

Analytical Mechanics: Variational Principles

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Agenda

- 1 Variational Principle in Statics
- 2 Variational Principle in Statics under Constraints
- 3 Variational Principle in Dynamics
- 4 Variational Principle in Dynamics under Constraints

Statics

Variation principle in statics

$$\text{minimize } I = U - W$$

under constraint

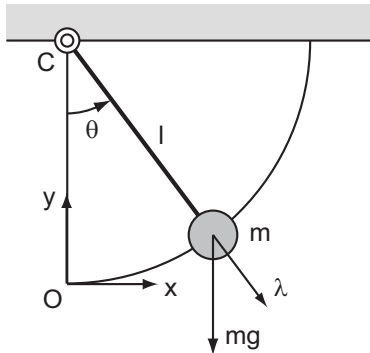
$$\text{minimize } I = U - W$$

$$\text{subject to } R = 0$$

Solutions

- analytically solve $\delta I = 0$
- numerical optimization (fminbnd or fmincon)

Example (simple pendulum)



simple pendulum of length l and mass m suspended at point C

τ : external torque around C , θ : angle around C

Given τ , derive θ at equilibrium.

Statics in variational form

U potential energy

W work done by external forces/torques

Variational principle in statics

Internal energy $I = U - W$ reaches to its minimum at equilibrium:

$$I = U - W \rightarrow \text{minimum}$$

Statics in variational form

Solutions:

① Solve

$$\text{minimize } I = U - W$$

analytically

② Solve

$$\text{minimize } I = U - W$$

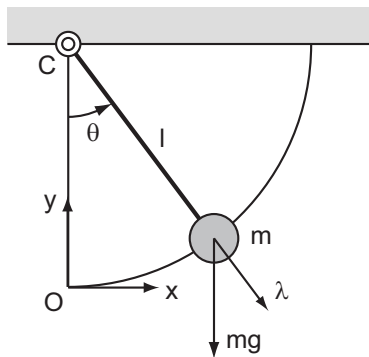
numerically

③ Solve

$$\delta I = 0$$

analytically

Example (simple pendulum)



$$U = mgl(1 - \cos \theta), \quad W = \tau \theta$$

$$l = mgl(1 - \cos \theta) - \tau \theta$$

Example (simple pendulum)

Solve

$$\text{minimize } I = mgl(1 - \cos \theta) - \tau \theta$$

analytically

↓

$$\frac{\partial I}{\partial \theta} = mgl \sin \theta - \tau = 0$$

Equilibrium of moment around C

Example (simple pendulum)

Solve

$$\begin{aligned} \text{minimize } I &= mgl(1 - \cos \theta) - \tau\theta \\ &(-\pi \leq \theta \leq \pi) \end{aligned}$$

numerically



Apply `fminbnd` to minimize a function numerically

Example (simple pendulum)

Sample Programs

- minimizing internal energy
- internal energy of simple pendulum

Example (simple pendulum)

Result

```
>> internal_energy_simple_pendulum_min
```

```
thetamin =
```

```
0.5354
```

```
Imin =
```

```
-0.0261
```

Example (simple pendulum)

Solve

$$\delta I = 0$$

analytically

↓

$$I = mgl(1 - \cos \theta) - \tau \theta$$

$$I + \delta I = mgl(1 - \cos(\theta + \delta\theta)) - \tau(\theta + \delta\theta)$$

Example (simple pendulum)

Note that $\cos(\theta + \delta\theta) = \cos\theta - (\sin\theta)\delta\theta$:

$$l = mgl(1 - \cos\theta) - \tau\theta$$

$$l + \delta l = mgl(1 - \cos\theta + (\sin\theta)\delta\theta) - \tau(\theta + \delta\theta)$$

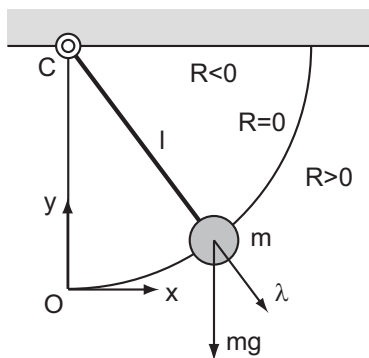
\Downarrow

$$\begin{aligned}\delta l &= mgl(\sin\theta)\delta\theta - \tau\delta\theta \\ &= (mgl \sin\theta - \tau)\delta\theta \equiv 0, \quad \forall \delta\theta\end{aligned}$$

\Downarrow

$$mgl \sin\theta - \tau = 0$$

Example (pendulum in Cartesian coordinates)



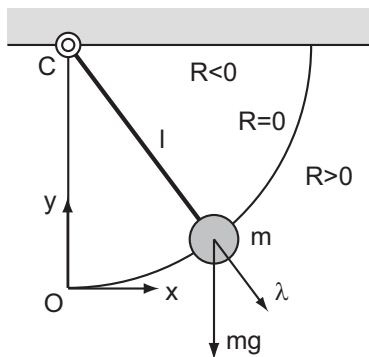
simple pendulum of length l and mass m suspended at point C

$[x, y]^T$: position of mass

$[f_x, f_y]^T$: external force applied to mass

Given $[f_x, f_y]^T$, derive $[x, y]^T$ at equilibrium.

Example (pendulum in Cartesian coordinates)



geometric constraint

distance between C and mass = l

$$R \triangleq \{x^2 + (y - l)^2\}^{1/2} - l = 0$$

Statics under single constraint

U potential energy

W work done by external forces/torques

R geometric constraint

Variational principle in statics

Internal energy $U - W$ reaches to its minimum at equilibrium under geometric constraint $R = 0$:

minimize $U - W$

subject to $R = 0$

Statics under single constraint

Solutions:

① Solve

$$\begin{aligned} &\text{minimize } U - W \\ &\text{subject to } R = 0 \end{aligned}$$

analytically

② Solve

$$\begin{aligned} &\text{minimize } U - W \\ &\text{subject to } R = 0 \end{aligned}$$

numerically

Statics under single constraint

Solve

$$\begin{aligned} &\text{minimize } U - W \\ &\text{subject to } R = 0 \end{aligned}$$

analytically

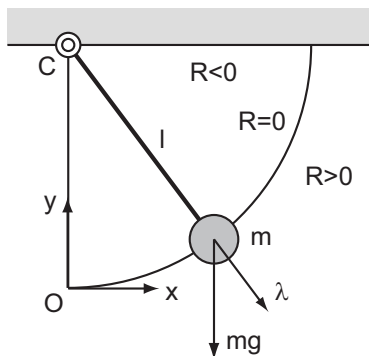
⇓

$$\begin{aligned} &\text{minimize } I = U - W - \lambda R \\ &\lambda: \quad \text{Lagrange's multiplier} \end{aligned}$$

⇓

$$\delta I = \delta(U - W - \lambda R) = 0$$

Example (pendulum in Cartesian coordinates)



$$U = mgy, \quad W = f_x x + f_y y$$

$$R = \{x^2 + (y - l)^2\}^{1/2} - l$$

Example (pendulum in Cartesian coordinates)

$$I = mgy - (f_x x + f_y y) - \lambda \left[\{x^2 + (y - l)^2\}^{1/2} - l \right]$$

