Approximations of non-smooth integral type functionals of one dimensional diffusion processes[*]

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Abstract

In this article, we obtain the weak and strong rates of convergence of time integrals of non-smooth functions of a one dimensional diffusion process. We propose the use the exact simulation scheme to simulate the process at discretization points. In particular, we also present the rates of convergence for the weak and strong error of approximation for the local time of a one dimensional diffusion process as an application of our method.

1 Introduction

Let $X = (X_t)_{t \in [0,T]}$ be a 1-dimensional diffusion process defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ as the solution of the stochastic differential equation (sde)

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t, \quad X_0 = x_0 \in \mathbb{R}, \ t \in [0,T], \tag{1.1}$$

where W is a one dimensional standard Brownian motion and the coefficients $b, \sigma : \mathbb{R} \to \mathbb{R}$ are bounded with bounded derivatives.

The problem of estimating $\mathbb{E}[f(F(X))]$, where $X = (X_t)_{t \in [0,T]}$, $f : \mathbb{R} \to \mathbb{R}$ and $F : C[0,T] \to \mathbb{R}$ with $C[0,T]$ is the set of continuous real valued functions over the time interval $[0,T]$, is of interest in the recent literature of weak approximations. For example, the cases of $F(X) = \int_0^T h(X_s)ds$ where h is in general a measurable bounded function and $F(X) = \max_{s \leq T} X_s$ are two typical examples. In both cases, one observes that the functional F is not regular and therefore the analysis of the error can not be carried out with classical techniques such as the ones exposed in [13] (see also [1, 16] for other related cases and techniques).

To the best of our knowledge, most of the previous results that evaluate a weak/strong rate of convergence in this setting assume that the functional F is Lipschitz or smooth with respect to supremum norm.

In this paper we study the rate of convergence of a numerical scheme to estimate the expectation of some path dependent irregular functionals of X. To be more precise, we want to estimate

$$I(f) = \mathbb{E}\Big[f\Big(\int_0^T h(X_s)ds\Big)\Big],$$

where f is a smooth function with polynomial growth at infinity and h is a function which is not necessarily smooth. For example, this is the case of $h(x) = I_{\{x \in A\}}$ where $A \subset \mathbb{R}$ or $h(x) = \delta_0(x)$, the Dirac delta distribution function at 0.

We remark that in our case, the functional $F : X \mapsto \int_0^T I_{\{X_s \in A\}}ds$ is not continuous in the sup-norm topology of $C[0,T]$. It is worth to note here that the path dependent random variable $\int_0^T I_{\{X_s \in A\}}ds$ is usually called the occupation time of X in A. In the case that $h(x) = \delta_0(x)$ then F stands for the local time of X up to time T at 0.

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Estimating how much time a diffusion spends on a set is an important problem in various applications. This is a classical problem which appears in many applied domains such as mathematical finance, queueing theory and biology.

For example, the occupation time of continuous diffusion processes plays an important role in pricing some type of occupation time derivatives like corridor option and eddoko option (see [7, 15, 2, 4, 5, 8] and references therein). Generally speaking, the price of such options depends on the amount of time that the continuous time price process, say X, stays in some designated intervals.

In this article we study the $L^p(\mathbb{P})$-strong approximation error

$$\left\{ \mathbb{E}\left[\left| \int_0^T h(X_s)ds - h(X_{\eta_n(s)})ds \right|^p \right] \right\}^{1/p} \tag{1.2}$$

and the weak approximation error

$$\left| \mathbb{E}\left[f\Big(\int_0^T h(X_s)ds \Big) - f\Big(\int_0^T h(X_{\eta_n(s)})ds \Big) \right] \right|.$$

Here we have used the notation $\eta_n(s) = \max\{t_i,\ t_i \le s\}$ for a uniform partition $\pi_n = \{t_i = \frac{iT}{n};\ i = 0, ..., n\}$ and h lies within a class of non-necessarily regular functions that admit $h(x) = I_{\{x \in A\}}$ or $h(x) = \delta_0(x)$ as an example.

As one needs to simulate X_{t_i} for different values of $i = 1, ..., n$, we concentrate on the one dimensional case and use the retrospective exact simulation method introduced in [3] to generate independent copies of discrete samples of X and then use the Monte Carlo method to approximate $I(f)$.

From a heuristic mathematical point of view, one may say that the time integral operator in F should regularize the properties of F. On the other hand, the fact that h is a non-regular function or even a Schwartz distribution function introduces a strong non-regular character in the integrand of F. The rates of convergence for the strong and weak approximation errors which appear in Theorems 2.3, 2.4 and 2.6 reflect the interplay between these two opposing characteristics.

We now proceed with a discussion of the method of analysis used for the problem and its relationship with other close results in the literature.

In the particular case that $f(x) = x$ the study of the weak rate of convergence has been done in [10] and [9] using basic estimates on the forward-backward Kolmogorov equation and a clever combination with classical techniques. This technique is not applicable to our case. In fact, if one applies a Taylor expansion to the error

$$\mathbb{E}\left[f\Big(\int_0^T I_{\{X_s \in A\}}ds \Big) - f\Big(\int_0^T I_{\{X_{\eta_n(s)} \in A\}}ds \Big) \right],$$

one quickly finds out that the irregularity of the functional $F(X) = \int_0^T I_{\{X_s \in A\}}ds$ appears in multiplicative form in the error and therefore the expansion only makes the problem more difficult to handle. This problem does not appear in the particular case that $f(x) = x$ and therefore the interchange between expectation and integral makes the problem somewhat easier.

In order to find the strong rate of convergence in (1.2) for moderate irregular functions h (e.g. the indicator function), we first consider a class $\mathcal{A}$ (defined before Proposition 2.1), which includes the indicator function. In particular, in the definition of property $\mathcal{A}(iii)$, it is important that uniform upper estimates of expectations of $\{h'_N\}_{N \in \mathbb{N}}$ are satisfied where $\{h_N\}_{N \in \mathbb{N}}$ is an approximation sequence of h.

Next, we assume uniform ellipticity for the diffusion coefficient σ in order to apply the Lamperti transform to X in (1.1) so that the problem is reduced to the consideration of a simpler process Y which has a unit diffusion coefficient. For Y we have uniform bounds on its Malliavin derivatives and Gaussian bounds on its density, both of which are needed in our proofs (see Section 3.1 and Remark 3.2 for more details).

Then our method uses the case $f(x) = x$ as a first building block and then a centering argument together with the Clark-Ocone formula allows us to obtain a first expression for the error. This formula explodes due to the stochastic derivative in the Clark-Ocone formula. On the other hand, one obtains some regularity due to the time integrals of conditional expectations which allow us to deal with the general case in a non-trivial manner.

The technique described above which is used in order to obtain the strong rate of convergence may have a wider applicability. In fact, it also serves to study of the corresponding weak type problem and further to investigate the approximations of local time.

This is our second step: to consider approximations for local times. We obtain the strong rates of convergence and the weak rates in the case where f is a polynomial function.

As mentioned before, the use of the Lamperti transformation and the uniform ellipticity assumption are crucial for the success of our method of proof. The cases where f satisfies weaker conditions, X is multi-dimensional or the diffusion σ is not uniformly elliptic remain as challenging problems.

Throughout the article, constants are denoted by C or K which may change from one line to the next and which are independent of the partition n, N and ϵ (to be introduced later as the approximation parameter for the Dirac delta function) but may depend on other parameters of the problem such as the time parameter $T > 0$, the coefficients of the sde or the initial point x_0. The time T is fixed throughout the article.

$C^1(A, B)$ denotes the space of once continuously differentiable functions from A to B. In the case that $A = B$, we use the notation $C^1(A) \equiv C^1(A, A)$ and $C^1 \equiv C^1(\mathbb{R})$. Similarly, $C_b^k(\mathbb{R})$ denotes the space of real valued bounded functions which are $k \in \mathbb{N}$ times continuously differentiable with bounded derivatives. $L^p(\mu)$ denotes the space of p-th power integrable functions with respect to the measure μ and which induces a norm denoted by $\| \cdot \|_{L^p(\mu)}$.

2 Main results

A function h is called *exponentially bounded* if there exist positive constants K_1, K_2 such that $|h(x)| \le K_1 e^{K_2|x|}$.

$C_{\exp}^k(\mathbb{R}, A)$ denotes the space of exponentially bounded functions taking values in A, which have continuous derivatives of any order up to k. In the case where $A = \mathbb{R}$, we may just write $C_{\exp}^k(\mathbb{R})$. In the particular case that $\alpha \in (0, 1]$ we also define $\hat{C}_{\exp}^\alpha(\mathbb{R}, A)$ as the space of α-Hölder exponentially bounded functions. Note in particular that $\hat{C}_{\exp}^1(\mathbb{R}, A) \neq C_{\exp}^1(\mathbb{R}, A)$.

Let $\mathcal{A}$ be a class of exponentially bounded functions $h : \mathbb{R} \to \mathbb{R}$ such that there exists a sequence of functions $(h_N)_{N \in \mathbb{N}} \subset C_{\exp}^3(\mathbb{R})$ satisfying:

$$\begin{cases} \mathcal{A}(i): & h_N \to h \text{ in } L_{loc}^1(\mathbb{R}), \\ \mathcal{A}(ii): & \sup_N |h_N(x)| + |h(x)| \le K_1 e^{K_2|x|} \text{ for some constant } K_1, K_2, \\ \mathcal{A}(iii): & K(h) := \sup_{N, u \in [0, CT]} \int |h_N'(x)| e^{-\frac{x^2}{u}} dx < \infty, \text{ for every positive constant } C. \end{cases}$$

Note that $K(h)$ defines some notion of norm which will appear in the error estimates. Clearly, $\hat{C}_{\exp}^1(\mathbb{R}) \subset \mathcal{A}$. The following Proposition shows that the class $\mathcal{A}$ is even larger.

Proposition 2.1. *$\mathcal{A}$ is a vector space on $\mathbb{R}$. Furthermore, $\mathcal{A}$ contains all monotone, exponentially bounded functions.*

Proof. It is obvious that $\mathcal{A}$ is a vector space on $\mathbb{R}$. Now we show that $\mathcal{A}$ contains all monotone, exponentially bounded functions. Indeed, let h be an non-decreasing, exponentially bounded function on $\mathbb{R}$. We introduce $(\rho_N)_{N \in \mathbb{N}}$ a sequence of mollifiers given by $\rho_N(x) = N\rho(Nx)$ with $\rho(x) = e^{\frac{1}{x^2-1}} I_{|x| < 1}$. Finally, we set $h_N(x) = \int h(x - y)\rho_N(y)dy$. Since $|h(x)| \le Ke^{K|x|}$ for some $K > 0$, we have

$$|h_N(x)| \le \int Ke^{K|x-y|}\rho_N(y)dy \le Ke^{K|x|} \int_0^{1/N} Ne^{K|y|} e^{\frac{1}{N^2y^2-1}} dy \le Ke^{K(|x|+1)}.$$

Therefore $(h_N)_{N \in \mathbb{N}}$ is uniformly exponentially bounded. Furthermore, since h is non-decreasing, so is h_N. It means that $h_N' \ge 0$ and

$$\int |h_N'(x)| e^{-\frac{x^2}{u}} dx = \int e^{-\frac{x^2}{u}} dh_N(x) = \int h_N(x) \frac{2x}{u} e^{-\frac{x^2}{u}} dx$$
$$\le \frac{C}{\sqrt{u}} \int e^{K|x| - \frac{x^2}{2u}} dx < \infty,$$

implies that $(h_N)_{N \in \mathbb{N}}$ satisfies $\mathcal{A}(iii)$. It remains to show that $(h_N)_{N \in \mathbb{N}}$ satisfies $\mathcal{A}(i)$. Indeed, for any $L > 0$,

we have

$$\int_{-L}^{L} |h_N(x) - h(x)|dx \le \int_{-L}^{L} dx \int |h(x-y) - h(x)|\rho_N(y)dy$$

$$= \int_{|z| \le 1} \Big(\int_{-L}^{L} |h(x - N^{-1}z) - h(x)|dx \Big) \rho(z)dz. \tag{2.1}$$

Note that monotonic functions are continuous almost everywhere and therefore $\int_{-L}^{L} |h(x-N^{-1}z)-h(x)|dx \to 0$ as $N \to \infty$ for all $z \in [-1, 1]$. This fact together with Lebesgue dominated convergence theorem implies that the last term of (2.1) tends to 0 as $N \to \infty$. Thus $h_N \to h$ in $L^1_{loc}(\mathbb{R})$. We conclude the proof of Proposition 2.1. $\qquad\square$

Remark 2.2. *This result implies that $h = 1_A \in \mathcal{A}$, where A is any finite union or intersection of intervals. Also any finite linear combination of indicator functions is an element of $\mathcal{A}$.*

Throughout this paper, we always suppose that the following assumption holds.

