Lower bounds for densities of Asian type stochastic differential equations

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August, 31 2009

Abstract

We obtain lower bounds for densities of solutions of certain hypoelliptic two dimensional stochastic differential equations where one of the components is the Lebesgue integral of the other. These results are non-trivial extensions of previous work of the authors. In particular, these type of equations are linked to the so-called Asian option set-up.

Keywords: lower bounds, density function, asian type sde's, Malliavin Calculus.

1 Introduction

We consider the bi-dimensional diffusion process solution of the equation

$$X_t^1 = x^1 + \int_0^t \sigma(X_s)dW_s + \int_0^t b_1(X_s)ds, \quad X_t^2 = x^2 + \int_0^t b_2(X_s)ds, \ t \in [0,T].$$

We assume that the coefficients σ, b_1 and b_2 are five times differentiable and have bounded derivatives but the functions themselves do not need to be bounded. Also notice that we have just one Brownian motion W driving the process $X = (X^1, X^2)$ so the ellipticity assumption fails at any point, and the strong Hörmander condition (based on the coefficients of the Brownian motion only) fails as well. But we assume that $\overline{\sigma}(x) = (\sigma(x), 0)$ and $[\overline{\sigma}, b](x)$ span $\mathbb{R}^2$ (here $[.,.]$ denotes the Lie bracket) so that the weak Hörmander's condition holds true. Therefore the law of $X_T(x)$ is absolutely continuous with respect to the Lebesgue measure and has a continuous density $p_T(x, y)$.

Our aim is to give lower bounds for $p_T(x, y)$. Such lower bounds have already been obtained by C.L.Fefferman, and A.Sanchez-Calle in [13] and [6], and by S. Kusuoka and D. Stroock in [7]. But the results in those papers do not cover the case of the weak Hörmander's condition.

On the same theme, S. Polidoro, A. Pascucci and U. Boscain in [12], [11] and [4] give an analytical approach to the problem discussed here. The lower bounds in those articles are analogues with the ones obtained here. Their framework is a little bit more general because they consider a diffusion process which has the same structure as ours but they work in an arbitrary dimension (we think that our approach works also in arbitrary dimension but this would ask some technical effort). On the other hand the framework here is more general that the analytical approach because in these articles, the representative example is $b_2(x) = x^1$ (or either a polynomial) with σ bounded.

Our approach is a probabilistic one based on the results in A. Kohatsu-Higa [8] and V. Bally [2]. Other methods can be found in P. Malliavin and E. Nualart [9] and F. Delarue and S. Menozzi [5]. We consider here the 2-dimensional case in order not to obscure the crucial arguments. We believe that similar arguments should be used in the general d-dimensional case. Also note that our elliptic hypothesis on σ will be only local (see (8)) while in the other articles mentioned above this is not the case.

Let us consider a fixed control $\phi \in L^2([0, T])$ and define the corresponding skeleton $x_t^i = x_t^i(\phi)$

$$x_t^1 = x^1 + \int_0^t \sigma(x_s)\phi_s ds + \int_0^t \hat{b}_1(x_s) ds, \quad x_t^2 = x^2 + \int_0^t b_2(x_s) ds, \qquad (1)$$

where $\hat{b}_1(x) = b_1(x) - \frac{1}{2}(\sigma\partial_1\sigma)(x)$. It is well known (see [1]) that the support of the diffusion process X_t is concentrated on the curves of the previous form. So a necessary condition for $p_T(x, y) > 0$ is that there exists a control ϕ such that $x_T^1(\phi) = y$ (this condition is not sufficient: see [1] and [3]). So in a first stage we assume that such a control exists and we prove that (under appropriate assumptions on σ and b_2, see (8), (9) and (24)) we have

$$p_T(x, y) \geq C_1 \exp\left(-C_2 \int_0^T \phi_t^2 H_t^2 dt\right)$$

where C_1 and C_2 are constants which depend on the bounds of the coefficients and H is an explicit function (see Theorem 14 and the comments right before Lemma 13).

Once this result is proved a second problem appears: how to find the control ϕ and how to compute $\int_0^T \phi_t^2 dt$. We solve this problem in two specific situations. First of all we assume that $|\sigma(z)| \geq \varepsilon > 0$ and $|\partial_1 b_2(z)| \geq \varepsilon > 0$. We call this case the "elliptic" case because the first component satisfies an ellipticity condition (although the whole diffusion does not). Then we prove (see Theorem 17) that

$$p_T(x,y) \geq C_1 \exp\left(-C_2\left(T + \frac{\left|y^2 - x^2 - b_2(x)T\right|^2}{T^3} + \frac{\left|y^1 - x^1\right|^2}{T}\right)\right).$$

This estimate is in the good range: if we consider the case $\sigma = 1, b_1 = 0$ and $b_2(x) = x^1$ then X_T is a Gaussian vector and the density of the law of this vector has upper and lower bounds of this form. The estimates in [4], [11] and [12] are of the same spirit. But in the estimates obtained under the strong Hörmander condition in [6],[7] and [13] the term $T^{-3} \times \left|y^2 - x^2 - b_2(x)T\right|^2$ does not appear.

As a representative example, we consider the "linear case". That is,

$$X_t^1 = x^1 + \int_0^t \bar{\sigma}(X_s) X_s^1 dW_s + \int_0^t \bar{b}_1(X_s) X_s^1 ds, \quad X_t^2 = x^2 + \int_0^t b_2(X_s) ds.$$

This framework is interesting in mathematical finance because it represents the model used for Asian type options. Under similar hypothesis we obtain a lower bound for the density (see Example 20). Note that, one can easily prove from the results in Yor [14] that in the particular case that $\bar{\sigma}(x) = 2$, $\bar{b}_1(x) = 2$ and $b_2(x) = x_1$ then $\lim_{y^2 \to \infty} y^2 \log\left(\left|y^2\right|^2 p_T(x,y)\right) < 0$. This result seems to indicate that the bound obtained here, (see Example 20 and also [12]) for this case, is general. The result of M. Yor is far more exact but it uses strongly the particular structure of the example while here the spirit is to develop a somewhat general theory.

Throughout the text we denote by $|\cdot|$ the Euclidean norm on the corresponding d-dimensional space $\mathbb{R}^d$. Given two positive definite invertible matrices A and B, we say that $A \geq B$ if $A - B$ is a positive definite matrix. $\mathcal{B}([a,b])$ stands for the Borelian sets on the interval $[a,b]$.

We use the following directional derivative notation $\partial_\sigma b = \sigma_1 \partial_1 b + \sigma_2 \partial_2 b$ for a 2-dimensional vector function $\sigma = (\sigma_1, \sigma_2) : \mathbb{R}^2 \to \mathbb{R}^2$ and a function $b : \mathbb{R}^2 \to \mathbb{R}^2$. Also we define $Lb := \frac{1}{2} \sum_{i,j=1}^2 \sigma_i \sigma_j \partial_i \partial_j b + \sum_{i=1}^2 b^i \partial_i b$.

2 Evolution sequences

In this section we recall the framework and some results from [2]. We consider a probability space $(\Omega, \mathcal{F}, P)$ with a filtration $\mathcal{F}_t, t \geq 0$ and a one dimensional Brownian motion $W_t, t \geq 0$. In order to define an "elliptic evolution sequence" we consider the following objects. We give a time grid $0 = t_0 < t_1 < ... < t_N = T$ and we define $\delta_k = t_k - t_{k-1}$. We consider also a space grid $y_k \in \mathbb{R}^d, k = 0, ..., N$. Moreover, we consider a sequence of positive definite and invertible deterministic $d \times d$ dimensional matrices M_k, $k = 1, ..., N$. To these matrices we associate the norms $|x|_k = \sqrt{\langle M_k^{-1} x, x \rangle}, x \in \mathbb{R}^d$ and we fix a sequence of numbers $H_k \geq 1$, $k = 1, ..., N$ such that for $k = 2, ..., N$

$$|x|_k \leq H_k |x|_{k-1} . \tag{2}$$

Moreover we fix a sequence of numbers $\rho_k \in (0,1), k = 1, ..., N$ such that

$$|y_k - y_{k-1}|_k \leq \frac{\rho_k}{4}. \tag{3}$$

We consider now a sequence of random variables $F_k, k = 0, ..., N$ such that F_k is $\mathcal{F}_{t_k}$ measurable and for $k = 1, ..., N$

$$F_k = F_{k-1} + \int_{t_{k-1}}^{t_k} h_k(s, \omega) dW_s + R_k. \tag{4}$$

Here $h_k : [t_{k-1}, t_k] \times \Omega \to R^d, k = 1, ..., N$ are $B([t_{k-1}, t_k]) \times \mathcal{F}_{t_{k-1}}$measurable and $E\left[\int_{t_{k-1}}^{t_k} |h_k(s, \omega)|^2 ds\right] < \infty$. So, conditionally to $\mathcal{F}_{t_{k-1}}$, $J_k := \int_{t_{k-1}}^{t_k} h_k(s, \omega) dW_s$ is a centered Gaussian random variable with covariance matrix

$$C^{ij}(J_k) = \int_{t_{k-1}}^{t_k} h_k^i(s, \omega) h_k^j(s, \omega) ds, \quad i, j \in \{1, ..., d\}.$$

R_k is a remainder which has to be small with respect to the Gaussian part. In order to express this assumption we need the following norms on the Wiener space. For a d dimensional functional $F : \Omega \to \mathbb{R}^d$ which is m

times differentiable in Malliavin sense (we refer to [10] and [2] for notations and definitions) we define

$$\left\|F\right\|_{k,m,p}=(\mathbb{E}_{t_{k-1}}\left[\left|F\right|^{p}\right])^{1/p}+\sum_{i=1}^{m}\left(\mathbb{E}_{t_{k-1}}\left[\left(\int_{[t_{k-1},t_k]^i}\left|D^i_{s_1,\dots,s_i}F\right|^2ds_1...ds_i\right)^{p/2}\right]\right)^{1/p}.$$

Here $\mathbb{E}_{t_{k-1}}$ designs the conditional expectation with respect to $\mathcal{F}_{t_{k-1}}$. Notice that $\left\|F\right\|_{k,m,p}$ was denoted by $\left\|F\right\|_{t_{k-1},\delta_k,m,p}$ in [2]. Finally we define the sets for $k=1,...,N$

$$A_k=\{\omega\in\Omega:\left|F_{i-1}-y_i\right|_i\leq\frac{\rho_i}{2},i=1,...,k\}\in\mathcal{F}_{t_{k-1}}.$$

In the following definition we will use two universal constants

$$p_d=2^{2(d+2)}p^*(d),\quad\text{and}\quad C_d=c^*(d)\mu(d+1)(2\pi)^{d/2}4^{3(d+3)^3}(d+1)^{d+3}.$$

Here $p^*(d)$ and $c^*(d)$ are given in Proposition 3 of [2] and $\mu(d+1)$ is given in equation (2) of [2]. They are universal constants depending on d only. In particular, that is the case of the constants C_d and p_d. We assume without loss of generality that $C_d\geq 1$. We are now able to give our definition.

