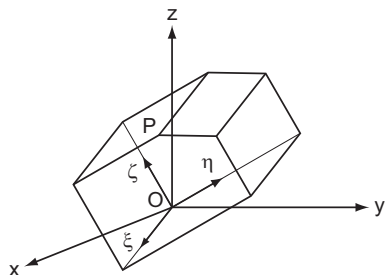
ξ, η, ζ : object coordinates of point P spatial coordinates of point P

$$\mathbf{x} = \xi \mathbf{a} + \eta \mathbf{b} + \zeta \mathbf{c} = \begin{bmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \end{bmatrix} \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix} = \mathbf{R} \boldsymbol{\xi}$$

R $\boldsymbol{\xi}$

Note: \mathbf{a} , \mathbf{b} , \mathbf{c} depend on time. ξ , η , ζ are independent of time.

Spatial and object coordinate systems



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R $\boldsymbol{\xi}$

Note: \mathbf{a} , \mathbf{b} , \mathbf{c} depend on time. ξ , η , ζ are independent of time.

Angular velocity vector in spatial rotation

rotation matrix R

$$R^T R = \begin{bmatrix} \mathbf{a}^T \\ \mathbf{b}^T \\ \mathbf{c}^T \end{bmatrix} \begin{bmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \end{bmatrix} = I_{3 \times 3} \quad (\text{unit matrix})$$

R is an orthogonal matrix

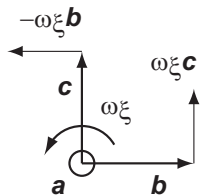
differentiating $R^T R = I_{3 \times 3}$ w.r.t time:

$$\dot{R}^T R + R^T \dot{R} = (R^T \dot{R})^T + (R^T \dot{R}) = O_{3 \times 3} \quad (\text{zero matrix})$$

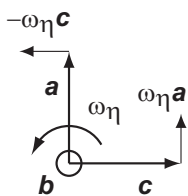
$R^T \dot{R}$ is a skew-symmetric matrix:

$$R^T \dot{R} = \begin{bmatrix} 0 & -\omega_\zeta & \omega_\eta \\ \omega_\zeta & 0 & -\omega_\xi \\ -\omega_\eta & \omega_\xi & 0 \end{bmatrix}$$

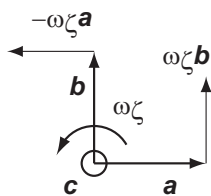
Angular velocity vector in spatial rotation



$$\omega_\xi = \mathbf{c}^T \dot{\mathbf{b}}$$



$$\omega_\eta = \mathbf{a}^T \dot{\mathbf{c}}$$



$$\omega_\zeta = \mathbf{b}^T \dot{\mathbf{a}}$$

$$R^T \dot{R} = \begin{bmatrix} \mathbf{a}^T \\ \mathbf{b}^T \\ \mathbf{c}^T \end{bmatrix} \begin{bmatrix} \dot{\mathbf{a}} & \dot{\mathbf{b}} & \dot{\mathbf{c}} \end{bmatrix} = \begin{bmatrix} \mathbf{a}^T \dot{\mathbf{a}} & \mathbf{a}^T \dot{\mathbf{b}} & \mathbf{a}^T \dot{\mathbf{c}} \\ \mathbf{b}^T \dot{\mathbf{a}} & \mathbf{b}^T \dot{\mathbf{b}} & \mathbf{b}^T \dot{\mathbf{c}} \\ \mathbf{c}^T \dot{\mathbf{a}} & \mathbf{c}^T \dot{\mathbf{b}} & \mathbf{c}^T \dot{\mathbf{c}} \end{bmatrix}$$

$$\dot{\mathbf{a}} = (\mathbf{a}^T \dot{\mathbf{a}})\mathbf{a} + (\mathbf{b}^T \dot{\mathbf{a}})\mathbf{b} + (\mathbf{c}^T \dot{\mathbf{a}})\mathbf{c} = \omega_\zeta \mathbf{b} - \omega_\eta \mathbf{c}$$

$$\dot{\mathbf{b}} = (\mathbf{a}^T \dot{\mathbf{b}})\mathbf{a} + (\mathbf{b}^T \dot{\mathbf{b}})\mathbf{b} + (\mathbf{c}^T \dot{\mathbf{b}})\mathbf{c} = \omega_\xi \mathbf{c} - \omega_\zeta \mathbf{a}$$

$$\dot{\mathbf{c}} = (\mathbf{a}^T \dot{\mathbf{c}})\mathbf{a} + (\mathbf{b}^T \dot{\mathbf{c}})\mathbf{b} + (\mathbf{c}^T \dot{\mathbf{c}})\mathbf{c} = \omega_\eta \mathbf{a} - \omega_\xi \mathbf{b}$$

Angular velocity vector in spatial rotation

angular velocity vector:

$$\boldsymbol{\omega} \triangleq \begin{bmatrix} \omega_\xi \\ \omega_\eta \\ \omega_\zeta \end{bmatrix}$$

Note that $\boldsymbol{\omega}$ is defined on object coordinate system.

$$\begin{aligned} (R^T \dot{R}) \boldsymbol{\xi} &= \begin{bmatrix} 0 & -\omega_\zeta & \omega_\eta \\ \omega_\zeta & 0 & -\omega_\xi \\ -\omega_\eta & \omega_\xi & 0 \end{bmatrix} \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix} = \begin{bmatrix} \omega_\eta \zeta - \omega_\zeta \eta \\ \omega_\zeta \xi - \omega_\xi \zeta \\ \omega_\xi \eta - \omega_\eta \xi \end{bmatrix} \\ &= \begin{bmatrix} \omega_\xi \\ \omega_\eta \\ \omega_\zeta \end{bmatrix} \times \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix} = \boldsymbol{\omega} \times \boldsymbol{\xi} \end{aligned}$$

Computing kinetic energy in spacial rotation

differentiating $\mathbf{x} = R\xi$ with respect to time:

$$\dot{\mathbf{x}} = \dot{R}\xi$$

quadratic form:

$$\begin{aligned}\dot{\mathbf{x}}^T \dot{\mathbf{x}} &= \xi^T \dot{R}^T \dot{R} \xi = \xi^T \dot{R}^T R R^T \dot{R} \xi \\ &= (\boldsymbol{\omega} \times \xi)^T (\boldsymbol{\omega} \times \xi) = (-\xi \times \boldsymbol{\omega})^T (-\xi \times \boldsymbol{\omega}) \\ &= (\xi \times \boldsymbol{\omega})^T (\xi \times \boldsymbol{\omega}) = ([\xi \times] \boldsymbol{\omega})^T ([\xi \times] \boldsymbol{\omega}) \\ &= \boldsymbol{\omega}^T [\xi \times]^T [\xi \times] \boldsymbol{\omega}\end{aligned}$$

where

$$[\xi \times] \triangleq \begin{bmatrix} 0 & -\zeta & \eta \\ \zeta & 0 & -\xi \\ -\eta & \xi & 0 \end{bmatrix}$$

Computing kinetic energy in spacial rotation

$$\begin{aligned} [\boldsymbol{\xi} \times]^T [\boldsymbol{\xi} \times] &= \begin{bmatrix} 0 & \zeta & -\eta \\ -\zeta & 0 & \xi \\ \eta & -\xi & 0 \end{bmatrix} \begin{bmatrix} 0 & -\zeta & \eta \\ \zeta & 0 & -\xi \\ -\eta & \xi & 0 \end{bmatrix} \\ &= \begin{bmatrix} \eta^2 + \zeta^2 & -\xi\eta & -\xi\zeta \\ -\eta\xi & \zeta^2 + \xi^2 & -\eta\zeta \\ -\zeta\xi & -\zeta\eta & \xi^2 + \eta^2 \end{bmatrix} \end{aligned}$$

kinetic energy of a rigid body:

$$\begin{aligned} T &= \sum_i \frac{1}{2} m_i \dot{\boldsymbol{x}}_i^T \dot{\boldsymbol{x}}_i = \sum_i \frac{1}{2} m_i \boldsymbol{\omega}^T [\boldsymbol{\xi}_i \times]^T [\boldsymbol{\xi}_i \times] \boldsymbol{\omega} \\ &= \frac{1}{2} \boldsymbol{\omega}^T J \boldsymbol{\omega} \end{aligned}$$

Computing kinetic energy in spacial rotation

inertia matrix

$$J \triangleq \sum_i m_i [\boldsymbol{\xi}_i \times]^T [\boldsymbol{\xi}_i \times] = \begin{bmatrix} J_\xi & J_{\xi\eta} & J_{\zeta\xi} \\ J_{\xi\eta} & J_\eta & J_{\eta\zeta} \\ J_{\zeta\xi} & J_{\eta\zeta} & J_\zeta \end{bmatrix}$$

where

$$J_\xi = \sum_i m_i (\eta_i^2 + \zeta_i^2), \quad J_\eta = \sum_i m_i (\zeta_i^2 + \xi_i^2),$$

$$J_\zeta = \sum_i m_i (\xi_i^2 + \eta_i^2),$$

$$J_{\xi\eta} = - \sum_i m_i \xi_i \eta_i, \quad J_{\eta\zeta} = - \sum_i m_i \eta_i \zeta_i, \quad J_{\zeta\xi} = - \sum_i m_i \zeta_i \xi_i$$

