

Statics in variational form

U potential energy
 W work done by external forces/torques

Variational principle in statics

Internal energy $I = U - W$ reaches to its minimum at equilibrium:

$$I = U - W \rightarrow \text{minimum}$$

Analytical Mechanics: Variational Principles

Shinichi Hirai

Dept. Robotics, Ritsumeikan Univ.

Agenda

- 1 Variational Principle in Statics
- 2 Variational Principle in Statics under Constraints
- 3 Variational Principle in Dynamics
- 4 Variational Principle in Dynamics under Constraints

Statics in variational form

Solutions:

1 Solve $\text{minimize } I = U - W$

analytically

2 Solve $\text{minimize } I = U - W$

numerically

3 Solve $\delta I = 0$

analytically

Statics

Variation principle in statics

$$\text{minimize } I = U - W$$

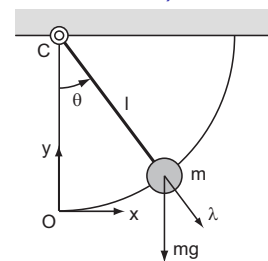
under constraint

$$\begin{aligned} &\text{minimize } I = U - W \\ &\text{subject to } R = 0 \end{aligned}$$

Solutions

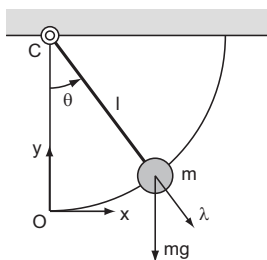
- analytically solve $\delta I = 0$
- numerical optimization (fminbnd or fmincon)

Example (simple pendulum)



$$\begin{aligned} U &= mgl(1 - \cos \theta), & W &= \tau \theta \\ I &= mgl(1 - \cos \theta) - \tau \theta \end{aligned}$$

Example (simple pendulum)



simple pendulum of length l and mass m suspended at point C
 τ : external torque around C , θ : angle around C
 Given τ , derive θ at equilibrium.

Example (simple pendulum)

Solve

$$\text{minimize } I = mgl(1 - \cos \theta) - \tau \theta$$

analytically

$$\Downarrow$$

$$\frac{\partial I}{\partial \theta} = mgl \sin \theta - \tau = 0$$

Equilibrium of moment around C

Example (simple pendulum)

Solve

$$\text{minimize } I = mgl(1 - \cos \theta) - \tau \theta$$

$$(-\pi \leq \theta \leq \pi)$$

numerically

↓

Apply `fminbnd` to minimize a function numerically

Example (simple pendulum)

Note that $\cos(\theta + \delta\theta) = \cos \theta - (\sin \theta)\delta\theta$:

$$I = mgl(1 - \cos \theta) - \tau \theta$$

$$I + \delta I = mgl(1 - \cos(\theta + \delta\theta)) - \tau(\theta + \delta\theta)$$

↓

$$\delta I = mgl(\sin \theta)\delta\theta - \tau \delta\theta$$

$$= (mgl \sin \theta - \tau)\delta\theta \equiv 0, \quad \forall \delta\theta$$

↓

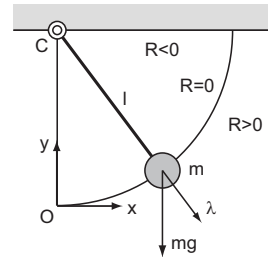
$$mgl \sin \theta - \tau = 0$$

Example (simple pendulum)

Sample Programs

- minimizing internal energy
- internal energy of simple pendulum

Example (pendulum in Cartesian coordinates)



simple pendulum of length l and mass m suspended at point C
 $[x, y]^T$: position of mass
 $[f_x, f_y]^T$: external force applied to mass
 Given $[f_x, f_y]^T$, derive $[x, y]^T$ at equilibrium.

Example (simple pendulum)

Result

```
>> internal_energy_simple_pendulum_min
```

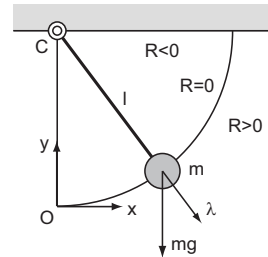
```
thetamin =
```

```
0.5354
```

```
Imin =
```

```
-0.0261
```

Example (pendulum in Cartesian coordinates)



geometric constraint

distance between C and mass = l

$$R \triangleq \{x^2 + (y - l)^2\}^{1/2} - l = 0$$

Example (simple pendulum)

Solve

$$\delta I = 0$$

analytically

↓

$$I = mgl(1 - \cos \theta) - \tau \theta$$

$$I + \delta I = mgl(1 - \cos(\theta + \delta\theta)) - \tau(\theta + \delta\theta)$$

Statics under single constraint

- U potential energy
- W work done by external forces/torques
- R geometric constraint

Variational principle in statics

Internal energy $U - W$ reaches to its minimum at equilibrium under geometric constraint $R = 0$:

$$\text{minimize } U - W$$

$$\text{subject to } R = 0$$

Statics under single constraint

Solutions:

- Solve

$$\begin{aligned} &\text{minimize } U - W \\ &\text{subject to } R = 0 \end{aligned}$$

analytically

- Solve

$$\begin{aligned} &\text{minimize } U - W \\ &\text{subject to } R = 0 \end{aligned}$$

numerically

Statics under single constraint

Solve

$$\begin{aligned} &\text{minimize } U - W \\ &\text{subject to } R = 0 \end{aligned}$$

analytically

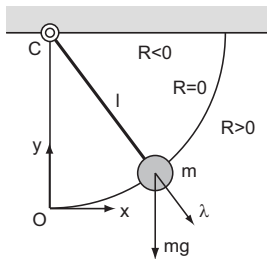
↓

$$\begin{aligned} &\text{minimize } I = U - W - \lambda R \\ &\lambda: \text{ Lagrange's multiplier} \end{aligned}$$

↓

$$\delta I = \delta(U - W - \lambda R) = 0$$

Example (pendulum in Cartesian coordinates)



$$\begin{aligned} U &= mgy, & W &= f_x x + f_y y \\ R &= \{x^2 + (y - l)^2\}^{1/2} - l \end{aligned}$$

Example (pendulum in Cartesian coordinates)

$$I = mgy - (f_x x + f_y y) - \lambda \left[\{x^2 + (y - l)^2\}^{1/2} - l \right]$$

Note that $\delta R = R_x \delta x + R_y \delta y$, where

$$\begin{aligned} R_x &\triangleq \frac{\partial R}{\partial x} = x \{x^2 + (y - l)^2\}^{-1/2} \\ R_y &\triangleq \frac{\partial R}{\partial y} = (y - l) \{x^2 + (y - l)^2\}^{-1/2} \end{aligned}$$

↓

$$\begin{aligned} \delta I &= mg \delta y - f_x \delta x - f_y \delta y - \lambda R_x \delta x - \lambda R_y \delta y \\ &= (-f_x - \lambda R_x) \delta x + (mg - f_y - \lambda R_y) \delta y \equiv 0, \quad \forall \delta x, \delta y \end{aligned}$$

Example (pendulum in Cartesian coordinates)

$$\begin{aligned} -f_x - \lambda R_x &= 0 \\ mg - f_y - \lambda R_y &= 0 \end{aligned}$$

↓

$$\begin{bmatrix} 0 \\ -mg \end{bmatrix} + \begin{bmatrix} f_x \\ f_y \end{bmatrix} + \lambda \begin{bmatrix} R_x \\ R_y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ -mg \end{bmatrix} \text{ grav. force, } \begin{bmatrix} f_x \\ f_y \end{bmatrix} \text{ ext. force, } \lambda \begin{bmatrix} R_x \\ R_y \end{bmatrix} \text{ constraint force}$$

gradient vector (\perp to $R = 0$)

Example (pendulum in Cartesian coordinates)

three equations w.r.t. three unknowns x , y , and λ :

$$\begin{aligned} -f_x - \lambda R_x &= 0 \\ mg - f_y - \lambda R_y &= 0 \\ R &= 0 \end{aligned}$$

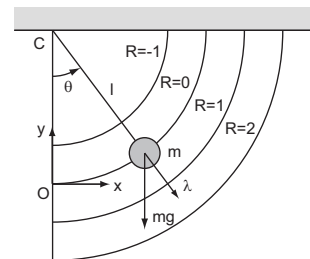
↓

we can determine position of mass $[x, y]^T$ and magnitude of constraint force λ

Example (pendulum in Cartesian coordinates)

Note

$$\begin{aligned} I &= U - W - \lambda R \\ &= U - (W + \lambda R) \end{aligned}$$



λ magnitude of a constraint force
 R distance along the force
 constraint force \perp
 contour $R = \text{constant}$
 λR work done by a constraint force

$W + \lambda R$ work done by external & constraint forces

Statics under single constraint

Solve

$$\begin{aligned} &\text{minimize } I = U - W \\ &\text{subject to } R = 0 \end{aligned}$$

numerically

↓

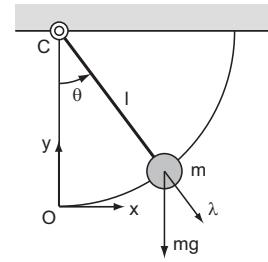
Apply `fmincon` to minimize a function numerically under constraints
 Note: "Optimization Toolbox" is needed to use `fmincon`

Example (pendulum in Cartesian coordinates)

Sample Programs

- minimizing internal energy (Cartesian)
- internal energy of simple pendulum (Cartesian)
- constraints

Example (simple pendulum)



simple pendulum of length l and mass m suspended at point C
 τ : external torque around C at time t , θ : angle around C at time t
 Derive the motion of the pendulum.