Definition 1 *The sequence $F_k, k=0,...,N$ is called an elliptic evolution sequence if there exists some constants $a_k\geq 1, k=0,...,N$ such that for every $k=1,...,N$ and every $\omega\in A_k$ one has*

$$i)\quad a_kM_k\geq C(J_k)\geq\frac{1}{a_k}M_k, \tag{5}$$

$$ii)\quad\left\|M_k^{-1/2}R_k\right\|_{k,d+2,p_d}\leq\left(C_da_k^{8(d+1)^2+\frac{1}{2}}e^{\frac{a_{k-1}(\rho_k+\rho_{k-1})^2}{8}}\right)^{-1},$$

In this case, we define

$$\theta=\ln(8^2(2\pi d)^{1/2})+\frac{1}{2N}\sum_{k=1}^{N}\ln(a_k)+\frac{1}{N}\sum_{k=2}^{N}\ln(\frac{H_k}{\rho_k})+\frac{1}{8dN}\sum_{k=2}^{N}a_{k-1}. \tag{6}$$

A slight modification of the main result from [2] gives the result that we will use in this paper. For some details of this proof see Appendix 6.

Theorem 2 *Let $F_k, k=0,...,N$ be an elliptic evolution sequence such that the law of F_N is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^d$ and has a continuous density p. Then*

$$p(y_N)\geq\frac{1}{4(2\pi)^{d/2}\sqrt{\det M_N}}\times e^{-N\times d\times\theta}. \tag{7}$$

3 General framework

We consider the two dimensional diffusion process

$$X_t = x + \int_0^t \sigma(X_s) dW_s + \int_0^t b(X_s) ds$$

with $\sigma, b : \mathbb{R}^2 \to \mathbb{R}^2$ five times differentiable functions. And W is an one dimensional Brownian motion. We denote by $\hat{b} = (\hat{b}_1, \hat{b}_2) = b - \frac{1}{2}\partial_\sigma \sigma$ the drift in the equation of X when written with a Stratonovich integral and by L its infinitesimal generator. We will also use the skeleton associated to this equation. That is, for a control $\phi \in L^2[0,T]$ we define $x_t = x_t(\phi) = (x_t^1, x_t^2)$ as the solution of the equation

$$x_t = x + \int_0^t \sigma(x_s)\phi_s ds + \int_0^t \hat{b}(x_s) ds.$$

We have to precise our hypothesis. The first one concerns the non degeneracy. We assume that for a fixed control function ϕ,

$$(\mathbf{H}_1(\phi)) \quad i) \quad \sigma_2(x) = 0, \forall x \in \mathbb{R}^2. \qquad (8)$$
$$ii) \text{There exist a function } \varepsilon_\phi : [0,\infty) \to (0,\infty) \text{ such that}$$
$$\min\{|\sigma_1(x_t)|, |\partial_\sigma b_2(x_t)|\} > \varepsilon_\phi(t), \quad t \geq 0.$$

Remark: Under the present setting condition $ii)$ is implied by the weak Hörmander condition. From now until Section 4 we assume that the control function ϕ is fixed.

We also assume that we have the following upper bounds. For a multi-index $\alpha = (\alpha_1, ..., \alpha_p) \in \{1,2\}^p$, $p \in \mathbb{N}$, we denote its length by $|\alpha| = p$ and $\partial_\alpha = \partial_{\alpha_1} ... \partial_{\alpha_p}$. We assume that for a fixed control function ϕ, there exist a function $C_\phi : [0,\infty) \to (0,\infty)$ and a constant $C_* > 1$ such that for any $\varepsilon \in [0,1]$ and $f = \sigma, b, \partial_\sigma b, \partial_\sigma \sigma, \sigma + \varepsilon \partial_\sigma b, Lb$ we have

$$(\mathbf{H}_2(\phi)) \quad i) \quad |f(x_t)| \leq C_\phi(t), \quad \forall t \geq 0, \qquad (9)$$
$$ii) \quad |\partial_\alpha f(x)| \leq C_*, \quad \forall x \in \mathbb{R}^2, \ 1 \leq |\alpha| \leq 5$$
$$iii) \quad C_\phi(t) \geq 1 \geq \varepsilon_\phi(t).$$

Notice that we assume that the derivatives of the coefficients are bounded but the coefficients themselves may have linear growth (the bound in $(\mathbf{H}_2(\phi))i)$

concerns $x_t, t \geq 0$ only). Note that the bounds like in $(\mathbf{H}_2(\phi))i)$ are also valid for the functions $\partial_\sigma b$ and $\sigma + \varepsilon\partial_\sigma b$ uniformly for $\varepsilon \in [0,1]$ as well as its derivatives up to order 5 if we only assume $|\sigma(x_t)| + |b(x_t)| \leq C_\phi(t)$. In fact, we have that

$$\begin{aligned} |\partial_\sigma b(x_t)| &\leq C_\phi(t)C_* \\ |(\sigma + \varepsilon\partial_\sigma b)(x_t)| &\leq C_\phi(t)(1+\varepsilon C_*) \\ |\partial_\sigma \sigma(x_t)| &\leq C_\phi(t)C_*. \end{aligned}$$

We have preferred this presentation which may be redundant as far as bounds are concerned. But this setting avoids writing cumbersome constants. A similar remark is valid about the hypothesis $|\partial_\alpha\partial_\sigma b(x)| \leq C_*$ in $(\mathbf{H}_2(\phi))ii)$.

3.1 Evolution sequence

We start this section with a description of the main goal in the first part of the article. We consider a time grid $0 = t_{-1} = t_0 < t_1 < ... < t_N$ with $t_{k+1} - t_k \leq 1$ and we denote for $k = 1, ..., N$

$$\delta_k = t_k - t_{k-1}, \quad C_k = \sup_{t_{k-2} \leq t \leq t_k} C_\phi(t), \quad \varepsilon_k = \min\{\varepsilon_\phi(t_{k-1}), \varepsilon_\phi(t_{k-2})\}.$$

Note that due to $(\mathbf{H}_2(\phi))$ we have that $\varepsilon_k \leq 1 \leq C_k$. Moreover we consider a sequence of numbers $\rho_k \leq 1$, $k = 1, ..., N$ such that

$$\rho_k := \sqrt{\max\{\delta_k, \int_{t_{k-1}}^{t_k} \phi_t^2 dt\}}. \tag{10}$$

Define the following quantities

$a_k = 196(C_k\varepsilon_k^{-1})^2$, $H_k = 6U_k^{3/2}(C_k\varepsilon_k^{-1})$ and $U_k = \max\{\delta_k, \delta_{k-1}\}/\min\{\delta_k, \delta_{k-1}\}$ for $k = 1, ..., N$.

We assume that for $\mu f_{k+1} \geq f_k \geq \mu^{-1} f_{k+1}$ for $f_k = \varepsilon_k, C_k$ and all k. Furthermore,we will prove in Lemma 10 that there exists a constant $C = C(p, T, C_*)$ such that for $u \in [t_{k-1}, t_k]$ and $|F_{i-1} - y_i|_i \leq \frac{\rho_i}{2}$ for $i = 1, ..., k$

$$\begin{aligned} \left\|X_u - X_{t_{k-1}}\right\|_{k,4,p} &\leq CC_k\rho_k, \\ \|X_u\|_{k,4,p} &\leq CC_k + |x_{t_{k-1}}|. \end{aligned}$$

Let $C^* \equiv C^*(\mu)$ be a positive constant such that

$$C^*(\mu) \leq \min \left\{ \left(CC_d 196^{\frac{145}{2}} \right)^{-1} e^{-\frac{49}{2}(C^*(\mu))^2(\mu^2+1)}, \sqrt{\frac{2}{9C_*}}, 1 \right\}. \qquad (11)$$

Such a constant exists because $x \to xe^{\frac{49}{2}(1+\mu^2)^2 x^2}$ is an increasing continuous function which is zero at $x = 0$. Moreover, denote

$$\overline{\sigma}_k = \sigma(x(t_{k-1})), \overline{b}_k = \partial_\sigma b(x(t_{k-1})) \quad \text{and} \quad \overline{c}_k = \overline{\sigma}_k + \overline{b}_k \frac{\delta_k}{2}$$

and we consider the matrices

$$\overline{N}_k = \sqrt{\delta_k} \begin{pmatrix} \overline{c}_{k,1} & \overline{b}_{k,1} \frac{\delta_k}{\sqrt{12}} \\ \overline{c}_{k,2} & \overline{b}_{k,2} \frac{\delta_k}{\sqrt{12}} \end{pmatrix}, \quad M_k = \overline{N}_k \times \overline{N}_k^*.$$

Finally we consider

$$F_k := X(t_k) - x(t_k), \quad k = 1, ..., N$$

and we notice that, if $x = X_0$ and $y = x(t_N)$ then

$$p_{t_N}(x, y) = p_{F_N}(0)$$

where p_{F_N} is the density of F_N. The goal in the first part of the article is to prove the following lower bound result for $p_{t_N}(x, y)$.

Theorem 3 *We suppose that $H_i(\phi), i = 1, 2$ hold and suppose that we have*

$$\rho_k \leq C^*(\varepsilon_k C_k^{-1})^{148}.$$

Then F_k $:, k = 1, ..., N$ is an elliptic evolution sequence with parametrs M_k, a_k, H_k, ρ_k and consequently we have

$$p_{t_N}(x, y) \geq \frac{1}{8\pi\sqrt{\det M_N}} \times e^{-2N\theta}$$

with θ defined in (6).

