Chapter 4
Computing Dynamic Deformation

4.1 Variational principle in dynamics

Let us calculate the dynamic deformation of an elastic body. We apply variational principle in dynamics for the calculation. Let $T$ and $U$ be kinetic and potential energies of the body. External forces applied to the body will deform the body. Let $W$ be work done by external forces. Geometric constraints imposed on the body causes the deformation of the body. Let $R$ be a collective vector of geometric constraints. Variational principle in dynamics insists that a geometrically admissible motion of a holonomic system between two configurations at specified times is natural if and only if the variation of action integral vanishes for any variations. This is equivalent to the Lagrange equations of motions. Lagrangian of a system is defined as:

$$\mathcal{L} = T - U + W + \lambda^T R$$  \hspace{1cm} (4.1.1)

where $\lambda$ denote a collective vector consisting of Lagrange multipliers corresponding to individual constraints.

In finite element approximation, deformation of an elastic body is described by nodal displacement vector $u_N$ and its time-derivative $\dot{u}_N$, implying that the Lagrangian is a function of vector $u_N$ and $\dot{u}_N$. Lagrange equation of motion and deformation is then described as follows:

$$\frac{\partial \mathcal{L}}{\partial u_N} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{u}_N} = 0$$  \hspace{1cm} (4.1.2)

A set of constraints $R = 0$ can be converted into a set of ordinary differential equations stabilizing the constraints:

$$\ddot{R} + 2\alpha \dot{R} + \alpha^2 R = 0$$  \hspace{1cm} (4.1.3)

where $\alpha$ is a positive constant. By solving the above two ordinary differential equations, we can compute dynamic deformation of a body.

Many algorithms for solving a set of ordinary differential equations (ODEs) have been proposed and available. We can apply such ODE solvers to the above ordinary differential equations. For example, MATLAB offers ODE solvers such as ode45 and ode15s. Solving a set of ordinary differential equations, we can obtain $u_N(t)$, which sketches dynamic deformation of a body during a given time period.
4.2 Dynamic deformation of one-dimensional soft body

Let us formulate the dynamic deformation of an regular-shaped elastic beam of its length \( L \). Assume that cross-sectional area \( A \), Young’s modulus \( E \), and density \( \rho \) are uniform along the beam, implying that they are constants. Dividing \([0, L]\) into four small regions, kinetic and strain potential energies of the beam are described as follows:

\[
T = \frac{1}{2} \dot{u}_N^T M u_N, \quad U = \frac{1}{2} u_N^T K u_N
\]

(see eqs. (2.2.16)(2.2.6)), where inertia matrix \( M \) is given in eq. (2.2.17) and stiffness matrix \( K \) is described in eq. (2.2.5). Assume that end point \( P(0) \) is fixed to space while an external force \( f \) is applied to end point \( P(L) \). Work done by the external force is then described as:

\[
W = f^T u_N
\]

where \( f = [0, 0, 0, 0, f]^T \). Since displacement of point \( P(0) \) should be equal to zero, the following geometric constraint must be satisfied:

\[
R = a^T u_N = 0
\]

where \( a = [1, 0, 0, 0, 0]^T \). Consequently, we have the following Lagrangian:

\[
\mathcal{L}(u, \dot{u}) = \frac{1}{2} \dot{u}_N^T M u_N - \frac{1}{2} u_N^T K u_N + f^T u_N + \lambda_a a^T u_N = 0
\] (4.2.1)

where \( \lambda_a \) is a Lagrange multiplier corresponding to a single constraint \( a^T u_N = 0 \). Since \( M \) and \( K \) are constant matrices, we have

\[
\frac{\partial \mathcal{L}}{\partial u_N} = -K u_N + f + \lambda_a a, \quad \frac{\partial \mathcal{L}}{\partial \dot{u}_N} = M \dot{u}_N
\]

which directly yields

\[
-K u_N + f + \lambda_a a - M \ddot{u}_N = 0
\] (4.2.2)