Example (pendulum in Cartesian coordinates)

Result:

```
>> internal_energy_pendulum_Cartesian_min
Local minimum found that satisfies the constraints.
```

<stopping criteria details>

```
qmin =
  1.4001
  3.4281
```

```
Imin =
 -0.4897
```

Dynamics in variational form

- T kinetic energy
- U potential energy
- W work done by external forces/torques

Lagrangian

$$\mathcal{L} = T - U + W$$

Lagrange equation of motion

$$\frac{\partial \mathcal{L}}{\partial \theta} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = 0$$

Statics under multiple constraints

- U potential energy
- W work done by external forces/torques
- R_1, R_2 geometric constraints

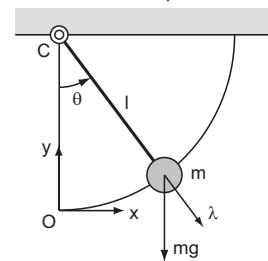
Variational principle in statics

Internal energy $U - W$ reaches to its minimum at equilibrium under geometric constraints $R_1 = 0$ and $R_2 = 0$:

$$\begin{aligned} &\text{minimize } U - W \\ &\text{subject to } R_1 = 0, \quad R_2 = 0 \end{aligned}$$

$$\begin{aligned} \delta I &= \delta(U - W - \lambda_1 R_1 - \lambda_2 R_2) = 0 \\ \lambda_1, \lambda_2 &: \text{Lagrange's multipliers} \end{aligned}$$

Example (simple pendulum)



$$\begin{aligned} T &= \frac{1}{2}(ml^2)\dot{\theta}^2 \\ U &= mgl(1 - \cos \theta), \quad W = \tau\theta \end{aligned}$$

Dynamics

Lagrangian

$$\begin{aligned} \mathcal{L} &= T - U + W \\ \mathcal{L} &= T - U + W + \lambda R \quad (\text{under constraint}) \end{aligned}$$

Lagrange equations of motion

$$\frac{\partial \mathcal{L}}{\partial \mathbf{q}} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} = \mathbf{0}$$

Solutions

- numerical ODE solver (ode45)
- constraint stabilization method (CSM)

Example (simple pendulum)

Lagrangian

$$\mathcal{L} = \frac{1}{2}(ml^2)\dot{\theta}^2 - mgl(1 - \cos \theta) + \tau\theta$$

partial derivatives

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \theta} &= -mgl \sin \theta + \tau, & \frac{\partial \mathcal{L}}{\partial \dot{\theta}} &= (ml^2)\dot{\theta} \\ \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\theta}} &= ml^2\ddot{\theta} \end{aligned}$$

Lagrange equation of motion

$$-mgl \sin \theta + \tau - ml^2\ddot{\theta} = 0$$

Example (simple pendulum)

Equation of the pendulum motion

$$ml^2\ddot{\theta} = -mgl \sin \theta + \tau$$

⇓

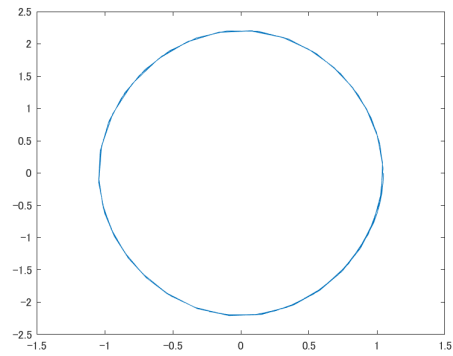
Canonical form of ordinary differential equation

$$\begin{aligned}\dot{\theta} &= \omega \\ \dot{\omega} &= \frac{1}{ml^2} (\tau - mgl \sin \theta)\end{aligned}$$

can be solved numerically by an ODE solver

Example (simple pendulum)

Result



Example (simple pendulum)

Sample Programs

- solve the equation of motion of simple pendulum
- equation of motion of simple pendulum
- external torque

Example (pendulum with viscous friction)

Assumptions

viscous friction around supporting point C works
viscous friction causes a negative torque around C
magnitude of the torque is proportional to angular velocity

$$\text{viscous friction torque} = -b\dot{\theta} \quad (b: \text{positive constant})$$

Replacing τ by $\tau - b\dot{\theta}$:

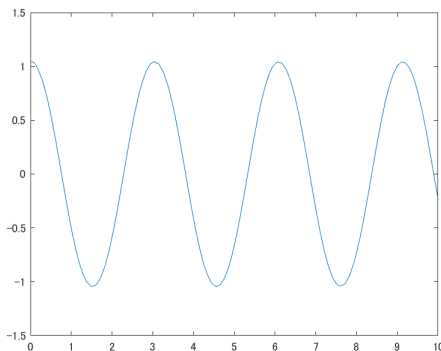
$$(ml^2)\ddot{\theta} = (\tau - b\dot{\theta}) - mgl \sin \theta$$

⇓

$$\begin{aligned}\dot{\theta} &= \omega \\ \dot{\omega} &= \frac{1}{ml^2} (\tau - b\omega - mgl \sin \theta)\end{aligned}$$

Example (simple pendulum)