Note: inertia matrix is constant (independent of time)

Lagrange equation of spatial rotation

generalized coordinates describing spatial rotation:

$$\mathbf{a} = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} c_x \\ c_y \\ c_z \end{bmatrix}$$

under geometric constraints:

$$R_1 = \mathbf{a}^T \mathbf{a} - 1 = 0, \quad R_2 = \mathbf{b}^T \mathbf{b} - 1 = 0, \quad R_3 = \mathbf{c}^T \mathbf{c} - 1 = 0, \\ Q_1 = \mathbf{b}^T \mathbf{c} = 0, \quad Q_2 = \mathbf{c}^T \mathbf{a} = 0, \quad Q_3 = \mathbf{a}^T \mathbf{b} = 0$$

kinetic energy

$$T = \frac{1}{2} \boldsymbol{\omega}^T \mathbf{J} \boldsymbol{\omega}$$

where

$$\omega_\xi = \mathbf{c}^T \dot{\mathbf{b}}, \quad \omega_\eta = \mathbf{a}^T \dot{\mathbf{c}}, \quad \omega_\zeta = \mathbf{b}^T \dot{\mathbf{a}}$$

Lagrange equation of spatial rotation

Lagrangian

$$L = T + \lambda_1 R_1 + \lambda_2 R_2 + \lambda_3 R_3 + \mu_1 Q_1 + \mu_2 Q_2 + \mu_3 Q_3$$

($\lambda_1, \lambda_2, \lambda_3, \mu_1, \mu_2, \mu_3$: Lagrange multipliers)

Lagrange equations of spacial rotation

$$\frac{\partial L}{\partial \mathbf{a}} - \frac{d}{dt} \frac{\partial L}{\partial \dot{\mathbf{a}}} = \mathbf{0},$$

$$\frac{\partial L}{\partial \mathbf{b}} - \frac{d}{dt} \frac{\partial L}{\partial \dot{\mathbf{b}}} = \mathbf{0},$$

$$\frac{\partial L}{\partial \mathbf{c}} - \frac{d}{dt} \frac{\partial L}{\partial \dot{\mathbf{c}}} = \mathbf{0}$$

Computing Lagrange equation of spatial rotation

kinetic energy

$$T = \frac{1}{2} \boldsymbol{\omega}^T J \boldsymbol{\omega}$$

derivative

$$\frac{dT}{d\boldsymbol{\omega}} = J\boldsymbol{\omega}$$

$$\frac{dT}{d\boldsymbol{\omega}} = \begin{bmatrix} \frac{\partial T}{\partial \omega_\xi} \\ \frac{\partial T}{\partial \omega_\eta} \\ \frac{\partial T}{\partial \omega_\zeta} \end{bmatrix}$$

Computing Lagrange equation of spatial rotation

angular velocities

$$\omega_\xi = \mathbf{c}^T \dot{\mathbf{b}}, \quad \omega_\eta = \mathbf{a}^T \dot{\mathbf{c}}, \quad \omega_\zeta = \mathbf{b}^T \dot{\mathbf{a}}$$

partial derivatives

$$\begin{aligned} \frac{\partial \omega_\xi}{\partial \mathbf{a}} &= \mathbf{0}, & \frac{\partial \omega_\eta}{\partial \mathbf{a}} &= \dot{\mathbf{c}}, & \frac{\partial \omega_\zeta}{\partial \mathbf{a}} &= \mathbf{0} \\ \frac{\partial \omega_\xi}{\partial \dot{\mathbf{a}}} &= \mathbf{0}, & \frac{\partial \omega_\eta}{\partial \dot{\mathbf{a}}} &= \mathbf{0}, & \frac{\partial \omega_\zeta}{\partial \dot{\mathbf{a}}} &= \mathbf{b} \end{aligned}$$

Computing Lagrange equation of spatial rotation

angular velocities

$$\omega_\xi = \mathbf{c}^T \dot{\mathbf{b}}, \quad \omega_\eta = \mathbf{a}^T \dot{\mathbf{c}}, \quad \omega_\zeta = \mathbf{b}^T \dot{\mathbf{a}}$$

partial derivatives

$$\begin{array}{l} \frac{\partial \omega_\xi}{\partial \mathbf{a}} = \mathbf{0}, \quad \frac{\partial \omega_\eta}{\partial \mathbf{a}} = \dot{\mathbf{c}}, \quad \frac{\partial \omega_\zeta}{\partial \mathbf{a}} = \mathbf{0}, \quad \frac{\partial \omega_\xi}{\partial \dot{\mathbf{a}}} = \mathbf{0}, \quad \frac{\partial \omega_\eta}{\partial \dot{\mathbf{a}}} = \mathbf{0}, \quad \frac{\partial \omega_\zeta}{\partial \dot{\mathbf{a}}} = \mathbf{b} \\ \frac{\partial \omega_\xi}{\partial \mathbf{b}} = \mathbf{0}, \quad \frac{\partial \omega_\eta}{\partial \mathbf{b}} = \mathbf{0}, \quad \frac{\partial \omega_\zeta}{\partial \mathbf{b}} = \dot{\mathbf{a}}, \quad \frac{\partial \omega_\xi}{\partial \dot{\mathbf{b}}} = \mathbf{c}, \quad \frac{\partial \omega_\eta}{\partial \dot{\mathbf{b}}} = \mathbf{0}, \quad \frac{\partial \omega_\zeta}{\partial \dot{\mathbf{b}}} = \mathbf{0} \\ \frac{\partial \omega_\xi}{\partial \mathbf{c}} = \dot{\mathbf{b}}, \quad \frac{\partial \omega_\eta}{\partial \mathbf{c}} = \mathbf{0}, \quad \frac{\partial \omega_\zeta}{\partial \mathbf{c}} = \mathbf{0}, \quad \frac{\partial \omega_\xi}{\partial \dot{\mathbf{c}}} = \mathbf{0}, \quad \frac{\partial \omega_\eta}{\partial \dot{\mathbf{c}}} = \mathbf{a}, \quad \frac{\partial \omega_\zeta}{\partial \dot{\mathbf{c}}} = \mathbf{0} \end{array}$$

Computing Lagrange equation of spatial rotation

dependency

$$T \leftarrow \omega_\xi, \omega_\eta, \omega_\zeta \leftarrow \mathbf{a}$$

\Downarrow

$$\frac{\partial T}{\partial \mathbf{a}} = \frac{\partial T}{\partial \omega_\xi} \frac{\partial \omega_\xi}{\partial \mathbf{a}} + \frac{\partial T}{\partial \omega_\eta} \frac{\partial \omega_\eta}{\partial \mathbf{a}} + \frac{\partial T}{\partial \omega_\zeta} \frac{\partial \omega_\zeta}{\partial \mathbf{a}}$$

$$= \left[\begin{array}{c|c|c} \frac{\partial \omega_\xi}{\partial \mathbf{a}} & \frac{\partial \omega_\eta}{\partial \mathbf{a}} & \frac{\partial \omega_\zeta}{\partial \mathbf{a}} \end{array} \right] \begin{bmatrix} \frac{\partial T}{\partial \omega_\xi} \\ \frac{\partial T}{\partial \omega_\eta} \\ \frac{\partial T}{\partial \omega_\zeta} \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{0} & \dot{\mathbf{c}} & \mathbf{0} \end{bmatrix} J \boldsymbol{\omega}$$

Computing Lagrange equation of spatial rotation

dependency

$$T \leftarrow \omega_\xi, \omega_\eta, \omega_\zeta \leftarrow \dot{\mathbf{a}}$$

\Downarrow

$$\frac{\partial T}{\partial \dot{\mathbf{a}}} = \frac{\partial T}{\partial \omega_\xi} \frac{\partial \omega_\xi}{\partial \dot{\mathbf{a}}} + \frac{\partial T}{\partial \omega_\eta} \frac{\partial \omega_\eta}{\partial \dot{\mathbf{a}}} + \frac{\partial T}{\partial \omega_\zeta} \frac{\partial \omega_\zeta}{\partial \dot{\mathbf{a}}}$$

$$= \left[\begin{array}{c|c|c} \frac{\partial \omega_\xi}{\partial \dot{\mathbf{a}}} & \frac{\partial \omega_\eta}{\partial \dot{\mathbf{a}}} & \frac{\partial \omega_\zeta}{\partial \dot{\mathbf{a}}} \end{array} \right] \begin{bmatrix} \frac{\partial T}{\partial \omega_\xi} \\ \frac{\partial T}{\partial \omega_\eta} \\ \frac{\partial T}{\partial \omega_\zeta} \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{b} \end{bmatrix} J\omega$$

Computing Lagrange equation of spatial rotation

partial derivative

$$\frac{\partial T}{\partial \dot{\mathbf{a}}} = [\mathbf{0} \quad \mathbf{0} \quad \mathbf{b}] J \boldsymbol{\omega}$$

time derivative:

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{\mathbf{a}}} = [\mathbf{0} \quad \mathbf{0} \quad \mathbf{b}] J \dot{\boldsymbol{\omega}} + [\mathbf{0} \quad \mathbf{0} \quad \dot{\mathbf{b}}] J \boldsymbol{\omega}$$

Computing Lagrange equation of spatial rotation

geometric constraints

$$R_1 = \mathbf{a}^T \mathbf{a} - 1 = 0, \quad R_2 = \mathbf{b}^T \mathbf{b} - 1 = 0, \quad R_3 = \mathbf{c}^T \mathbf{c} - 1 = 0,$$
$$Q_1 = \mathbf{b}^T \mathbf{c} = 0, \quad Q_2 = \mathbf{c}^T \mathbf{a} = 0, \quad Q_3 = \mathbf{a}^T \mathbf{b} = 0$$

partial derivatives

$$\frac{\partial R_1}{\partial \mathbf{a}} = 2\mathbf{a}, \quad \frac{\partial R_2}{\partial \mathbf{a}} = \mathbf{0}, \quad \frac{\partial R_3}{\partial \mathbf{a}} = \mathbf{0}$$
$$\frac{\partial Q_1}{\partial \mathbf{a}} = \mathbf{0}, \quad \frac{\partial Q_2}{\partial \mathbf{a}} = \mathbf{c}, \quad \frac{\partial Q_3}{\partial \mathbf{a}} = \mathbf{b}$$

Computing Lagrange equation of spatial rotation

contributions to Lagrange equation of motion w.r.t. \mathbf{a} :

$$\begin{aligned}\frac{\partial L}{\partial \mathbf{a}} &= \frac{\partial T}{\partial \mathbf{a}} + \lambda_1 \frac{\partial R_1}{\partial \mathbf{a}} + \cdots + \mu_3 \frac{\partial Q_3}{\partial \mathbf{a}} \\ &= \begin{bmatrix} \mathbf{0} & \dot{\mathbf{c}} & \mathbf{0} \end{bmatrix} J\boldsymbol{\omega} + 2\lambda_1 \mathbf{a} + \mu_2 \mathbf{c} + \mu_3 \mathbf{b} \\ \frac{d}{dt} \frac{\partial L}{\partial \dot{\mathbf{a}}} &= \frac{d}{dt} \frac{\partial T}{\partial \dot{\mathbf{a}}} \\ &= \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{b} \end{bmatrix} J\dot{\boldsymbol{\omega}} + \begin{bmatrix} \mathbf{0} & \mathbf{0} & \dot{\mathbf{b}} \end{bmatrix} J\boldsymbol{\omega}\end{aligned}$$

Lagrange equation of motion w.r.t. \mathbf{a} :

$$\begin{aligned}\begin{bmatrix} \mathbf{0} & \dot{\mathbf{c}} & \mathbf{0} \end{bmatrix} J\boldsymbol{\omega} - \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{b} \end{bmatrix} J\dot{\boldsymbol{\omega}} - \begin{bmatrix} \mathbf{0} & \mathbf{0} & \dot{\mathbf{b}} \end{bmatrix} J\boldsymbol{\omega} \\ + 2\lambda_1 \mathbf{a} + \mu_2 \mathbf{c} + \mu_3 \mathbf{b} = \mathbf{0}\end{aligned}$$

Computing Lagrange equation of spatial rotation

Lagrange equation of motion w.r.t. \mathbf{a} , \mathbf{b} , \mathbf{c} :

$$\begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{b} \end{bmatrix} J\dot{\boldsymbol{\omega}} + \begin{bmatrix} \mathbf{0} & -\dot{\mathbf{c}} & \dot{\mathbf{b}} \end{bmatrix} J\boldsymbol{\omega} - 2\lambda_1\mathbf{a} - \mu_2\mathbf{c} - \mu_3\mathbf{b} = \mathbf{0}$$

$$\begin{bmatrix} \mathbf{c} & \mathbf{0} & \mathbf{0} \end{bmatrix} J\dot{\boldsymbol{\omega}} + \begin{bmatrix} \dot{\mathbf{c}} & \mathbf{0} & -\dot{\mathbf{a}} \end{bmatrix} J\boldsymbol{\omega} - 2\lambda_2\mathbf{b} - \mu_3\mathbf{a} - \mu_1\mathbf{c} = \mathbf{0}$$

$$\begin{bmatrix} \mathbf{0} & \mathbf{a} & \mathbf{0} \end{bmatrix} J\dot{\boldsymbol{\omega}} + \begin{bmatrix} -\dot{\mathbf{b}} & \dot{\mathbf{a}} & \mathbf{0} \end{bmatrix} J\boldsymbol{\omega} - 2\lambda_3\mathbf{c} - \mu_1\mathbf{b} - \mu_2\mathbf{a} = \mathbf{0}$$

\mathbf{c}^T (2nd eq.) (note: $\mathbf{c}^T\mathbf{c} = 1$, $\mathbf{c}^T\dot{\mathbf{c}} = 0$, and $-\mathbf{c}^T\dot{\mathbf{a}} = \mathbf{a}^T\dot{\mathbf{c}} = \omega_\eta$)

$$\begin{bmatrix} 1 & 0 & 0 \end{bmatrix} J\dot{\boldsymbol{\omega}} + \begin{bmatrix} 0 & 0 & \omega_\eta \end{bmatrix} J\boldsymbol{\omega} - \mu_1 = 0$$

\mathbf{b}^T (3rd eq.) (note: $\mathbf{b}^T\mathbf{a} = 0$, $\mathbf{b}^T\dot{\mathbf{b}} = 0$, and $\mathbf{b}^T\dot{\mathbf{a}} = \omega_\zeta$)

$$\begin{bmatrix} 0 & 0 & 0 \end{bmatrix} J\dot{\boldsymbol{\omega}} + \begin{bmatrix} 0 & \omega_\zeta & 0 \end{bmatrix} J\boldsymbol{\omega} - \mu_1 = 0$$

\mathbf{c}^T (2nd eq.) - \mathbf{b}^T (3rd eq.)

$$\begin{bmatrix} 1 & 0 & 0 \end{bmatrix} J\dot{\boldsymbol{\omega}} + \begin{bmatrix} 0 & -\omega_\zeta & \omega_\eta \end{bmatrix} J\boldsymbol{\omega} = 0$$

Computing Lagrange equation of spatial rotation

$$\mathbf{c}^T(\text{2nd eq.}) - \mathbf{b}^T(\text{3rd eq.})$$

$$\begin{bmatrix} 1 & 0 & 0 \end{bmatrix} J\dot{\boldsymbol{\omega}} + \begin{bmatrix} 0 & -\omega_\zeta & \omega_\eta \end{bmatrix} J\boldsymbol{\omega} = 0$$

$$\mathbf{a}^T(\text{3rd eq.}) - \mathbf{c}^T(\text{1st eq.})$$

$$\begin{bmatrix} 0 & 1 & 0 \end{bmatrix} J\dot{\boldsymbol{\omega}} + \begin{bmatrix} \omega_\zeta & 0 & -\omega_\xi \end{bmatrix} J\boldsymbol{\omega} = 0$$

$$\mathbf{b}^T(\text{1st eq.}) - \mathbf{a}^T(\text{2nd eq.})$$

$$\begin{bmatrix} 0 & 0 & 1 \end{bmatrix} J\dot{\boldsymbol{\omega}} + \begin{bmatrix} -\omega_\eta & \omega_\xi & 0 \end{bmatrix} J\boldsymbol{\omega} = 0$$

Euler's equation of rotation

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} J\dot{\omega} + \begin{bmatrix} 0 & -\omega_{\zeta} & \omega_{\eta} \\ \omega_{\zeta} & 0 & -\omega_{\xi} \\ -\omega_{\eta} & \omega_{\xi} & 0 \end{bmatrix} J\omega = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$J\dot{\omega} + [\omega \times] J\omega = \mathbf{0}$$

Euler's equation of rotation

$$J\dot{\omega} + \omega \times J\omega = \mathbf{0}$$

Dynamic equations describing spacial rotation

12 state variables

$$\boldsymbol{\omega} = \begin{bmatrix} \omega_\xi \\ \omega_\eta \\ \omega_\zeta \end{bmatrix}, \quad \mathbf{a} = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} c_x \\ c_y \\ c_z \end{bmatrix}$$

12 equations (6 differential eqs. and 6 algebraic eqs.)

$$J\dot{\boldsymbol{\omega}} = -\boldsymbol{\omega} \times J\boldsymbol{\omega},$$

$$\mathbf{c}^T \dot{\mathbf{b}} = \omega_\xi, \quad \mathbf{a}^T \dot{\mathbf{c}} = \omega_\eta, \quad \mathbf{b}^T \dot{\mathbf{a}} = \omega_\zeta,$$

$$\mathbf{a}^T \mathbf{a} = 1, \quad \mathbf{b}^T \mathbf{b} = 1, \quad \mathbf{c}^T \mathbf{c} = 1,$$

$$\mathbf{a}^T \mathbf{b} = 0, \quad \mathbf{b}^T \mathbf{c} = 0, \quad \mathbf{c}^T \mathbf{a} = 0$$