Note that $\delta R = R_x \delta x + R_y \delta y$, where

$$R_x \triangleq \frac{\partial R}{\partial x} = x \{x^2 + (y - l)^2\}^{-1/2}$$

$$R_y \triangleq \frac{\partial R}{\partial y} = (y - l) \{x^2 + (y - l)^2\}^{-1/2}$$

↓

$$\begin{aligned} \delta I &= mg \delta y - f_x \delta x - f_y \delta y - \lambda R_x \delta x - \lambda R_y \delta y \\ &= (-f_x - \lambda R_x) \delta x + (mg - f_y - \lambda R_y) \delta y \equiv 0, \quad \forall \delta x, \delta y \end{aligned}$$

Example (pendulum in Cartesian coordinates)

$$\begin{aligned} -f_x - \lambda R_x &= 0 \\ mg - f_y - \lambda R_y &= 0 \end{aligned}$$

⇓

$$\begin{bmatrix} 0 \\ -mg \end{bmatrix} + \begin{bmatrix} f_x \\ f_y \end{bmatrix} + \lambda \begin{bmatrix} R_x \\ R_y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ -mg \end{bmatrix} \text{ grav. force, } \begin{bmatrix} f_x \\ f_y \end{bmatrix} \text{ ext. force, } \lambda \underbrace{\begin{bmatrix} R_x \\ R_y \end{bmatrix}}_{\text{gradient vector } (\perp \text{ to } R = 0)} \text{ constraint force}$$

Example (pendulum in Cartesian coordinates)

three equations w.r.t. three unknowns x , y , and λ :

$$\begin{aligned} -f_x - \lambda R_x &= 0 \\ mg - f_y - \lambda R_y &= 0 \\ R &= 0 \end{aligned}$$

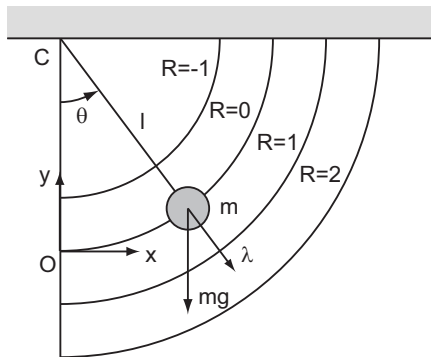
⇓

we can determine position of mass $[x, y]^T$ and magnitude of constraint force λ

Example (pendulum in Cartesian coordinates)

Note

$$\begin{aligned} I &= U - W - \lambda R \\ &= U - (W + \lambda R) \end{aligned}$$



λ magnitude of a constraint force
 R distance along the force

constraint force \perp
contour $R = \text{constant}$

λR work done by a constraint force

$W + \lambda R$ work done by external & constraint forces

Statics under single constraint

Solve

$$\text{minimize } I = U - W$$

$$\text{subject to } R = 0$$

numerically



Apply `fmincon` to minimize a function numerically under constraints

Note: "Optimization Toolbox" is needed to use `fmincon`

Example (pendulum in Cartesian coordinates)

Sample Programs

- minimizing internal energy (Cartesian)
- internal energy of simple pendulum (Cartesian)
- constraints

Example (pendulum in Cartesian coordinates)

Result:

```
>> internal_energy_pendulum_Cartesian_min
```

```
Local minimum found that satisfies the constraints.
```

```
<stopping criteria details>
```

```
qmin =
```

```
1.4001
```

```
3.4281
```

```
Imin =
```

```
-0.4897
```

Statics under multiple constraints

U potential energy

W work done by external forces/torques

R_1, R_2 geometric constraints

Variational principle in statics

Internal energy $U - W$ reaches to its minimum at equilibrium under geometric constraints $R_1 = 0$ and $R_2 = 0$:

minimize $U - W$

subject to $R_1 = 0, R_2 = 0$

$$\delta I = \delta(U - W - \lambda_1 R_1 - \lambda_2 R_2) = 0$$

λ_1, λ_2 : Lagrange's multipliers

Dynamics

Lagrangian

$$\mathcal{L} = T - U + W$$

$$\mathcal{L} = T - U + W + \lambda R \quad (\text{under constraint})$$

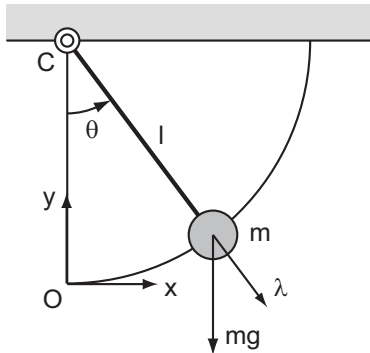
Lagrange equations of motion

$$\frac{\partial \mathcal{L}}{\partial \mathbf{q}} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} = \mathbf{0}$$

Solutions

- numerical ODE solver (ode45)
- constraint stabilization method (CSM)

Example (simple pendulum)



simple pendulum of length l and mass m suspended at point C
 τ : external torque around C at time t , θ : angle around C at time t
Derive the motion of the pendulum.

Dynamics in variational form

T kinetic energy

U potential energy

W work done by external forces/torques

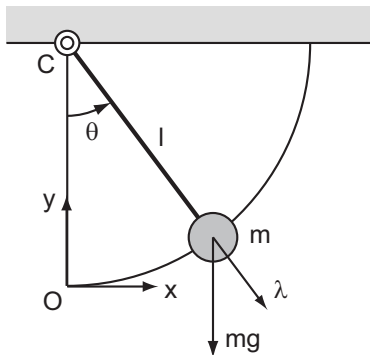
Lagrangian

$$\mathcal{L} = T - U + W$$

Lagrange equation of motion

$$\frac{\partial \mathcal{L}}{\partial \theta} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = 0$$

Example (simple pendulum)



$$T = \frac{1}{2}(ml^2)\dot{\theta}^2$$

$$U = mgl(1 - \cos \theta), \quad W = \tau \theta$$

Example (simple pendulum)

Lagrangian

$$\mathcal{L} = \frac{1}{2}(ml^2)\dot{\theta}^2 - mgl(1 - \cos \theta) + \tau\theta$$

partial derivatives

$$\frac{\partial L}{\partial \theta} = -mgl \sin \theta + \tau, \quad \frac{\partial L}{\partial \dot{\theta}} = (ml^2)\dot{\theta}$$
$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} = ml^2\ddot{\theta}$$

Lagrange equation of motion

$$-mgl \sin \theta + \tau - ml^2\ddot{\theta} = 0$$

Example (simple pendulum)

Equation of the pendulum motion

$$ml^2\ddot{\theta} = -mgl \sin \theta + \tau$$



Canonical form of ordinary differential equation

$$\dot{\theta} = \omega$$

$$\dot{\omega} = \frac{1}{ml^2} (\tau - mgl \sin \theta)$$

can be solved numerically by an ODE solver

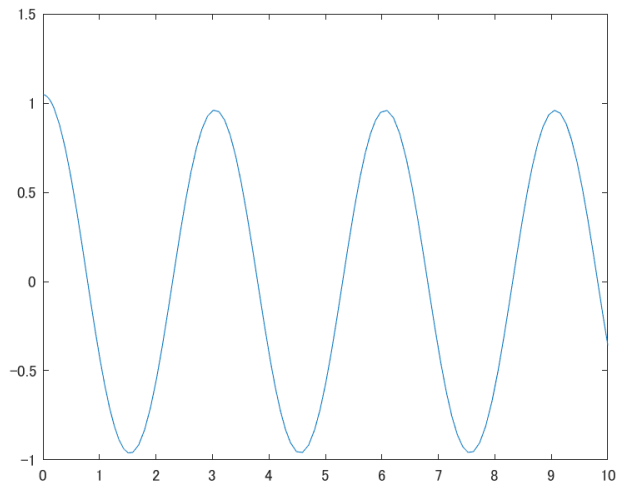
Example (simple pendulum)