Result



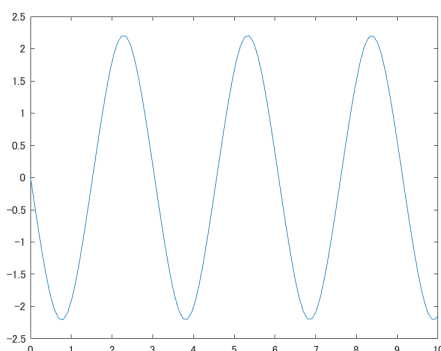
Example (pendulum with viscous friction)

Sample Programs

- solve the equation of motion of damped pendulum
- equation of motion of damped pendulum
- external torque

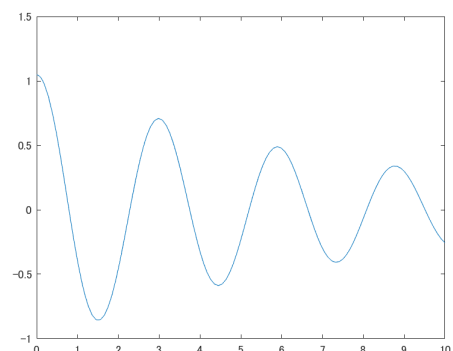
Example (simple pendulum)

Result



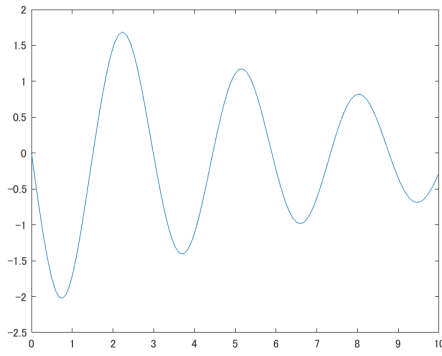
Example (pendulum with viscous friction)

Result



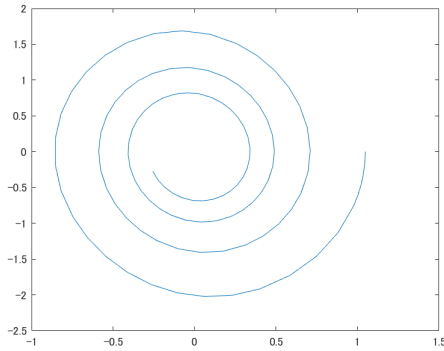
Example (pendulum with viscous friction)

Result

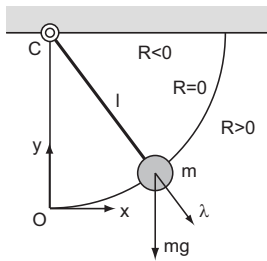


Example (pendulum with viscous friction)

Result



Example (pendulum in Cartesian coordinates)



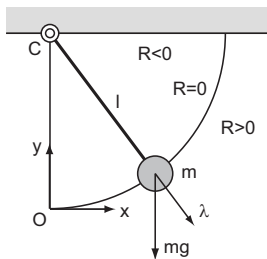
simple pendulum of length l and mass m suspended at point C

$[x, y]^T$: position of mass at time t

$[f_x, f_y]^T$: external force applied to mass at time t

Derive the motion of the pendulum in Cartesian coordinates.

Example (pendulum in Cartesian coordinates)



geometric constraint

distance between C and mass = l

$$R \triangleq \{x^2 + (y - l)^2\}^{1/2} - l = 0$$

Dynamics under single constraint

T kinetic energy

U potential energy

W work done by external forces/torques

Lagrangian

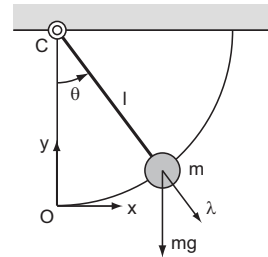
$$\mathcal{L} = T - U + W + \lambda R$$

Lagrange equations of motion

$$\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}} = 0$$

$$\frac{\partial \mathcal{L}}{\partial y} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{y}} = 0$$

Example (pendulum in Cartesian coordinates)



$$T = \frac{1}{2} m \{\dot{x}^2 + \dot{y}^2\}$$

$$U = mgy, \quad W = f_x x + f_y y$$

$$R = \{x^2 + (y - l)^2\}^{1/2} - l$$

Example (pendulum in Cartesian coordinates)

Lagrangian

$$\mathcal{L} = \frac{1}{2} m \{\dot{x}^2 + \dot{y}^2\} - mgy + f_x x + f_y y + \lambda \left[\{x^2 + (y - l)^2\}^{1/2} - l \right]$$

partial derivatives

$$\frac{\partial \mathcal{L}}{\partial x} = f_x + \lambda R_x, \quad \frac{\partial \mathcal{L}}{\partial \dot{x}} = m \dot{x}$$

$$\frac{\partial \mathcal{L}}{\partial y} = -mg + f_y + \lambda R_y, \quad \frac{\partial \mathcal{L}}{\partial \dot{y}} = m \dot{y}$$