The proof is given through a sequence of lemmas. The basic decomposition that leads to (4) is given by the following lemma.

We take $y_k = 0$. In particular the relation (3) trivially holds true.

Lemma 4 *With the previous definitions, we have that (4) is satisfied with*

$$F_k - F_{k-1} = J_k + R_k$$

where

$$J_k = \sigma(X_{t_{k-1}})\Delta_k + \partial_\sigma b(X_{t_{k-1}})\overline{\Delta}_k,$$
$$\Delta_k := W(t_k) - W(t_{k-1}), \quad \overline{\Delta}_k := \int_{t_{k-1}}^{t_k} (t_k - s)dW_s$$

and

$$\begin{aligned} R_k &:= \sum_{i=1}^{6} R_{k,i} \qquad (12) \\ &= \int_{t_{k-1}}^{t_k} (\sigma(X_s) - \sigma(X_{t_{k-1}}))dW_s + \int_{t_{k-1}}^{t_k} (\partial_\sigma b(X_s) - \partial_\sigma b(X_{t_{k-1}}))(t_k - s)dW_s \\ &+ \int_{t_{k-1}}^{t_k} Lb(X_s)(t_k - s)ds - \int_{t_{k-1}}^{t_k} (\sigma(x_s)\phi_s + \frac{1}{2}\partial_\sigma\sigma(x_s))ds \\ &- \int_{t_{k-1}}^{t_k} (b(x_s) - b(x_{t_{k-1}}))ds + (b(X_{t_{k-1}}) - b(x_{t_{k-1}}))\delta_k. \end{aligned}$$

Proof. It is enough to note that

$$\begin{aligned} &\int_{t_{k-1}}^{t_k} b(X_s)ds \\ &= b(X_{t_{k-1}})\delta_k + \int_{t_{k-1}}^{t_k} (b(X_s) - b(X_{t_{k-1}}))ds \\ &= b(X_{t_{k-1}})\delta_k + \int_{t_{k-1}}^{t_k} \left(\int_{t_{k-1}}^{s} \partial_\sigma b(X_r)dW_r + \int_{t_{k-1}}^{s} Lb(X_r)dr \right) ds \\ &= b(X_{t_{k-1}})\delta_k + \int_{t_{k-1}}^{t_k} \partial_\sigma b(X_r)(t_k - r)dW_r + \int_{t_{k-1}}^{s} Lb(X_r)(t_k - r)dr, \\ &\int_{t_{k-1}}^{t_k} \partial_\sigma b(X_r)(t_k - r)dW_r \\ &= \partial_\sigma b(X_{t_{k-1}}) \int_{t_{k-1}}^{t_k} (t_k - r)dW_r + \int_{t_{k-1}}^{t_k} \left(\partial_\sigma b(X_r) - \partial_\sigma b(X_{t_{k-1}})\right)(t_k - r)dW_r. \end{aligned}$$

Furthermore,

$$x_{t_k}-x_{t_{k-1}} = b(x_{t_{k-1}})\delta_k+\int_{t_{k-1}}^{t_k}(\sigma(x_s)\phi_s-\frac{1}{2}\partial_\sigma\sigma(x_s))ds+\int_{t_{k-1}}^{t_k}(b(x_s)-b(x_{t_{k-1}}))ds. \tag{13}$$

So we have the above decomposition if we note that

$$F_k - F_{k-1} = (X_{t_k} - X_{t_{k-1}}) - (x_{t_k} - x_{t_{k-1}}).$$

□

3.2 Covariance matrices

For $x, y \in \mathbb{R}^2$ we define

$$[x, y] = \langle x, y^{\perp}\rangle = x_1y_2 - x_2y_1.$$

Notice that $[x, y] = 0$, $y \neq 0$ implies that x and y are colinear. Also one has that

$$|[x, y]| \leq |x|\,|y|\,.$$

We denote for $k = 1, ..., N$

$$\sigma_k = \sigma(X_{t_{k-1}}), \quad b_k = \partial_\sigma b(X_{t_{k-1}}), \quad c_k = \sigma_k + b_k\frac{\delta_k}{2}$$

$$d_k = [c_k, b_k] = [\sigma_k, b_k].$$

The context will determine the difference between the vector valued random variables b_k and σ_k as defined above and the functions b_k and σ_k, $k = 1, 2$ used in (1). We will denote the components of the above random vector by $f_k = (f_{k,1}, f_{k,2})$ for $f = b, c, d, \sigma$. We compute for $k = 1, ..., N$:

$$\mathbb{E}_{t_{k-1}}\left[J_{k,i}^2\right] = \delta_k\left(c_{k,i}^2 + b_{k,i}^2\frac{\delta_k^2}{12}\right), \quad i = 1, 2,$$

$$\mathbb{E}_{t_{k-1}}\left[J_{k,1}\times J_{k,2}\right] = \delta_k\left(c_{k,1}c_{k,2} + b_{k,1}b_{k,2}\frac{\delta_k^2}{12}\right).$$

So the conditional covariance matrix of the random vector J_k is

$$C(J_k) = N_k \times N_k^*$$

where

$$N_k := \sqrt{\delta_k}\left(\begin{array}{cc} c_{k,1} & b_{k,1}\times\frac{\delta_k}{\sqrt{12}} \\ c_{k,2} & b_{k,2}\times\frac{\delta_k}{\sqrt{12}} \end{array}\right)$$

and N_k^* denotes the transpose of N_k. We define now the deterministic matrix M_k which will control the covariance matrix $C(J_k)$. The idea is to we replace the random variable $X_{t_{k-1}}$ by the corresponding deterministic point $x_{t_{k-1}}$. We denote for $k=1,...,N$

$$\overline{d}_k = [\overline{c}_k, \overline{b}_k] = [\overline{\sigma}_k, \overline{b}_k].$$

Notice that if $\bar{d}_k \neq 0$ then $\bar{N}_k$ is invertible and we have

$$\overline{N}_k^{-1} = \frac{\sqrt{12}}{\delta_k^{3/2}\overline{d}_k}\left(\begin{array}{cc} \overline{b}_{k,2}\times\frac{\delta_k}{\sqrt{12}} & -\overline{b}_{k,1}\times\frac{\delta_k}{\sqrt{12}} \\ -\overline{c}_{k,2} & \overline{c}_{k,1} \end{array}\right).$$

Note that under hypotheses (8) and (9), we have $|\overline{b}_k| \leq C_k$, $|\overline{\sigma}_k| \leq C_k$, $|\overline{c}_k| \leq C_k$ and $\varepsilon_k^2 \leq |\overline{d}_k| \leq |\overline{\sigma}_k||\overline{b}_k| \leq C_k^2$.

3.3 Properties of the norm $|\cdot|_k$

As stated in Section 2, we define for $k=1,...,N$

$$|x|_k^2 = \left\langle M_k^{-1}x, x\right\rangle = \left|\overline{N}_k^{-1}x\right|^2.$$

Therefore

$$|x|_k^2 = \frac{1}{\delta_k \overline{d}_k^2}\left([x,\overline{b}_k]^2 + \frac{12}{\delta_k^2}[x,\overline{c}_k]^2\right).$$

Similarly

$$\left\langle C(J_k)^{-1}x, x\right\rangle = \frac{1}{\delta_k d_k^2}\left([x,b_k]^2 + \frac{12}{\delta_k^2}[x,c_k]^2\right).$$

We will use the following two inequalities of norms.

Lemma 5 *We have for $k=1,...,N$*

$$|x|^2 \leq \left(\frac{\delta_k}{\sqrt{12}}\left|\overline{b}_k\right| + |\overline{c}_k|\right)^2 |x|_k^2\,\delta_k, \tag{14}$$

and for $x = (x_1, x_2)$

$$|x|_k^2 \leq 24 C_k^2 \frac{\left|\left(x_1 \frac{\delta_k}{\sqrt{2}}, \sqrt{2} x_2\right)\right|^2}{\delta_k^3 \overline{d}_k^2}. \tag{15}$$

Proof. Let $\alpha, \beta \in \mathbb{R}^2$. We solve the system of equations $y_1 = [x, \alpha], y_2 = [x, \beta]$ and we obtain

$$x = \frac{1}{[\beta, \alpha]}(y_1\beta_1 - y_2\alpha_1, y_1\beta_2 - y_2\alpha_2) = \frac{1}{[\beta, \alpha]}(y_1\beta - y_2\alpha).$$

It follows that

$$\begin{aligned} [\beta, \alpha]^2 |x|^2 &= |y_1\beta|^2 + |y_2\alpha|^2 - 2y_1y_2 \langle \alpha, \beta \rangle \leq |y_1\beta|^2 + |y_2\alpha|^2 + 2|y_1\beta||y_2\alpha| \\ &= (|y_1\beta| + |y_2\alpha|)^2 \leq |y|^2 (|\beta| + |\alpha|)^2. \end{aligned}$$

So we have

$$|x|^2 \leq \frac{[x, \alpha]^2 + [x, \beta]^2}{[\beta, \alpha]^2}(|\beta| + |\alpha|)^2. \tag{16}$$

Using (16) with $\alpha = \overline{b}_k$ and $\beta = \frac{\sqrt{12}}{\delta_k}\overline{c}_k$ we obtain

$$|x|^2 \leq \frac{\left(\frac{\delta_k}{\sqrt{12}}\left|\overline{b}_k\right| + |\overline{c}_k|\right)^2}{\overline{d}_k^2}\left([x, \overline{b}_k]^2 + \frac{12}{\delta_k^2}[x, \overline{c}_k]^2\right) = \left(\frac{\delta_k}{\sqrt{12}}\left|\overline{b}_k\right| + |\overline{c}_k|\right)^2 |x|_k^2 \, \delta_k.$$

Now we proceed with the proof of (15). Let $v_k^i = \left(\overline{c}_{k,i}, \overline{b}_{k,i}\frac{\delta_k}{\sqrt{12}}\right)$, $i = 1, 2$. Then clearly,

$$|x|_k^2 = 12\delta_k^{-3}\overline{d}_k^{-2}\left(x_1^2 \left|v_k^2\right|^2 - 2x_1x_2 \left\langle v_k^1, v_k^2 \right\rangle + x_2^2 \left|v_k^1\right|^2\right).$$

Then the result follows because (here, we use that $\sigma_2 \equiv 0$)

$$\left|v_k^2\right|^2 \leq C_k^2 \frac{\delta_k^2}{2}$$
$$\left|v_k^1\right|^2 \leq 2C_k^2$$

And this yields the result.□

3.4 Proof of the property (2)

We start this section establishing a general matrix inequality that will help us prove the property (2). First, we give a preparatory lemma from linear algebra. Its proof is left to the reader.