Sample Programs

- solve the equation of motion of simple pendulum
- equation of motion of simple pendulum
- external torque

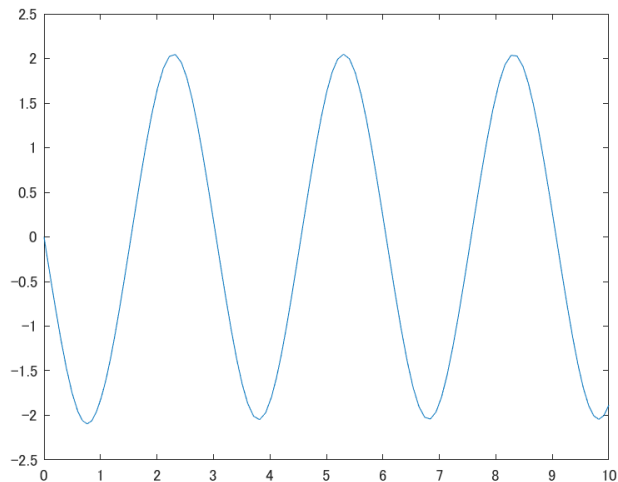
Example (simple pendulum)

Result



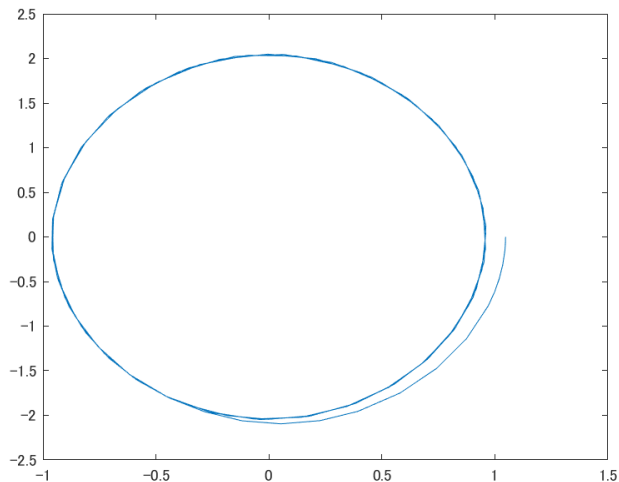
Example (simple pendulum)

Result



Example (simple pendulum)

Result



Example (pendulum with viscous friction)

Assumptions

viscous friction around supporting point C works

viscous friction causes a negative torque around C

magnitude of the torque is proportional to angular velocity

$$\text{viscous friction torque} = -b\dot{\theta} \quad (b: \text{positive constant})$$

Replacing τ by $\tau - b\dot{\theta}$:

$$(ml^2)\ddot{\theta} = (\tau - b\dot{\theta}) - mgl \sin \theta$$

⇓

$$\dot{\theta} = \omega$$

$$\dot{\omega} = \frac{1}{ml^2} (\tau - b\omega - mgl \sin \theta)$$

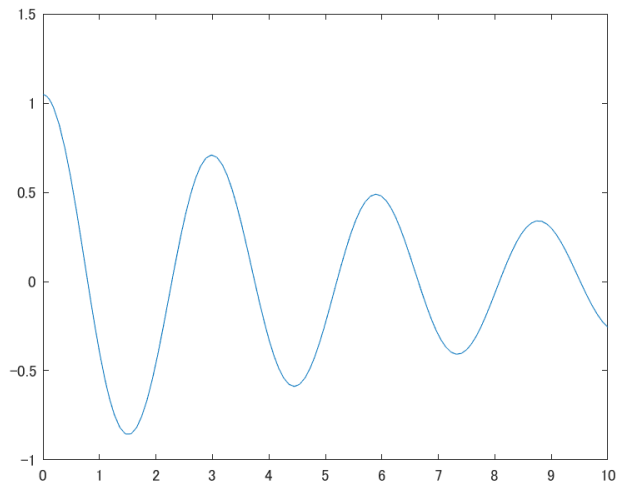
Example (pendulum with viscous friction)

Sample Programs

- solve the equation of motion of damped pendulum
- equation of motion of damped pendulum
- external torque

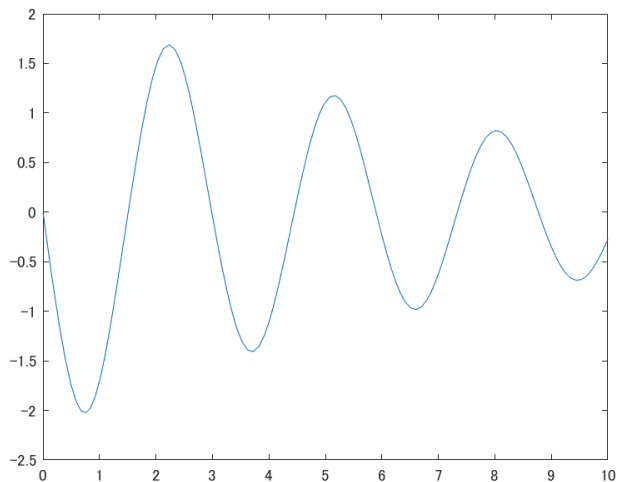
Example (pendulum with viscous friction)

Result



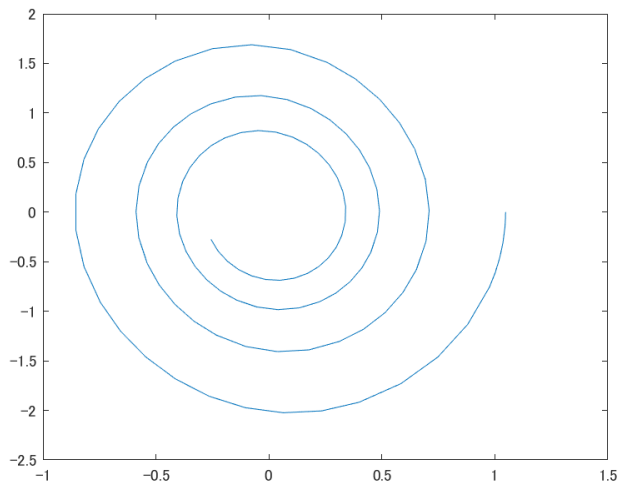
Example (pendulum with viscous friction)

Result

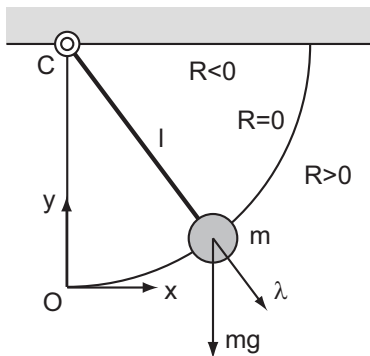


Example (pendulum with viscous friction)

Result



Example (pendulum in Cartesian coordinates)



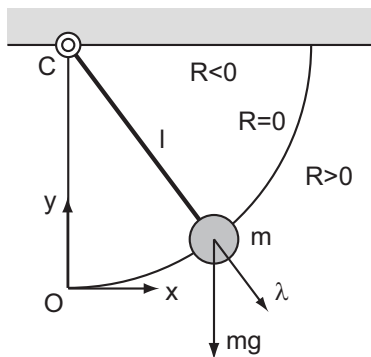
simple pendulum of length l and mass m suspended at point C

$[x, y]^T$: position of mass at time t

$[f_x, f_y]^T$: external force applied to mass at time t

Derive the motion of the pendulum in Cartesian coordinates.