Lagrange equations of motion

$$f_x + \lambda R_x - m \ddot{x} = 0$$

$$-mg + f_y + \lambda R_y - m \ddot{y} = 0$$

Example (pendulum in Cartesian coordinates)

Lagrange equations of motion

$$\begin{bmatrix} 0 \\ -mg \end{bmatrix} + \begin{bmatrix} f_x \\ f_y \end{bmatrix} + \lambda \begin{bmatrix} R_x \\ R_y \end{bmatrix} + \left\{ -m \begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} \right\} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

gravitational external constraint inertial

dynamic equilibrium among forces

Example (pendulum in Cartesian coordinates)

three equations w.r.t. three unknowns x , y , and λ :

$$\begin{aligned} m\ddot{x} &= f_x + \lambda R_x \\ m\ddot{y} &= -mg + f_y + \lambda R_y \\ R &= 0 \end{aligned}$$

Example (pendulum in Cartesian coordinates)

three equations w.r.t. three unknowns x , y , and λ :

$$\begin{aligned} m\ddot{x} &= f_x + \lambda R_x \\ m\ddot{y} &= -mg + f_y + \lambda R_y \\ R &= 0 \end{aligned}$$

Mixture of differential and algebraic equations

↓

Difficult to solve the mixture of differential and algebraic equations

Constraint stabilization method (CSM)

Constraint stabilization

convert algebraic eq. to its almost equivalent differential eq.

$$\text{algebraic eq. } R = 0$$

↓

$$\begin{aligned} \text{differential eq. } \ddot{R} + 2\alpha\dot{R} + \alpha^2 R &= 0 \\ (\alpha: \text{large positive constant}) \end{aligned}$$

critical damping (converges to zero most quickly)

Constraint stabilization method (CSM)

Dynamic equation of motion under geometric constraint:

$$\begin{aligned} \text{differential eq. } \frac{\partial \mathcal{L}}{\partial \mathbf{q}} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} &= \mathbf{0} \\ \text{algebraic eq. } R &= 0 \end{aligned}$$

↓

$$\begin{aligned} \text{differential eq. } \frac{\partial \mathcal{L}}{\partial \mathbf{q}} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} &= \mathbf{0} \\ \text{differential eq. } \ddot{R} + 2\alpha\dot{R} + \alpha^2 R &= 0 \end{aligned}$$

can be solved numerically by an ODE solver.

Computing equation for constraint stabilization

Assume R depends on x and y : $R(x, y) = 0$

Differentiating $R(x, y)$ w.r.t time t :

$$\dot{R} = \frac{\partial R}{\partial x} \frac{dx}{dt} + \frac{\partial R}{\partial y} \frac{dy}{dt} = R_x \dot{x} + R_y \dot{y}$$

Differentiating $R_x(x, y)$ and $R_y(x, y)$ w.r.t time t :

$$\begin{aligned} \dot{R}_x &= \frac{\partial R_x}{\partial x} \frac{dx}{dt} + \frac{\partial R_x}{\partial y} \frac{dy}{dt} = R_{xx} \dot{x} + R_{xy} \dot{y} \\ \dot{R}_y &= \frac{\partial R_y}{\partial x} \frac{dx}{dt} + \frac{\partial R_y}{\partial y} \frac{dy}{dt} = R_{yx} \dot{x} + R_{yy} \dot{y} \end{aligned}$$

Second order time derivative:

$$\begin{aligned} \ddot{R} &= (\dot{R}_x \dot{x} + R_x \ddot{x}) + (\dot{R}_y \dot{y} + R_y \ddot{y}) \\ &= (R_{xx} \dot{x} + R_{xy} \dot{y}) \dot{x} + R_x \ddot{x} + (R_{yx} \dot{x} + R_{yy} \dot{y}) \dot{y} + R_y \ddot{y} \end{aligned}$$

Computing equation for constraint stabilization

Second order time derivative:

$$\ddot{R} = \begin{bmatrix} R_x & R_y \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} + \begin{bmatrix} \dot{x} & \dot{y} \end{bmatrix} \begin{bmatrix} R_{xx} & R_{xy} \\ R_{yx} & R_{yy} \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix}$$

Equation to stabilize constraint:

$$\begin{aligned} - \begin{bmatrix} R_x & R_y \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} &= \begin{bmatrix} \dot{x} & \dot{y} \end{bmatrix} \begin{bmatrix} R_{xx} & R_{xy} \\ R_{yx} & R_{yy} \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} \\ &\quad + 2\alpha(R_x \dot{x} + R_y \dot{y}) + \alpha^2 R \end{aligned}$$

$$\Downarrow \quad \mathbf{v}_x \triangleq \dot{x}, \mathbf{v}_y \triangleq \dot{y}$$

$$\begin{aligned} - \begin{bmatrix} R_x & R_y \end{bmatrix} \begin{bmatrix} \dot{v}_x \\ \dot{v}_y \end{bmatrix} &= \begin{bmatrix} v_x & v_y \end{bmatrix} \begin{bmatrix} R_{xx} & R_{xy} \\ R_{yx} & R_{yy} \end{bmatrix} \begin{bmatrix} v_x \\ v_y \end{bmatrix} \\ &\quad + 2\alpha(R_x v_x + R_y v_y) + \alpha^2 R \end{aligned}$$