Lagrange equation of forced spatial rotation

external force \mathbf{f} is applied to point $P(\xi, \eta, \zeta)$:

$$W = \mathbf{f}^T R\xi \quad (R\xi \text{ denotes displacement vector of point } P)$$

$$\frac{\partial W}{\partial a_x} = \mathbf{f}^T \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xi = \mathbf{f}^T \begin{bmatrix} \xi \\ 0 \\ 0 \end{bmatrix} = [\xi \ 0 \ 0] \mathbf{f}$$

$$\frac{\partial W}{\partial a_y} = \mathbf{f}^T \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xi = \mathbf{f}^T \begin{bmatrix} 0 \\ \xi \\ 0 \end{bmatrix} = [0 \ \xi \ 0] \mathbf{f}$$

$$\frac{\partial W}{\partial a_z} = \mathbf{f}^T \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \xi = \mathbf{f}^T \begin{bmatrix} 0 \\ 0 \\ \xi \end{bmatrix} = [0 \ 0 \ \xi] \mathbf{f}$$

$$\implies \frac{\partial W}{\partial \mathbf{a}} = \xi \mathbf{f}$$

Lagrange equation of forced spatial rotation

partial derivatives of W w.r.t. \mathbf{a} , \mathbf{b} , and \mathbf{c} :

$$\frac{\partial W}{\partial \mathbf{a}} = \xi \mathbf{f}, \quad \frac{\partial W}{\partial \mathbf{b}} = \eta \mathbf{f}, \quad \frac{\partial W}{\partial \mathbf{c}} = \zeta \mathbf{f}$$

Lagrangian

$$L = T + \lambda_1 R_1 + \lambda_2 R_2 + \lambda_3 R_3 + \mu_1 Q_1 + \mu_2 Q_2 + \mu_3 Q_3 + W$$

Lagrange equation of motion w.r.t. \mathbf{a} , \mathbf{b} , and \mathbf{c} :

$$\begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{b} \end{bmatrix} J\dot{\omega} + \begin{bmatrix} \mathbf{0} & -\dot{\mathbf{c}} & \dot{\mathbf{b}} \end{bmatrix} J\omega - 2\lambda_1 \mathbf{a} - \mu_2 \mathbf{c} - \mu_3 \mathbf{b} = \xi \mathbf{f}$$

$$\begin{bmatrix} \mathbf{c} & \mathbf{0} & \mathbf{0} \end{bmatrix} J\dot{\omega} + \begin{bmatrix} \dot{\mathbf{c}} & \mathbf{0} & -\dot{\mathbf{a}} \end{bmatrix} J\omega - 2\lambda_2 \mathbf{b} - \mu_3 \mathbf{a} - \mu_1 \mathbf{c} = \eta \mathbf{f}$$

$$\begin{bmatrix} \mathbf{0} & \mathbf{a} & \mathbf{0} \end{bmatrix} J\dot{\omega} + \begin{bmatrix} -\dot{\mathbf{b}} & \dot{\mathbf{a}} & \mathbf{0} \end{bmatrix} J\omega - 2\lambda_3 \mathbf{c} - \mu_1 \mathbf{b} - \mu_2 \mathbf{a} = \zeta \mathbf{f}$$

Lagrange equation of forced spatial rotation

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} J\dot{\boldsymbol{\omega}} + \begin{bmatrix} 0 & -\omega_{\zeta} & \omega_{\eta} \\ \omega_{\zeta} & 0 & -\omega_{\xi} \\ -\omega_{\eta} & \omega_{\xi} & 0 \end{bmatrix} J\boldsymbol{\omega} = \begin{bmatrix} \eta \mathbf{c}^T \mathbf{f} - \zeta \mathbf{b}^T \mathbf{f} \\ \zeta \mathbf{a}^T \mathbf{f} - \xi \mathbf{c}^T \mathbf{f} \\ \xi \mathbf{b}^T \mathbf{f} - \eta \mathbf{a}^T \mathbf{f} \end{bmatrix}$$

external torque:

$$\boldsymbol{\tau} = \begin{bmatrix} \tau_{\xi} \\ \tau_{\eta} \\ \tau_{\zeta} \end{bmatrix} \triangleq \begin{bmatrix} \eta \mathbf{c}^T \mathbf{f} - \zeta \mathbf{b}^T \mathbf{f} \\ \zeta \mathbf{a}^T \mathbf{f} - \xi \mathbf{c}^T \mathbf{f} \\ \xi \mathbf{b}^T \mathbf{f} - \eta \mathbf{a}^T \mathbf{f} \end{bmatrix} = \begin{bmatrix} \eta f_{\zeta} - \zeta f_{\eta} \\ \zeta f_{\xi} - \xi f_{\zeta} \\ \xi f_{\eta} - \eta f_{\xi} \end{bmatrix} = \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix} \times \begin{bmatrix} f_{\xi} \\ f_{\eta} \\ f_{\zeta} \end{bmatrix}$$

Euler's equation of rotation with external torque

$$J\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times J\boldsymbol{\omega} = \boldsymbol{\tau}$$

Rotation matrix using quaternion

Definition of quaternion

$$\mathbf{q} = \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

where

$$\mathbf{q}^T \mathbf{q} = q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1$$

Rotation matrix using quaternion

$$R(\mathbf{q}) = \begin{bmatrix} 2(q_0^2 + q_1^2) - 1 & 2(q_1 q_2 - q_0 q_3) & 2(q_1 q_3 + q_0 q_2) \\ 2(q_1 q_2 + q_0 q_3) & 2(q_0^2 + q_2^2) - 1 & 2(q_2 q_3 - q_0 q_1) \\ 2(q_1 q_3 - q_0 q_2) & 2(q_2 q_3 + q_0 q_1) & 2(q_0^2 + q_3^2) - 1 \end{bmatrix}$$

Describing column vectors of rotation matrix

column vectors \mathbf{a} , \mathbf{b} , and \mathbf{c} of rotation matrix $R(\mathbf{q})$:

$$\mathbf{a} = \begin{bmatrix} 2(q_0^2 + q_1^2) - 1 \\ 2(q_1q_2 + q_0q_3) \\ 2(q_1q_3 - q_0q_2) \end{bmatrix} = \begin{bmatrix} q_0 & q_1 & -q_2 & -q_3 \\ q_3 & q_2 & q_1 & q_0 \\ -q_2 & q_3 & -q_0 & q_1 \end{bmatrix} \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix} \triangleq \mathbf{A}\mathbf{q}$$

$$\mathbf{b} = \begin{bmatrix} 2(q_1q_2 - q_0q_3) \\ 2(q_0^2 + q_2^2) - 1 \\ 2(q_2q_3 + q_0q_1) \end{bmatrix} = \begin{bmatrix} -q_3 & q_2 & q_1 & -q_0 \\ q_0 & -q_1 & q_2 & -q_3 \\ q_1 & q_0 & q_3 & q_2 \end{bmatrix} \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix} \triangleq \mathbf{B}\mathbf{q}$$

$$\mathbf{c} = \begin{bmatrix} 2(q_1q_3 + q_0q_2) \\ 2(q_2q_3 - q_0q_1) \\ 2(q_0^2 + q_3^2) - 1 \end{bmatrix} = \begin{bmatrix} q_2 & q_3 & q_0 & q_1 \\ -q_1 & -q_0 & q_3 & q_2 \\ q_0 & -q_1 & -q_2 & q_3 \end{bmatrix} \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix} \triangleq \mathbf{C}\mathbf{q}$$

Describing column vectors of rotation matrix

$$\begin{aligned} A^T A &= \begin{bmatrix} 1 - q_1^2 & q_0 q_1 & q_1 q_3 & -q_1 q_2 \\ q_0 q_1 & 1 - q_0^2 & -q_0 q_3 & q_0 q_2 \\ q_1 q_3 & -q_0 q_3 & 1 - q_3^2 & q_2 q_3 \\ -q_1 q_2 & q_0 q_2 & q_2 q_3 & 1 - q_2^2 \end{bmatrix}, & (A^T A) \mathbf{q} &= \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix} \\ B^T B &= \begin{bmatrix} 1 - q_2^2 & -q_2 q_3 & q_0 q_2 & q_1 q_2 \\ -q_2 q_3 & 1 - q_3^2 & q_0 q_3 & q_1 q_3 \\ q_0 q_2 & q_0 q_3 & 1 - q_0^2 & -q_0 q_1 \\ q_1 q_2 & q_1 q_3 & -q_0 q_1 & 1 - q_1^2 \end{bmatrix}, & (B^T B) \mathbf{q} &= \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix} \\ C^T C &= \begin{bmatrix} 1 - q_3^2 & q_2 q_3 & -q_1 q_3 & q_0 q_3 \\ q_2 q_3 & 1 - q_2^2 & q_1 q_2 & -q_0 q_2 \\ -q_1 q_3 & q_1 q_2 & 1 - q_1^2 & q_0 q_1 \\ q_0 q_3 & -q_0 q_2 & q_0 q_1 & 1 - q_0^2 \end{bmatrix}, & (C^T C) \mathbf{q} &= \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix} \end{aligned}$$