Lemma 6 *Assume that for positive definite, symmetric and invertible $d \times d$ matrices A and B we have that $x'Ax \le x'Bx$ for all $x \in \mathbb{R}^d$. Then $x'A^{-1}x \ge x'B^{-1}x$ for all $x \in \mathbb{R}^d$.*

Lemma 7 *Assume that for the matrices $I_i = \begin{pmatrix} \alpha_i & \gamma_i \\ 0 & \beta_i \end{pmatrix}$, $i = 1, 2$, there exists positive constants $\theta > 0$, $\nu > 0$ such that the following inequalities are satisfied*

$$|\alpha_1| \le \theta\,|\alpha_2|\,,\ |\beta_1| \le \theta\,|\beta_2|$$
$$|\gamma_1| \le \nu\,|\beta_1|\,,\ |\gamma_2| \le \nu\,|\beta_2|\,.$$

Then $|I_1 x| \le \theta(1 + 2\nu)\,|I_2 x|$ for all $x \in \mathbb{R}^2$.

Proof. We consider the auxiliary matrices

$$J_i = \begin{pmatrix} \alpha_i & 0 \\ 0 & \beta_i \end{pmatrix}, \quad K_i = \begin{pmatrix} 0 & \gamma_i \\ 0 & 0 \end{pmatrix}$$

and we notice that

$$|K_1 x| = |\gamma_1 x_2| \le \nu\theta\,|\beta_2 x_2| \le \nu\theta\,|I_2 x| \quad \textit{and} \quad |K_2 x| = |\gamma_2 x_2| \le \nu\,|\beta_2 x_2| \le \nu\,|I_2 x|\,.$$

Then

$$|I_1 x| \le |J_1 x| + |K_1 x| \le \theta\,|J_2 x| + |K_1 x| \le \theta(|I_2 x| + |K_2 x|) + |K_1 x| \le \theta(1+2\nu)\,|I_2 x|\,.$$

□

Now we apply the above Lemma to our particular case to obtain the property (2).

Lemma 8 *Assume hypotheses $(\mathbf{H}_1(\phi))$ and $(\mathbf{H}_2(\phi))$. Furthermore assume*

$$\rho_k^2 \le \frac{2\varepsilon_k}{9C_* C_k} \tag{17}$$

where ρ_k is given by (10). Then

$$|x|_k \le H_k\, |x|_{k-1} \tag{18}$$

with

$$\begin{aligned} H_k &= 6U_k^{3/2}(C_k\varepsilon_k^{-1}),\\ U_k &= \max\{\delta_k,\delta_{k-1}\}/\min\{\delta_k,\delta_{k-1}\}. \end{aligned}$$

Proof. First note that Lemma 6 gives us the following sequence of statements

$$\left|\overline{N}_{k-1}x\right| \le H_k \left|\overline{N}_k x\right| \Longrightarrow \langle M_{k-1}x,x\rangle \le H_k^2 \langle M_k x,x\rangle \Rightarrow |x|_k \le H_k\,|x|_{k-1}\,.$$

Furthermore, note that

$$\overline{N}_k = \sqrt{\delta_k}\bar{I}_k Q$$

where (here we use that $\sigma_2 \equiv 0$)

$$\bar{I}_k := \begin{pmatrix} \overline{\sigma}_{k,1} & \bar{b}_{k,1}\delta_k \\ 0 & \bar{b}_{k,2}\delta_k \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & 0 \\ \frac{1}{2} & \frac{1}{\sqrt{12}} \end{pmatrix}.$$

Therefore (18) will follow if we prove that

$$U_k^{1/2}\left|\bar{I}_{k-1}x\right| \le H_k \left|\bar{I}_k x\right|. \tag{19}$$

To obtain this result, we will apply Lemma 7 with $\alpha_1 = \overline{\sigma}_{k-1,1}$, $\beta_1 = \bar{b}_{k-1,2}\delta_{k-1}$, $\gamma_1 = \bar{b}_{k-1,1}\delta_{k-1}$ and $\alpha_2 = \overline{\sigma}_{k,1}$, $\beta_2 = \bar{b}_{k,2}\delta_k$, $\gamma_2 = \bar{b}_{k,1}\delta_k$. To prove the required inequalities in Lemma 7, we need to consider

1. From (13), we have $\left|x_{t_k} - x_{t_{k-1}}\right| \le \frac{9}{2}C_k\rho_k^2$. In fact,

$$\begin{aligned} &\left|x_{t_k} - x_{t_{k-1}}\right| \\ &= \left|b(x_{t_{k-1}})\delta_k + \int_{t_{k-1}}^{t_k}(\sigma(x_s)\phi_s - \frac{1}{2}\partial_\sigma\sigma(x_s))ds + \int_{t_{k-1}}^{t_k}(b(x_s) - b(x_{t_{k-1}}))ds\right| \\ &\le C_k\left(\frac{7}{2}\delta_k + \int_{t_{k-1}}^{t_k}|\phi_s|\,ds\right) \le C_k\left(\frac{7}{2}\delta_k + \sqrt{\delta_k}\rho_k\right). \end{aligned} \tag{20}$$

2. Using $(\mathbf{H}_2(\phi))$ we have that for $f_\cdot = \bar{\sigma}_{\cdot,i}, \bar{b}_{\cdot,i}$, $i = 1, 2$ then

$$|f_k - f_{k-1}| \leq \frac{9}{2} C_* C_k \rho_k^2.$$

3. Similarly, from $(\mathbf{H}_1(\phi))$ and $(\mathbf{H}_2(\phi))$, we have for $f_\cdot = \bar{\sigma}_{\cdot,1}, \bar{b}_{\cdot,2}$, that $|f_k| \geq \varepsilon_k$ and

$$\left| \frac{f_{k-1}}{f_k} - 1 \right| \leq \frac{9}{2} C_* C_k \rho_k^2 \varepsilon_k^{-1}.$$

Using (17), we obtain that

$$|f_{k-1}| \leq \left(1 + \frac{9}{2} C_* C_k \rho_k^2 \varepsilon_k^{-1} \right) |f_k| \leq 2 \, |f_k| \, .$$

Therefore, this yields

$$\begin{aligned} \left|\overline{\sigma}_{k-1,1}\right| &\leq 2 \left|\overline{\sigma}_{k,1}\right| , \\ \left|\bar{b}_{k-1,2}\delta_{k-1}\right| &\leq 2U_k \left|\bar{b}_{k,2}\delta_k\right| . \end{aligned}$$

Since $\left|\bar{b}_{k-1,2}\right| \wedge \left|\bar{b}_{k,2}\right| \geq \varepsilon_k$ and $\left|\bar{b}_{k-1,1}\right| \wedge \left|\bar{b}_{k,1}\right| \leq C_k$ we also obtain

$$\begin{aligned} \left|\bar{b}_{k-1,1}\delta_{k-1}\right| &\leq C_k \varepsilon_k^{-1} \left|\bar{b}_{k-1,2}\delta_{k-1}\right| , \\ \left|\bar{b}_{k,1}\delta_k\right| &\leq C_k \varepsilon_k^{-1} \left|\bar{b}_{k,2}\delta_k\right| . \end{aligned}$$

Therefore we apply Lemma 7 with $\theta = 2U_k$ and $\nu = C_k \varepsilon_k^{-1}$. This gives

$$\left|\bar{I}_{k-1}x\right| \leq 2U_k \left(1 + 2C_k \varepsilon_k^{-1}\right) \left|\bar{I}_k x\right| .$$

From here (19) follows and therefore we obtain (18).□

3.5 Proof of property (5)

In a similar fashion we now prove that (5), $a_k M_k \geq C(J_k) \geq \frac{1}{a_k} M_k$, is satisfied. We recall that

$$A_k = \{\omega \in \Omega : \left|X_{t_{i-1}} - x_{t_{i-1}}\right|_i \leq \frac{\rho_i}{2}, i = 1, ..., k\}.$$

Lemma 9 *Assume hypotheses* $(\mathbf{H}_1(\phi))$ *and* $(\mathbf{H}_2(\phi))$ *and (17). Then the matrix inequality* $a_k M_k \geq C(J_k) \geq \frac{1}{a_k} M_k$, $k = 1, ..., N$ *is satisfied on* A_k *with*

$$a_k = 196(C_k \varepsilon_k^{-1})^2.$$

Proof. In order to prove that $C(J_k) \leq a_k M_k$, it is enough to prove that

$$|N_k x| \leq \sqrt{a_k} \left|\bar{N}_k x\right|.$$

As in the proof of Lemma 8, we will apply Lemma 7 with $\alpha_1 = \sigma_{k,1}$, $\alpha_2 = \bar{\sigma}_{k,1}$, $\beta_1 = b_{k,2}$, $\beta_2 = \bar{b}_{k,2}$, $\gamma_1 = b_{k,1}$ and $\gamma_2 = \bar{b}_{k,1}$. In order to prove the inequalities required in Lemma 7, we prove the following properties.