Example (pendulum in Cartesian coordinates)



geometric constraint

distance between C and mass = l

$$R \triangleq \{x^2 + (y - l)^2\}^{1/2} - l = 0$$

Dynamics under single constraint

- T kinetic energy
- U potential energy
- W work done by external forces/torques

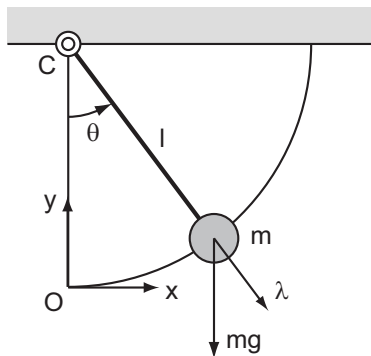
Lagrangian

$$\mathcal{L} = T - U + W + \lambda R$$

Lagrange equations of motion

$$\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}} = 0$$
$$\frac{\partial \mathcal{L}}{\partial y} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{y}} = 0$$

Example (pendulum in Cartesian coordinates)



$$T = \frac{1}{2}m\{\dot{x}^2 + \dot{y}^2\}$$

$$U = mgy, \quad W = f_x x + f_y y$$

$$R = \{x^2 + (y - l)^2\}^{1/2} - l$$

Example (pendulum in Cartesian coordinates)

Lagrangian

$$\mathcal{L} = \frac{1}{2}m\{\dot{x}^2 + \dot{y}^2\} - mgy + f_x x + f_y y + \lambda \left[\{x^2 + (y - l)^2\}^{1/2} - l \right]$$

partial derivatives

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial x} &= f_x + \lambda R_x, & \frac{\partial \mathcal{L}}{\partial \dot{x}} &= m\dot{x} \\ \frac{\partial \mathcal{L}}{\partial y} &= -mg + f_y + \lambda R_y, & \frac{\partial \mathcal{L}}{\partial \dot{y}} &= m\dot{y} \end{aligned}$$

Lagrange equations of motion

$$\begin{aligned} f_x + \lambda R_x - m\ddot{x} &= 0 \\ -mg + f_y + \lambda R_y - m\ddot{y} &= 0 \end{aligned}$$

Example (pendulum in Cartesian coordinates)

Lagrange equations of motion

$$\begin{array}{ccccccc} \begin{bmatrix} 0 \\ -mg \end{bmatrix} & + & \begin{bmatrix} f_x \\ f_y \end{bmatrix} & + & \lambda \begin{bmatrix} R_x \\ R_y \end{bmatrix} & + & \left\{ -m \begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} \right\} & = & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \text{gravitational} & & \text{external} & & \text{constraint} & & \text{inertial} & & \end{array}$$

dynamic equilibrium among forces

Example (pendulum in Cartesian coordinates)

three equations w.r.t. three unknowns x , y , and λ :

$$m\ddot{x} = f_x + \lambda R_x$$

$$m\ddot{y} = -mg + f_y + \lambda R_y$$

$$R = 0$$

Example (pendulum in Cartesian coordinates)

three equations w.r.t. three unknowns x , y , and λ :

$$m\ddot{x} = f_x + \lambda R_x$$

$$m\ddot{y} = -mg + f_y + \lambda R_y$$

$$R = 0$$

Mixture of **differential** and **algebraic** equations



Difficult to solve the mixture of differential and algebraic equations

Constraint stabilization method (CSM)

Constraint stabilization

convert algebraic eq. to its almost equivalent differential eq.

$$\text{algebraic eq. } R = 0$$



$$\text{differential eq. } \ddot{R} + 2\alpha\dot{R} + \alpha^2 R = 0$$

(α : large positive constant)

critical damping (converges to zero most quickly)

Constraint stabilization method (CSM)

Dynamic equation of motion under geometric constraint:

$$\text{differential eq. } \frac{\partial \mathcal{L}}{\partial \mathbf{q}} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} = \mathbf{0}$$

$$\text{algebraic eq. } R = 0$$

↓

$$\text{differential eq. } \frac{\partial \mathcal{L}}{\partial \mathbf{q}} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} = \mathbf{0}$$

$$\text{differential eq. } \ddot{R} + 2\alpha\dot{R} + \alpha^2 R = 0$$

can be solved numerically by an ODE solver.

Computing equation for constraint stabilization

Assume R depends on x and y : $R(x, y) = 0$

Differentiating $R(x, y)$ w.r.t time t :

$$\dot{R} = \frac{\partial R}{\partial x} \frac{dx}{dt} + \frac{\partial R}{\partial y} \frac{dy}{dt} = R_x \dot{x} + R_y \dot{y}$$

Differentiating $R_x(x, y)$ and $R_y(x, y)$ w.r.t time t :

$$\dot{R}_x = \frac{\partial R_x}{\partial x} \frac{dx}{dt} + \frac{\partial R_x}{\partial y} \frac{dy}{dt} = R_{xx} \dot{x} + R_{xy} \dot{y}$$

$$\dot{R}_y = \frac{\partial R_y}{\partial x} \frac{dx}{dt} + \frac{\partial R_y}{\partial y} \frac{dy}{dt} = R_{yx} \dot{x} + R_{yy} \dot{y}$$

Second order time derivative:

$$\begin{aligned} \ddot{R} &= (\dot{R}_x \dot{x} + R_x \ddot{x}) + (\dot{R}_y \dot{y} + R_y \ddot{y}) \\ &= (R_{xx} \dot{x} + R_{xy} \dot{y}) \dot{x} + R_x \ddot{x} + (R_{yx} \dot{x} + R_{yy} \dot{y}) \dot{y} + R_y \ddot{y} \end{aligned}$$

Computing equation for constraint stabilization

Second order time derivative:

$$\ddot{R} = \begin{bmatrix} R_x & R_y \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} + \begin{bmatrix} \dot{x} & \dot{y} \end{bmatrix} \begin{bmatrix} R_{xx} & R_{xy} \\ R_{yx} & R_{yy} \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix}$$

Equation to stabilize constraint:

$$-\begin{bmatrix} R_x & R_y \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} = \begin{bmatrix} \dot{x} & \dot{y} \end{bmatrix} \begin{bmatrix} R_{xx} & R_{xy} \\ R_{yx} & R_{yy} \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} + 2\alpha(R_x\dot{x} + R_y\dot{y}) + \alpha^2 R$$

$$\Downarrow \quad v_x \triangleq \dot{x}, \quad v_y \triangleq \dot{y}$$

$$-\begin{bmatrix} R_x & R_y \end{bmatrix} \begin{bmatrix} \dot{v}_x \\ \dot{v}_y \end{bmatrix} = \begin{bmatrix} v_x & v_y \end{bmatrix} \begin{bmatrix} R_{xx} & R_{xy} \\ R_{yx} & R_{yy} \end{bmatrix} \begin{bmatrix} v_x \\ v_y \end{bmatrix} + 2\alpha(R_x v_x + R_y v_y) + \alpha^2 R$$