Describing column vectors of rotation matrix

$$A^T B = \begin{bmatrix} -q_1 q_2 & -q_1 q_3 & q_0 q_1 & q_1^2 - 1 \\ q_0 q_2 & q_0 q_3 & 1 - q_0^2 & -q_0 q_1 \\ q_2 q_3 & q_3^2 - 1 & -q_0 q_3 & -q_1 q_3 \\ 1 - q_2^2 & -q_2 q_3 & q_0 q_2 & q_1 q_2 \end{bmatrix}, \quad (A^T B) \mathbf{q} = \begin{bmatrix} -q_3 \\ q_2 \\ -q_1 \\ q_0 \end{bmatrix}$$
$$B^T C = \begin{bmatrix} -q_2 q_3 & q_2^2 - 1 & -q_1 q_2 & q_0 q_2 \\ 1 - q_3^2 & q_2 q_3 & -q_1 q_3 & q_0 q_3 \\ q_0 q_3 & -q_0 q_2 & q_0 q_1 & 1 - q_0^2 \\ q_1 q_3 & -q_1 q_2 & q_1^2 - 1 & -q_0 q_1 \end{bmatrix}, \quad (B^T C) \mathbf{q} = \begin{bmatrix} -q_1 \\ q_0 \\ q_3 \\ -q_2 \end{bmatrix}$$
$$C^T A = \begin{bmatrix} -q_1 q_3 & q_0 q_3 & q_3^2 - 1 & -q_2 q_3 \\ q_1 q_2 & -q_0 q_2 & -q_2 q_3 & q_2^2 - 1 \\ 1 - q_1^2 & q_0 q_1 & q_1 q_3 & -q_1 q_2 \\ q_0 q_1 & 1 - q_0^2 & -q_0 q_3 & q_0 q_2 \end{bmatrix}, \quad (C^T A) \mathbf{q} = \begin{bmatrix} -q_2 \\ -q_3 \\ q_0 \\ q_1 \end{bmatrix}$$

Describing column vectors of rotation matrix

$$\mathbf{a}^T \mathbf{a} = (\mathbf{A}\mathbf{q})^T (\mathbf{A}\mathbf{q}) = \mathbf{q}^T \mathbf{A}^T \mathbf{A} \mathbf{q} = \mathbf{q}^T \mathbf{q} = 1$$

$$\mathbf{b}^T \mathbf{b} = (\mathbf{B}\mathbf{q})^T (\mathbf{B}\mathbf{q}) = \mathbf{q}^T \mathbf{B}^T \mathbf{B} \mathbf{q} = \mathbf{q}^T \mathbf{q} = 1$$

...

$$\mathbf{a}^T \mathbf{b} = (\mathbf{A}\mathbf{q})^T (\mathbf{B}\mathbf{q}) = \mathbf{q}^T \mathbf{A}^T \mathbf{B} \mathbf{q} = \begin{bmatrix} q_0 & q_1 & q_2 & q_3 \end{bmatrix} \begin{bmatrix} -q_3 \\ q_2 \\ -q_1 \\ q_0 \end{bmatrix} = 0$$

$$\mathbf{b}^T \mathbf{c} = (\mathbf{B}\mathbf{q})^T (\mathbf{C}\mathbf{q}) = \mathbf{q}^T \mathbf{B}^T \mathbf{C} \mathbf{q} = \begin{bmatrix} q_0 & q_1 & q_2 & q_3 \end{bmatrix} \begin{bmatrix} -q_1 \\ q_0 \\ q_3 \\ -q_2 \end{bmatrix} = 0$$

...

Describing column vectors of rotation matrix

Assume that \mathbf{q} depends on time.
time derivative of quaternion:

$$\dot{\mathbf{q}} = \begin{bmatrix} \dot{q}_0 \\ \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix}$$

Note that matrices $R(\mathbf{q})$, A , B , and C depend on time.

Describing angular velocity vector using quaternion

time derivatives of column vectors \mathbf{a} , \mathbf{b} , and \mathbf{c} :

$$\dot{\mathbf{a}} = \dot{A}\mathbf{q} + A\dot{\mathbf{q}} = A\dot{\mathbf{q}} + A\dot{\mathbf{q}} = 2A\dot{\mathbf{q}}$$

$$\dot{\mathbf{b}} = \dot{B}\mathbf{q} + B\dot{\mathbf{q}} = B\dot{\mathbf{q}} + B\dot{\mathbf{q}} = 2B\dot{\mathbf{q}}$$

$$\dot{\mathbf{c}} = \dot{C}\mathbf{q} + C\dot{\mathbf{q}} = C\dot{\mathbf{q}} + C\dot{\mathbf{q}} = 2C\dot{\mathbf{q}}$$

angular velocities:

$$\omega_{\xi} = \mathbf{c}^T \dot{\mathbf{b}} = \mathbf{q}^T C^T 2B\dot{\mathbf{q}} = 2(B^T C\mathbf{q})^T \dot{\mathbf{q}}$$

$$\omega_{\eta} = \mathbf{a}^T \dot{\mathbf{c}} = \mathbf{q}^T A^T 2C\dot{\mathbf{q}} = 2(C^T A\mathbf{q})^T \dot{\mathbf{q}}$$

$$\omega_{\zeta} = \mathbf{b}^T \dot{\mathbf{a}} = \mathbf{q}^T B^T 2A\dot{\mathbf{q}} = 2(A^T B\mathbf{q})^T \dot{\mathbf{q}}$$

Describing angular velocity vector using quaternion

angular velocities:

$$\omega_{\xi} = 2 \begin{bmatrix} -q_1 & q_0 & q_3 & -q_2 \end{bmatrix} \dot{\mathbf{q}}$$

$$\omega_{\eta} = 2 \begin{bmatrix} -q_2 & -q_3 & q_0 & q_1 \end{bmatrix} \dot{\mathbf{q}}$$

$$\omega_{\zeta} = 2 \begin{bmatrix} -q_3 & q_2 & -q_1 & q_0 \end{bmatrix} \dot{\mathbf{q}}$$

angular velocity vector

$$\boldsymbol{\omega} = 2H\dot{\mathbf{q}}$$

where

$$H \triangleq \begin{bmatrix} -q_1 & q_0 & q_3 & -q_2 \\ -q_2 & -q_3 & q_0 & q_1 \\ -q_3 & q_2 & -q_1 & q_0 \end{bmatrix}$$

Describing Lagrangian using quaternion

Lagrangian

$$L = T + \lambda_{\text{quat}} Q_{\text{quat}}$$

kinetic energy of a rotating rigid body:

$$T = \frac{1}{2} \boldsymbol{\omega}^T J \boldsymbol{\omega} = \frac{1}{2} (2H\dot{\mathbf{q}})^T J (2H\dot{\mathbf{q}}) = 2\dot{\mathbf{q}}^T (H^T J H) \dot{\mathbf{q}}$$

or

$$T = \frac{1}{2} \boldsymbol{\omega}^T J \boldsymbol{\omega} = \frac{1}{2} (-2\dot{H}\mathbf{q})^T J (-2\dot{H}\mathbf{q}) = 2\mathbf{q}^T (\dot{H}^T J \dot{H}) \mathbf{q}$$

constraint on quaternion:

$$Q_{\text{quat}} = \mathbf{q}^T \mathbf{q} - 1$$

λ_{quat} : Lagrange multiplier

Computing Lagrange equation

partial derivatives of T w.r.t. $\dot{\mathbf{q}}$ and \mathbf{q} :

$$\frac{\partial T}{\partial \dot{\mathbf{q}}} = 4(H^T JH)\dot{\mathbf{q}}, \quad \frac{\partial T}{\partial \mathbf{q}} = 4(\dot{H}^T J\dot{H})\mathbf{q}$$

since $\dot{H}\mathbf{q} = -H\dot{\mathbf{q}}$

$$\frac{\partial T}{\partial \mathbf{q}} = 4(\dot{H}^T J)\dot{H}\mathbf{q} = 4(\dot{H}^T J)(-H\dot{\mathbf{q}}) = -4(\dot{H}^T JH)\dot{\mathbf{q}}$$

time derivative of $\partial T / \partial \dot{\mathbf{q}}$:

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{\mathbf{q}}} = 4(H^T JH)\ddot{\mathbf{q}} + 4(\dot{H}^T JH + H^T J\dot{H})\dot{\mathbf{q}}$$

since $\dot{H}\dot{\mathbf{q}} = \mathbf{0}$ yields $(H^T J\dot{H})\dot{\mathbf{q}} = (H^T J)\dot{H}\dot{\mathbf{q}} = \mathbf{0}$

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{\mathbf{q}}} = 4(H^T JH)\ddot{\mathbf{q}} + 4(\dot{H}^T JH)\dot{\mathbf{q}}$$

Computing Lagrange equation

contribution of kinetic energy T to Lagrange equation:

$$\begin{aligned}\frac{\partial T}{\partial \mathbf{q}} - \frac{d}{dt} \frac{\partial T}{\partial \dot{\mathbf{q}}} &= -4(\dot{H}^T JH)\dot{\mathbf{q}} - \left\{ 4(H^T JH)\ddot{\mathbf{q}} + 4(\dot{H}^T JH)\dot{\mathbf{q}} \right\} \\ &= -4(H^T JH)\ddot{\mathbf{q}} - 8(\dot{H}^T JH)\dot{\mathbf{q}}\end{aligned}$$

contribution of constraint Q_{quat} to Lagrange equation:

$$\frac{\partial Q_{\text{quat}}}{\partial \mathbf{q}} - \frac{d}{dt} \frac{\partial Q_{\text{quat}}}{\partial \dot{\mathbf{q}}} = 2\mathbf{q}$$