Assumption (H): f has two derivatives and its second derivative is bounded. $b \in C_b^3(\mathbb{R}), \sigma \in C_b^4(\mathbb{R})$ and $\sigma(x) \ge \sigma_0 > 0$ for all $x \in \mathbb{R}$.

Our main results are:

Theorem 2.3 (Strong rates). *Suppose (H). Then for any $p \ge 1$, there exists a positive constant C such that the following upper bounds for the strong error of approximation are valid.*

(i) Let $h \in \mathcal{A}$. Then

$$\mathbb{E}\left[\Big| \int_0^T h(X_s)ds - \int_0^T h(X_{\eta_n(s)})ds \Big|^{2p} \right] \le \frac{C}{n^{p+\frac{1}{2}}}.$$

(ii) Let $h \in \hat{C}_{\exp}^\alpha(\mathbb{R})$ for some $\alpha \in (0, 1]$. Then

$$\mathbb{E}\left[\Big| \int_0^T h(X_s)ds - \int_0^T h(X_{\eta_n(s)})ds \Big|^{2p} \right] \le \begin{cases} C\frac{\log^{2p}(n)}{n^{2p}}, & \text{if } \alpha = 1; \\ \frac{C}{n^{(1+\alpha)p}}, & \text{if } \alpha \in (0, 1). \end{cases}$$

Theorem 2.4 (Weak rates). *Suppose (H) and that $h \in \mathcal{A}$. Then there exists a positive constant C such that*

$$\left| \mathbb{E}\Big[f\Big(\int_0^T h(X_s)ds \Big) \Big] - \mathbb{E}\Big[f\Big(\int_0^T h(X_{\eta_n(s)})ds \Big) \Big] \right| \le C\frac{\log(n)}{n}.$$

Remark 2.5.

(i) The strong rate in Theorem 2.3(i) is optimal as it can be verified in the case that $b = 0$ and $\sigma = 1$(see Proposition 2.3 in [16]). Similarly, the weak rate in Theorem 2.4 is optimal up to the factor $\log(n)$ in the uniformly elliptic case (see e.g. [10] or [13]).

(ii) When $\alpha = 1$ in Theorem 2.3, the strong approximation error is of order 1, i.e,

$$\Big\| \int_0^T h(X_s)ds - \int_0^T h(X_{\eta_n(s)})ds \Big\|_{L^{2p}(\mathbb{P})} \le C\frac{\log(n)}{n}.$$

This result is compatible with the result of Theorem 1.1 in [12].

(iii) $\mathcal{A} - \hat{C}_{\exp}^\alpha(\mathbb{R}) \ne \emptyset$ and $\hat{C}_{\exp}^\alpha(\mathbb{R}) - \mathcal{A} \ne \emptyset$ for any $\alpha \in (0, 1)$. However, note that $h(x) := \sin(\exp(x^3)) \in C_{\exp}^1(\mathbb{R})$ but $h \notin \mathcal{A}$.

(iv) The $L^2(\mathbb{P})$ $(p = 1)$ bound of the strong error estimate in Theorem 2.3 (i) for $h \in \mathcal{A}$ is larger than the estimate in Theorem 2.3 (ii) for $h \in \hat{C}_{\exp}^\alpha(\mathbb{R})$ iff $\alpha > \frac{1}{2}$. On the other hand, if we consider the general $L^p(\mathbb{P})$ norms, the strong rate of convergence estimate for functions $h \in \hat{C}_{\exp}^\alpha(\mathbb{R})$ and $h \in \mathcal{A}$ are $\frac{1+\alpha}{2}$ and $\frac{1}{2}$ as $p \to \infty$, respectively. It means that the bound of the former case is smaller than the one of the latter case when we consider higher order moments.

Finally, we present some applications about the strong and weak approximation for the local time of X which is defined as

$$L_t(x) = \lim_{\delta \to 0} \frac{1}{2\delta} \int_0^t I_{|X_s - x| \leq \delta} ds.$$

Denote $\phi_\epsilon(x) = \frac{1}{\sqrt{2\pi\epsilon}} e^{-x^2/2\epsilon}$ for $\epsilon > 0$. We have

Theorem 2.6. *There exists a constant $C > 0$ such that*

$$\mathbb{E}\left[\left(L_T(0) - \int_0^T \phi_{\frac{1}{n}}(X_{\eta_n(s)}) ds \right)^2 \right] \leq C \frac{\log(n)}{\sqrt{n}}, \tag{2.2}$$

$$\left| \mathbb{E}\left[L_T(0) - \int_0^T \phi_{\frac{1}{n}}(X_{\eta_n(s)}) ds \right] \right| \leq C \frac{\log(n)}{\sqrt{n}}. \tag{2.3}$$

Moreover, if $x_0 \neq 0$ then

$$\left| \mathbb{E}\left[L_T(0) - \int_0^T \phi_{\frac{1}{n}}(X_{\eta_n(s)}) ds \right] \right| \leq C \frac{\log(n)}{n|x_0|}. \tag{2.4}$$

Remark 2.7. *(i) The estimate (2.2) can be used to show the statistical consistence of the statistic $\int_0^T \phi_{\frac{1}{n}}(X_{\eta_n(s)}) ds$ for the estimation of the local time $L_T(0)$. On the other hand, it has been shown in [11] that the process $n^{\frac{1}{4}}\left(L_t(0) - \int_0^t \phi_{\frac{1}{n}}(X_{\eta_n(s)}) ds \right)$ converges stably in law to some non-degenerate process. This implies that the strong rate obtained in (2.2) is optimal up to the $\log(n)$ term.*

(ii) Estimates (2.3) and (2.4) give the rates of weak approximation for $\mathbb{E}[f(L_T(0))]$ when $f(x) = x$. The rate in (2.3) is almost optimal since one can easily verify that when X is a standard Brownian motion starting at 0 (i.e. $b = x_0 = 0$, $\sigma = 1$) then

$$\mathbb{E}\left[L_T(0) - \int_0^T \phi_{\frac{1}{n}}(X_{\eta_n(s)}) ds \right] = \frac{\sqrt{T}}{\sqrt{2\pi}} \left(2 - \sum_{i=1}^n \frac{1}{\sqrt{in}} \right) \geq \frac{\sqrt{T}}{\sqrt{2\pi n}}.$$

Following the same method of proof as the one presented here, we may obtain the same rates in (2.3) and (2.4) when f is any polynomial function. However, the problem of estimating the rate of weak approximation for general function f is still open.

(iii) The rate in (2.4) is better than (2.3) because in the particular case that the starting point of X is zero then the crossings of X at smaller times around zero increase significantly and therefore the approximation method deteriorates.

3   Proofs

We start with the following observation: we set $\mathcal{S}(x) = \int_0^x \frac{1}{\sigma(y)} dy$ and $Y_t = \mathcal{S}(X_t)$. That is, using the standard Lamperti transform (see [14]), one deduces from Itô formula that

$$dY_t = \hat{b}(Y_t) dt + dW_t, \quad 0 \leq t \leq T, \tag{3.1}$$

where $\hat{b}(x) = \tilde{b}(\mathcal{S}^{-1}(x))$ with $\tilde{b} = \frac{b}{\sigma} - \frac{1}{2}\sigma'$ and $\mathcal{S}^{-1}$ denotes the inverse function of $\mathcal{S}$. It is straightforward to verify that if b and σ satisfy assumption (H) then $\hat{b} \in C_b^3(\mathbb{R})$ and $\mathcal{S}$ has bounded derivatives up to order 4. Moreover, we have the following result.

Proposition 3.1. *Assume (H).*

1. If $h \in \mathcal{A}$ then $h \circ \mathcal{S}^{-1} \in \mathcal{A}$.

2. If h is α-Hölder continuous then $h \circ \mathcal{S}^{-1}$ is also α-Hölder continuous.

Therefore, it is enough to prove Theorems 2.3 and 2.4 for the case $\sigma = 1$. In the case of local times a similar reduction will be applied.

Proof of Proposition 3.1. The desired results follow from the fact that both $\mathcal{S}$, $\mathcal{S}^{-1}$ are increasing differentiable functions and that there exist two positive constants K_1, K_2 such that $K_1|x| \le |\varphi(x)| \le K_2|x|$ for $\varphi = \mathcal{S}$, $\mathcal{S}^{-1}$ and any $x \in \mathbb{R}$. $\qquad\square$

This proposition reduces the proof of Theorem 2.3 to the proof of the same statements where h and X are replaced by $h \circ \mathcal{S}^{-1}$ and Y.

3.1 Malliavin Calculus tools

We refer the reader to [17] for an introduction of Malliavin calculus tools and related definitions and notations. Let X be the solution of the sde

$$X_t = x_0 + \int_0^t b(X_s)ds + W_t, \tag{3.2}$$

where W is a one dimensional standard Wiener process and $b \in C_b^3(\mathbb{R})$.

1. $\mathcal{E}_s = \exp\left(\int_0^s b'(X_u)du\right)$. Since $b \in C_b^2(\mathbb{R})$ then we have that $\mathcal{E}_s + \mathcal{E}_s^{-1} \le C$ for a positive constant C and all $(s,\omega) \in [0,T] \times \Omega$. Furthermore, we have that $d\mathcal{E}_s = b'(X_s)\mathcal{E}_s ds$.

2. For any $t > u > 0$ and $\varphi \in C^1$, the Malliavin derivative of $\varphi(X_t)$ is given by

$$D_u(\varphi(X_t)) = \varphi'(X_t)\mathcal{E}_t\mathcal{E}_u^{-1}.$$

   In particular, due to the previous item and the hypothesis (H) there exists a positive constant C such that $D_u b(X_t) \le C$ for all $u \le t$ and any $\omega \in \Omega$.

3. $\mathbb{P}_t(A) = \mathbb{P}(A|\mathcal{F}_t)$ and $\mathbb{E}_t[F] \equiv \mathbb{E}[F|\mathcal{F}_t]$. Conditional L^p-norms are denoted by

$$\|F\|_{r,n,p}^p := \mathbb{E}_r\left[|F|^p + \sum_{k=1}^n \|D^k F\|_{L_2([r,T]^k)}^p\right].$$

   When the context is clear, we simplify the notation in the case that $n = 0$ by denoting $\|\cdot\|_{r,p} \equiv \|\cdot\|_{r,0,p}$

4. Conditional duality formula:

$$\mathbb{E}_t\left[\int_t^{t+h} D_s F u_s ds\right] = \mathbb{E}_t\left[F\delta_{t,t+h}(u)\right], \tag{3.3}$$

   where $\delta_{u,t}$ is Skorohod integral in the time interval $[u,t]$.

5. Some properties of the Skorohod integral: for any random variable F and a random process u

$$\delta(Fu) = F\delta(u) - \int_0^T (D_t F)u_t dt$$

   as long as both sides of the equation make sense in $L^2(\mathbb{P})$. Furthermore, if u is an adapted process then $\delta(u) = \int_0^T u_t dW_t$.

6. We will repeatedly use the following integration by parts (IBP) argument:
   Let Z_k, $k \le 3$, be $\mathcal{F}_t$-measurable random variables satisfying $\mathbb{E}_r[|D_\alpha^k Z_k|^p] \le C(k,p)$ for some positive deterministic constant $C(k,p)$ and for any multi-index $\alpha = (\alpha_1,...,\alpha_k) \in [r,t]^k$ (D_α^k denotes the Malliavin k-th order derivative at $(\alpha_1,...,\alpha_k)$). Then we have

$$D_u f(X_t) = f'(X_t)\mathcal{E}_t\mathcal{E}_u^{-1} \text{ or } f'(X_t)Z_1 = D_u f(X_t)Z_1\mathcal{E}_u\mathcal{E}_t^{-1}$$

   for $t > u > 0$. Furthermore, we have the following integration by parts formula

$$\mathbb{E}_r[f'(X_t)Z_1] = \frac{1}{t-r}\mathbb{E}_r\left[\int_r^t D_u f(X_t)Z_1\mathcal{E}_u\mathcal{E}_t^{-1}du\right]$$

$$= \frac{1}{t-r}\mathbb{E}_r\left[f(X_t)\delta_{r,t}\left(Z_1\mathcal{E}.\mathcal{E}_t^{-1}\right)\right] = \mathbb{E}_r[f(X_t)H_{r,t}(X_t,Z_1)], \tag{3.4}$$

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with $H_{r,t}(X_t, Z_1) = \frac{1}{t-r}\delta_{r,t}\left(Z_1\mathcal{E}.\mathcal{E}_t^{-1}\right)$. For higher order derivatives, we define inductively $H_{r,t}^k(X_t, Z^k) = H_{r,t}^{k-1}(X_t, Z_kH_{r,t}(X_t, Z^{k-1}))$ for $k = 2, 3$, with $H_{r,t}^1(X_t, \cdot) = H_{r,t}(X_t, \cdot)$ and $Z^k = (Z_1, \ldots, Z_k)$, we have the following properties:

$$\mathbb{E}_r\left[H_{r,t}^k(X_t, Z^k)\right] = 0, \tag{3.5}$$

$$\|H_{r,t}^k(X_t, Z^k)\|_{r,q} \leq \frac{C(k,p)}{(t-r)^{\frac{k}{2}}}. \tag{3.6}$$

These estimates are obtained by applying the norm properties in Propositions 1.5.6 – 1.5.7 in [17] together with the explicit expressions above.