1. We prove first that $\left|X_{t_{k-1}} - x_{t_{k-1}}\right| \leq C_k \rho_k^2$ for $\omega \in A_k$. As $\omega \in A_k$ we know that $\left|X_{t_{k-1}} - x_{t_{k-1}}\right|_k = |F_{k-1}|_k \leq \frac{\rho_k}{2}$ and using (14) and (9) we have that

$$\begin{aligned} \left|X_{t_{k-1}} - x_{t_{k-1}}\right| &\leq \left(\frac{\delta_k \left|\bar{b}_k\right|}{\sqrt{12}} + |\bar{c}_k|\right) \frac{\rho_k}{2} \delta_k^{1/2}, \\ &< C_k \rho_k^2. \end{aligned} \tag{21}$$

2. Using (9), we have that $|\bar{f}_k - f_k| \leq C_* \left|X_{t_{k-1}} - x_{t_{k-1}}\right| \leq C_* C_k \rho_k^2$, for $f_\cdot = b_{\cdot,i}, \sigma_{\cdot,i}$, $i = 1, 2$. Furthermore, $|f_k| \leq C_k(1 + C_* C_k \rho_k^2)$.

3. On A_k, we have that

$$|\sigma_{k,1}| \geq |\bar{\sigma}_{k,1}| - C_* C_k \rho_k^2 \geq \varepsilon_k - C_* C_k \rho_k^2 \geq \frac{\varepsilon_k}{2}.$$

Since $\left|\bar{b}_{k,2}\right| \geq \varepsilon_k$ we obtain

$$|b_{k,2}| \geq \varepsilon_k - C_* C_k \rho_k^2 \geq \frac{1}{2} \varepsilon_k.$$

3. Let $f_\cdot = \sigma_{\cdot,1}, b_{\cdot,2}$, then

$$\left|\frac{f_k}{\bar{f}_k} - 1\right| \leq C_* C_k \rho_k^2 \varepsilon_k^{-1} \leq \frac{2}{9}$$

or equivalently

$$|f_k| \leq \left(1 + C_* C_k \rho_k^2 \varepsilon_k^{-1}\right) \left|\bar{f}_k\right| \leq \frac{11}{9} \left|\bar{f}_k\right|$$

Using (17) and $C_k \varepsilon_k^{-1} \geq 1$, we have that the inequalities in Lemma 7 are satisfied with $\theta = 2$ and $\nu = 3 C_k \varepsilon_k^{-1}$. Finally one has that

$$|I_k x| \leq 2(1 + 6 C_k \varepsilon_k^{-1}) \left|\bar{I}_k x\right|$$

Although the constants change slightly, the other inequality is obtained exactly in the same way. We still have $\theta = 2$ and $\nu = 3 C_k \varepsilon_k^{-1}$.$\square$

3.6 The Remainder

We will now give estimations for the remainder. Before doing so we give some estimates needed in the next lemma. The proofs are standard and can be obtained by suitable modifications of similar statements in [2] or [8]. In this section, $C \equiv C(p,T,C_*)$ denotes a constant bigger than C_* that depends exclusively on T, C_* and an integer $p \geq 2$ chosen independently of the other parameters.

Lemma 10 *Suppose that the hypotheses* $(\mathbf{H}_1(\phi))$ *and* $(\mathbf{H}_2(\phi))$ *hold true. For* $u \in [t_{k-1}, t_k]$ *and* $\omega \in A_k$ *there exists a positive deterministic constant* $C \equiv C(p,T,C_*)$*, which is increasing in* T*, such that*

$$\left\| X_u - X_{t_{k-1}} \right\|_{k,4,p} \leq C C_k \rho_k,$$
$$\left\| X_u \right\|_{k,4,p} \leq C C_k + |x_{t_{k-1}}|.$$

Proof. It is enough to consider the following decomposition

$$X_t - x_t = X_{t_{k-1}} - x_{t_{k-1}} + \sum_{i=1}^{5} I_i(t)$$

with

$$I_1(t) = \int_{t_k}^t (\sigma(X_s) - \sigma(x_s)) dW_s, \quad I_2(t) = \int_{t_k}^t (b(X_s) - b(x_s)) ds,$$
$$I_3(t) = \int_{t_k}^t \sigma(x_s) dW_s, \quad I_4(t) = \int_{t_k}^t b(x_s) ds, \quad I_5(t) = -\int_{t_k}^t (\sigma(x_s)\phi_s + \widehat{b}(x_s)) ds.$$

We have the following estimates

$$\|I_1(t)\|_{k,0,p} \leq C_* \delta_k^{\frac{1}{2}-\frac{1}{p}} \left(\int_{t_k}^t \|X_s - x_s\|_{k,0,p}^p \, ds \right)^{1/p},$$
$$\|I_2(t)\|_{k,0,p} \leq C_* \delta_k^{1-\frac{1}{p}} \left(\int_{t_k}^t \|X_s - x_s\|_{k,0,p}^p \, ds \right)^{1/p}$$

and $\|I_i(t)\|_{k,0,p} \leq 2C_k(\sqrt{\delta_k} + \rho_k), i = 3,4,5$. Since $\omega \in A_k$ we also have $\left| X_{t_{k-1}} - x_{t_{k-1}} \right| \leq C_k \rho_k^2$ (see 1. in the proof of Lemma 9). We conclude that

$$\|X_t - x_t\|_{k,0,p}^p \leq C \left\{ C_k^p (\sqrt{\delta_k} + \rho_k)^p + (2C_*)^p \int_{t_k}^t \|X_s - x_s\|_{k,0,p}^p \, ds \right\}$$

with C an universal constant which depends on p only. Then using Gronwall's lemma we obtain $\|X_t - x_t\|_{k,0,p}^p \leq CC_k^p(\sqrt{\delta_k}+\rho_k)^p$ where C is a constant that only depends on p, C_* and (increasingly in) T.

Using (21) and (20), we obtain

$$\begin{aligned}\left\|X_u - X_{t_{k-1}}\right\|_{k,0,p} &\leq \|X_u - x_u\|_{k,0,p} + \left|x_u - x_{t_{k-1}}\right| + C_k\rho_k^2, \\ &\leq CC_k(\sqrt{\delta_k} + \rho_k) + \frac{11}{2}C_k\rho_k^2, \\ &\leq CC_k\rho_k. \end{aligned} \tag{22}$$

Similarly,

$$\begin{aligned}\left\|X_u - X_{t_{k-1}}\right\|_{k,4,p} &\leq CC_k\rho_k \\ \|X_u\|_{k,4,p} &\leq CC_k + |x_{t_{k-1}}|.\end{aligned}$$

□

Lemma 11 *Suppose that the hypothesis* $(\mathbf{H}_1(\phi))$ *and* $(\mathbf{H}_2(\phi))$ *hold true. For any integer* $p \geq 2$ *there exists some universal constant* $C = C(C_*, T, p)$*, which is increasing in* T*, such that for* $\omega \in A_k$

$$\left\|M_k^{-1/2}R_k\right\|_{k,4,p} \leq C\frac{C_k^3\rho_k}{\varepsilon_k^2}.$$

Proof. We use the definition for R_k in (12) and also Lemma 5 together with estimates of the conditional Sobolev norms of $R_{k,i}$. Recall that $\overline{d}_k \geq \varepsilon_k^2$ and $\overline{c}_{k,2} = \overline{b}_{k,2}\frac{\delta_k}{2}$ (because $\sigma_2 = 0$). So for every random vector $V = (V^1, V^2)$ which is four times differentiable in Malliavin sense we have

$$\left\|M_k^{-1/2}V\right\|_{k,4,p} \leq \frac{4\sqrt{3}C_k}{\delta_k^{1/2}\varepsilon_k^2}(\left\|V^1\right\|_{k,4,p} + \frac{1}{\delta_k}\left\|V^2\right\|_{k,4,p}).$$

Using Lemma 10, (10), (20), (21) and (14), we obtain

$$\begin{aligned}\left\|R_{k,1}^1\right\|_{k,4,p} + \left\|R_{k,4}^1\right\|_{k,4,p} &\leq CC_k\rho_k\delta_k^{1/2}, \\ \sum_{i=1}^{2}(\left\|R_{k,2}^i\right\|_{k,4,p} + \left\|R_{k,5}^i\right\|_{k,4,p} + \left\|R_{k,6}^i\right\|_{k,4,p}) &\leq CC_k\rho_k\delta_k^{3/2}, \\ \left\|R_{k,3}^i\right\|_{k,4,p} &\leq CC_k^2\delta_k^2\end{aligned}$$

Since $\sigma_2 = 0$ we have $R^2_{k,1} = R^2_{k,4} = 0$ and so the result follows.$\Box$

The actual value of the constant $C = C(C_*, T, p) > 1$ appearing in the Lemma 11 may change from one line to the next, but it always remains a constant that depends only on C_*, T and p. Due to the increasing property of $C(C_*, T, p)$ in T, we will consider in what follows a time $T' > T$. So that, our estimates are not valid for times T going to infinity. Now we verify that condition ii) in Definition 1 is satisfied. That is, we prove Theorem 3. For the reader's convenience, we restate it here.

Theorem 12 *Suppose that* $(\mathbf{H}_1(\phi))$ *and* $(\mathbf{H}_2(\phi))$*. Furthermore, assume that there exists a constant* $\mu \geq 1$*, independent of* k *such that* $\mu f_{k+1} \geq f_k \geq \mu^{-1} f_{k+1}$ *for* $f_k = \varepsilon_k, C_k$ *and all* k*. There exists a positive constant* $C^*(\mu)$ *such that* $C^*(\mu) \leq \min\left\{\left(CC_d 196^{\frac{145}{2}}\right)^{-1} e^{-\frac{49}{2}(C^*(\mu))^2(\mu^2+1)}, \sqrt{\frac{2}{9C_*}}, 1\right\}$ *for which we assume*

$$\rho_k \leq C^*(\mu) \left(\frac{\varepsilon_k}{C_k}\right)^{148} \tag{23}$$

then $F_n, n \leq N$ *is an elliptic evolution sequence with*

$$a_k = 196 (C_k \varepsilon_k^{-1})^2$$

and

$$H_k = 6 U_k^{3/2} (C_k \varepsilon_k^{-1}).$$

with $U_k = \max\{\delta_k, \delta_{k-1}\} / \min\{\delta_k, \delta_{k-1}\}$ *for* $k = 1, ..., N$ *where* θ *is defined as in (6). Furthermore, we obtain*

$$p_T(x, y) \geq \frac{1}{8\pi\sqrt{\det M_N}} \times e^{-2N\theta}.$$

Proof: The existence of the constant $C^*(\mu)$ follows from the increasing property of the function $f(x) = xe^{196(\mu^2+1)x^2}$ for $x \geq 0$. Then for $p = 148$.Then we have

$$\begin{aligned}
\frac{1}{8} a_{k-1} (\rho_k + \rho_{k-1})^2 = \frac{1}{8} (a^{\frac{1}{2}}_{k-1} \rho_k + a^{\frac{1}{2}}_{k-1} \rho_{k-1})^2 &\leq \frac{49}{2} (C_{k-1} \varepsilon^{-1}_{k-1} \rho_{k-1} + \mu^2 C_k \varepsilon_k^{-1} \rho_k)^2 \\
&\leq \frac{49}{2} C^*(\mu)^2 ((C^{-1}_{k-1} \varepsilon_{k-1})^{p-1} + \mu^2 (C_k^{-1} \varepsilon_k)^{p-1})^2. \\
&\leq \frac{49}{2} C^*(\mu)^2 \left(1 + \mu^2\right)^2.
\end{aligned}$$