Example (pendulum in Cartesian coordinates)

Equation for stabilizing constraint $R(x, y) = 0$:

$$-R_x \dot{v}_x - R_y \dot{v}_y = C(x, y, v_x, v_y)$$

where

$$C(x, y, v_x, v_y) = \begin{bmatrix} v_x & v_y \end{bmatrix} \begin{bmatrix} R_{xx} & R_{xy} \\ R_{yx} & R_{yy} \end{bmatrix} \begin{bmatrix} v_x \\ v_y \end{bmatrix} \\ + 2\alpha(R_x v_x + R_y v_y) + \alpha^2 R$$

In this example

$$P = \{x^2 + (y - l)^2\}^{-1/2}, \quad R_x = xP, \quad R_y = (y - l)P \\ R_{xx} = P - x^2 P^3, \quad R_{yy} = P - (y - l)^2 P^3 \\ R_{xy} = R_{yx} = -x(y - l)P^3$$

Example (pendulum in Cartesian coordinates)

Combining equations of motion and equation for constraint stabilization:

$$\begin{aligned}\dot{x} &= v_x \\ \dot{y} &= v_y \\ \begin{bmatrix} m & & -R_x \\ & m & -R_y \\ -R_x & -R_y & \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \lambda \end{bmatrix} &= \begin{bmatrix} f_x \\ -mg + f_y \\ C(x, y, v_x, v_y) \end{bmatrix}\end{aligned}$$

five equations w.r.t. five unknown variables x , y , v_x , v_y and λ

given x , y , v_x , $v_y \implies \dot{x}$, \dot{y} , \dot{v}_x , \dot{v}_y

This canonical ODE can be solved numerically by an ODE solver.

Example (pendulum in Cartesian coordinates)

Let $\mathbf{x} = [x, y]^T$. Introducing gradient vector

$$\mathbf{g} = \begin{bmatrix} R_x \\ R_y \end{bmatrix}$$

yields

$$\dot{R} = \mathbf{g}^T \dot{\mathbf{x}}$$

Introducing Hessian matrix

$$H = \begin{bmatrix} R_{xx} & R_{xy} \\ R_{yx} & R_{yy} \end{bmatrix}$$

yields

$$\ddot{R} = \mathbf{g}^T \ddot{\mathbf{x}} + \dot{\mathbf{x}}^T H \dot{\mathbf{x}}$$

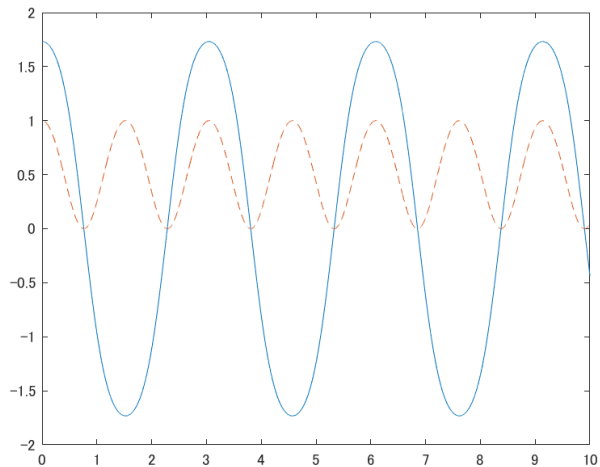
Example (pendulum in Cartesian coordinates)

Sample Programs

- solve the equation of motion of simple pendulum (Cartesian)
- equation of motion of simple pendulum (Cartesian)

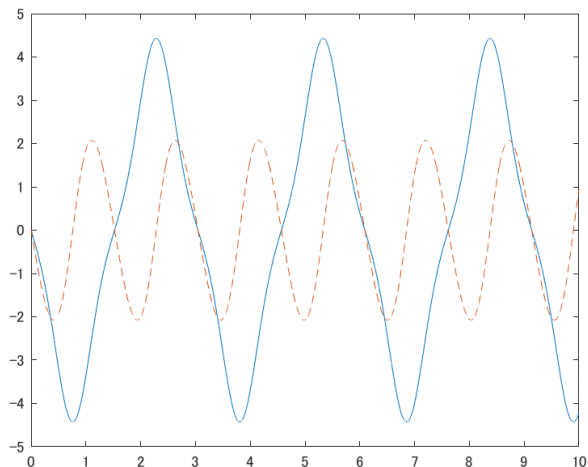
Example (pendulum in Cartesian coordinates)

$t-x, y$



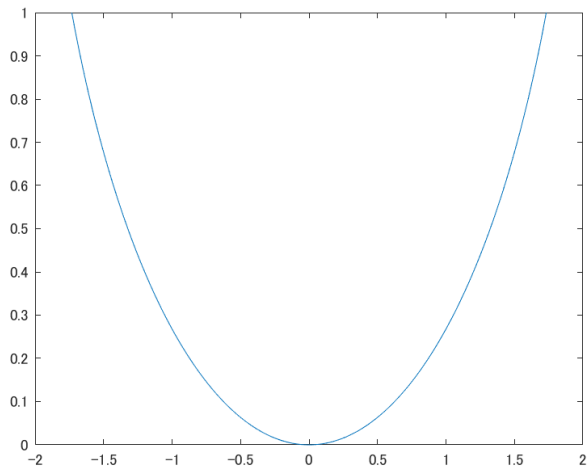
Example (pendulum in Cartesian coordinates)

$t-v_x, v_y$



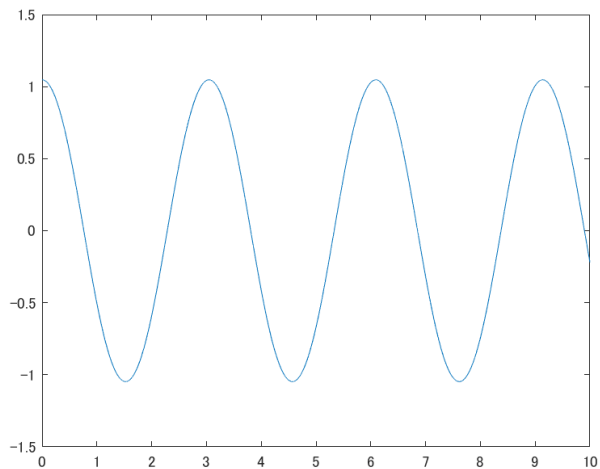
Example (pendulum in Cartesian coordinates)