Lagrange equation of motion

$$-4(H^T JH)\ddot{\mathbf{q}} - 8(\dot{H}^T JH)\dot{\mathbf{q}} + 2\lambda_{\text{quat}}\mathbf{q} = \mathbf{0}_4$$

Dynamic equations describing spatial rotation

multiply H to Lagrange equation of motion:

$$-4H(H^T JH)\ddot{\mathbf{q}} - 8H(\dot{H}^T JH)\dot{\mathbf{q}} + 2\lambda_{\text{quat}} H\mathbf{q} = \mathbf{0}_3$$

since $HH^T = I_{3 \times 3}$ and $H\mathbf{q} = \mathbf{0}$:

$$JH\ddot{\mathbf{q}} + 2(H\dot{H}^T JH)\dot{\mathbf{q}} = \mathbf{0}_3$$

matrix J is regular:

$$H\ddot{\mathbf{q}} = -2J^{-1}(H\dot{H}^T JH)\dot{\mathbf{q}}$$

since $\dot{Q}_{\text{quat}} = 2\mathbf{q}^T \dot{\mathbf{q}}$ and $\ddot{Q}_{\text{quat}} = 2\mathbf{q}^T \ddot{\mathbf{q}} + 2\dot{\mathbf{q}}^T \dot{\mathbf{q}}$, equation for stabilizing constraint $Q_{\text{quat}} = 0$ is given by

$$-\mathbf{q}^T \ddot{\mathbf{q}} = r(\mathbf{q}, \dot{\mathbf{q}})$$

where

$$r(\mathbf{q}, \dot{\mathbf{q}}) \triangleq \dot{\mathbf{q}}^T \dot{\mathbf{q}} + 2\nu \mathbf{q}^T \dot{\mathbf{q}} + \frac{1}{2} \nu^2 (\mathbf{q}^T \mathbf{q} - 1) \quad (\nu: \text{positive constant})$$

Dynamic equations describing spatial rotation

differential equations:

$$\begin{aligned} -\mathbf{q}^T \ddot{\mathbf{q}} &= r(\mathbf{q}, \dot{\mathbf{q}}) \\ H\ddot{\mathbf{q}} &= -2J^{-1}(H\dot{H}^T JH)\dot{\mathbf{q}} \end{aligned}$$

combining the two equations:

$$\begin{aligned} \begin{bmatrix} -\mathbf{q}^T \\ H \end{bmatrix} \ddot{\mathbf{q}} &= \begin{bmatrix} r(\mathbf{q}, \dot{\mathbf{q}}) \\ -2J^{-1}(H\dot{H}^T JH)\dot{\mathbf{q}} \end{bmatrix} \\ \hat{H}\ddot{\mathbf{q}} &= \begin{bmatrix} r(\mathbf{q}, \dot{\mathbf{q}}) \\ -2J^{-1}(H\dot{H}^T JH)\dot{\mathbf{q}} \end{bmatrix} \end{aligned}$$

where

$$\hat{H} \triangleq \begin{bmatrix} -\mathbf{q}^T \\ H \end{bmatrix}$$

Dynamic equations describing spatial rotation

$$\begin{aligned}\hat{H}\hat{H}^T &= \begin{bmatrix} -\mathbf{q}^T \\ H \end{bmatrix} \begin{bmatrix} -\mathbf{q} & H^T \end{bmatrix} = \begin{bmatrix} \mathbf{q}^T\mathbf{q} & -(H\mathbf{q})^T \\ -H\mathbf{q} & HH^T \end{bmatrix} \\ &= \begin{bmatrix} 1 & \mathbf{0}_3^T \\ \mathbf{0}_3 & I_{3\times 3} \end{bmatrix} = I_{4\times 4}\end{aligned}$$

matrix \hat{H} is orthogonal:

$$\begin{aligned}\ddot{\mathbf{q}} &= \hat{H}^T \begin{bmatrix} r(\mathbf{q}, \dot{\mathbf{q}}) \\ -2J^{-1}(H\dot{H}^T JH)\dot{\mathbf{q}} \end{bmatrix} \\ &= \begin{bmatrix} -\mathbf{q} & H^T \end{bmatrix} \begin{bmatrix} r(\mathbf{q}, \dot{\mathbf{q}}) \\ -2J^{-1}(H\dot{H}^T JH)\dot{\mathbf{q}} \end{bmatrix} \\ &= -r(\mathbf{q}, \dot{\mathbf{q}})\mathbf{q} - 2H^T J^{-1}(H\dot{H}^T JH)\dot{\mathbf{q}}\end{aligned}$$

Dynamic equations describing spatial rotation

$$\begin{aligned} 2H\dot{H}^T &= 2 \begin{bmatrix} -q_1 & q_0 & q_3 & -q_2 \\ -q_2 & -q_3 & q_0 & q_1 \\ -q_3 & q_2 & -q_1 & q_0 \end{bmatrix} \begin{bmatrix} -\dot{q}_1 & -\dot{q}_2 & -\dot{q}_3 \\ \dot{q}_0 & -\dot{q}_3 & \dot{q}_2 \\ \dot{q}_3 & \dot{q}_0 & -\dot{q}_1 \\ -\dot{q}_2 & \dot{q}_1 & \dot{q}_0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & -\omega_\zeta & \omega_\eta \\ \omega_\zeta & 0 & -\omega_\xi \\ -\omega_\eta & \omega_\xi & 0 \end{bmatrix} = [\boldsymbol{\omega} \times] = [(2H\dot{\mathbf{q}}) \times] \end{aligned}$$

matrix $H\dot{H}^T$ represents outer product with $H\dot{\mathbf{q}}$

$$\ddot{\mathbf{q}} = -r(\mathbf{q}, \dot{\mathbf{q}})\mathbf{q} - 2H^T J^{-1} (H\dot{H}^T JH)\dot{\mathbf{q}}$$

\Downarrow

$$\ddot{\mathbf{q}} = -r(\mathbf{q}, \dot{\mathbf{q}})\mathbf{q} - 2H^T J^{-1} \{ (H\dot{\mathbf{q}}) \times (JH\dot{\mathbf{q}}) \}$$

Dynamic equations describing spatial rotation

Equation of rotation

4 generalized coordinates:

$$\mathbf{q} = \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

4 differential equations:

$$\ddot{\mathbf{q}} = -r(\mathbf{q}, \dot{\mathbf{q}})\mathbf{q} - 2H^T J^{-1} \{ (H\dot{\mathbf{q}}) \times (JH\dot{\mathbf{q}}) \}$$

Dynamic equations describing spatial rotation

Canonical form for numerical computation

$$\dot{\mathbf{q}} = \mathbf{p}$$

$$\dot{\mathbf{p}} = -r(\mathbf{q}, \mathbf{p})\mathbf{q} - 2H^T J^{-1} \{ (H\mathbf{p}) \times (JH\mathbf{p}) \}$$

state variable vector

$$\mathbf{s} = \begin{bmatrix} \mathbf{q} \\ \mathbf{p} \end{bmatrix}$$

Comparison summary

- a set of rotation matrix elements
 - ▶ 12 state variables (9 for orientation and 3 for angular velocity)
 - ▶ 12 equations (6 differential + 6 algebraic)
- **quaternion**
 - ▶ 4 generalized coordinates (8 state variables)
 - ▶ quadratic expressions, no trigonometric functions
 - ▶ no singularity, implying no gimbal lock or no instability
- a set of Euler angles
 - ▶ 3 generalized coordinates (6 state variables)
 - ▶ trigonometric functions
 - ▶ singularity, causing gimbal lock or instability

Lagrange equation of forced spatial rotation

$\boldsymbol{\tau}_{\text{quat}} = [\tau_0, \tau_1, \tau_2, \tau_3]^T$ a set of generalized torques corresponding to quaternion $\mathbf{q} = [q_0, q_1, q_2, q_3]^T$

$$W = \boldsymbol{\tau}_{\text{quat}}^T \mathbf{q} = \tau_0 q_0 + \tau_1 q_1 + \tau_2 q_2 + \tau_3 q_3$$

contribution of work W to Lagrange equation:

$$\frac{\partial W}{\partial \mathbf{q}} - \frac{d}{dt} \frac{\partial W}{\partial \dot{\mathbf{q}}} = \boldsymbol{\tau}_{\text{quat}}$$

Lagrange equation of motion:

$$-4(H^T JH)\ddot{\mathbf{q}} - 8(\dot{H}^T JH)\dot{\mathbf{q}} + 2\lambda_{\text{quat}} \mathbf{q} + \boldsymbol{\tau}_{\text{quat}} = \mathbf{0}_4$$

$$\ddot{\mathbf{q}} = -r(\mathbf{q}, \dot{\mathbf{q}})\mathbf{q} - 2H^T J^{-1} \left\{ (H\dot{\mathbf{q}}) \times (JH\dot{\mathbf{q}}) - \frac{1}{8} H\boldsymbol{\tau}_{\text{quat}} \right\}$$