Furthermore using Lemma 11, we have

$$\left\|M_k^{-1/2}R_k\right\|_{k,4,p} \leq C\frac{C_k^3\rho_k}{\varepsilon_k^2} \leq CC^*(\mu)\left(\frac{\varepsilon_k}{C_k}\right)^{p-3} \leq \frac{1}{C_d\left(196(C_k\varepsilon_k^{-1})^2\right)^{\frac{145}{2}}e^{\frac{49}{2}(C^*(\mu))^2(1+\mu^2)^2}}$$
$$\leq \left(C_d a_k^{\frac{145}{2}} e^{\frac{a_{k-1}(\rho_k+\rho_{k-1})^2}{8}}\right)^{-1}.$$

and therefore condition ii) in Definition 1 is satisfied and furthermore due to the choice of $C^*(\mu)$, we also have that Lemma 8 and Theorem 2 can be applied.□

4 Construction of an evolution sequence associated to a given skeleton

Up to now we supposed that the time grid $t_k, k \in N$ was given. Now we will give a specific construction of the skeleton ϕ and the associated time grid. We introduce the following class of functions. For $\mu \geq 1$ and $h_* > 0$, define the set of functions Λ_{μ,h_*} as the functions $f : (0,\infty) \to \mathbb{R}$ which verify

$$|f(s)| \leq \mu\,|f(t)| \quad if \quad |t-s| \leq h_*.$$

The control ϕ to be used in the examples will belong to this class. This class is useful in order to simplify the lower bounds for the density of X. We make the following supplementary assumption.

($\mathbf{H}_3(\phi)$) We assume that $\varepsilon_\phi, C_\phi, \varepsilon_\phi C_\phi^{-1} \in \Lambda_{\mu,h_*}$ for some $\mu \geq 1$ and $h_* > 0$. Moreover we assume that there exists a control $\overline{\phi}$ such that $\left|\overline{\phi}_t\right| \geq |\phi_t|$ and $\overline{\phi} \in \Lambda_{\overline{\mu},h_*}$ for some $\overline{\mu} \geq 1$.

We define

$$h_\phi(t) := \min\left\{h_*, C^*(\mu)\left(\frac{\varepsilon_\phi(t)}{C_\phi(t)}\right)^{148}\right\}. \tag{24}$$

where C^* is the universal constant given in ($\mathbf{H}_2(\phi)$)ii) and C_* is given in (9). Note that under hypothesis ($\mathbf{H}_3(\phi)$) , $h_\phi \in \Lambda_{\mu^*,h_*}$ for $\mu^* = \mu^{296}$ and that due to (23) we have that $h_\phi(t) < 1$. This function will serve to define the grid.

We construct now the time grid: we put $t_0 = 0$ and if t_k is given we denote $h_k = h_\phi(t_k)$ and we define t_{k+1} by

$$t_{k+1} = (t_k + h_k) \wedge \inf\{t > t_k : \int_{t_k}^{t} \overline{\phi}_s^2 ds > h_k\}.$$

Notice that $\delta_k := t_{k+1} - t_k \leq h_k$ and $\int_{t_k}^{t_{k+1}} \phi_s^2 ds \leq \int_{t_k}^{t_{k+1}} \overline{\phi}_s^2 ds \leq h_k$, therefore $\rho_k = h_k$ (see (10)) which together with (24) will allow the application of Theorem 3.

Notice also that we may choose $\overline{\phi} = \phi$ and this seems to be the natural choice, but in some cases it may be simpler to choose another $\overline{\phi}$ for which $\overline{\mu}$ is simpler to compute. For example if we know that $|\phi(t)| \leq Q$ then we may take $\overline{\phi}(t) = Q$ and then $\overline{\mu} = 1$.

Lemma 13 *Suppose that* $(\mathbf{H}_1(\phi)), (\mathbf{H}_2(\phi))$ *and* $(\mathbf{H}_3(\phi))$ *hold true. Then* $(F_k)_{k\in N}$ *is an elliptic evolution sequence. We define* $N_T := \sup\{k : t_k \leq T\}$. *Then*

$$N_T \leq \mu^* \int_0^T \frac{1 + \overline{\phi}_t^2}{h_\phi(t)} dt. \tag{25}$$

Then for all non-negative integers a, b *and* c *and any strictly increasing positive function* f, *we have*

$$\sum_{k=0}^{N_T - 1} f\left(\frac{C_k^a}{\varepsilon_k^b h_k^c}\right) \leq \mu^* \int_0^T \frac{1 + \overline{\phi}_t^2}{h_\phi(t)} f\left(\mu^{a+b} (\mu^*)^c \frac{C_\phi(t)^a}{\varepsilon_\phi(t)^b h_\phi(t)}\right) dt. \tag{26}$$

Furthermore, $U_k \leq \mu^* \overline{\mu}^4$.

Proof. Using the definition of N_T, we have that

$$\int_0^T \frac{1 + \overline{\phi}_t^2}{h_\phi(t)} dt \geq \sum_{k=1}^{N_T} \int_{t_{k-1}}^{t_k} \frac{1 + \overline{\phi}_t^2}{h_\phi(t)} dt. \tag{27}$$

Since $h_\phi \in \Lambda_{\mu^*, h_*}$ and $t_k - t_{k-1} \leq h_{k-1} \leq h_*$ we have

$$\int_{t_{k-1}}^{t_k} \frac{1 + \overline{\phi}_t^2}{h_\phi(t)} dt \geq \frac{1}{\mu^* h_{k-1}} \int_{t_{k-1}}^{t_k} (1 + \overline{\phi}_t^2) dt \geq \frac{1}{\mu^* h_{k-1}} h_{k-1} = \frac{1}{\mu^*}.$$

Appling this to (27), it follows that

$$\int_0^T \frac{1+\overline{\phi}_t^2}{h_\phi(t)}dt \geq \frac{N_T}{\mu^*}$$

and (25) is proved. The proof of (26) is analogue. From here, it follows in particular that N_T is finite.

The estimate for U_k is a consequence of the following inequality: $\delta_k \leq \mu^*\overline{\mu}^4\delta_{k-1}$. In order to prove it, we assume first that $\delta_{k-1} = h_{k-1} = h_\phi(t_{k-1})$. Since $h_\phi(t_{k-1}) \geq (\mu^*)^{-1} h_\phi(t_k) = (\mu^*)^{-1} h_k \geq (\mu^*)^{-1} \delta_k$, our inequality is proved. Assume now by contradiction that $\delta_{k-1} < h_{k-1}$ and $\mu^*\overline{\mu}^4\delta_{k-1} < \delta_k$. Then $\int_{t_{k-1}}^{t_k} \overline{\phi}_s^2 ds = h_{k-1} = h_\phi(t_{k-1})$. Furthermore,

$$\begin{aligned}\int_{t_k}^{t_k+\mu^*\overline{\mu}^4\delta_{k-1}} \overline{\phi}_s^2 ds &\geq \overline{\mu}^{-2}\overline{\phi}(t_k)^2 \times \mu^*\overline{\mu}^4\delta_{k-1} \geq \mu^* \int_{t_{k-1}}^{t_{k-1}+\delta_{k-1}} \overline{\phi}_s^2 ds = \mu^* h_\phi(t_{k-1}) \\ &\geq h_\phi(t_k) = h_k.\end{aligned}$$

This leads to a contradiction and therefore it proves that $\mu^*\overline{\mu}^4\delta_{k-1} \geq \delta_k$. For the reverse inequality note that if $\delta_k = h_k$ then we have as before that $\delta_{k-1} \leq h_{k-1} \leq \mu^* h_k = \mu^*\delta_k$. Then assume by contradiction that $\delta_k < h_k$ and $\delta_{k-1} > \mu^*\overline{\mu}^4\delta_k$ then $\int_{t_k}^{t_{k+1}} \overline{\phi}_s^2 ds = h_k$ and

$$\int_{t_{k-1}}^{t_{k-1}+\mu^*\overline{\mu}^4\delta_k} \overline{\phi}_s^2 ds \geq \overline{\mu}^{-2}\overline{\phi}(t_k)^2 \times \mu^*\overline{\mu}^4\delta_k \geq \mu^* \int_{t_k}^{t_k+\delta_k} \overline{\phi}_s^2 ds = \mu^* h_k \geq h_{k-1}$$

which again leads to a contradiction. □

Theorem 14 *Suppose that the hypothesis* $(\mathbf{H}_1(\phi)), (\mathbf{H}_2(\phi))$ *and* $(\mathbf{H}_3(\phi))$ *hold true (see (8),(9) and (24)). Let* $y = x_T$*, where* $x_t, t \geq 0$ *is the skeleton associated to the control* ϕ*. Then*

$$p_T(x,y) \geq K_1(\mu,\mu^*) \left(\frac{C_\phi(T)}{\varepsilon_\phi(T)}\right)^{220} \exp\left(-K_2(\mu,\mu^*) \int_0^T \left(1+\overline{\phi}_t^2\right)\left(\frac{C_\phi(t)}{\varepsilon_\phi(t)}\right)^{150} dt\right).$$

where

$$\begin{aligned}K_1(\mu,\mu^*) &= \frac{\sqrt{3}}{4\pi\,(\mu^*)^{3/2}\,\mu^2}, \\ K_2(\mu,\mu^*) &= \frac{2\mu^*}{h_* \wedge C^*(\mu)}\left(\ln(2^9 \times 21\pi^{1/2}\,(\mu^*)^{5/2}\,\bar{\mu}^6\mu^4\,(h_* \wedge C^*(\mu))^{-1}) + 150 + 49\mu^4\right).\end{aligned}$$