$x-y$



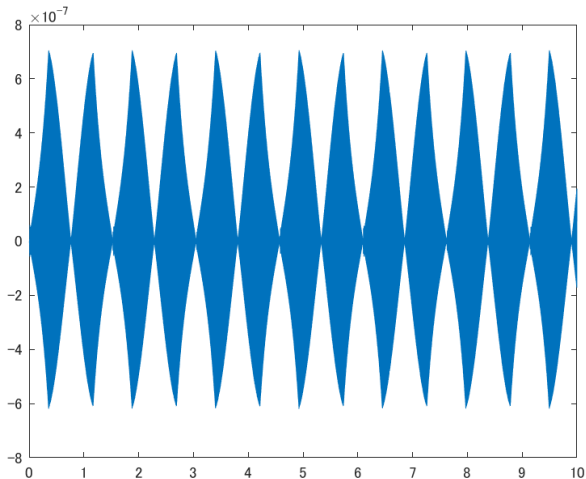
Example (pendulum in Cartesian coordinates)

t -computed θ



Example (pendulum in Cartesian coordinates)

t -constraint R



Notice

Lagrangian

$$\begin{aligned}\mathcal{L} &= T - U + W + \lambda R \\ &= T - (U - W - \lambda R) \\ &= T - I\end{aligned}$$

Lagrangian is equal to the difference between kinetic energy and internal energy under a constraint

Summary

Variational principles

- statics $I = U - W$
- statics under constraint $I = U - W - \lambda R$

$$\delta I \equiv 0$$

- dynamics $\mathcal{L} = T - U + W$
- dynamics under constraint $\mathcal{L} = T - U + W + \lambda R$

$$\frac{\partial \mathcal{L}}{\partial \mathbf{q}} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} = \mathbf{0}$$

- constraint stabilization method

Summary

How to solve a static problem

Solve (nonlinear) equations originated from variation

or

Numerically minimize internal energy

How to solve a dynamic problem

Step 1 Derive Lagrange equations of motion **analytically**

Step 2 Solve the derived equations **numerically**

Report

Report # 1 due date : Oct. 23 (Mon) 1:00 AM

Simulate the dynamic motion of a pendulum under viscous friction described with Cartesian coordinates x and y . Apply constraint stabilization method to convert the constraint into its almost equivalent ODE, then apply any ODE solver to solve a set of ODEs (equations of motion and equation for constraint stabilization) numerically.

Submit your report in pdf format to manaba+R

File name should be:

student number (11 digits) your name (without space).pdf

For example 12345678901HiraiShinichi.pdf

Report

Report # 2 due date : Oct. 30 (Mon) 1:00 AM

Assume that a system is described by four coordinates q_1 through q_4 . Two constraints R_1 and R_2 are imposed on the system. Let $\mathbf{q} = [q_1, q_2, q_3, q_4]^T$ and $\mathbf{R} = [R_1, R_2]^T$. Let \mathbf{g}_1 and H_1 be gradient vector and Hessian matrix related to R_1 while \mathbf{g}_2 and H_2 be gradient vector and Hessian matrix related to R_2 . Let J be Jacobian given by

$$J = \begin{bmatrix} \partial R_1 / \partial q_1 & \partial R_1 / \partial q_2 & \partial R_1 / \partial q_3 & \partial R_1 / \partial q_4 \\ \partial R_2 / \partial q_1 & \partial R_2 / \partial q_2 & \partial R_2 / \partial q_3 & \partial R_2 / \partial q_4 \end{bmatrix}$$

Show the following equations:

$$\dot{\mathbf{R}} = J\dot{\mathbf{q}}$$

$$\ddot{\mathbf{R}} = J\ddot{\mathbf{q}} + \begin{bmatrix} \dot{\mathbf{q}}^T H_1 \dot{\mathbf{q}} \\ \dot{\mathbf{q}}^T H_2 \dot{\mathbf{q}} \end{bmatrix}$$

Appendix: Variational calculus

Small virtual deviation of variables or functions.

$$y = x^2$$

Let us change **variable** x to $x + \delta x$, then variable y changes to $y + \delta y$ accordingly.

$$\begin{aligned}y + \delta y &= (x + \delta x)^2 \\ &= x^2 + 2x \delta x + (\delta x)^2 \\ &= x^2 + 2x \delta x\end{aligned}$$

Thus

$$\delta y = 2x \delta x$$

Appendix: Variational calculus

Small virtual deviation of variables or functions.

$$I = \int_0^T \{x(t)\}^2 dt$$

Let us change **function** $x(t)$ to $x(t) + \delta x(t)$, then variable I changes to $I + \delta I$ accordingly.

$$\begin{aligned} I + \delta I &= \int_0^T \{x(t) + \delta x(t)\}^2 dt \\ &= \int_0^T \{x(t)\}^2 + 2x(t) \delta x(t) dt \end{aligned}$$

Thus

$$\delta I = \int_0^T 2x(t) \delta x(t) dt$$

Appendix: Variational calculus

Variational operator δ

$\delta\theta$ virtual deviation of variable θ

$\delta f(\theta)$ virtual deviation of function $f(\theta)$

$$\delta f(\theta) = f'(\theta)\delta\theta$$

virtual increment of variable $\theta \rightarrow \theta + \delta\theta$

increment of function $f(\theta) \rightarrow f(\theta + \delta\theta) = f(\theta) + f'(\theta)\delta\theta$
 $f(\theta) \rightarrow f(\theta) + \delta f(\theta)$

simple examples

$$\delta(5x) = 5 \delta x \quad \delta x^2 = 2x \delta x$$

$$\delta \sin \theta = (\cos \theta) \delta \theta, \quad \delta \cos \theta = (-\sin \theta) \delta \theta$$

Appendix: Variational calculus

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$$\delta \sin \theta = (\cos \theta) \delta \theta, \quad \delta \cos \theta = (-\sin \theta) \delta \theta$$

Appendix: Variational calculus

assume that θ depends on time t

virtual increment of function $\theta(t) \rightarrow \theta(t) + \delta\theta(t)$

$$\begin{aligned}\frac{d\theta}{dt} &\rightarrow \frac{d}{dt}(\theta + \delta\theta) = \frac{d\theta}{dt} + \frac{d}{dt}\delta\theta \\ \int \theta dt &\rightarrow \int (\theta + \delta\theta) dt = \int \theta dt + \int \delta\theta dt\end{aligned}$$

variation of derivative and integral

$$\begin{aligned}\delta \frac{d\theta}{dt} &= \frac{d}{dt}\delta\theta \\ \delta \int \theta dt &= \int \delta\theta dt\end{aligned}$$

variational operator and differential/integral operator can commute

Appendix: Lagrange multiplier method

converts minimization (maximization) under conditions into minimization (maximization) without conditions.

$$\begin{aligned} &\text{minimize } f(\mathbf{x}) \\ &\text{subject to } g(\mathbf{x}) = 0 \end{aligned}$$

⇓

$$\text{minimize } l(\mathbf{x}, \lambda) = f(\mathbf{x}) + \lambda g(\mathbf{x})$$

⇓

$$\begin{aligned} \frac{\partial l}{\partial \mathbf{x}} &= \frac{\partial f}{\partial \mathbf{x}} + \lambda \frac{\partial g}{\partial \mathbf{x}} = \mathbf{0} \\ \frac{\partial l}{\partial \lambda} &= g(\mathbf{x}) = 0 \end{aligned}$$

Appendix: Lagrange multiplier method (example)

Length of each edge of a cube is given by x , y , and z .