Example (pendulum in Cartesian coordinates)

Equation for stabilizing constraint $R(x, y) = 0$:

$$-R_x \dot{v}_x - R_y \dot{v}_y = C(x, y, v_x, v_y)$$

where

$$\begin{aligned} C(x, y, v_x, v_y) &= \begin{bmatrix} v_x & v_y \end{bmatrix} \begin{bmatrix} R_{xx} & R_{xy} \\ R_{yx} & R_{yy} \end{bmatrix} \begin{bmatrix} v_x \\ v_y \end{bmatrix} \\ &\quad + 2\alpha(R_x v_x + R_y v_y) + \alpha^2 R \end{aligned}$$

and

$$\begin{aligned} P &= \{x^2 + (y-l)^2\}^{-1/2}, \quad R_x = xP, \quad R_y = (y-l)P \\ R_{xx} &= P - x^2 P^3, \quad R_{yy} = P - (y-l)^2 P^3 \\ R_{xy} &= R_{yx} = -x(y-l)P^3 \end{aligned}$$

Example (pendulum in Cartesian coordinates)

Combining equations of motion and equation for constraint stabilization:

$$\begin{aligned} \dot{x} &= v_x \\ \dot{y} &= v_y \\ \begin{bmatrix} m & & -R_x \\ & m & -R_y \\ -R_x & -R_y & \end{bmatrix} \begin{bmatrix} \dot{v}_x \\ \dot{v}_y \\ \lambda \end{bmatrix} &= \begin{bmatrix} f_x \\ -mg + f_y \\ C(x, y, v_x, v_y) \end{bmatrix} \end{aligned}$$

five equations w.r.t. five unknown variables x , y , v_x , v_y and λ

given $x, y, v_x, v_y \implies \dot{x}, \dot{y}, \dot{v}_x, \dot{v}_y$

This canonical ODE can be solved numerically by an ODE solver.

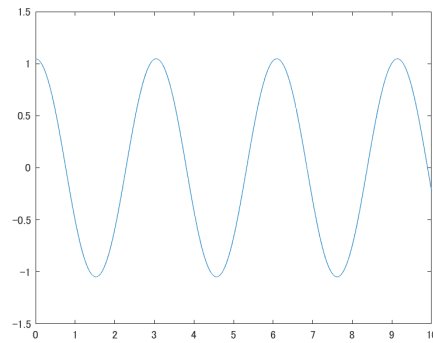
Example (pendulum in Cartesian coordinates)

Sample Programs

- solve the equation of motion of simple pendulum (Cartesian)
- equation of motion of simple pendulum (Cartesian)

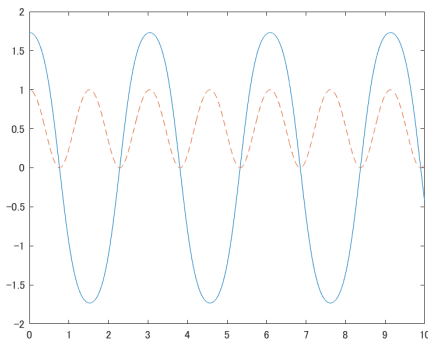
Example (pendulum in Cartesian coordinates)

t -computed θ



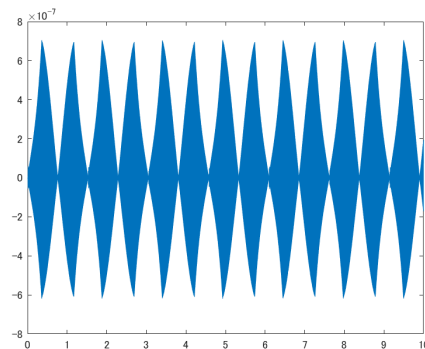
Example (pendulum in Cartesian coordinates)

t - x, y



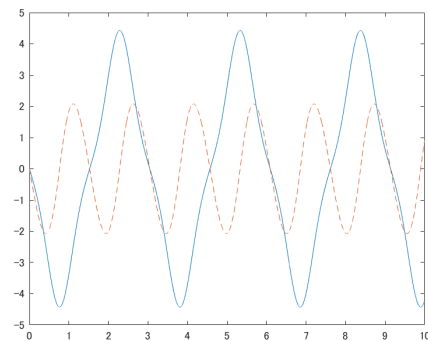
Example (pendulum in Cartesian coordinates)

t -constraint R



Example (pendulum in Cartesian coordinates)

t - v_x, v_y



Notice

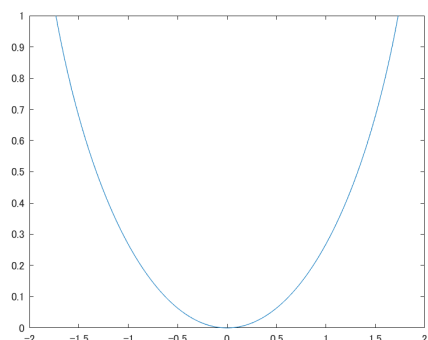
Lagrangian

$$\begin{aligned}\mathcal{L} &= T - U + W + \lambda R \\ &= T - (U - W - \lambda R) \\ &= T - I\end{aligned}$$