Remark 3.2. *The fact that coefficients are bounded allows us to obtain the property that $\mathcal{E}$ is bounded (property 1.) and also the Gaussian bounds on the density (see* (3.6) *and* (4.1)*). In fact, the proofs use strongly these two facts although one can find other ways to deal with the expressions related to $\mathcal{E}$ using the Malliavin calculus integration by parts formulas even in the case that $\mathcal{E}$ is only bounded in $L^p(\mathbb{P})$. On the other hand, the fact that the density estimates are Gaussian is extremely important and it becomes the key element which forces us to impose the boundedness assumptions on the coefficients of the sde.*

A first result related with our problem is the following particular case of weak rate of convergence.

Lemma 3.3 (Theorem 2.5 [10]). *Suppose that h is exponentially bounded. Then*

$$\left|\mathbb{E}\left[\int_0^T h(X_s)ds - \int_0^T h(X_{\eta_n(s)})ds\right]\right| \leq \frac{C\log(n)}{n}.$$

Although the result in [10] uses the Euler scheme as approximating method, the proof is much simpler in our case and it can also be carried out in the fashion described in [10][1].

3.2 Preliminary estimations based on the IBP formula

Recall that throughout this section, we suppose that $\sigma = 1$.

Lemma 3.4. *Suppose that $\zeta \in C_{\exp}^2(\mathbb{R})$. Then there exists a constant C depending on X such that the estimate*

$$\left|\mathbb{E}_r\left[(\zeta'b)'(X_v)\mathcal{E}_v\right]\right| \leq \frac{C}{v-r}\|\zeta(X_v) - \mathbb{E}_r[\zeta(X_v)]\|_{r,2} \leq \frac{C}{v-r}\|\zeta(X_v)\|_{r,2}$$

holds for any $0 \leq r \leq v \leq T$. Furthermore if $|\zeta(x)| \leq Ke^{K|x|}$ for some $K > 0$, then there exists a constant C depending on K and the coefficients of X such that $\|\zeta(X_v)\|_{r,2} \leq Ce^{K|X_r|}$.

Proof. For the first part, it follows from the IBP formula (3.4), the definition of $C_{\exp}^2(\mathbb{R})$ and the estimates in (3.6) with $Z_1 = \mathcal{E}_v$ and $Z_2 = b(X_v)$ so that

$$\begin{aligned}
\left|\mathbb{E}_r\left[(\zeta'b)'(X_v)\mathcal{E}_v\right]\right| &= \left|\mathbb{E}_r\left[\zeta'b(X_v)H_{r,v}(X_v, \mathcal{E}_v)\right]\right| \\
&= \left|\mathbb{E}_r\left[\zeta(X_v)H_{r,v}(X_v, b(X_v)H_{r,v}(X_v, \mathcal{E}_v))\right]\right| \\
&= \left|\mathbb{E}_r\left[(\zeta(X_v) - \mathbb{E}_r(\zeta(X_v)))H_{r,v}(X_v, b(X_v)H_{r,v}(X_v, \mathcal{E}_v))\right]\right|,
\end{aligned}$$

where we have used, in the last equality, the zero-mean property (3.5) of H. Then, by using the moment estimate (3.6), we get

$$\left|\mathbb{E}_r\left[(\zeta'b)'(X_v)\mathcal{E}_v\right]\right| \leq C\|\zeta(X_v) - \mathbb{E}_r[\zeta(X_v)]\|_{r,2}\|H_{r,v}^2(X_v, Z^2)\|_{r,2} \leq \frac{C}{v-r}\|\zeta(X_v) - \mathbb{E}_r[\zeta(X_v)]\|_{r,2}.$$

Furthermore, if $|\zeta(x)| \leq Ke^{K|x|}$ for some $K > 0$ then the fact that $\zeta(X_v) \leq Ke^{K|X_v-X_r|}e^{K|X_r|}$ together with (4.1) gives the desired property. $\square$

---

[1]In fact, [10] claims that the error is bounded by $\frac{C}{n}$. However the proof given in [10] only gives the slightly bigger bound $\frac{C\log(n)}{n}$. The authors provided us with an alternative proof which gives the correct bound. For our results, either bound give the same final rate.

Lemma 3.5. *(i) Assume that $\zeta \in C_{\exp}^1(\mathbb{R})$. For any $v > r \geq 0$, there exists a constant C such that*

$$\mathbb{E}_r\big[(\zeta(X_v) - \mathbb{E}_r[\zeta(X_v)])^2\big] \leq C\Big(\int |\zeta'(z)| \exp\Big(-\frac{C(z - X_r)^2}{v - r}\Big)dz\Big)^2.$$

(ii) Assume that $\zeta \in \hat{C}_{\exp}^\alpha(\mathbb{R})$ for some $\alpha \in (0, 1]$. Then there exists a constant C (depending on X and α) such that

$$\mathbb{E}_r\big[(\zeta(X_v) - \mathbb{E}_r[\zeta(X_v)])^2\big] \leq C(v - r)^\alpha.$$

The nature of the above estimates (i) and (ii) are the same. In the above statements, this is not so explicit because the derivative of ζ is not necessarily bounded. Later, we will see that after taking expectations this term is of order $\sqrt{v - r}$. The same remark can be made about Lemma 3.6.

Proof. (i) Fix $r < v$, and for each $s \in [r, v]$, we denote

$$u(s, x) = \mathbb{E}[\zeta(X_v)|X_s = x].$$

Then $u \in C^{1,2}([0, v] \times \mathbb{R}, \mathbb{R})$, $u(v, X_v) = \zeta(X_v)$ and $u(r, X_r) = \mathbb{E}_r[\zeta(X_v)]$. Furthermore, since $\zeta(X_v)$ is integrable and $u(s, X_s) = \mathbb{E}[\zeta(X_v)|X_s]$, $(u(s, X_s))_{s \in [r,v]}$ is a martingale. Hence, it follows from Itô's formula that

$$\zeta(X_v) - \mathbb{E}_r[\zeta(X_v)] = u(v, X_v) - u(r, X_r) = \int_r^v \partial_x u(s, X_s)dW_s.$$

Therefore,

$$\mathbb{E}_r\big[|\zeta(X_v) - \mathbb{E}_r[\zeta(X_v)]|^2\big] = \mathbb{E}_r\Big[\int_r^v |\partial_x u(s, X_s)|^2 ds\Big]. \tag{3.7}$$

Then we have

$$\Big(\mathbb{E}_r[|\partial_x u(s, X_s)|^2]\Big)^{1/2} = \Big(\mathbb{E}_r[|\mathbb{E}_s[\zeta'(X_v)\mathcal{E}_v\mathcal{E}_s^{-1}]|^2]\Big)^{1/2}$$

$$\leq C\int \Big(\mathbb{E}_r\Big[\Big(\frac{|\zeta'(z)|}{\sqrt{v - s}}\exp\Big(-\frac{C(z - X_s)^2}{v - s}\Big)\Big)^2\Big]\Big)^{1/2}dz$$

$$= \frac{C}{\sqrt{v - s}}\int |\zeta'(z)|\Big(\mathbb{E}_r\Big[\exp\Big(-\frac{C(z - X_s)^2}{v - s}\Big)\Big]\Big)^{1/2}dz.$$

In the above, we have used that

- $\mathcal{E}_v\mathcal{E}_s^{-1} \leq C$

- The Gaussian bound on the transition density of X in Lemma 4.1.

- The generalized Minkowski inequality.

$$\Big\|\int F(x_1, x_2)\mu_1(dx_1)\Big\|_{\mathbb{L}^q(\mu_2)} \leq \int \|F(x_1, x_2)\|_{\mathbb{L}^q(\mu_2)}\mu_1(dx_1) \; ; \; q \geq 1, \tag{3.8}$$

applied with $\mu_1(dx_1) = dx_1$, $\mu_2(dx_2) = \mathbb{P}_r(dx_2)$, $F(x_1, x_2) = \frac{\zeta'(x_1)}{\sqrt{v-s}}\exp\Big(-\frac{C(z - X_s(x_2))^2}{v-s}\Big)$ and $q = 2$.

Then, we get using Lemma 4.2

$$\mathbb{E}_r[|\partial_x u(s, X_s)|^2] \leq \frac{C}{\sqrt{(v - s)(v - r)}}\Big(\int |\zeta'(z)|\exp\Big(-\frac{C(z - X_r)^2}{v - r}\Big)dz\Big)^2. \tag{3.9}$$

Plugging (3.9) into (3.7) we obtain

$$\mathbb{E}_r[|\zeta(X_v) - \mathbb{E}_r[\zeta(X_v)]|^2] \leq \int_r^v \frac{C}{\sqrt{(v - s)(v - r)}}\Big(\int |\zeta'(z)|\exp\Big(-\frac{C(z - X_r)^2}{v - r}\Big)dz\Big)^2 ds$$

$$\leq C\Big(\int |\zeta'(z)|\exp\Big(-\frac{C(z - X_r)^2}{v - r}\Big)dz\Big)^2,$$

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which completes the proof of Lemma 3.5-(i).

(ii) If we suppose that $\zeta \in \hat{C}^\alpha_{\exp}(\mathbb{R})$ for some $\alpha \in (0, 1]$, then

$$
\mathbb{E}_r\left[(\zeta(X_v) - \mathbb{E}_r[\zeta(X_v)])^2\right] \leq \mathbb{E}_r\left[(\zeta(X_v) - \zeta(X_r))^2\right]
$$
$$
\leq C\mathbb{E}_r\left[|X_v - X_r|^{2\alpha}\right].
$$

Writing $X_v - X_r = \int_r^v b(X_t)dt + \int_r^v \sigma(X_t)dW_t$ and using the boundedness of b and σ yields

$$
\mathbb{E}_r\left[(\zeta(X_v) - \mathbb{E}_r[\zeta(X_v)])^2\right] \leq C(v - r)^\alpha.
$$

$\square$

Lemma 3.6. *(i) Assume that $\zeta \in C^3_{\exp}(\mathbb{R})$. For $u \in (0, \eta_n(s))$, it holds that*

$$
\left|\mathbb{E}_u\left[\zeta'(X_s)\mathcal{E}_s - \zeta'(X_{\eta_n(s)})\mathcal{E}_{\eta_n(s)}\right]\right| \leq C \int_{\eta_n(s)}^s \left(\frac{\|\zeta(X_v)\|_{u,2}}{v - u} + \frac{1}{(v - u)^{\frac{3}{2}}} \int |\zeta'(z)| \exp\left(-\frac{C(z - X_u)^2}{v - u}\right)dz\right)dv.
$$

(ii) Furthermore, assume that $\zeta \in C^3_{\exp}(\mathbb{R}) \cap \hat{C}^\alpha_{\exp}(\mathbb{R})$ for some $\alpha \in (0, 1]$. Then, for $u \in (0, \eta_n(s))$

$$
\left|\mathbb{E}_u\left[\zeta'(X_s)\mathcal{E}_s - \zeta'(X_{\eta_n(s)})\mathcal{E}_{\eta_n(s)}\right]\right| \leq C \int_{\eta_n(s)}^s (v - u)^{\frac{\alpha - 3}{2}} dv.
$$

Proof. (i) Using Itô's formula, we have

$$
\mathbb{E}_u\left[\zeta'(X_s)\mathcal{E}_s - \zeta'(X_{\eta_n(s)})\mathcal{E}_{\eta_n(s)}\right] = \int_{\eta_n(s)}^s \mathbb{E}_u\left[(\zeta'b)'(X_v)\mathcal{E}_v + \frac{1}{2}\zeta'''(X_v)\mathcal{E}_v\right]dv.
$$

Then Lemma 3.4 yields

$$
\left|\mathbb{E}_u\left[(\zeta'b)'(X_v)\mathcal{E}_v\right]\right| \leq \frac{C}{v - u}\|\zeta(X_v)\|_{u,2}.
$$

For $0 \leq r \leq v \leq T$, we have for $Z_1 = \mathcal{E}_v$ and $Z_2 = Z_3 = 1$ in (3.5) and (3.6) together with Lemma 3.5

$$
\begin{aligned}
|\mathbb{E}_u[\zeta'''(X_v)\mathcal{E}_v]| &= |\mathbb{E}_u[\zeta(X_v)H^3_{u,v}(X_v, Z^3)]| \\
&= |\mathbb{E}_u[(\zeta(X_v) - \mathbb{E}_u[\zeta(X_v)])H^3_{u,v}(X_v, Z^3)]| \\
&\leq \|\zeta(X_v) - \mathbb{E}_u[\zeta(X_v)]\|_{u,2}\|H^3_{u,v}(X_v, Z^3)\|_{u,2} \\
&\leq \frac{C}{(v - u)^{\frac{3}{2}}} \int |\zeta'(z)| \exp\left(-\frac{C(z - X_u)^2}{v - u}\right)dz.
\end{aligned}
$$

(ii) The estimate when $\zeta \in C^3_{\exp}(\mathbb{R}) \cap \hat{C}^\alpha_{\exp}(\mathbb{R})$ readily follows from the above proof and Lemma 3.5-(ii). $\square$

We are now in a position to prove all the results mentioned in Section 2. Recall that due to Proposition 3.1, we may assume without loss of generality that $\sigma = 1$ in the proof of Theorems 2.3 and 2.4.