Equation for stabilizing constraint \( a^T u_N = 0 \) is given by

\[
a^T \ddot{u}_N + 2\alpha a^T \dot{u}_N + a^T a u_N = 0
\] (4.2.3)

where \( \alpha \) is a positive constant. Introducing \( v_N = \dot{u}_N \), the above two ordinary differential equations turn into

\[
M \ddot{v}_N - a\lambda_a = -K u_N + f
\]
\[
-a^T \ddot{v}_N = 2\alpha a^T v_N + \alpha^2 a^T u_N
\]

Combining the above two equations, we have

\[
\begin{bmatrix}
M & -a \\
-a^T & 0
\end{bmatrix}
\begin{bmatrix}
\dot{v}_N \\
\lambda_a
\end{bmatrix}
= 
\begin{bmatrix}
-K u_N + f \\
C(u_N, v_N)
\end{bmatrix}
\] (4.2.4)

where \( C(u_N, v_N) = 2\alpha a^T v_N + \alpha^2 a^T u_N \). Note that the coefficient matrix of the above equation is constant. Given \( u_N \) and \( v_N \), we can calculate the right-side vector of the above equation, implying that solving the above linear equation yields \( \dot{v}_N \). Consequently, given \( u_N \) and \( v_N \), we can calculate their time derivatives \( \ddot{u}_N \) and \( \ddot{v}_N \), which offers a canonical form of ordinary differential equations. Any ODE solver is available to solve the canonical form of ordinary differential equations numerically.
Example Let us calculate the dynamic deformation of an elastic beam of length $L$. Divide region $[0, L]$ into five small regions. During time interval $[0, t_{\text{push}}]$, one end $P(0)$ of the beam is in contact with the floor and a pushing force $f_{\text{push}}$ is applied to the other end $P(L)$ to shrink the beam. During $[t_{\text{push}}, t_{\text{end}}]$, the applied force is released. A reaction force exerts to the contacting end as long as the end is in contact with the floor. Penalty method is applied to calculate the reaction force. Namely,

$$
\text{reaction force} = \begin{cases} 
-k_{\text{floor}}Au(0) & u(0) \leq 0 \\
0 & u(0) > 0
\end{cases}
$$

Figure 4.1 shows a calculation result with $L = 10$ cm, $A = 2$ cm$^2$, $E = 50$ kPa, $c = 0.2$ kPa·s, $\rho = 1.0$ g/cm$^3$, $f_{\text{push}} = 2.0$ N, $t_{\text{push}} = 0.2$ s, and $k_{\text{floor}} = 0.1$ MPa/cm. Deformation of the beam during $[t_{\text{push}}, t_{\text{end}}]$ and jumping motion during $[t_{\text{push}}, t_{\text{end}}]$ are calculated properly.

4.3 Dynamic deformation of two-dimensional soft body

Let us formulate the dynamic deformation of a two-dimensional elastic body specified by $S$. Assume that Lamé’s constants $\lambda, \mu$, density $\rho$, and width $h$ are uniform over the body, implying that they are constants. Approximating region $S$ by a finite number of triangles and letting $u_N$ be nodal displacement vector, kinetic and strain potential energies are described as:

$$
T = \frac{1}{2} u_N^T M \dot{u}_N, \quad U = \frac{1}{2} u_N^T K u_N
$$

where $M$ denote the inertia matrix and $K$ represent the stiffness matrix. Work done by external forces can be formulated as:

$$
W = f^T u_N
$$

A set of geometric constraints imposed on the body can be described as

$$
R = A^T u_N - b(t) = 0
$$
where $b(t)$ is a collective vector specifying the position of constrained points at time $t$. Consequently, we have the following Lagrangian:

$$\mathcal{L}(u, \dot{u}) = T - U + W + \lambda^T R$$

$$= \frac{1}{2} u_N^T M \dot{u}_N - \frac{1}{2} u_N^T K u_N + f^T u_N + \lambda^T (A^T u_N - b(t)), \quad (4.3.1)$$

where $\lambda$ is a collective vector of Langrange multipliers corresponding to a set of constraints. Since $M$ and $K$ are constant matrices, we have