Lagrangian is equal to the difference between kinetic energy and internal energy under a constraint

Example (pendulum in Cartesian coordinates)

x - y



Summary

Variational principles

- statics $I = U - W$
- statics under constraint $I = U - W - \lambda R$

$$\delta I \equiv 0$$

- dynamics $\mathcal{L} = T - U + W$
- dynamics under constraint $\mathcal{L} = T - U + W + \lambda R$

$$\frac{\partial \mathcal{L}}{\partial \mathbf{q}} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} = \mathbf{0}$$

- constraint stabilization method

Summary

How to solve a static problem

Solve (nonlinear) equations originated from variation
or
Numerically minimize internal energy

How to solve a dynamic problem

Step 1 Derive Lagrange equations of motion **analytically**

Step 2 Solve the derived equations **numerically**

Report

Report # 1 due date : Nov. 2 (Mon)

Simulate the dynamic motion of a pendulum under viscous friction described with Cartesian coordinates x and y . Apply constraint stabilization method to convert the constraint into its almost equivalent ODE, then apply any ODE solver to solve a set of ODEs (equations of motion and equation for constraint stabilization) numerically.

Submit your report in pdf format to manaba+R

File name should be:

student number (11 digits) your name (without space).pdf

For example 12345678901HiraiShinichi.pdf

Appendix: Variational calculus

Small virtual deviation of variables or functions.

$$y = x^2$$

Let us change **variable** x to $x + \delta x$, then variable y changes to $y + \delta y$ accordingly.

$$\begin{aligned} y + \delta y &= (x + \delta x)^2 \\ &= x^2 + 2x \delta x + (\delta x)^2 \\ &= x^2 + 2x \delta x \end{aligned}$$

Thus

$$\delta y = 2x \delta x$$

Appendix: Variational calculus

Small virtual deviation of variables or functions.

$$I = \int_0^T \{x(t)\}^2 dt$$

Let us change **function** $x(t)$ to $x(t) + \delta x(t)$, then variable I changes to $I + \delta I$ accordingly.

$$\begin{aligned} I + \delta I &= \int_0^T \{x(t) + \delta x(t)\}^2 dt \\ &= \int_0^T \{x(t)\}^2 + 2x(t) \delta x(t) dt \end{aligned}$$

Thus

$$\delta I = \int_0^T 2x(t) \delta x(t) dt$$

Appendix: Variational calculus

Variational operator δ

$\delta\theta$ virtual deviation of variable θ

$\delta f(\theta)$ virtual deviation of function $f(\theta)$

$$\delta f(\theta) = f'(\theta)\delta\theta$$

virtual increment of variable $\theta \rightarrow \theta + \delta\theta$

increment of function $f(\theta) \rightarrow f(\theta + \delta\theta) = f(\theta) + f'(\theta)\delta\theta$
 $f(\theta) \rightarrow f(\theta) + \delta f(\theta)$

simple examples

$$\delta(5x) = 5 \delta x \quad \delta x^2 = 2x \delta x$$

$$\delta \sin \theta = (\cos \theta) \delta \theta, \quad \delta \cos \theta = (-\sin \theta) \delta \theta$$

Appendix: Variational calculus

Variational operator δ

$\delta\theta$ virtual deviation of variable θ

$\delta f(\theta)$ virtual deviation of function $f(\theta)$

$$\delta f(\theta) = f'(\theta)\delta\theta$$

virtual increment of variable $\theta \rightarrow \theta + \delta\theta$

increment of function $f(\theta) \rightarrow f(\theta + \delta\theta) = f(\theta) + f'(\theta)\delta\theta$
 $f(\theta) \rightarrow f(\theta) + \delta f(\theta)$

simple examples

$$\delta(5x) = 5 \delta x \quad \delta x^2 = 2x \delta x$$

$$\delta \sin \theta = (\cos \theta) \delta \theta, \quad \delta \cos \theta = (-\sin \theta) \delta \theta$$

Appendix: Variational calculus

assume that θ depends on time t

virtual increment of function $\theta(t) \rightarrow \theta(t) + \delta\theta(t)$

$$\begin{aligned} \frac{d\theta}{dt} &\rightarrow \frac{d}{dt}(\theta + \delta\theta) = \frac{d\theta}{dt} + \frac{d}{dt}\delta\theta \\ \int \theta dt &\rightarrow \int (\theta + \delta\theta) dt = \int \theta dt + \int \delta\theta dt \end{aligned}$$

variation of derivative and integral

$$\begin{aligned} \delta \frac{d\theta}{dt} &= \frac{d}{dt}\delta\theta \\ \delta \int \theta dt &= \int \delta\theta dt \end{aligned}$$

variational operator and differential/integral operator can commute

Appendix: Lagrange multiplier method

converts minimization (maximization) under conditions into minimization (maximization) without conditions.