3.3 Proof of Theorem 2.3(i)

As $h \in \mathcal{A}$, there exists a sequence of smooth functions $\{h_N\}_{N \in \mathbb{N}}$ converging to h with the properties stated in the definition of the space $\mathcal{A}$. Define

$$
S_n = \int_0^T (h(X_s) - h(X_{\eta_n(s)}))ds,
$$

$$
S_{n,N} = \int_0^T (h_N(X_s) - h_N(X_{\eta_n(s)}))ds.
$$

Before continuing, we provide as a guide a brief description of the line of proof. We will first prove the convergence of the regularizing sequence $\mathbb{E}|S_{n,N}|^{2p}$ to $\mathbb{E}|S_n|^{2p}$.

Then, we write $S_{n,N}$ as the sum of the centered term $(S_{n,N} - \mathbb{E}[S_{n,N}])$ and the expectation $\mathbb{E}[S_{n,N}]$. The

latter is easier to handle due to Lemma 3.3. The former can be written, thanks to the Clark-Ocone formula, as an Itô integral of $\mathbb{E}_u[D_u S_{n,N}]$.

This term will be a time integral of the conditional expectation of $(h'_N(X_s) - h'_N(X_{\eta_n(s)}))$. We control this term (uniformly in N) through conditional IBP thanks to Lemma 3.6.

In other words, the way to understand and solve this problem is to observe that the derivatives of h_N explode and that the conditional expectation $\mathbb{E}_u$ (combined with the time integral) will play the role of smoothing $D_u S_{n,N}$ uniformly in N finally controlling the error.

In particular, we have to estimate multiple time-integrals of negative powers of the time-variables, which finally will give the rate of convergence.

Now, let us go into the details of the proof. We first show that

$$\lim_{N \to \infty} \mathbb{E}[|S_{n,N}|^{2p}] = \mathbb{E}[|S_n|^{2p}]. \tag{3.10}$$

Indeed, we have using Hölder's inequality, Fubini's theorem and (4.1)

$$\mathbb{E}\left[\left|\int_0^T (h_N(X_s) - h(X_s))ds\right|^{2p}\right] \leq C \int_0^T \mathbb{E}\left[|h_N(X_s) - h(X_s)|^{2p}\right] ds$$

$$\leq C \int_0^T ds \int |h_N(x) - h(x)|^{2p} \frac{e^{-\frac{Cx^2}{s}}}{\sqrt{s}} dx.$$

Recall that as $h \in \mathcal{A}$, then for any $\epsilon > 0$ there exists a compact set $\mathbb{K}$, such that

$$\int_0^T ds \int_{\mathbb{K}^c} |h_N(x) - h(x)|^{2p} \frac{e^{-\frac{Cx^2}{s}}}{\sqrt{s}} dx \leq C \int_0^T ds \int_{\mathbb{K}^c} e^{2pK_2|x|} \frac{e^{-\frac{Cx^2}{s}}}{\sqrt{s}} dx < \epsilon.$$

Therefore, since $h_N \to h$ in $L^1_{loc}(\mathbb{R})$, applying Lebesgue dominated convergence theorem, we get

$$\lim_{N \to \infty} \mathbb{E}\left[\left|\int_0^T (h_N(X_s) - h(X_s))ds\right|^{2p}\right] = 0.$$

A similar argument shows

$$\lim_{N \to \infty} \mathbb{E}\left[\left|\int_0^T (h_N(X_{\eta_n(s)}) - h(X_{\eta_n(s)}))ds\right|^{2p}\right] = 0.$$

Therefore $\lim_{N \to \infty} \mathbb{E}\left[|S_{n,N} - S_n|^{2p}\right] = 0$ and therefore the triangle inequality for $L^p(\mathbb{P})$ gives the convergence of $2p$ moments.

Next, it follows from Clark-Ocone formula that

$$\mathbb{E}\left[|S_{n,N} - \mathbb{E}[S_{n,N}]|^{2p}\right] = \mathbb{E}\left[\left|\int_0^T \mathbb{E}_u[D_u S_{n,N}]dW_u\right|^{2p}\right] \leq C\mathbb{E}\left[\left|\int_0^T |\mathbb{E}_u[D_u S_{n,N}]|^2 du\right|^p\right], \tag{3.11}$$

where the last inequality follows from the BDG (Burkholder-Davis-Gundy) inequality. Using the chain rule property of the Malliavin derivative and the fact that $\mathcal{E}$ and $\mathcal{E}^{-1}$ are uniformly bounded, we have

$$\mathbb{E}\left[|S_{n,N} - \mathbb{E}[S_{n,N}]|^{2p}\right] = \mathbb{E}\left[\left|\int_0^T \left(\mathcal{E}_u^{-1}\mathbb{E}_u\left[\int_0^T h'_N(X_s)\mathcal{E}_s I_{s \geq u} - h'_N(X_{\eta_n(s)})\mathcal{E}_{\eta_n(s)} I_{\eta_n(s) \geq u}ds\right]\right)^2 du\right|^p\right]$$

$$\leq Cn^{p-1}(T^1_{n,N} + T^2_{n,N}),$$

where

$$T^1_{n,N} = \sum_{i=0}^{n-1} \mathbb{E}\left[\left|\int_{t_i}^{t_{i+1}} \left(\int_u^{t_{i+1}} \mathbb{E}_u\left[|h'_N(X_s)|\right] ds\right)^2 du\right|^p\right],$$

$$T^2_{n,N} = \sum_{i=0}^{n-1} \mathbb{E}\left[\left|\int_{t_i}^{t_{i+1}} \left(\int_{t_{i+1}}^T \mathbb{E}_u\left[h'_N(X_s)\mathcal{E}_s - h'_N(X_{\eta_n(s)})\mathcal{E}_{\eta_n(s)}\right] ds\right)^2 du\right|^p\right].$$

Here we have used that $[0, T) = \cup_{i=0}^{n-1} [t_i, t_{i+1})$ and Hölder's inequality for sums. We will show that there exists a positive constant independent of N and n such that

$$T_{n,N}^1 + T_{n,N}^2 \leq Cn^{-2p+1/2}.$$

Therefore the proof finishes by noting that

$$\mathbb{E}[|S_{n,N}|^{2p}] \leq C\left(|\mathbb{E}[S_{n,N}]|^{2p} + \mathbb{E}\left[|S_{n,N} - \mathbb{E}[S_{n,N}]|^{2p}\right]\right).$$

Therefore we finish using Lemma 3.3 to estimate $\mathbb{E}[S_{n,N}]$ in (3.11) and noting that $\lim_{N\to\infty} \mathbb{E}[|S_{n,N}|^{2p}] = \mathbb{E}[|S_n|^{2p}]$ which gives the estimate for $\mathbb{E}[|S_n|^{2p}]$.

3.3.1 Estimate for $T_{n,N}^1$

First we write

$$\mathbb{E}_u[|h_N'(X_s)|] = \int |h_N'(x)| p_{s-u}(X_u, x) dx$$

$$\leq C\int |h_N'(x)| \frac{1}{\sqrt{s-u}} \exp\left(-\frac{C(x-X_u)^2}{s-u}\right) dx.$$

This estimate together with Hölder and generalized Minkowski inequality (3.8) with $\mu_1(ds, dx) = ds dx$, $\mu_2 = \mathbb{P}$ and $q = 2p$ yields

$$T_{n,N}^1 \leq C\sum_{i=0}^{n-1} \mathbb{E}\left[\left|\int_{t_i}^{t_{i+1}} du\left(\int_u^{t_{i+1}} ds \int |h_N'(x)| \frac{1}{\sqrt{s-u}} \exp\left(-\frac{C(x-X_u)^2}{s-u}\right) dx\right)^2\right|^p\right]$$

$$\leq Cn^{1-p}\sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \mathbb{E}\left[\left|\int_u^{t_{i+1}} ds \int |h_N'(x)| \frac{1}{\sqrt{s-u}} \exp\left(-\frac{C(x-X_u)^2}{s-u}\right) dx\right|^{2p}\right] du$$

$$\leq Cn^{1-p}\sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \left\{\int_u^{t_{i+1}} ds \int \left(\mathbb{E}\left[\left|h_N'(x) \frac{1}{\sqrt{s-u}} \exp\left(-\frac{C(x-X_u)^2}{s-u}\right)\right|^{2p}\right]\right)^{1/2p} dx\right\}^{2p} du$$

$$= Cn^{1-p}\sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \left\{\int_u^{t_{i+1}} ds \int |h_N'(x)| \frac{1}{\sqrt{s-u}} \left(\mathbb{E}\left[\exp\left(-\frac{2pC(x-X_u)^2}{s-u}\right)\right]\right)^{1/2p} dx\right\}^{2p} du.$$

Since h satisfies $\mathcal{A}(iii)$, we obtain after using (4.2),

$$T_{n,N}^1 \leq CK(h)^{2p}n^{1-p}\sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \left(\int_u^{t_{i+1}} (s-u)^{(1-2p)/4p} s^{-1/4p} ds\right)^{2p} du$$

$$\leq CK(h)^{2p}n^{-2p+1/2}.$$

In this last estimate, we have used in the above integral that $s^{-1/4p} \leq u^{-1/4p}$, direct integration and then $(t_i - u) \leq \frac{T}{n}$.

3.3.2 Estimate for $T_{n,N}^2$

Thanks to Lemma 3.6(i), we have $T_{n,N}^2 \leq C(U_{n,N,1} + U_{n,N,2})$ where

$$U_{n,N,1} = \sum_{i=0}^{n-1} \mathbb{E}\left[\left|\int_{t_i}^{t_{i+1}} \left(\sum_{j=i+1}^{n-1} \int_{t_j}^{t_{j+1}} ds \int_{t_j}^s \frac{\|h_N(X_v)\|_{u,2}}{v-u} dv\right)^2 du\right|^p\right],$$

$$U_{n,N,2} = \sum_{i=0}^{n-1} \mathbb{E}\left[\left|\int_{t_i}^{t_{i+1}} \left(\sum_{j=i+1}^{n-1} \int_{t_j}^{t_{j+1}} ds \int_{t_j}^s \frac{1}{(v-u)^{\frac{3}{2}}} \int |h_N'(z)| \exp\left(-\frac{C(z-X_u)^2}{v-u}\right) dz dv\right)^2 du\right|^p\right].$$

(3.12)

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For the first term, using Fubini's theorem, $(t_{j+1} - v) \leq \frac{T}{n}$ and Hölder's inequality, we estimate

$$U_{n,N,1} = \sum_{i=0}^{n-1} \mathbb{E}\left[\left|\int_{t_i}^{t_{i+1}} \left(\sum_{j=i+1}^{n-1} \int_{t_j}^{t_{j+1}} \frac{(t_{j+1}-v)\|h_N(X_v)\|_{u,2}}{v-u}dv\right)^2 du\right|^p\right]$$

$$\leq Cn^{-2p} \sum_{i=0}^{n-1} \mathbb{E}\left[\left|\int_{t_i}^{t_{i+1}} \left(\int_{t_{i+1}}^{T} \frac{\|h_N(X_v)\|_{u,2}}{v-u}dv\right)^2 du\right|^p\right]$$

$$\leq Cn^{1-3p} \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \mathbb{E}\left[\left(\int_{t_{i+1}}^{T} \frac{\|h_N(X_v)\|_{u,2}}{v-u}dv\right)^{2p}\right] du.$$