$$\frac{\partial \mathcal{L}}{\partial u_N} = -K u_N + f + \lambda \alpha, \quad \frac{\partial \mathcal{L}}{\partial \dot{u}_N} = M \ddot{u}_N$$

which directly yields

$$-K u_N + f + \lambda \alpha - M \ddot{u}_N = 0 \quad (4.3.2)$$

Equation for stabilizing constraint $A u_N = 0$ is given by

$$(A^T \ddot{u}_N - \ddot{b}(t)) + 2\alpha (A^T \dot{u}_N - \dot{b}(t)) + \alpha^2 (A^T u_N - b(t)) = 0, \quad (4.3.3)$$

where $\alpha$ is a positive constant. Introducing $v_N = \dot{u}_N$, the above two ordinary differential equations collectively turn into

$$\begin{bmatrix} M & -A \\ -A^T & \lambda \end{bmatrix} \begin{bmatrix} \dot{v}_N \\ \lambda \end{bmatrix} = \begin{bmatrix} -K u_N + f \\ C(u_N, v_N) \end{bmatrix} \quad (4.3.4)$$

where

$$C(u_N, v_N) = -\ddot{b}(t) + 2\alpha (A^T v_N - \dot{b}(t)) + \alpha^2 (A^T u_N - b(t)).$$

The above equation provides a canonical form of ordinary differential equations, which can be solved numerically by any ODE solver.

**Damping forces** Let us introduce damping forces caused by viscosity of robot body material. Let $c$ be *viscous modulus* of the material. Linear isotropic viscosity can be characterized by two constants

$$\lambda^{vis} = \frac{\nu c}{(1 + \nu)(1 - 2\nu)}, \quad \mu^{vis} = \frac{c}{2(1 + \nu)},$$

where $\nu$ denote Poisson’s ratio. Then, a set of damping forces at nodal points is described by

$$-B \ddot{u}_N$$

where

$$B = \lambda^{vis} J_\lambda + \mu^{vis} J_\mu$$

is referred to as *damping matrix*. Replacing elastic forces $-K u_N$ in eq. (4.3.4) by viscoelastic forces $-K u_N - B u_N$, we can compute the dynamic deformation of a viscoelastic body.
Example (deforming body)  Let us calculate the dynamic deformation of a two-dimensional elastic square body of width \( w \) shown in Fig. 4.2. Let us divide the square region into \( 3 \times 3 \times 2 \) triangles. During time interval \([0, t_{push}]\), the bottom of the body is fixed to the floor and edge \( P_{14}P_{15} \) moves downward at a constant velocity \( v_{push} \). During \([t_{push}, t_{hold}]\), the bottom remains fixed and edge \( P_{14}P_{15} \) keeps its position. During \([t_{hold}, t_{end}]\), the bottom remains fixed while \( P_{14}P_{15} \) is released.

During \([0, t_{push}]\), the following constraints are imposed on the square body:

\[
\begin{align*}
  u_1 &= u_2 = u_3 = u_4 = 0 \\
  u_{14} &= u_{15} = 0 + v_{push}t
\end{align*}
\]

where \( v_{push} = [0, -v_{push}]^T \). Matrix

\[
A^T = \begin{bmatrix}
I & \cdots & \cdots & \cdots \\
I & \cdots & \cdots & \cdots \\
I & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & I \\
1 & 2 & 3 & 4 & 14 & 15-th block columns
\end{bmatrix}
\]

specifies the nodal point displacements under constraints, that is,

\[
A^T u_N = \begin{bmatrix}
u_1 \\
u_2 \\
u_3 \\
u_4 \\
u_{14} \\
u_{15}
\end{bmatrix}
\]

A collective vector specifying the position of constrained points at time \( t \) is then given by

\[
b(t) = b_0 + b_1 t
\]
where

\[
\begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
\end{bmatrix}, \quad \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
v_{\text{push}} \\
v_{\text{push}} \\
\end{bmatrix}
\]

Noting that \(\dot{b}(t) = b_1\) and \(\ddot{b}(t) = 0\), we find

\[
C(u_N, v_N) = 2\alpha (A^T v_N - b_1) + \alpha^2 (A^T u_N - (b_0 + b_1 t)).
\]