Determine x , y , and z that minimizes the surface of the cube when the cube volume is constantly specified by a^3 :

$$\text{minimize } S(x, y, z) = 2xy + 2yz + 2zx$$

$$\text{subject to } R(x, y, z) \triangleq xyz - a^3 = 0$$

Introducing Lagrange multiplier λ , the above conditional minimization can be converted into the following unconditional minimization:

$$\begin{aligned} \text{minimize } I(x, y, z, \lambda) &= S(x, y, z) + \lambda R(x, y, z) \\ &= 2xy + 2yz + 2zx + \lambda(xyz - a^3) \end{aligned}$$

Appendix: Lagrange multiplier method (example)

Calculating partial derivatives:

$$\frac{\partial I}{\partial x} = 2y + 2z - \lambda yz = 0 \quad (1)$$

$$\frac{\partial I}{\partial y} = 2z + 2x - \lambda zx = 0 \quad (2)$$

$$\frac{\partial I}{\partial z} = 2x + 2y - \lambda xy = 0 \quad (3)$$

$$\frac{\partial I}{\partial \lambda} = xyz - a^3 = 0 \quad (4)$$

Calculating (1) $\cdot x -$ (2) $\cdot y$, we have

$$z(x - y) = 0,$$

which directly yields $x = y$. Similarly, we have $y = z$ and $z = x$. Consequently, we concludes $x = y = z = a$.

Appendix: ODE solver

Let us solve **van del Pol equation**:

$$\ddot{x} - 2(1 - x^2)\dot{x} + x = 0$$

Canonical form:

$$\dot{x} = v$$

$$\dot{v} = 2(1 - x^2)v - x$$

State variable vector:

$$\mathbf{q} = \begin{bmatrix} x \\ v \end{bmatrix}$$

Appendix: ODE solver (MATLAB)

File `van_der_Pol.m` describes the canonical form:

```
function dotq = van_der_Pol (t,q)
    x = q(1);
    v = q(2);
    dotx = v;
    dotv = 2*(1-x^2)*v - x;
    dotq = [dotx; dotv];
end
```

File name `van_der_Pol` should coincide with function name `van_der_Pol`.

Appendix: ODE solver (MATLAB)

File `van_der_Pol_solve.m` solves van der Pol equation numerically:

```
timestep=0.00:0.10:10.00;  
qinit=[2.00;0.00];  
[time,q]=ode45(@van_der_Pol,timestep,qinit);  
  
% line style    solid -    broken -.    chain --    dotted :  
plot(time,q(:,1),'-', time,q(:,2),'-.');
```

Appendix: ODE solver (MATLAB)

```
>> time
```

```
time =
```

```
0
```

```
0.1000
```

```
0.2000
```

```
0.3000
```

```
0.4000
```

```
>> q
```

```
q =
```

```
2.0000      0
```

```
1.9917  -0.1504
```

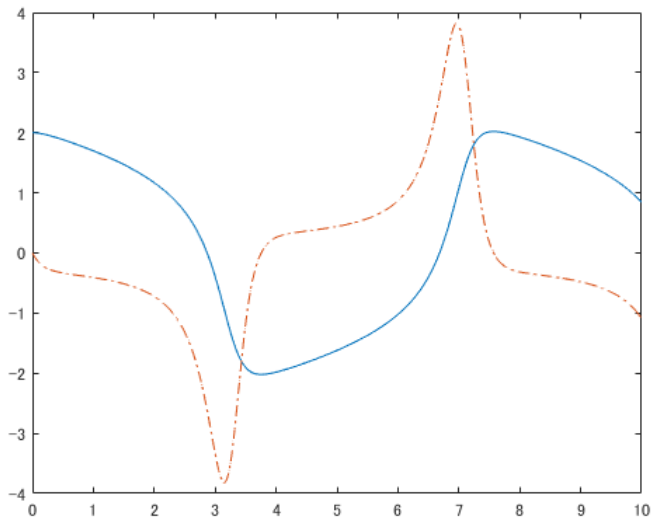
```
1.9721  -0.2338
```

```
1.9461  -0.2822
```

```
1.9163  -0.3125
```

The first and second columns corresponds to x and v .

Appendix: ODE solver (MATLAB)



Dynamics in variational form

Lagrangian $\mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}, t)$ w.r.t.

a set of generalized coordinates \mathbf{q} and its time derivative $\dot{\mathbf{q}}$
time integral of Lagrangian:

$$\text{action integral} = \int_{t_1}^{t_2} \mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}, t) dt$$

Variational principle in dynamics

variation of action integral vanishes

for any geometrically admissible variation of \mathbf{q}

$$\text{V.I.} \triangleq \int_{t_1}^{t_2} \delta \mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}, t) dt \equiv 0$$

for any $\delta \mathbf{q}$ satisfying $\delta \mathbf{q}(t_1) = 0$ and $\delta \mathbf{q}(t_2) = 0$

Dynamics in variational form

Lagrangian $\mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}, t)$ w.r.t.

a set of generalized coordinates \mathbf{q} and its time derivative $\dot{\mathbf{q}}$
time integral of Lagrangian:

$$\text{action integral} = \int_{t_1}^{t_2} \mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}, t) dt$$

Variational principle in dynamics

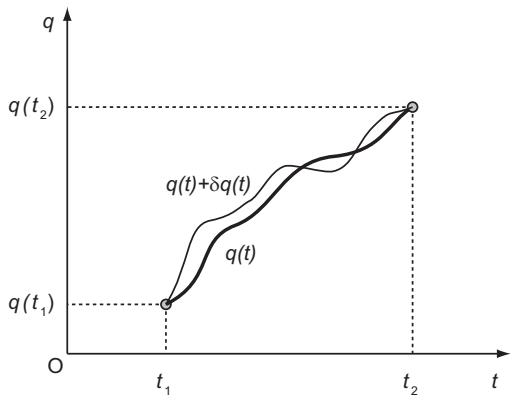
variation of action integral vanishes

for any **geometrically admissible variation of \mathbf{q}**

$$\text{V.I.} \triangleq \int_{t_1}^{t_2} \delta \mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}, t) dt \equiv 0$$

for any **$\delta \mathbf{q}$ satisfying $\delta \mathbf{q}(t_1) = 0$ and $\delta \mathbf{q}(t_2) = 0$**

Dynamics in variational form



Lagrangian corresponding to $\mathbf{q} + \delta \mathbf{q}$:

$$\mathcal{L}(\mathbf{q} + \delta \mathbf{q}, \dot{\mathbf{q}} + \delta \dot{\mathbf{q}}, t) = \mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}, t) + \left(\frac{\partial \mathcal{L}}{\partial \mathbf{q}} \right)^T \delta \mathbf{q} + \left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} \right)^T \delta \dot{\mathbf{q}}$$

Dynamics in variational form

variation of $L(\mathbf{q}, \dot{\mathbf{q}}, t)$:

$$\delta \mathcal{L} = \left(\frac{\partial \mathcal{L}}{\partial \mathbf{q}} \right)^T \delta \mathbf{q} + \left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} \right)^T \delta \dot{\mathbf{q}}$$

time integral of the second term:

$$\begin{aligned} \int_{t_1}^{t_2} \left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} \right)^T \delta \dot{\mathbf{q}} dt &= \int_{t_1}^{t_2} \left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} \right)^T \frac{d}{dt} \delta \mathbf{q} dt \\ &= \left[\left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} \right)^T \delta \mathbf{q} \right]_{t=t_1}^{t=t_2} - \int_{t_1}^{t_2} \left(\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} \right)^T \delta \mathbf{q} dt \\ &= 0 \quad \text{since } \delta \mathbf{q}(t_1) = 0 \text{ and } \delta \mathbf{q}(t_2) = 0 \\ &= \int_{t_1}^{t_2} \left(-\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} \right)^T \delta \mathbf{q} dt \end{aligned}$$