Proof. We have using the properties of the functions in the class Λ that

$$\begin{aligned}\det M_{N_T} &\leq \delta_{N_T}^3 \left|\overline{c}_{N_T}\right|^2 \left|\overline{b}_{N_T}\right|^2 /12 \\ &\leq (\mu^*)^3 \mu^4 h_\phi^3(T) C_\phi^4(T)/12.\end{aligned}$$

Furthermore using the definitions for H_k, a_k and ρ_k together with (6) and (7) as well as from Theorem 12 and Lemma 13, we obtain that

$$\begin{aligned}N_T\left|\theta\right| &\leq N_T \ln(2^7\pi^{1/2}) + \sum_{k=1}^{N_T} \ln(\frac{a_k^{1/2} H_k}{\rho_k}) + \frac{1}{16}\sum_{k=1}^{N_T-1} a_k \\ &= N_T \ln(2^9\times 21\pi^{1/2}) + \sum_{k=1}^{N_T} \ln\left(\frac{(C_k\varepsilon_k^{-1})^2 U_k^{3/2}}{h_k}\right) + 49\sum_{k=1}^{N_T-1}(C_k\varepsilon_k^{-1})^2 \\ &\leq \mu^*\int_0^T \frac{1+\overline{\phi}_t^2}{h_\phi(t)}\left(\left(C_{\mu,\bar{\mu}}^1 + \ln\left(\frac{C_\phi(t)^2}{\varepsilon_\phi(t)^2 h_\phi(t)}\right)\right) + 49\mu^4\frac{C_\phi(t)^2}{\varepsilon_\phi(t)^2}\right)dt\end{aligned}$$

with $C_{\mu,\bar{\mu}}^1 = \ln(2^9 \times 21\pi^{1/2}\left(\mu^*\right)^{5/2}\bar{\mu}^6\mu^4)$. From here it follows that for $C_\mu^2 = 2 + 49\mu^4$

$$\begin{aligned}p(y_N) &\geq \frac{\sqrt{3}}{4\pi\left(\mu^*\right)^{3/2}\mu^2 h_\phi^{3/2}(T) C_\phi^2(T)} \\ &\quad\times\exp\left(-2\mu^*\int_0^T \frac{1+\overline{\phi}_t^2}{h_\phi(t)}\left(\left(C_{\mu,\bar{\mu}}^1 + \ln\left(\frac{1}{h_\phi(t)}\right)\right) + C_\mu^2\frac{C_\phi(t)^2}{\varepsilon_\phi(t)^2}\right)dt\right).\end{aligned}$$

Furthermore if we define $\tilde{C}(\mu) = h_* \wedge C^*(\mu) \leq 1$, we have that

$$h_\phi(t) \geq \tilde{C}(\mu)\left(\frac{\varepsilon_\phi(t)}{C_\phi(t)}\right)^{148}.$$

Therefore we have that

$$\begin{aligned}p(y_N) &\geq \frac{\sqrt{3}C_\phi^{220}(T)}{4\pi\left(\mu^*\right)^{3/2}\mu^2\varepsilon_\phi^{222}(T)} \\ &\quad\times\exp\left(-2\mu^*\int_0^T \frac{1+\overline{\phi}_t^2}{\tilde{C}(\mu)}\left(\frac{C_\phi(t)}{\varepsilon_\phi(t)}\right)^{150}\left(C_{\mu,\bar{\mu}}^1 - \ln\tilde{C}(\mu) + 148 + C_\mu^2\right)dt\right).\end{aligned}$$

Therefore the result follows.□

5 Construction of skeletons

In this section we consider the construction of skeletons satisfying the equation

$$x_t^1 = x^1 + \int_0^t (\sigma(x_s)\phi_s + b_1(x_s))ds, \quad x_t^2 = x^2 + \int_0^t b_2(x_s)ds. \tag{28}$$

We fix $x, y \in \mathbb{R}^2$, $T > 0$ and we want to construct control functions ϕ that generate solutions to the above ordinary differential equation (28) $x_t = x_t(\phi)$, with $x_0(\phi) = x = (x^1, x^2), x_T(\phi) = y = (y^1, y^2)$. First, consider any differentiable function z_t, $0 \leq t \leq T$ which verifies

$$z_0 = b_2(x), \quad z_T = b_2(y), \quad \int_0^T z_t dt = y^2 - x^2. \tag{29}$$

Then we define

$$x_t^2 = x^2 + \int_0^t z_s ds \tag{30}$$

and we denote $f(t, a) = b_2(a, x_t^2)$. We suppose that b_2 is once differentiable and

$$\min \left\{ \inf_{x \in \mathbb{R}^2} \partial_1 b_2(x), \inf_{x \in \mathbb{R}^2} \sigma(x) \right\} \geq \varepsilon_0 > 0. \tag{31}$$

Although this restriction is stronger than (8), we will relax it in the next subsection.

By (31) we have $\partial_a f(t, a) = \partial_1 b_2(a, x_t^2) \geq \varepsilon_0 > 0$ so the function $a \to f(t, a)$ is invertible for every $t \geq 0$. We denote by $g(t, a) = f^{-1}(t, a)$ the corresponding inverse function. Therefore, we have that $f(t, g(t, a)) = g(t, f(t, a)) = a$. Then we define

$$x_t^1 = g(t, z_t), \tag{32}$$

$$\phi_t = \frac{\partial_t z_t - (b_2 \partial_2 b_2)(x_t) - (b_1 \partial_1 b_2)(x_t)}{(\sigma \partial_1 b_2)(x_t)} = \frac{\partial_t z_t - \partial_b b_2(x_t)}{\partial_\sigma b_2(x_t)}. \tag{33}$$

We sometimes denote by $x_t(z), \phi_t(z)$ the functions constructed in this way.

Lemma 15 *We suppose that (31) holds and that z satisfies (29) for some fixed $x = (x^1, x^2), y = (y^1, y^2) \in \mathbb{R}^2$ and $T > 0$. Then the function $x_t = (x_t^1, x_t^2)$ defined by (30) and (32) satisfies that $x_0 = x = (x^1, x^2)$, $x_T = y = (y^1, y^2)$ and equation (28) holds with the control $\phi_t = \phi_t(z)$ defined by (33).*

Proof. First, it is clear that

$$x_0 = (f^{-1}(0, b_2(x)), x^2) = x \quad \text{and} \quad x_T = (f^{-1}(T, b_2(y)), x_T^2) = y.$$

Next, we have $\partial_a f(t,a) = \partial_1 b_2(a, x_t^2)$ and $\partial_t f(t,a) = \partial_2 b_2(a, x_t^2)\partial_t x_t^2 = \partial_2 b_2(a, x_t^2) z_t$. Taking derivatives with respect to t in the equality $f(t, g(t,a)) = a$ we obtain

$$0 = \partial_t(f(t, g(t,a))) = (\partial_t f)(t, g(t,a)) + (\partial_a f)(t, g(t,a))\partial_t g(t,a)$$

which gives

$$\partial_t g(t,a) = -\frac{(\partial_t f)(t, g(t,a))}{(\partial_a f)(t, g(t,a))} = -\frac{\partial_2 b_2(g(t,a), x_t^2) z_t}{\partial_1 b_2(g(t,a), x_t^2)}.$$

And by the inverse function theorem, we have that

$$\partial_a g(t,a) = \frac{1}{(\partial_a f)(t, g(t,a))} = \frac{1}{\partial_1 b_2(g(t,a), x_t^2)}.$$

We compute now

$$\begin{aligned}\partial_t x_t^1 &= \partial_t(g(t, z_t)) = (\partial_t g)(t, z_t) + (\partial_a g)(t, z_t)\partial_t z_t \\ &= \frac{\partial_t z_t - \partial_2 b_2(g(t, z_t), x_t^2) z_t}{\partial_1 b_2(g(t, z_t), x_t^2)} = \frac{\partial_t z_t - (b_2 \partial_2 b_2)(x_t)}{\partial_1 b_2(x_t)}\end{aligned}$$

the last equality being a consequence of $g(t, z_t) = x_t^1$ and $z_t = b_2(x_t)$.

The equation (28) reads $\partial_t x_t^1 = \phi_t \sigma(x_t) + b_1(x_t)$ which, in view of the previous equality, amounts to

$$\phi_t \sigma(x_t) + b_1(x_t) = \frac{\partial_t z_t - (b_2 \partial_2 b_2)(x_t)}{\partial_1 b_2(x_t)}.$$

And this is true by the definition of ϕ. $\square$

From now on, we assume global ellipticity and boundedness (that is, (31) and (34) below). Nevertheless, we will see later, that the core of the argument in the general case is in the following proof.

Lemma 16 *Assume (31) and for $\varepsilon \in (0,1)$*

i) $|\partial_\alpha f(x)| \le C_* \quad \forall x \in \mathbb{R}^2$ *for* $f = \sigma, b, \partial_\sigma b, \partial_b b_2, \partial_\sigma \sigma, \sigma + \varepsilon \partial_\sigma b, Lb$ *and* $|\alpha| \le 5$. (34)

Then there exists a function z^* generating a control $\phi = \phi(z^*)$ which gives a solution $x_t = (x_t^1, x_t^2)$ of (30) and (32) which satisfies that $x_0 = x = (x^1, x^2)$, $x_T = y = (y^1, y^2)$ and holds equation (28) with

$$\int_0^T \phi_t^2 dt \leq \frac{64}{\varepsilon_0^4}(C_*^2 T + \frac{\left|y^2 - x^2 - b_2(x)T\right|^2}{T^3} + \frac{\left|b_2(y) - b_2(x)\right|^2}{T}).$$

$$\int_0^T \frac{1+\phi_t^2}{h_\phi(t)} f\left(\frac{C_\phi(t)^a}{\varepsilon_\phi(t)^b h_\phi(t)^c}\right) dt \leq C(C_*, \varepsilon_0) \int_0^T \phi_t^2 dt$$

for any increasing positive function f.