Using generalized Minkowski inequality (3.8) with $\mu_1(dv) = dv$, $\mu_2 = \mathbb{P}$ and $q = 2p$, we get

$$U_{n,N,1} \leq Cn^{1-3p} \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \left\{\int_{t_{i+1}}^{T} \left(\mathbb{E}\left[\left(\frac{\|h_N(X_v)\|_{u,2}}{v-u}\right)^{2p}\right]\right)^{1/2p} dv\right\}^{2p} du.$$

From Assumption $\mathcal{A}(ii)$ and Lemma 4.1 we deduce that $\sup_{0\leq u\leq v\leq T} \sup_N \mathbb{E}[\|h_N(X_v)\|_{u,2}^{2p}] \leq C \sup_{u\in[0,T]} \mathbb{E}[e^{2pK_2|X_u|}] < \infty$. Then Lemma 4.5(i) gives

$$U_{n,N,1} \leq Cn^{1-3p} \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \left(\log \frac{T-u}{t_{i+1}-u}\right)^{2p} du \leq C\frac{\log^{2p}(n)}{n^{3p-1}}. \tag{3.13}$$

We now turn to the evaluation of $U_{n,N,2}$. From (3.12), integrating with respect to s and using that $t_{j+1} - v \leq \frac{T}{n}$, we obtain

$$U_{n,N,2} \leq Cn^{-2p} \sum_{i=0}^{n-1} \mathbb{E}\left[\left|\int_{t_i}^{t_{i+1}} \left(\int_{t_{i+1}}^{T} \frac{dv}{(v-u)^{\frac{3}{2}}} \int |h'_N(z)| \exp\left(-\frac{C(z-X_u)^2}{v-u}\right)dz\right)^2 du\right|^p\right].$$

Applying generalized Minkowski inequality (3.8) three times (with different definitions for μ_1, μ_2, q and F) and Lemma 4.2, we have

$$\left\|\int_{t_i}^{t_{i+1}} \left(\int_{t_{i+1}}^{T} \frac{dv}{(v-u)^{\frac{3}{2}}} \int |h'_N(z)| \exp\left(-\frac{C(z-X_u)^2}{v-u}\right)dz\right)^2 du\right\|_{L^p(\mathbb{P})}$$

$$\leq \int_{t_i}^{t_{i+1}} \left(\mathbb{E}\left[\left|\int_{t_{i+1}}^{T} \frac{dv}{(v-u)^{\frac{3}{2}}} \int |h'_N(z)| \exp\left(-\frac{C(z-X_u)^2}{v-u}\right)dz\right|^{2p}\right]\right)^{\frac{1}{p}} du$$

$$\leq \int_{t_i}^{t_{i+1}} \left(\int_{t_{i+1}}^{T} \left(\mathbb{E}\left[\left|\frac{1}{(v-u)^{\frac{3}{2}}} \int |h'_N(z)| \exp\left(-\frac{C(z-X_u)^2}{v-u}\right)dz\right|^{2p}\right]\right)^{\frac{1}{2p}} dv\right)^2 du$$

$$= \int_{t_i}^{t_{i+1}} \left(\int_{t_{i+1}}^{T} \frac{1}{(v-u)^{\frac{3}{2}}} \int |h'_N(z)| \left(\mathbb{E}\left[\exp\left(-\frac{2pC(z-X_u)^2}{v-u}\right)\right]\right)^{\frac{1}{2p}} dzdv\right)^2 du$$

$$\leq C \int_{t_i}^{t_{i+1}} \left(\int_{t_{i+1}}^{T} (v-u)^{\frac{1-6p}{4p}} v^{-\frac{1}{4p}} \int |h'_N(z)| \exp\left(-\frac{Cz^2}{v}\right)dzdv\right)^2 du$$

$$\leq CK(h)^2 \int_{t_i}^{t_{i+1}} \left(\int_{t_{i+1}}^{T} (v-u)^{\frac{1-6p}{4p}} t_{i+1}^{-\frac{1}{4p}} dv\right)^2 du.$$

where the last inequality follows from the property $\mathcal{A}(iii)$. A direct computation shows

$$\int_{t_i}^{t_{i+1}} \left(\int_{t_{i+1}}^{T} (v-u)^{\frac{1-6p}{4p}} t_{i+1}^{-\frac{1}{4p}} dv\right)^2 du \leq C \int_{t_i}^{t_{i+1}} (t_{i+1}-u)^{\frac{1-2p}{2p}} t_{i+1}^{-\frac{1}{2p}} du = Cn^{-\frac{1}{2p}} t_{i+1}^{-\frac{1}{2p}}.$$

Therefore, we obtain

$$U_{n,N,2} \leq CK(h)^{2p}n^{-2p}\sum_{i=0}^{n-1}t_{i+1}^{-1/2}n^{-1/2} \leq CK(h)^{2p}n^{-2p+1/2}\int_0^T \frac{dt}{\sqrt{t}}.$$

This concludes the proof of Theorem 2.3. □

3.4 Proof of Theorem 2.3(ii)

The structure of the proof of Theorem 2.3(ii) is similar to that of Theorem 2.3(i) except that we will use Lemma 3.5(ii) and Lemma 3.6(ii) instead of Lemma 3.5(i) and Lemma 3.6(i).

For each $N > 0$, denote

$$h_N(x) = \int h(y)\frac{N}{\sqrt{2\pi}}\exp\Big(-\frac{N^2(x-y)^2}{2}\Big)dy = \int h(x+\frac{z}{N})\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz.$$

We note that for fixed N, $h_N \in C^k_{\exp}(\mathbb{R})$ for any $k \in \mathbb{N}$ and if $h \in \hat{C}^\alpha_{\exp}(\mathbb{R})$ then

$$|h_N(x) - h_N(y)| \leq \int \Big|h\big(x+\frac{z}{N}\big) - h\big(y+\frac{z}{N}\big)\Big|\frac{e^{-z^2/2}}{\sqrt{2\pi}}dz$$

$$\leq C_h\int |x-y|^\alpha \frac{e^{-z^2/2}}{\sqrt{2\pi}}dz = C_h|x-y|^\alpha.$$

Proof. We use similar ideas as in the proof of Theorem 2.3(i) and therefore we also borrow the notations from that proof. First, note that

$$|h_N(x) - h(x)| = \Big|\int (h(x+N^{-1}z) - h(x))\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz\Big| \leq CN^{-\alpha}.$$

Therefore, we have

$$\mathbb{E}\left[|S_n - S_{n,N}|^{2p}\right] = \mathbb{E}\left[\Big(\int_0^T (h_N(B_s) - h_N(B_{\eta_n(s)}) - h(B_s) + h(B_{\eta_n(s)}))ds\Big)^{2p}\right]$$

$$\leq C\mathbb{E}\left[\Big(\int_0^T (h_N(B_s) - h(B_s))ds\Big)^{2p}\right] + C\mathbb{E}\left[\Big(\int_0^T (h_N(B_{\eta_n(s)}) - h(B_{\eta_n(s)}))ds\Big)^{2p}\right]$$

$$\leq CN^{-2p\alpha}.$$

Then as in the proof of Theorem 2.3(i), we have

$$\lim_{N\to\infty} \mathbb{E}[|S_{n,N}|^{2p}] = \mathbb{E}[|S_n|^{2p}]. \tag{3.14}$$

Next, in a similar way as in the proof of Theorem 2.3(i) we have that

$$\mathbb{E}\left[|S_{n,N} - \mathbb{E}[S_{n,N}]|^{2p}\right] \leq Cn^{p-1}(\hat{T}^1_{n,N} + \hat{T}^2_{n,N}),$$

where

$$\hat{T}^1_{n,N} = \sum_{i=0}^{n-1}\mathbb{E}\left[\left|\int_{t_i}^{t_{i+1}}\Big(\int_u^{t_{i+1}}\mathbb{E}_u\left[h'_N(X_s)\mathcal{E}_s\right]ds\Big)^2 du\right|^p\right],$$

$$\hat{T}^2_{n,N} = \sum_{i=0}^{n-1}\mathbb{E}\left[\left|\int_{t_i}^{t_{i+1}}\Big(\int_{t_{i+1}}^T\mathbb{E}_u\left[h'_N(X_s)\mathcal{E}_s - h'_N(X_{\eta_n(s)})\mathcal{E}_{\eta_n(s)}\right]ds\Big)^2 du\right|^p\right].$$

We remark here that h does not necessarily belong to $\mathcal{A}$. Instead, the upper estimate will be obtained using Lemma 3.5(ii) and Lemma 3.6(ii).

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a) Here, we estimate $\hat{T}_{n,N}^1$. From the proof of Lemma 3.4, Lemma 3.5(ii) and the above estimate, the following inequalities are straighforward:

$$\left| \mathbb{E}_u \left[h_N'(X_s)\mathcal{E}_s \right] \right| \leq \frac{C}{(s-u)^{\frac{1}{2}}} \| h_N(X_s) - \mathbb{E}_u(h_N(X_s)) \|_{u,2} \leq C(s-u)^{\frac{\alpha-1}{2}}.$$

Thus

$$\hat{T}_{n,N}^1 \leq C \sum_{i=0}^{n-1} \left| \int_{t_i}^{t_{i+1}} \left(\int_u^{t_{i+1}} (s-u)^{\frac{\alpha-1}{2}} ds \right)^2 du \right|^p \leq Cn^{1-p(2+\alpha)}.$$

b) Now, we estimate $\hat{T}_{n,N}^2$. Thanks to Lemma 3.6(ii), we have integrating with respect to s,

$$\hat{T}_{n,N}^2 \leq C \sum_{i=0}^{n-1} \left| \int_{t_i}^{t_{i+1}} \left(\sum_{j=i+1}^{n-1} \int_{t_j}^{t_{j+1}} ds \int_{t_j}^s (v-u)^{\frac{\alpha-3}{2}} dv \right)^2 du \right|^p$$

$$\leq Cn^{-2p} \sum_{i=0}^{n-1} \left| \int_{t_i}^{t_{i+1}} \left(\int_{t_{i+1}}^T (v-u)^{\frac{\alpha-3}{2}} dv \right)^2 du \right|^p.$$

i) If $\alpha \in (0,1)$, we have

$$\hat{T}_{n,N}^2 \leq Cn^{-2p} \sum_{i=0}^{n-1} \left| \int_{t_i}^{t_{i+1}} (t_{i+1}-u)^{\alpha-1} du \right|^p \leq Cn^{1-p(2+\alpha)}.$$

ii) If $\alpha = 1$, we have by Lemma 4.5(i)

$$\hat{T}_{n,N}^2 \leq Cn^{-2p} \sum_{i=0}^{n-1} \left| \int_{t_i}^{t_{i+1}} \left(\log\left(\frac{T-u}{t_{i+1}-u}\right) \right)^2 du \right|^p \leq C\frac{\log^{2p}(n)}{n^{3p-1}}.$$

Therefore,

$$\mathbb{E}[|S_{n,N} - \mathbb{E}(S_{n,N})|^{2p}] \leq \begin{cases} C\frac{\log^{2p}(n)}{n^{2p}} & \text{if } \alpha = 1 \\ \frac{C}{n^{p(\alpha+1)}} & \text{if } \alpha \in (0,1), \end{cases}$$

where C is a constant which is independent of the values of N and n. This fact together with Lemma 3.3 and equation (3.14) yields the desired result. $\qquad\square$

3.5   Proof of Theorem 2.4

Once the hard work of proving the strong rate of convergence has been achieved, we use this result and the method of proof in order to obtain the weak error of convergence. First, we apply a Taylor expansion of order two and then we apply the strong rate result on the second order term. For the first order term, we have to proceed as before using Clark-Ocone's formula. This will lead to a double explosion terms (see (3.16)) as the derivatives of h_N will appear twice. But time integrals and iterated conditional expectations appear as in the proof of Theorem 2.3(i) except that now they appear twice. So that a similar argument gives the rate of convergence.