minimize $f(\mathbf{x})$
subject to $g(\mathbf{x}) = 0$

↓

minimize $I(\mathbf{x}, \lambda) = f(\mathbf{x}) + \lambda g(\mathbf{x})$

↓

$$\begin{aligned} \frac{\partial I}{\partial \mathbf{x}} &= \frac{\partial f}{\partial \mathbf{x}} + \lambda \frac{\partial g}{\partial \mathbf{x}} = \mathbf{0} \\ \frac{\partial I}{\partial \lambda} &= g(\mathbf{x}) = 0 \end{aligned}$$

Appendix: Lagrange multiplier method (example)

Length of each edge of a cube is given by x , y , and z . Determine x , y , and z that minimizes the surface of the cube when the cube volume is constantly specified by a^3 :

$$\begin{aligned} \text{minimize } & S(x, y, z) = 2xy + 2yz + 2zx \\ \text{subject to } & R(x, y, z) \triangleq xyz - a^3 = 0 \end{aligned}$$

Introducing Lagrange multiplier λ , the above conditional minimization can be converted into the following unconditional minimization:

$$\begin{aligned} \text{minimize } I(x, y, z, \lambda) &= S(x, y, z) + \lambda R(x, y, z) \\ &= 2xy + 2yz + 2zx + \lambda(xyz - a^3) \end{aligned}$$

Appendix: Lagrange multiplier method (example)

Calculating partial derivatives:

$$\frac{\partial I}{\partial x} = 2y + 2z - \lambda yz = 0 \quad (1)$$

$$\frac{\partial I}{\partial y} = 2z + 2x - \lambda zx = 0 \quad (2)$$

$$\frac{\partial I}{\partial z} = 2x + 2y - \lambda xy = 0 \quad (3)$$

$$\frac{\partial I}{\partial \lambda} = xyz - a^3 = 0 \quad (4)$$

Calculating (1) \cdot x - (2) \cdot y , we have

$$z(x - y) = 0,$$

which directly yields $x = y$. Similarly, we have $y = z$ and $z = x$.

Consequently, we conclude $x = y = z = a$.

Appendix: ODE solver

Let us solve **van der Pol** equation:

$$\ddot{x} - 2(1 - x^2)\dot{x} + x = 0$$

Canonical form:

$$\begin{aligned} \dot{x} &= v \\ \dot{v} &= 2(1 - x^2)v - x \end{aligned}$$

State variable vector:

$$\mathbf{q} = \begin{bmatrix} x \\ v \end{bmatrix}$$

Appendix: ODE solver (MATLAB)

File `van_der_Pol.m` describes the canonical form:

```
function dotq = van_der_Pol (t,q)
    x = q(1);
    v = q(2);
    dotx = v;
    dotv = 2*(1-x^2)*v - x;
    dotq = [dotx; dotv];
end
```

File name `van_der_Pol` should coincide with function name `van_der_Pol`.

Appendix: ODE solver (MATLAB)

File `van_der_Pol_solve.m` solves van der Pol equation numerically:

```
timestep=0.00:0.10:10.00;
qinit=[2.00;0.00];
[time,q]=ode45(@van_der_Pol,timestep,qinit);

% line style  solid -   broken -.   chain --   dotted :
plot(time,q(:,1),'-', time,q(:,2),'-.'.');
```

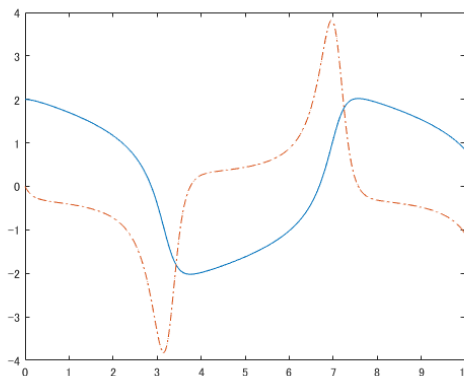
Appendix: ODE solver (MATLAB)

```
>> time
time =
     0
 0.1000
 0.2000
 0.3000
 0.4000

>> q
q =
 2.0000     0
 1.9917  -0.1504
 1.9721  -0.2338
 1.9461  -0.2822
 1.9163  -0.3125
```

The first and second columns corresponds to x and v .

Appendix: ODE solver (MATLAB)



Appendix: ODE solver (MATLAB)

File `van_der_Pol.m` describes the canonical form:

```
function dotq = van_der_Pol (t,q)
    x = q(1);
    v = q(2);
    dotx = v;
    dotv = 2*(1-x^2)*v - x;
    dotq = [dotx; dotv];
end
```

File name `van_der_Pol` should coincide with function name `van_der_Pol`.