During \([t_{\text{push}}, t_{\text{hold}}]\), we find

\[
\begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
v_{\text{push}} t_{\text{push}} \\
v_{\text{push}} t_{\text{push}} \\
\end{bmatrix}, \quad \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
v_{\text{push}} t_{\text{push}} \\
v_{\text{push}} t_{\text{push}} \\
\end{bmatrix}
\]
During $[t_{\text{hold}}, t_{\text{end}}]$, we have the following constraints:

$$u_1 = u_2 = u_3 = u_4 = 0,$$

which yields,

$$A^T = \begin{bmatrix}
I & I & \cdots \\
I & I & \cdots \\
I & I & \cdots 
\end{bmatrix},$$

$$b_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad b_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$
Example (jumping body) Let us simulate the jumping of a two-dimensional elastic square body. Divide the square region into $3 \times 3 \times 2$ triangles (Fig. 4.2). During $[0, t_{\text{push}}]$, the bottom of the body is fixed to the floor and edge $P_{14}P_{15}$ moves downward at a constant velocity $v_{\text{push}}$. During $[t_{\text{push}}, t_{\text{hold}}]$, the bottom remains fixed and edge $P_{14}P_{15}$ keeps its position. During $[t_{\text{hold}}, t_{\text{end}}]$, all constraints are released, but reaction forces are applied to the bottom region contacting to the floor. Penalty method calculates a reaction force applied to bottom edge $P_iP_j$. Let $P(x)$ be any point in the edge and $[u(x), v(x)]^T$ denote its displacement. Assume that contact pressure at point $P(x)$ be given by

$$p(x) = \begin{cases} -k_{\text{floor}}v(x) & v(x) \leq 0 \\ 0 & v(x) > 0 \end{cases}$$

Let us convert contact pressure over contact region into equivalent forces $f_i, f_j$ at point $P_i, P_j$. Assume that forces $f_i, f_j$ are perpendicular to $P_iP_j$. Forces $f_i, f_j$ are determined so that the total force and total moment around $P_i$ or $P_j$ coincide with each other. Calculating
equivalent forces $f_i$, $f_j$, we have

\begin{align*}
\text{if } v_i \leq 0 \text{ and } v_j \leq 0 & \quad f_i = \frac{wh}{3} p_i + \frac{wh}{6} p_j, & f_j = \frac{wh}{6} p_i + \frac{wh}{3} p_j \\
\text{if } v_i \leq 0 \text{ and } v_j > 0 & \quad f_i = \frac{wh}{2} \left( \frac{-v_i}{v_j - v_i} \right) p_i - f_j, & f_j = \frac{wh}{6} \left( \frac{-v_i}{v_j - v_i} \right)^2 p_i \\
\text{if } v_i > 0 \text{ and } v_j \leq 0 & \quad f_i = \frac{wh}{6} \left( \frac{v_j}{v_j - v_i} \right)^2 p_j, & f_j = \frac{wh}{2} \left( \frac{v_j}{v_j - v_i} \right) p_j - f_i \\
\text{if } v_i > 0 \text{ and } v_j > 0 & \quad f_i = 0, & f_j = 0
\end{align*}

where $p_i = p(x_i)$, $p_j = p(x_j)$, and $w = x_j - x_i$ (see Problem 1). Figure 4.5 shows a snapshot of the computation result with $w = 30 \text{ cm}$, $h = 1 \text{ cm}$, $E = 1.0 \text{ MPa}$, $c = 40 \text{ Pa} \cdot \text{s}$, $\nu = 0.48$, $\rho = 1.0 \text{ g/cm}^3$, $t_{\text{push}} = 0.5 \text{ s}$, $t_{\text{hold}} = 1.0 \text{ s}$, $v_{\text{push}} = 16 \text{ cm/s}$, and $k_{\text{floor}} = 2.0 \text{ kPa/cm}$. Figure 4.6 shows a snapshot of the computation result under a finer mesh; the square region consists of $9 \times 9 \times 2$ triangles. These computation results demonstrate that deformation can be simulated properly. Additionally, computation results depend on mesh.