Using Taylor's expansion, we define the bounded random variable A_2

$$A_2 = \begin{cases} \frac{f(I)-f(I_n)-f'(I)(I-I_n)}{(I-I_n)^2} & \text{if } I \neq I_n \\ f''(I) & \text{if } I = I_n. \end{cases}$$

where $I = \int_0^T h(X_s)ds$, $I_n = \int_0^T h(X_{\eta_n(s)})ds$. Then we have

$$\mathbb{E}\left[f\left(\int_0^T h(X_s)ds \right) \right] - \mathbb{E}\left[f\left(\int_0^T h(X_{\eta_n(s)})ds \right) \right]$$

$$= \mathbb{E}\left[f'\left(\int_0^T h(X_s)ds \right) \int_0^T (h(X_s) - h(X_{\eta_n(s)}))ds \right] + \mathbb{E}\left[A_2 \left(\int_0^T (h(X_s) - h(X_{\eta_n(s)}))ds \right)^2 \right].$$

Since A_2 is bounded, then from Theorem 2.3(i), the second term is bounded as follows

$$\left| \mathbb{E}\left[A_2 \left(\int_0^T (h(X_s) - h(X_{\eta_n(s)}))ds \right)^2 \right] \right| \leq C\mathbb{E}\left[\left(\int_0^T (h(X_s) - h(X_{\eta_n(s)}))ds \right)^2 \right]$$

$$\leq Cn^{-\frac{3}{2}}.$$

Since f'' is bounded, there is a constant C such that $|f'(x)| \leq C(|x|+1)$ for all $x \in \mathbb{R}$. Thus, for any $\beta \geq 1$,

$$\sup_N \mathbb{E}\left[\left| f'\left(\int_0^T h_N(X_s)ds \right) \right|^\beta \right] \leq C \sup_N \mathbb{E}\left[\left| \int_0^T h_N(X_s)ds \right|^\beta \right] + C \leq C \int_0^T \mathbb{E}[e^{K_2\beta|X_s|}]ds + C < \infty. \quad (3.15)$$

This estimate is also valid for h instead of h_N. Therefore, by following a similar argument as in the beginning of the proof of Theorem 2.3(i), we obtain

$$\mathbb{E}\left[f'\left(\int_0^T h(X_s)ds \right) \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} (h(X_s) - h(X_{\eta_n(s)}))ds \right]$$

$$= \lim_{N \to \infty} \mathbb{E}\left[f'\left(\int_0^T h_N(X_s)ds \right) \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} (h_N(X_s) - h_N(X_{t_i}))ds \right].$$

Using Clark-Ocone formula, we write

$$\mathbb{E}\left[f'\left(\int_0^T h_N(X_s)ds \right) \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} (h_N(X_s) - h_N(X_{t_i}))ds \right]$$

$$= \mathbb{E}\left[f'\left(\int_0^T h_N(X_s)ds \right) \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \mathbb{E}[h_N(X_s) - h_N(X_{t_i})]ds \right]$$

$$+ \mathbb{E}\left[f'\left(\int_0^T h_N(X_s)ds \right) \left(\sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \int_0^s \mathbb{E}_r[D_r(h_N(X_s) - h_N(X_{t_i}))]dW_r ds \right) \right]$$

$$= S_1^{n,N} + S_2^{n,N}.$$

It follows from Lemma 3.3 and the estimate (3.15) that

$$\lim_{N \to \infty} |S_1^{n,N}| = O(n^{-1}).$$

Using the chain rule for Malliavin derivatives, we write the second term as follows

$$S_2^{n,N} = \mathbb{E}\left[f'\left(\int_0^T h_N(X_s)ds \right) \left(\sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \int_0^{t_i} \mathbb{E}_r\left[h_N'(X_s)\mathcal{E}_s - h_N'(X_{t_i})\mathcal{E}_{t_i} \right] \mathcal{E}_r^{-1}dW_r ds \right) \right]$$

$$+ \mathbb{E}\left[f'\left(\int_0^T h_N(X_s)ds \right) \left(\sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \int_{t_i}^s \mathbb{E}_r[h_N'(X_s)\mathcal{E}_s]\mathcal{E}_r^{-1}dW_r ds \right) \right]$$

$$= S_{2,a}^{n,N} + S_{2,b}^{n,N}.$$

We estimate first $S_{2,a}^{n,N}$. It follows from interchanging integrals, Fubini's theorem and the duality formula (3.3) with $F = f'\left(\int_0^T h_N(X_s)ds \right)$ and $u_r = \mathbb{E}_r\left[h_N'(X_s)\mathcal{E}_s \right] \mathcal{E}_r^{-1}$ that

$$S_{2,a}^{n,N} = \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \mathbb{E}\left[f'\left(\int_0^T h_N(X_u)du \right) \int_0^{t_i} \mathcal{E}_r^{-1}\mathbb{E}_r[h_N'(X_s)\mathcal{E}_s - h_N'(X_{t_i})\mathcal{E}_{t_i}]dW_r \right]ds$$

$$= \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \mathbb{E}\left[\int_0^{t_i} f''\left(\int_0^T h_N(X_u)du \right) \int_r^T h_N'(X_u)\mathcal{E}_u\mathcal{E}_r^{-2}\mathbb{E}_r[h_N'(X_s)\mathcal{E}_s - h_N'(X_{t_i})\mathcal{E}_{t_i}]dudr \right]ds.$$

We deduce from the boundedness of $\mathcal{E}, \mathcal{E}^{-1}$ and f'' that

$$|S_{2,a}^{n,N}| \leq C \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \int_0^{t_i} \int_r^T \mathbb{E}\left[|h_N'(X_u)||\mathbb{E}_r\left[h_N'(X_s)\mathcal{E}_s - h_N'(X_{t_i})\mathcal{E}_{t_i}\right]|\right] du\, dr\, ds. \tag{3.16}$$

It follows from Lemma 3.6(i) that

$$|S_{2,a}^{n,N}| \leq C \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \int_0^{t_i} \int_r^T \mathbb{E}\left[|h_N'(X_u)| \int_{t_i}^s \frac{\|h_N(X_v)\|_{r,2}}{v-r} dv\right] du\, dr\, ds$$

$$+ \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \int_0^{t_i} \int_r^T \mathbb{E}\left[|h_N'(X_u)| \int_{t_i}^s \frac{dv}{(v-r)^{\frac{3}{2}}} \int |h_N'(z)| \exp\left(-\frac{C(z-X_r)^2}{v-r}\right) dz\right] du\, dr\, ds$$

$$= U_1^{n,N} + U_2^{n,N}.$$

Now, for any $u > r \geq 0$, consider

$$\mathbb{E}_r[|h_N'(X_u)|] \leq C \int |h_N'(x)| \frac{1}{\sqrt{u-r}} \exp\left(-\frac{C(x-X_r)^2}{u-r}\right) dx, \tag{3.17}$$

and, using Lemma 3.4,

$$\mathbb{E}\left[\exp\left(-\frac{C(x-X_r)^2}{u-r}\right) \|h_N(X_v)\|_{r,2}\right] \leq C\mathbb{E}\left[\exp\left(-\frac{C(x-X_r)^2}{u-r}\right) \exp(K|X_r|)\right]$$

$$\leq C \exp(K|x|)\mathbb{E}\left[\exp\left(-\frac{C(x-X_r)^2}{u-r}\right) \exp(K|x-X_r|)\right]$$

$$\leq C \exp(K|x|)\mathbb{E}\left[\exp\left(-\frac{C(x-X_r)^2}{u-r}\right)\right],$$

from which, and from inequalities (4.2) and (3.17), we get

$$\mathbb{E}\left[|h_N'(X_u)|\|h_N(X_v)\|_{r,2}\right] = \mathbb{E}\left[\mathbb{E}_r\left[|h_N'(X_u)|\|h_N(X_v)\|_{r,2}\right]\right]$$

$$\leq C \int |h_N'(x)| \exp(K|x|) \frac{\exp(-\frac{Cx^2}{u})}{\sqrt{u}} dx \leq C \int |h_N'(x)| \frac{\exp(-\frac{Cx^2}{u})}{\sqrt{u}} dx.$$

Then, we have, using $\mathcal{A}(iii)$ and Lemma 4.5(ii)

$$U_1^{n,N} \leq C \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \int_0^{t_i} \int_r^T \int_{t_i}^s \frac{1}{v-r} \int |h_N'(x)| \frac{\exp(-\frac{Cx^2}{u})}{\sqrt{u}} dx\, dv\, du\, dr\, ds \leq CK(h) \frac{\log(n)}{n}.$$

Next we evaluate $U_2^{n,N}$. Applying (3.17) and Lemma 4.4 on $[r,T]^2$ with $a_1(v) = 1(t_i \leq v \leq s)(v-r)^{-\frac{3}{2}}$ and $a_2(u) = 1(r \leq u \leq T)(u-r)^{-\frac{1}{2}}$ and $\mathcal{A}(iii)$, we get from Lemma 4.5(iii)

$$U_2^{n,N} \leq CK(h)^2 \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} ds \int_0^{t_i} dr \int_r^T du \int_{t_i}^s (v-r)^{-1}(u-r)^{-\frac{1}{2}} v^{-\frac{1}{2}} dv \leq CK(h)^2 \frac{\log(n)}{n}.$$

We estimate $S_{2,b}^{n,N}$. Again, thanks to the duality formula (3.3), we have

$$S_{2,b}^{n,N} = \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \mathbb{E}\left[f'\left(\int_0^T h_N(X_u) du\right) \int_{t_i}^s \mathbb{E}_r[h_N'(X_s)\mathcal{E}_s\mathcal{E}_r^{-1}] dW_r\right] ds$$

$$= \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \mathbb{E}\left[\int_{t_i}^s f''\left(\int_0^T h_N(X_u) du\right) \int_r^T h_N'(X_v)\mathcal{E}_v\mathcal{E}_r^{-1}\mathbb{E}_r[h_N'(X_s)\mathcal{E}_s\mathcal{E}_r^{-1}] dv\, dr\right] ds.$$

Thanks to the boundedness of f'' and $\mathcal{E}_s$, we have

$$|S_{2,b}^{n,N}| \leq C \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} ds \int_{t_i}^s dr \int_r^T \mathbb{E}\left[|h_N'(X_v)|\mathbb{E}_r[|h_N'(X_s)|]\right] dv$$

$$= C \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} ds \int_{t_i}^s dr \int_r^T \mathbb{E}\left[\mathbb{E}_r\left[|h_N'(X_v)|\right]\mathbb{E}_r\left[|h_N'(X_s)|\right]\right] dv.$$

Using (3.17) and Lemma 4.4 on $[r,T]^2$ with $a_1(s) = 1(r \leq s \leq t_{i+1})(s-r)^{-\frac{1}{2}}$ and $a_2(v) = 1(r \leq v \leq T)(v-r)^{-\frac{1}{2}}$ and Lemma 4.5(iv), we have that

$$|S_{2,b}^{n,N}| \leq CK(h)^2 \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} ds \int_{t_i}^{s} dr \int_r^T \frac{1}{\sqrt{s(v-r)}} dv \leq \frac{CK(h)^2}{n}.$$

This concludes the proof of Theorem 2.4. $\qquad\square$

3.6 Proof of Theorem 2.6

The proof is divided into two parts. We first consider the error of the continuous approximation $L_T(0) \approx \int_0^T \phi_\epsilon(X_s)ds$. The error of the discrete approximation $\int_0^T \phi_\epsilon(X_s)ds \approx \int_0^T \phi_\epsilon(X_{\eta_n(s)})ds$ is considered in the second part.

For the moment, we deal with the case of general σ, although in Lemma 3.9 we will make the assumption that $\sigma = 1$. Later, we will see that after applying Lamperti's transformation we can reduce our study to this case.

Lemma 3.7. *There exists a constant $C > 0$ such that the inequality*

$$\mathbb{E}\left[\left(L_T(0) - \int_0^T \phi_\epsilon(X_s)ds\right)^2\right] \leq C\epsilon^{1/2}|\log(\epsilon)|. \tag{3.18}$$

holds for any $\epsilon \in (0,1)$.