Dynamics in variational form

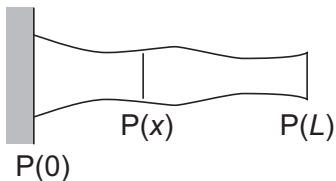
Variation of action integral:

$$\begin{aligned} \text{V.I.} &= \int_{t_1}^{t_2} \left(\frac{\partial \mathcal{L}}{\partial \mathbf{q}} \right)^T \delta \mathbf{q} dt + \int_{t_1}^{t_2} \left(-\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} \right)^T \delta \mathbf{q} dt \\ &= \int_{t_1}^{t_2} \left(\frac{\partial \mathcal{L}}{\partial \mathbf{q}} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} \right)^T \delta \mathbf{q} dt \equiv 0 \quad \forall \delta \mathbf{q} \\ &\quad \Downarrow \end{aligned}$$

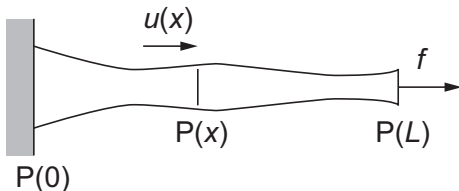
Lagrange equation of motion

$$\frac{\partial \mathcal{L}}{\partial \mathbf{q}} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} = \mathbf{0}$$

Example (extension of beam)



natural shape at time 0



deformed shape at time t

Deformation at time t is described by **function $u(x, t)$** ($0 \leq x \leq L$)

Example (extension of beam)

E : Young's modulus at point $P(x)$

A : Cross-sectional area at point $P(x)$

ρ : density at point $P(x)$

Assumption: $A(x)$, $E(x)$, and $\rho(x)$ do not change despite of extension, *i.e.* axial deformation is negligible.

Kinetic energy, elastic potential energy, and work done by external force

$$T = \int_0^L \frac{1}{2} \rho A \left(\frac{\partial u}{\partial t} \right)^2 dx$$

$$U = \int_0^L \frac{1}{2} EA \left(\frac{\partial u}{\partial x} \right)^2 dx$$

$$W = f u(L, t)$$

Example (extension of beam)

Lagrangian

$$\begin{aligned}\mathcal{L} &= T - U + W \\ &= \int_0^L \frac{1}{2} \rho A \left(\frac{\partial u}{\partial t} \right)^2 dx - \int_0^L \frac{1}{2} EA \left(\frac{\partial u}{\partial x} \right)^2 dx + f u(L, t)\end{aligned}$$

Variation of Lagrangian

$$\delta \mathcal{L} = \int_0^L \rho A \frac{\partial u}{\partial t} \frac{\partial}{\partial t} \delta u dx - \int_0^L EA \frac{\partial u}{\partial x} \frac{\partial}{\partial x} \delta u dx + f \delta u(L, t)$$

Recall

$$\begin{aligned}\delta U &= \int_0^L EA \frac{\partial u}{\partial x} \frac{\partial}{\partial x} \delta u dx \\ &= EA \frac{\partial u}{\partial x} \delta u \Big|_{x=L} - \int_0^L \frac{\partial}{\partial x} \left(EA \frac{\partial u}{\partial x} \right) \delta u dx\end{aligned}$$

Example (extension of beam)

Time-integral of the first term of $\delta\mathcal{L}$:

$$\begin{aligned} & \int_{t_1}^{t_2} \int_0^L \rho A \frac{\partial u}{\partial t} \frac{\partial}{\partial t} \delta u \, dx \, dt \\ &= \int_0^L \int_{t_1}^{t_2} \rho A \frac{\partial u}{\partial t} \frac{\partial}{\partial t} \delta u \, dt \, dx \\ &= \int_0^L \left\{ \left[\rho A \frac{\partial u}{\partial t} \delta u \right]_{t=t_1}^{t=t_2} - \int_{t_1}^{t_2} \frac{\partial}{\partial t} \left(\rho A \frac{\partial u}{\partial t} \right) \delta u \, dt \right\} dx \\ &= \int_0^L \left\{ - \int_{t_1}^{t_2} \rho A \frac{\partial^2 u}{\partial t^2} \delta u \, dt \right\} dx \\ &= \int_{t_1}^{t_2} \int_0^L \left\{ - \rho A \frac{\partial^2 u}{\partial t^2} \delta u \right\} dx \, dt \end{aligned}$$

Example (extension of beam)

$$\begin{aligned} \text{V.I.} &= \int_{t_1}^{t_2} \delta \mathcal{L} dt \\ &= \int_{t_1}^{t_2} \int_0^L \left\{ -\rho A \frac{\partial^2 u}{\partial t^2} + \frac{\partial}{\partial x} \left(EA \frac{\partial u}{\partial x} \right) \right\} \delta u dx dt \\ &\quad + \int_{t_1}^{t_2} \left\{ -EA \frac{\partial u}{\partial x} \Big|_{x=L} + f \right\} \delta u(L, t) dt \end{aligned}$$

should be equal to 0 for any $\delta u(x, t)$

⇓

$$-\rho A \frac{\partial^2 u}{\partial t^2} + \frac{\partial}{\partial x} \left(EA \frac{\partial u}{\partial x} \right) = 0, \quad -EA \frac{\partial u}{\partial x} \Big|_{x=L} + f = 0$$

Example (extension of beam)

Equation of deformation (partial differential equation)

$$\rho A \frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x} \left(EA \frac{\partial u}{\partial x} \right)$$

Boundary conditions

$$u(0, t) = 0$$

$$E(L, t)A(L, t) \frac{du}{dx}(L, t) = f(t)$$

Initial conditions (example)

$$u(x, 0) = 0, \quad \forall x \in [0, L]$$

$$\frac{du}{dt}(x, 0) = 0, \quad \forall x \in [0, L]$$

Example (extension of beam)

Assume that E , A , and ρ are constant:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

where $c = \sqrt{E/\rho}$

Given function $f(x)$, let

$$u(x, t) = f(x - ct)$$

Then

$$\begin{aligned}\frac{\partial u}{\partial x} &= f'(x - ct), & \frac{\partial^2 u}{\partial x^2} &= f''(x - ct), \\ \frac{\partial u}{\partial t} &= f'(x - ct)(-c), & \frac{\partial^2 u}{\partial t^2} &= f''(x - ct)(-c)^2\end{aligned}$$

Thus, $f(x - ct)$ is one solution of the PDE.

Example (extension of beam)

Given function $g(x)$, let

$$u(x, t) = g(x + ct)$$

Then

$$\partial u / \partial x = g'(x + ct), \quad \partial^2 u / \partial x^2 = g''(x + ct),$$

$$\partial u / \partial t = g'(x + ct)(+c), \quad \partial^2 u / \partial t^2 = g''(x + ct)(+c)^2$$

Thus, $g(x + ct)$ is one solution of the PDE.

Rayleigh wave

Solution $f(x - ct)$: wave propagating at speed $+c$

Solution $g(x + ct)$: wave propagating at speed $-c$