Lagrange equation of forced spatial rotation

Principle of virtual works

$$\boldsymbol{\omega} = 2H\dot{\mathbf{q}} \quad \Rightarrow \quad \boldsymbol{\tau}_{\text{quat}} = (2H)^T \boldsymbol{\tau}$$

since $H\boldsymbol{\tau}_{\text{quat}} = 2HH^T \boldsymbol{\tau} = 2\boldsymbol{\tau}$, equation of rotation turns into:

$$\ddot{\mathbf{q}} = -r(\mathbf{q}, \dot{\mathbf{q}})\mathbf{q} - 2H^T J^{-1} \left\{ (H\dot{\mathbf{q}}) \times (JH\dot{\mathbf{q}}) - \frac{1}{4}\boldsymbol{\tau} \right\}$$

Lagrange equation of forced spatial rotation

Canonical form for numerical computation

$$\dot{\mathbf{q}} = \mathbf{p}$$

$$\dot{\mathbf{p}} = -r(\mathbf{q}, \mathbf{p})\mathbf{q} - 2\mathbf{H}^T \mathbf{J}^{-1} \left\{ (\mathbf{H}\mathbf{p}) \times (\mathbf{J}\mathbf{H}\mathbf{p}) - \frac{1}{4}\boldsymbol{\tau} \right\}$$

state variable vector

$$\mathbf{s} = \begin{bmatrix} \mathbf{q} \\ \mathbf{p} \end{bmatrix}$$

Summary

Planar rotation

- described by angle θ and angular velocity ω
- Lagrangian formulation yields equation of planar rotation

Spatial rotation

- described by rotation matrix R under geometric constraints
- Lagrangian approach derives Euler's equation of rotation
- equation of forced rotation

Quaternion

- a set of four variables under one constraint
- differential eq. w.r.t. quaternion describing spacial rotation

Quaternion

Report #4 due date : Dec. 14 (Mon.)

- (1) Show that $R(\mathbf{q})$ is orthogonal.
- (2) Show $\dot{A}\mathbf{q} = A\dot{\mathbf{q}}$, $\dot{B}\mathbf{q} = B\dot{\mathbf{q}}$, and $\dot{C}\mathbf{q} = C\dot{\mathbf{q}}$.
- (3) Show $H\mathbf{q} = \mathbf{0}$.
- (4) Show $\dot{H}\dot{\mathbf{q}} = \mathbf{0}$.
- (5) Show $H\dot{\mathbf{q}} = -\dot{H}\mathbf{q}$ and $\boldsymbol{\omega} = -2\dot{H}\mathbf{q}$.
- (6) Show $HH^T = I_{3 \times 3}$.
- (7) Show $H^T H\dot{\mathbf{q}} = \dot{\mathbf{q}}$ and $\dot{\mathbf{q}} = (1/2)H^T\boldsymbol{\omega}$.

Dynamic Simulation of Rotation

Report #5 due date : Dec. 28 (Mon.)

Let us compute rotation of a rigid cylindrical body of which inertia matrix is given by $J = \text{diag}\{2.00, 100.00, 100.00\}$. The rigid body remains still at $t = 0$. External torque $\tau_\xi = 10.00$ is applied to the rigid body during $t \in [0, 5]$, then external torque $\tau_\eta = 4.00$ is applied to the rigid body during $t \in [5, 10]$. Simulate the rotation of the body to show how quaternions change according to time during $t \in [0, 20]$. Illustrate how the direction of the central axis of the cylindrical body (ζ -axis) changes according to time.

Appendix: Vector Calculus

Let \mathbf{x} and \mathbf{y} are three-dimensional vectors given as

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

Inner product between \mathbf{x} and \mathbf{y} is described as

$$\mathbf{x}^T \mathbf{y} = x_1 y_1 + x_2 y_2 + x_3 y_3$$

Thus, partial derivatives of the inner product with respect to column vectors \mathbf{x} and \mathbf{y} are given as follows:

$$\frac{\partial(\mathbf{x}^T \mathbf{y})}{\partial \mathbf{x}} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \mathbf{y}, \quad \frac{\partial(\mathbf{x}^T \mathbf{y})}{\partial \mathbf{y}} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{x}$$

Since the inner product is a scalar, the above derivatives are three-dimensional column vectors.

Appendix: Vector Calculus

Outer product between \mathbf{x} and \mathbf{y} :

$$\begin{aligned}\mathbf{x} \times \mathbf{y} &= \begin{bmatrix} x_2 y_3 - x_3 y_2 \\ x_3 y_1 - x_1 y_3 \\ x_1 y_2 - x_2 y_1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \\ &\triangleq [\mathbf{x} \times] \mathbf{y}\end{aligned}$$

Note that $[\mathbf{x} \times]$ is a 3×3 skew-symmetric matrix.

$$[\mathbf{x} \times] = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}$$

Appendix: Vector Calculus

Partial derivatives of the outer product with respect to row vectors \mathbf{x}^T and \mathbf{y}^T :

$$\frac{\partial(\mathbf{x} \times \mathbf{y})}{\partial \mathbf{x}^T} = \begin{bmatrix} 0 & y_3 & -y_2 \\ -y_3 & 0 & y_1 \\ y_2 & -y_1 & 0 \end{bmatrix} = [-\mathbf{y} \times],$$
$$\frac{\partial(\mathbf{x} \times \mathbf{y})}{\partial \mathbf{y}^T} = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix} = [\mathbf{x} \times]$$

Since the outer product is a three-dimensional column vector, the above derivatives are 3×3 matrices.

Appendix: Vector Calculus

Let \mathbf{x} be a three-dimensional vector and A be a 3×3 symmetric matrix independent of \mathbf{x} . Quadratic form is described as:

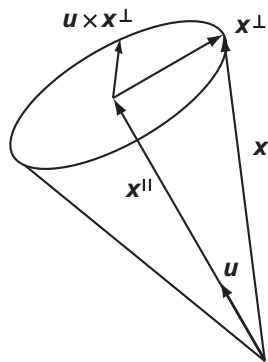
$$\begin{aligned}\mathbf{x}^T A \mathbf{x} &= \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \\ &= a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + 2a_{12}x_1x_2 + 2a_{13}x_1x_3 + 2a_{23}x_2x_3\end{aligned}$$

Partial derivative of the quadratic form with respect to \mathbf{x} is:

$$\begin{aligned}\frac{\partial \mathbf{x}^T A \mathbf{x}}{\partial \mathbf{x}} &= \begin{bmatrix} 2a_{11}x_1 + 2a_{12}x_2 + 2a_{13}x_3 \\ 2a_{22}x_2 + 2a_{12}x_1 + 2a_{23}x_3 \\ 2a_{33}x_3 + 2a_{13}x_1 + 2a_{23}x_2 \end{bmatrix} \\ &= 2A\mathbf{x}\end{aligned}$$

Since the quadratic form is a scalar, the above derivative is a three-dimensional column vector.

Appendix: deriving quaternion description



$\mathbf{u} = [u_x, u_y, u_z]^T$ unit vector
 $R(\mathbf{u}, \alpha)$ rotation around \mathbf{u} by angle α

\mathbf{x} arbitrary vector
decompose \mathbf{x} into two components

$$\mathbf{x} = \mathbf{x}^{\parallel} + \mathbf{x}^{\perp}$$
$$\mathbf{x}^{\parallel} \parallel \mathbf{u}, \quad \mathbf{x}^{\perp} \perp \mathbf{u}$$

\mathbf{x}^{\parallel} is the projection of \mathbf{x} to a line specified by unit vector \mathbf{u} :

$$\mathbf{x}^{\parallel} = (\mathbf{u}^T \mathbf{x}) \mathbf{u} = (\mathbf{u} \mathbf{u}^T) \mathbf{x},$$
$$\mathbf{x}^{\perp} = \mathbf{x} - \mathbf{x}^{\parallel} = (\mathbf{I}_{3 \times 3} - \mathbf{u} \mathbf{u}^T) \mathbf{x}$$

vectors \mathbf{u} , \mathbf{x}^{\perp} , and $\mathbf{u} \times \mathbf{x}^{\perp}$ form a right-handed coordinate system

Appendix: deriving quaternion description

rotation $R(\mathbf{u}, \alpha)$ transforms \mathbf{x}^{\parallel} into itself:

$$R\mathbf{x}^{\parallel} = \mathbf{x}^{\parallel}.$$

rotation $R(\mathbf{u}, \alpha)$ transforms \mathbf{x}^{\perp} into $C_{\alpha}\mathbf{x}^{\perp} + S_{\alpha}\mathbf{u} \times \mathbf{x}^{\perp}$:

$$R\mathbf{x}^{\perp} = C_{\alpha}\mathbf{x}^{\perp} + S_{\alpha}\mathbf{u} \times \mathbf{x}^{\perp}$$