Proof. Denote by $p_t(x_0, x)$ the transition density of X_t. Since

$$\mathbb{E}[L_T(0)^2] = 2 \int_0^T \int_0^s p_u(x_0, 0)p_{s-u}(0,0)duds,$$

$$\mathbb{E}\left[L_T(0) \int_0^T \phi_\epsilon(X_s)ds\right] = \int_0^T ds \int_s^T du \int \phi_\epsilon(x)p_s(x_0,x)p_{u-s}(x,0)dx$$
$$+ \int_0^T ds \int_0^s du \int \phi_\epsilon(x)p_u(x_0,0)p_{s-u}(0,x)dx,$$

$$\mathbb{E}\left[\left(\int_0^T \phi_\epsilon(X_s)ds\right)^2\right] = 2\int_0^T ds \int_s^T du \int\int \phi_\epsilon(x)\phi_\epsilon(y)p_s(x_0,x)p_{u-s}(x,y)dxdy,$$

we decompose the left hand side of (3.18) as $2S_1 + 2S_2$ where

$$S_1 = -\int_0^T ds \int_0^s du \int p_u(x_0,0)\phi_\epsilon(x)\big(p_{s-u}(0,x) - p_{s-u}(0,0)\big)dx,$$

$$S_2 = \int_0^T ds \int_s^T du \int \left(\int \phi_\epsilon(y)p_{u-s}(x,y)dy - p_{u-s}(x,0)\right)\phi_\epsilon(x)p_s(x_0,x)dx.$$

We first rewrite S_1 as follows

$$S_1 = -\int_0^T ds \int_0^s du \int p_u(x_0,0)\phi_\epsilon(x) \int_0^x \frac{\partial p_{s-u}}{\partial y}(0,y)dydx.$$

Using the estimate (4.1) and the definition of ϕ_ϵ, we have

$$|S_1| \leq C \int_0^T ds \int_0^s du \int \frac{1}{\sqrt{u}} \frac{e^{-\frac{Cx^2}{\epsilon}}}{\sqrt{\epsilon}} \int_0^{|x|} \frac{e^{-\frac{Cy^2}{s-u}}}{s-u} dydx \leq C\epsilon^{1/2}|\log(\epsilon)|,$$

where the last estimate follows from Lemma 4.5(vi). To estimate S_2, we write

$$S_2 = \int_0^T ds \int_s^T du \int\int\int_0^y \phi_\epsilon(y)\frac{\partial p_{u-s}}{\partial z}(x,z)\phi_\epsilon(x)p_s(x_0,x)dzdydx.$$

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Thus, as in the estimation of S_1, we have

$$|S_2| \leq C \int_0^T ds \int_s^T du \int_0^\infty dy \phi_\epsilon(y) \int_0^y dz \int \frac{e^{-\frac{C(x-z)^2}{u-s}}}{u-s} \phi_\epsilon(x) \frac{e^{-\frac{C(x-x_0)^2}{s}}}{\sqrt{s}} dx$$

$$+ C \int_0^T ds \int_s^T du \int_{-\infty}^0 dy \phi_\epsilon(y) \int_y^0 dz \int \frac{e^{-\frac{C(x-z)^2}{u-s}}}{u-s} \phi_\epsilon(x) \frac{e^{-\frac{C(x-x_0)^2}{s}}}{\sqrt{s}} dx$$

$$\leq C \int_0^T ds \int_s^T du \int dy \phi_\epsilon(y) \int_0^{|y|} \frac{1}{\sqrt{s(u-s)(u-s+\epsilon)}} dz.$$

Using Lemma 4.5(v) we obtain

$$|S_2| \leq C\epsilon^{\frac{1}{2}} |\log(\epsilon)|.$$

This finishes the proof of Lemma 3.7. $\square$

Lemma 3.8. *Suppose that $\hat{\phi}_\epsilon \in C^1(\mathbb{R}, \mathbb{R}_+)$ such that there exists $\epsilon > 0$ satisfying*

$$|\hat{\phi}_\epsilon(x)| + \sqrt{\epsilon}|\hat{\phi}'_\epsilon(x)| \leq C \frac{e^{-\frac{Cx^2}{\epsilon}}}{\sqrt{\epsilon}}. \tag{3.19}$$

Then

$$\left| \mathbb{E}\Big[\int_0^T \hat{\phi}_\epsilon(X_s) ds \Big] - \mathbb{E}\Big[\int_0^T \hat{\phi}_\epsilon(X_{\eta_n(s)}) ds \Big] \right| \leq C \frac{\log(n)}{n\sqrt{\epsilon}}. \tag{3.20}$$

Moreover, if $x_0 \neq 0$ then

$$\left| \mathbb{E}\Big[\int_0^T \hat{\phi}_\epsilon(X_s) ds \Big] - \mathbb{E}\Big[\int_0^T \hat{\phi}_\epsilon(X_{\eta_n(s)}) ds \Big] \right| \leq C \frac{\log(n)}{n|x_0|}. \tag{3.21}$$

Proof. The estimate (3.20) is deduced as in Theorem 2.5 in [10], using the uniform estimate $|\hat{\phi}_\epsilon(x)| \leq \frac{C}{\sqrt{\epsilon}}$. We need only to show (3.21) for $x_0 \neq 0$,

$$\left| \mathbb{E}\Big[\int_0^T \hat{\phi}_\epsilon(X_s) ds \Big] - \mathbb{E}\Big[\int_0^T \hat{\phi}_\epsilon(X_{\eta_n(s)}) ds \Big] \right|$$

$$\leq \int_0^{\frac{T}{n}} \mathbb{E}[\hat{\phi}_\epsilon(X_s) + \hat{\phi}_\epsilon(X_{\eta_n(s)})] ds + \int_{\frac{T}{n}}^T \left| \mathbb{E}[\hat{\phi}_\epsilon(X_s) - \hat{\phi}_\epsilon(X_{\eta_n(s)})] \right| ds$$

$$\leq \int_0^{\frac{T}{n}} \Big(\int \hat{\phi}_\epsilon(x) p_s(x_0, x) dx + \hat{\phi}_\epsilon(x_0) \Big) ds + \int_{\frac{T}{n}}^T ds \int dy \hat{\phi}_\epsilon(y) \int_{\eta_n(s)}^s |\partial_u p_u(x_0, y)| du.$$

Using the Gaussian bound for the transition density p (Lemma 4.1), hypothesis (3.19) and the Chapman-Kolmogorov property for Gaussian kernels, we get

$$\left| \mathbb{E}\Big[\int_0^T \hat{\phi}_\epsilon(X_s) ds \Big] - \mathbb{E}\Big[\int_0^T \hat{\phi}_\epsilon(X_{\eta_n(s)}) ds \Big] \right|$$

$$\leq \int_0^{\frac{T}{n}} \frac{e^{-\frac{Cx_0^2}{s+\epsilon}}}{\sqrt{s+\epsilon}} ds + \frac{T\hat{\phi}_\epsilon(x_0)}{n} + \int_{\frac{T}{n}}^T \int_{\eta_n(s)}^s \frac{e^{-\frac{Cx_0^2}{u+\epsilon}}}{u\sqrt{u+\epsilon}} du ds$$

$$\leq C\Big(\frac{T}{n|x_0|} + \frac{1}{|x_0|} \int_{\frac{T}{n}}^T ds \int_{\eta_n(s)}^s \frac{1}{u} du \Big)$$

$$\leq C \frac{\log(n)}{n|x_0|},$$

where the second inequality follows from the estimate: $e^{-\frac{x_0^2}{u}} \leq \frac{\sqrt{u}}{|x_0|}$ for any $u > 0$ and the last estimate from Fubini's Theorem. $\square$

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Lemma 3.9. *Let $\hat{\phi}_\epsilon : \mathbb{R} \to \mathbb{R}$ satisfying (3.19). Assume that $\sigma = 1$. For any $p \geq 1$*

$$\mathbb{E}\left[\left(\int_0^T \hat{\phi}_\epsilon(X_s)ds - \int_0^T \hat{\phi}_\epsilon(X_{\eta_n(s)})ds\right)^{2p}\right] \leq \frac{C}{\epsilon^p n^{2p-\frac{1}{2}}}.$$

Proof. Denote

$$S_n = \int_0^T \hat{\phi}_\epsilon(X_s)ds - \int_0^T \hat{\phi}_\epsilon(X_{\eta_n(s)})ds.$$

As in the proof of Theorem 2.3(i), using the Clark-Ocone formula and the BDG inequality, we have

$$\mathbb{E}\left[|S_n - \mathbb{E}[S_n]|^{2p}\right] \leq Cn^{p-1}(T^1_{n,\epsilon} + T^2_{n,\epsilon}),$$

where

$$T^1_{n,\epsilon} = \sum_{i=0}^{n-1} \mathbb{E}\left[\left|\int_{t_i}^{t_{i+1}}\left(\int_u^{t_{i+1}} \mathbb{E}_u\left[|\hat{\phi}'_\epsilon(X_s)|\right]ds\right)^2 du\right|^p\right],$$

$$T^2_{n,\epsilon} = \sum_{i=0}^{n-1} \mathbb{E}\left[\left|\int_{t_i}^{t_{i+1}}\left(\int_{t_{i+1}}^T \mathbb{E}_u\left[\hat{\phi}'_\epsilon(X_s)\mathcal{E}_s - \hat{\phi}'_\epsilon(X_{\eta_n(s)})\mathcal{E}_{\eta_n(s)}\right]ds\right)^2 du\right|^p\right].$$

Next, we will show that

$$T^1_{n,\epsilon} + T^2_{n,\epsilon} \leq C\epsilon^{-p} n^{-2p+\frac{1}{2}}.$$

Therefore the proof of the Lemma finishes by using Lemma 3.8. The estimate of $T^1_{n,\epsilon}$ is similar to the one in the proof of Theorem 2.3 (Section 3.3.1). In fact, due to condition (3.19), we obtain that $\int |\hat{\phi}'_\epsilon(x)|dx \leq \frac{C}{\sqrt{\epsilon}}$. Therefore $K(\hat{\phi}_\epsilon) \leq \frac{C}{\sqrt{\epsilon}}$ and hence,

$$T^1_n \leq \frac{C}{\epsilon^p n^{2p-\frac{1}{2}}}.$$

To estimate $T^2_{n,\epsilon}$, we notice that since $\|\hat{\phi}_\epsilon(X_v)\|_{u,2} \leq C\epsilon^{-\frac{1}{4}}(v-u+\epsilon)^{-\frac{1}{4}}e^{-\frac{CX_u^2}{v-u+\epsilon}}$ and proceeding as in Section 3.3.2, applying Lemma 3.6 for $\zeta = \hat{\phi}_\epsilon$ yields $T^2_{n,N} \leq C(U_{n,\epsilon,1} + U_{n,\epsilon,2})$ where

$$U_{n,\epsilon,1} = \sum_{i=0}^{n-1} \mathbb{E}\left[\left|\int_{t_i}^{t_{i+1}}\left(\sum_{j=i+1}^{n-1}\int_{t_j}^{t_{j+1}}ds\int_{t_j}^s (v-u)^{-1}\|\hat{\phi}_\epsilon(X_v)\|_{u,2}dv\right)^2 du\right|^p\right],$$

$$U_{n,\epsilon,2} = \sum_{i=0}^{n-1} \mathbb{E}\left[\left|\int_{t_i}^{t_{i+1}}\left(\sum_{j=i+1}^{n-1}\int_{t_j}^{t_{j+1}}ds\int_{t_j}^s \frac{1}{(v-u)^{3/2}}\int |\hat{\phi}'_\epsilon(z)|\exp\left(-\frac{C(z-X_u)^2}{v-u}\right)dzdv\right)^2 du\right|^p\right].$$

The proof continues along the same lines of (3.13) in Section 3.3.2 after noticing that $\mathbb{E}[\|\hat{\phi}_\epsilon(X_v)\|^{2p}_{u,2}] \leq C\epsilon^{-\frac{p}{2}}(v-u+\epsilon)^{\frac{1-p}{2}}\frac{e^{-\frac{CX_0^2}{v+\epsilon}}}{\sqrt{v+\epsilon}}$ in order to obtain

$$U_{n,\epsilon,1} \leq \frac{C}{n^{3p-1}}\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}\left\{\int_{t_{i+1}}^T (v-u)^{-1}\epsilon^{-1/4}(v-u+\epsilon)^{(1-p)/4p}(v+\epsilon)^{-1/4p}\exp\left(-\frac{CX_0^2}{v+\epsilon}\right)dv\right\}^{2p}du$$

$$\leq \frac{C}{\epsilon^p n^{3p-1}}\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}\left\{\int_{t_{i+1}}^T (v-u)^{-1}dv\right\}^{2p}du.$$

We have using Lemma 4.5(i) that

$$U_{n,\epsilon,1} \leq \frac{C}{\epsilon^p n^{3p-1}}\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}\left(\log\left(\frac{T-u}{t_{i+1}-u}\right)\right)^{2p}du \leq C\frac{\log^{2p}(n)}{\epsilon^p n^{3p-1}}.$$

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The evaluation of $U_{n,\epsilon,2}$ is also done similarly, to obtain

$$U_{n,\epsilon,2} \leq \frac{C}{\epsilon^p n^{2p+\frac{1}{2}}} \sum_{i=0}^{n-1} t_{i+1}^{-\frac{1}{2}} \leq \frac{C}{\epsilon^p n^{2p-\frac{1}{2}}}.$$

This concludes the proof of Lemma 3.9. $\qquad\square$

Lemma 3.10. *For any $p \geq 1$*

$$\mathbb{E}\left[\left(\int_0^T \phi_\epsilon(X_s)ds - \int_0^T \phi_\epsilon(X_{\eta_n(s)})ds\right)^{2p}\right] \leq \frac{C}{\epsilon^p n^{2p-\frac{1}{2}}}.$$

Proof. Denote $\hat{\phi}_\epsilon = \phi_\epsilon \circ \mathcal{S}^{-1}$ where $\mathcal{S}$ is defined at the beginning of Section 3. It is straightforward to verify that $\hat{\phi}_\epsilon$ satisfies condition (3.19). Applying Lemma 3.9 for the diffusion process Y, defined by (3.1), we obtain the desired result. $\qquad\square$

Choosing the optimal value for ϵ, i.e., $\epsilon = n^{-1}$ in Lemmas 3.7, 3.8 and 3.10, we conclude the proof of Theorem 2.6.

Conclusion

Considering the weak and strong rate for multidimensional occupation time remains an open and difficult problem because, in the technique presented in this paper, it is essential that the same process X is considered in Lemma 3.6 at different times s and $\eta_n(s)$. In the general multidimensional case, if one considers $\mathbb{E}_u[\zeta'(X_s)\mathcal{E}_s - \zeta'(\bar{X}_{\eta_n(s)})\bar{\mathcal{E}}_{\eta_n(s)}]$ where the bars denote (Euler-Maruyama) approximation process, then the estimates are not easy to obtain. This topic as well as possible extensions to Multilevel Monte Carlo (MLMC) methods will be treated in future research. In particular, the fact that the strong rates obtained here for the approximation of local times are slower than the classical rates for smooth functionals imply that the MLMC methods have to be implemented taking into account both strong and weak rates of convergence.

In the case of general weak approximation problem for local times (i.e. non polynomial function f), one may find ways of obtaining non-optimal rates using the strong rate obtained in Theorem 2.6.

Another aspect of interest is the non-uniform elliptic case. For example in the hypoelliptic case. the estimates in time that appear in Lemma 4.1 are not valid. In fact, the terms $t - s$ are affected by higher powers which make the problem difficult to handle.

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4 Appendix

4.1 Some simple inequalities related to Gaussian densities

We first recall the following well-known Gaussian estimate for the transition probability density of diffusion X.

Lemma 4.1. *([6], Chapter 9) Let X be defined by (1.1). Under the condition (H), the Markov process X admits a transition probability density $p(s, x; t, y) = p_{t-s}(x, y)$ satisfying*

$$|\partial_x^m \partial_y^n p_{t-s}(x,y)| \leq \frac{C}{(t-s)^{\frac{m+n+1}{2}}} \exp\Big(-\frac{(y-x)^2}{C(t-s)}\Big), \tag{4.1}$$

for all $m, n \geq 0$, $m + n \leq 2$ and for some positive constant C which does not depend on s, t.

The following inequality can be easily obtained from a Gaussian bound on the transition density of X, and the fact that a convolution of Gaussian densities is still a Gaussian density.

Lemma 4.2.

$$\mathbb{E}_r\left[\exp\Big(-C\frac{(z-X_s)^2}{v-s}\Big)\right] \leq C\frac{\sqrt{v-s}}{\sqrt{v-r}}\exp\Big(-\frac{C(z-X_r)^2}{v-r}\Big). \tag{4.2}$$

Lemma 4.3. *Suppose that $u \geq v > r > 0$ and $x, y \in \mathbb{R}$, then*

$$\frac{ux^2 + vy^2 - 2rxy}{uv - r^2} \geq \frac{x^2 + y^2}{u + v}. \tag{4.3}$$

Proof. The proof of the following Lemma is algebraic and straightforward. In fact, multiply both sides of the inequality by $(uv-r^2)(u+v)$ and then simplify in order to obtain the inequality $(ux-ry)^2+(vy-rx)^2 \geq 0$. □

Lemma 4.4. *Suppose that $\zeta \in C^1_{\exp}(\mathbb{R})$. Let $a_i : (u, v) \to \mathbb{R}_+$, $i = 1, 2$ be two integrable functions. Define for $i = 1, 2$*

$$I_i = \int_u^v a_i(s) \int |\zeta'(x)| \exp\left(-\frac{C(x - X_u)^2}{s - u}\right) dx ds$$

$$\mathbb{E}\left[I_1 I_2\right] \leq C \int_{[u,v]^2} \frac{a_1(s) a_2(t) \sqrt{s - u}}{\sqrt{s}} \left(\int |\zeta'(x)| \exp\left(-C\frac{x^2}{s + t}\right) dx\right)^2 ds dt.$$

Proof. Expanding the square and Fubini's theorem, (4.1), straightforward calculations with Gaussian kernels and (4.3) it is enough to note that there exists a positive constant C such that

$$\mathbb{E}\left[\int_{\mathbb{R}^2} |\zeta'(x)||\zeta'(y)| \exp\left(-\frac{C(x - X_u)^2}{s - u}\right) \exp\left(-\frac{C(y - X_u)^2}{t - u}\right) dy dx\right]$$

$$\leq \frac{C}{\sqrt{u}} \int_{\mathbb{R}^3} |\zeta'(x)\zeta'(y)| \exp\left(-\frac{C(x - z)^2}{s - u} - \frac{C(y - z)^2}{t - u} - \frac{Cz^2}{u}\right) dz dx dy$$

$$= C\sqrt{(s - u)(t - u)} \int_{\mathbb{R}^2} \frac{|\zeta'(x)\zeta'(y)|}{\sqrt{st - u^2}} \exp\left(-C\frac{sy^2 + tx^2 - 2uxy}{st - u^2}\right) dx dy$$

$$\leq C\sqrt{(s - u)(t - u)} \int_{\mathbb{R}^2} \frac{|\zeta'(x)\zeta'(y)|}{\sqrt{st - u^2}} \exp\left(-C\frac{x^2 + y^2}{s + t}\right) dx dy.$$

Finally one uses the inequality $st - u^2 \geq s(t - u)$ to finish the proof. $\qquad\square$

4.2 Some error estimates for Riemman sums

In this section we give various estimates for error terms that appear in various Riemann like sums throughout the article.

Lemma 4.5. *We have the following estimates for a positive constant C independent of n and ϵ,*

(i) $\int_{t_i}^{t_{i+1}} \left(\log\left(\frac{T - u}{t_{i+1} - u}\right)\right)^{2p} du \leq C\frac{\log^{2p}(n)}{n}$, *for any $p \geq 1$.*

(ii) $\int_{t_i}^{t_{i+1}} \int_0^{t_i} \int_r^T \int_{t_i}^s \frac{1}{(v - r)\sqrt{u}} dv du dr ds \leq C\frac{\log(n)}{n^2}$,

(iii) $\sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} ds \int_0^{t_i} dr \int_r^T du \int_{t_i}^s (v - r)^{-1}(u - r)^{-\frac{1}{2}} v^{-\frac{1}{2}} dv \leq C\frac{\log(n)}{n}$,

(iv) $\sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} ds \int_{t_i}^s dr \int_r^T \frac{1}{\sqrt{s(v - r)}} dv \leq \frac{C}{n}$,

(v) $\int_0^T ds \int_s^T du \int dy \phi_\epsilon(y) \int_0^{|y|} \frac{1}{\sqrt{s(u - s)(u - s + \epsilon)}} dz \leq C\epsilon^{\frac{1}{2}} |\log(\epsilon)|$.

(vi) $\int_0^T ds \int_0^s du \int u^{-\frac{1}{2}} \frac{e^{-\frac{Cx^2}{\epsilon}}}{\sqrt{\epsilon}} \int_0^{|x|} \frac{e^{-\frac{Cy^2}{s - u}}}{s - u} dy dx \leq C\epsilon^{\frac{1}{2}} |\log(\epsilon)|$.

Proof. All the proofs follow by explicit integration when possible and then bounding the terms either by T or $\frac{T}{n}$. We only remark explicit points where the calculation has to be carefully done.

Proof of (i). Using a change of variable $x = \frac{T - t_{i+1}}{t_{i+1} - u}$, we write

$$\int_{t_i}^{t_{i+1}} \left(\log\left(\frac{T - u}{t_{i+1} - u}\right)\right)^{2p} du = \frac{T(n - i - 1)}{n} \int_{n-i-1}^\infty x^{-2} \log^{2p}(x + 1) dx \leq C\frac{\log^{2p}(n)}{n},$$

where the last inequality is obtained by applying integration by parts formula $[2p] + 1$ times and the trivial inequality $\frac{1}{x+1} < \frac{1}{x}$ for $x > 0$.

Proof of (ii). One integrates directly wrt u. The result is bounded by $\sqrt{T}$. Then by Fubini's theorem one carries out the integral wrt s first and then wrt r and v respectively. This gives the estimate.

Proof of (iii). Integrate first wrt u. Secondly, by Fubini integrate wrt s and bound the result by n^{-1}. Integrate wrt r to obtain the function

$$\frac{1}{n}\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}\frac{1}{\sqrt{v}}\log\left(\frac{v}{v-t_i}\right)dv$$

$$\leq \frac{1}{n}\sum_{i=1}^{n-1}\frac{1}{\sqrt{t_i}}\int_{t_i}^{t_{i+1}}\log\left(\frac{v}{v-t_i}\right)dv$$

$$\leq \frac{C\log(n)}{n}+\frac{1}{n}\sum_{i=1}^{n-1}\frac{1}{\sqrt{t_i}}(t_{i+1}\log(t_{i+1})-t_i\log(t_i)).$$

The Proof of (iv) is straightforward.

Proof of (v). Integrating with dz and dy, we obtain

$$\int_0^T ds\int_s^T du\int dy\epsilon^{-\frac{1}{2}}e^{-\frac{y^2}{2\epsilon}}\int_0^{|y|}\frac{1}{\sqrt{s(u-s)(u-s+\epsilon)}}dz$$

$$= C\epsilon^{\frac{1}{2}}\int_0^T ds\int_s^T\frac{1}{\sqrt{s(u-s)(u-s+\epsilon)}}du$$

By using a change of variables $x=\frac{u-s}{\epsilon}$ we obtain

$$\int_s^T\frac{1}{\sqrt{(u-s)(u-s+\epsilon)}}du=\int_0^{\frac{T-s}{\epsilon}}\frac{1}{\sqrt{x(x+1)}}dx\leq C\left(1+\log\left(\frac{T-s}{\epsilon}\right)\right),$$

hence

$$\int_0^T ds\int_s^T\frac{1}{\sqrt{s(u-s)(u-s+\epsilon)}}du\leq C\int_0^T\frac{1}{\sqrt{s}}\left(1+\log\left(\frac{T-s}{\epsilon}\right)\right)ds\leq C|\log(\epsilon)|.$$

Proof of (vi).

$$\int_0^T ds\int_0^s du\int u^{-\frac{1}{2}}\frac{e^{-\frac{Cx^2}{\epsilon}}}{\sqrt{\epsilon}}\int_0^{|x|}\frac{e^{-\frac{Cy^2}{s-u}}}{s-u}dydx$$

$$= \int_0^T ds\int_0^{(s-\epsilon)\vee 0}du\int u^{-\frac{1}{2}}\frac{e^{-\frac{Cx^2}{\epsilon}}}{\sqrt{\epsilon}}\int_0^{|x|}\frac{e^{-\frac{Cy^2}{s-u}}}{s-u}dydx$$

$$+ \int_0^T ds\int_{(s-\epsilon)\vee 0}^s du\int u^{-\frac{1}{2}}\frac{e^{-\frac{Cx^2}{\epsilon}}}{\sqrt{\epsilon}}\int_0^{|x|}\frac{e^{-\frac{Cy^2}{s-u}}}{s-u}dydx$$

$$= J_1 + J_2.$$

We have

$$J_1\leq C\int_0^T ds\int_0^{(s-\epsilon)\vee 0}du\int u^{-\frac{1}{2}}\frac{e^{-\frac{Cx^2}{\epsilon}}}{\sqrt{\epsilon}}|x|(s-u)^{-1}dx$$

$$\leq C\epsilon^{\frac{1}{2}}\int_0^T ds\int_0^{(s-\epsilon)\vee 0}u^{-\frac{1}{2}}(s-u)^{-1}du\leq C\epsilon^{\frac{1}{2}}|\log(\epsilon)|,$$

and

$$J_2\leq C\int_0^T ds\int_{(s-\epsilon)\vee 0}^s du\int u^{-\frac{1}{2}}\frac{e^{-\frac{Cx^2}{\epsilon}}}{\sqrt{\epsilon}}(s-u)^{-\frac{1}{2}}dx$$

$$\leq C\int_0^T ds\int_{(s-\epsilon)\vee 0}^s u^{-\frac{1}{2}}(s-u)^{-\frac{1}{2}}du\leq C\epsilon^{\frac{1}{2}}.$$

$\square$