Thus

$$R\mathbf{x} = R(\mathbf{x}^{\parallel} + \mathbf{x}^{\perp}) = \mathbf{x}^{\parallel} + C_{\alpha}\mathbf{x}^{\perp} + S_{\alpha}\mathbf{u} \times \mathbf{x}^{\perp}.$$

since $\mathbf{u} \times \mathbf{x}^{\parallel} = \mathbf{0}$

$$\mathbf{u} \times \mathbf{x}^{\perp} = \mathbf{u} \times (\mathbf{x}^{\parallel} + \mathbf{x}^{\perp}) = \mathbf{u} \times \mathbf{x} = [\mathbf{u} \times] \mathbf{x}$$

$$\begin{aligned} R\mathbf{x} &= (\mathbf{u}\mathbf{u}^T)\mathbf{x} + C_{\alpha}(I_{3 \times 3} - \mathbf{u}\mathbf{u}^T)\mathbf{x} + S_{\alpha}[\mathbf{u} \times] \mathbf{x} \\ &= \{C_{\alpha}I_{3 \times 3} + (1 - C_{\alpha})\mathbf{u}\mathbf{u}^T + S_{\alpha}[\mathbf{u} \times]\} \mathbf{x}. \end{aligned}$$

Appendix: deriving quaternion description

rotation around unit vector \mathbf{u} by angle α :

$$\begin{aligned} R &= C_\alpha I_{3 \times 3} + (1 - C_\alpha) \mathbf{u} \mathbf{u}^T + S_\alpha [\mathbf{u} \times] \\ &= \begin{bmatrix} C_\alpha + \bar{C}_\alpha u_x^2 & \bar{C}_\alpha u_x u_y - S_\alpha u_z & \bar{C}_\alpha u_x u_z + S_\alpha u_y \\ \bar{C}_\alpha u_y u_x + S_\alpha u_z & C_\alpha + \bar{C}_\alpha u_y^2 & \bar{C}_\alpha u_y u_z - S_\alpha u_x \\ \bar{C}_\alpha u_z u_x - S_\alpha u_y & \bar{C}_\alpha u_z u_y + S_\alpha u_x & C_\alpha + \bar{C}_\alpha u_z^2 \end{bmatrix} \end{aligned}$$

where $\bar{C}_\alpha = 1 - C_\alpha$

Define $q_0 = \cos(\alpha/2)$:

$$C_\alpha = 2q_0^2 - 1, \quad \bar{C}_\alpha = 2 \sin^2 \frac{\alpha}{2}, \quad S_\alpha = 2q_0 \sin \frac{\alpha}{2}$$

Define $[q_1, q_2, q_3]^T = \sin(\alpha/2)[u_x, u_y, u_z]^T$:

$$R = \begin{bmatrix} 2(q_0^2 + q_1^2) - 1 & 2(q_1 q_2 - q_0 q_3) & 2(q_1 q_3 + q_0 q_2) \\ 2(q_1 q_2 + q_0 q_3) & 2(q_0^2 + q_2^2) - 1 & 2(q_2 q_3 - q_0 q_1) \\ 2(q_1 q_3 - q_0 q_2) & 2(q_2 q_3 + q_0 q_1) & 2(q_0^2 + q_3^2) - 1 \end{bmatrix}$$

Appendix: algebraic description of quaternion

R : a sequence of rotations P followed by Q

P, Q, R : denoted by quaternions

$\mathbf{p} = [p_0, p_1, p_2, p_3]^T$, $\mathbf{q} = [q_0, q_1, q_2, q_3]^T$, $\mathbf{r} = [r_0, r_1, r_2, r_3]^T$

$$R(\mathbf{r}) = R(\mathbf{p})R(\mathbf{q}),$$

\Updownarrow

$$\begin{bmatrix} r_0 \\ r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} p_0 q_0 - p_1 q_1 - p_2 q_2 - p_3 q_3 \\ p_0 q_1 + p_1 q_0 + p_2 q_3 - p_3 q_2 \\ p_0 q_2 + p_2 q_0 + p_3 q_1 - p_1 q_3 \\ p_0 q_3 + p_3 q_0 + p_1 q_2 - p_2 q_1 \end{bmatrix} \quad (1)$$

Appendix: algebraic description of quaternion

Define numbers of which units are given by 1, i , j , and k .

Four units satisfy:

$$i^2 = j^2 = k^2 = -1,$$

$$ij = k, \quad jk = i, \quad ki = j,$$

$$ji = -k, \quad kj = -i, \quad ik = -j$$

Multiplication among i , j , and k circulates but does not commute.

Numbers p , q , r :

$$p = p_0 + p_1i + p_2j + p_3k,$$

$$q = q_0 + q_1i + q_2j + q_3k,$$

$$r = r_0 + r_1i + r_2j + r_3k$$

Appendix: algebraic description of quaternion

Product pq :

$$\begin{aligned}r &= pq = (p_0 + p_1i + p_2j + p_3k)(q_0 + q_1i + q_2j + q_3k) \\&= (p_0q_0 - p_1q_1 - p_2q_2 - p_3q_3) \\&\quad + (p_0q_1 + p_1q_0 + p_2q_3 - p_3q_2)i \\&\quad + (p_0q_2 + p_2q_0 + p_3q_1 - p_1q_3)j \\&\quad + (p_0q_3 + p_3q_0 + p_1q_2 - p_2q_1)k\end{aligned}$$

\Updownarrow

$$\begin{bmatrix} r_0 \\ r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} p_0q_0 - p_1q_1 - p_2q_2 - p_3q_3 \\ p_0q_1 + p_1q_0 + p_2q_3 - p_3q_2 \\ p_0q_2 + p_2q_0 + p_3q_1 - p_1q_3 \\ p_0q_3 + p_3q_0 + p_1q_2 - p_2q_1 \end{bmatrix} \quad (2)$$

(1) and (2) are equivalent each other.

Appendix: Euler angles

a set of 3-2-3 Euler angles:

$$\begin{aligned} R(\phi, \theta, \psi) &= R_3(\phi)R_2(\theta)R_3(\psi) \\ &= \begin{bmatrix} C_\phi & -S_\phi & & \\ S_\phi & C_\phi & & \\ & & & 1 \end{bmatrix} \begin{bmatrix} C_\theta & & S_\theta & \\ & 1 & & \\ -S_\theta & & C_\theta & \end{bmatrix} \begin{bmatrix} C_\psi & -S_\psi & & \\ S_\psi & C_\psi & & \\ & & & 1 \end{bmatrix} \end{aligned}$$

Rotation matrix

$$\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & & \\ 1/\sqrt{2} & 1/\sqrt{2} & & \\ & & & 1 \end{bmatrix}$$

corresponds to an infinite number of sets of Euler angles satisfying $\theta = 0$ and $\phi + \psi = \pi/4$. This **singularity** causes

- gimbal lock
- instability in solving equation of rotation

Appendix: Euler angles

a set of 1-2-3 Euler angles:

$$\begin{aligned} R(\phi, \theta, \psi) &= R_1(\phi)R_2(\theta)R_3(\psi) \\ &= \begin{bmatrix} 1 & & \\ & C_\phi & -S_\phi \\ & S_\phi & C_\phi \end{bmatrix} \begin{bmatrix} C_\theta & & S_\theta \\ & 1 & \\ -S_\theta & & C_\theta \end{bmatrix} \begin{bmatrix} C_\psi & -S_\psi & \\ S_\psi & C_\psi & \\ & & 1 \end{bmatrix} \end{aligned}$$

Rotation matrix

$$\begin{bmatrix} & & & 1 \\ & 1/\sqrt{2} & 1/\sqrt{2} & \\ -1/\sqrt{2} & 1/\sqrt{2} & & \end{bmatrix}$$

corresponds to an infinite number of sets of Euler angles satisfying $\theta = \pi/2$ and $\phi + \psi = \pi/4$.

Any set of Euler angles has singularity.

Appendix: Euler angles

computing angular velocity vector for a set of 3-2-3 Euler angles:

$$R^T = R_3^T(\psi)R_2^T(\theta)R_3^T(\phi)$$

$$\dot{R} = \dot{R}_3(\phi)R_2(\theta)R_3(\psi) + R_3(\phi)\dot{R}_2(\theta)R_3(\psi) + R_3(\phi)R_2(\theta)\dot{R}_3(\psi)$$

$$\begin{aligned} [\boldsymbol{\omega} \times] &= R^T \dot{R} \\ &= R_3^T(\psi)R_2^T(\theta)R_3^T(\phi)\dot{R}_3(\phi)R_2(\theta)R_3(\psi) \\ &\quad + R_3^T(\psi)R_2^T(\theta)\dot{R}_2(\theta)R_3(\psi) \\ &\quad + R_3^T(\psi)\dot{R}_3(\psi) \end{aligned}$$

angular velocity vector:

$$\begin{bmatrix} \omega_\xi \\ \omega_\eta \\ \omega_\zeta \end{bmatrix} = \begin{bmatrix} -S_\theta C_\psi \\ S_\theta S_\psi \\ C_\theta \end{bmatrix} \dot{\phi} + \begin{bmatrix} S_\psi \\ C_\psi \\ 0 \end{bmatrix} \dot{\theta} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \dot{\psi}$$