Proof. We construct now a function z which verifies (29) and which is linear on $[0, T/2]$ and on $[T/2, T]$. We fix $q \in \mathbb{R}$ and we ask that $z_{T/2} = q$. Since $z_0 = b_2(x)$ and $z_T = b_2(y)$ we have

$$\int_0^T z_t dt = \frac{T}{2}(q + \frac{b_2(x) + b_2(y)}{2}).$$

We solve the equation $y^2 - x^2 = \int_0^T z_t dt$ and we find

$$q = \frac{y^2 - x^2}{T/2} - \frac{b_2(x) + b_2(y)}{2}.$$

And we denote by z_t^* the function corresponding to this value of q. Furthermore,

$$\partial_t z_t^* = \frac{q - b_2(x)}{T/2} 1_{(0,T/2)}(t) + \frac{b_2(y) - q}{T/2} 1_{(T/2,T)}(t).$$

Therefore

$$|\partial_t z_t^*| \leq \frac{4\left|y^2 - x^2 - b_2(x)T\right|}{T^2} + \frac{3\left|b_2(y) - b_2(x)\right|}{T}.$$

From here it follows that

$$\begin{aligned} |\phi_t(z^*)| &= \left|\frac{\partial_t z_t^* - \partial_b b_2(x_t)}{\partial_\sigma b_2(x_t)}\right| \\ &\leq \frac{1}{\varepsilon_0^2}\left(\frac{4\left|y^2 - x^2 - b_2(x)T\right|}{T^2} + \frac{3\left|b_2(y) - b_2(x)\right|}{T} + C_*\right). \end{aligned}$$

The second estimate follows by noting that $\varepsilon_\phi(t) = \varepsilon_0$, $C_\phi(t) = C_*$ and $h_\phi(t) = C(1, C_*)$.□

Therefore we obtain the following result by direct application of Lemma 16 to Theorem 14.

Theorem 17 *Assume the conditions (31) and (34) then the density of X is bounded below as follows:*

$$p_T(x,y) \geq \frac{1}{C(C_*,\varepsilon_0)} \exp\left(-C(C_*,\varepsilon_0)\left(T + \frac{\left|y^2 - x^2 - b_2(x)T\right|^2}{T^3} + \frac{\left|b_2(y) - b_2(x)\right|^2}{T}\right)\right).$$

The above result may seem restrictive due to condition (31). Nevertheless we claim that this assumption is needed just to define the path z^*. Finding a similar path in non-uniform elliptic cases is possible. Similarly, one could replace condition (34) by a localized version on the range of $x(\phi(z^*))$. This is in part the objective of the next section.

5.1 An optimal version of the lower bound result

So far, we have only used a particular skeleton in order to obtain the lower bounds for the densities of X. One could decide to optimize over the possible paths satisfying certain conditions. This is, in general, a procedure that will not lead to explicit results and can only be treated on a case by case basis. Nevertheless one can leave the optimization procedure unsolved and obtain a lower bound. This is done in this section.

For this, we consider as before the equations (28).We fix $x = (x^1, x^2)$ and we assume that

$$|\sigma(x)| \wedge |\partial_1 b_2(x)| > 0. \tag{35}$$

The component $x_t^1(z)$ is constructed in the following Lemma.

Lemma 18 *Assume (35). Let $z : [0,\infty) \to \mathbb{R}$ be a deterministic differentiable function such that $z_0 = b_2(x)$. Let $x_t^2(z) \in \mathbb{R}^2, t \in [0,\infty)$ be defined as $x_t^2(z) = x^2 + \int_0^t z_s ds$. Then we have*
A. There exists an unique time $\tau \in (0,\infty]$ and a unique continuous curve $x_t^1 \in \mathbb{R}, t \in [0,\tau)$ such that $x_0^1 = x^1$ and

$$\begin{aligned} i) &\quad b_2(x_t^1, x_t^2(z)) = z_t \quad 0 \leq t < \tau, \\ ii) &\quad \left|\partial_1 b_2(x_t^1, x_t^2(z))\right| > 0 \quad 0 \leq t < \tau, \\ iii) &\quad \lim_{t\uparrow\tau} \partial_1 b_2(x_t^1, x_t^2(z)) = 0 \quad if \quad \tau < \infty. \end{aligned}$$

We denote by $\tau_b(z)$ the above time and by $x_t^1(z)$ the above curve. We also denote $\tau_\sigma(z) = \inf\{t \leq \tau_b(z) : |\sigma(x_t^1(z))| = 0\}$ and $\tau(z) = \tau_b(z) \wedge \tau_\sigma(z)$.

B. Moreover the unique curve $x_t^1(z), t \in [0,\tau)$ constructed in A. together with $x_t^2(z)$, $t \in [0,\tau)$ is the unique solution of (28) with the control ϕ given by

$$\phi_t(z) = \frac{\partial_t z_t - \partial_b b_2(x_t(z))}{\partial_\sigma b_2(x_t(z))}.$$

Proof. A. Step 1. We prove that there exists $\tau' > 0$ and a continuous curve $x_t^1 \in R, t \in [0,\tau')$ such that $i)$ and $ii)$ hold true.

Define $f(t,a) := b_2(a, x_t^2(z))$ and $\delta := \frac{1}{2}\left|\partial_1 b_2(x)\right|$. By assumption (35), we have that $|\partial_a f(0,x^1)| = |\partial_1 b_2(x)| > 0$. Therefore by a continuity argument, there exists $\eta > 0$ such that for $0 \le t < \eta$ and $|a - x^1| < \eta$ we have $|\partial_a f(t,a)| > \delta$. Therefore for each $0 \le t < \eta$, the function $f(t,.) : B_\eta(x^1) \to U_t =: f(t, B_\eta(x^1))$ is a bijection.

Moreover, since $|\partial_a f(t,a)| > \delta$ for $a \in B_\eta(x^1)$, we may find $r(\delta) > 0$ (which depends on δ but not on t) such that $B_{r(\delta)}(f(t,x^1)) \subset U_t$. Next, we define $\tau' = \eta \wedge \inf\{t : |f(t,x^1) - z_t| \ge r(\delta)\}$. We have $z_0 = b_2(x) = f(0,x^1)$ and by continuity $\tau' > 0$. For every $t < \tau'$, we have that $z_t \in B_{r(\delta)}(f(t,x^1))$ so we may define

$$x_t^1 = g(t, z_t)$$

where $g(t,.) : B_{r(\delta)}(f(t,x^1)) \to B_\eta(x^1)$ is the inverse of $f(t,.)$. By the definition of x_t^1 we have $b_2(x_t^1, x_t^2(z)) = f(t,x_t^1) = f(t, g(t,z_t)) = z_t$. And for $t \le \tau' \le \eta$ we have $x_t^1 \in B_\eta(x^1)$ so that $|\partial_a f(t,x_t^1)| > \delta > 0$. Therefore, the properties $i)$ and $ii)$ are satisfied. Finally, for $t = 0$ we have $x_0^1 = g(0,z_0) = g(0, f(0,x^1)) = x^1$.

Step 2. In this step we prove the uniqueness of x_t^1 for $t \le \tau'$ satisfying $i)$ and $ii)$ above. For this, consider another continuous curve $y_t, t \le \tau'$ such that $y_0 = x^1$ and $b_2(y_t, x_t^1(z)) = z_t$ and $|\partial_1 b_2(y_t, x_t^1(z))| > 0$ hold for $t \le \tau'$. Let $\theta = \inf\{t : y_t \ne x_t^1\}$.

As before, note that there exists $\eta > 0$ such that for all $t < \eta$ and for all a such that $|a - x^1| < \eta$ we have that $|\partial_a f(t,a)| > \delta$. Define $\mu_X := \inf\{t; x_t \notin B_\eta(x^1)\}$ and similarly $\mu_Y = \inf\{t; y_t \notin B_\eta(x^1)\}$. If $\mu_X < \theta$ then due to the bijection property of f then it follows that $x_t = y_t$ up to τ. Otherwise, assume that $\mu_X > \theta$ then $\mu_Y > \theta$ and then there $\varepsilon > 0$ such that $y_t, x_t^1 \in B_\eta(x^1)$ for $\theta \le t \le \theta + \varepsilon$. And using $i)$ we have $f(t,y_t) = f(t,x_t^1) = z_t$. Since $f(t,.)$ is injective on $B_\eta(x^1)$ we get $y_t = x_t^1$, which is in contradiction with the definition of θ.

Step 3. Let $\tau = \sup\{\tau' : \exists x_t^1, t \le \tau' \text{ which satisfies } i), ii)\}$. Using the uniqueness property proved at Step 2 we may construct a path $\{x_t^1, t < \tau\}$

which satisfies $i)$ and $ii)$. Suppose that $\tau < \infty$ and let us prove by contradiction that $\lim_{t\uparrow\tau} \partial_1 b_2(x_t^1, x_t^2(z)) = 0$.

In fact, otherwise there exists $\delta > 0$ and a sequence $t_n \uparrow \tau$ such that $\left|\partial_1 b_2(x_{t_n}^1, x_{t_n}^2(z))\right| > \delta$. We fix n and we come back to the reasoning from Step 1 with x replaced by $x_n = (x_{t_n}^1, x_{t_n}^2(z))$. Since $|\partial_1 b_2(x_n)| > \delta$, we may find $\eta > 0$ such that for $t_n \leq t \leq t_n+\eta$ and $|a - x_n^1| < \eta$ we have $|\partial_1 b_2(a, x_n^2)| > \frac{1}{2}\delta$. Notice that η depends on the Lipschitz constant of $\partial_1 b_2$ and on $\sup_{t\leq\tau} |z_t|$ but not on n. We recall that $f(t,a) = b_2(a, x_t^2(z))$ and the function $f(t,.) : B_\eta(x_n^1) \to U_t =: f(t, B_\eta(x_n^1)), t_n \leq t \leq t_n + \eta$ is a bijection.

Moreover, since $|\partial_a f(t,a)| > \delta/2$ for $a \in B_\eta(x_n^1)$ and $t_n \leq t \leq t_n + \eta$, we may find $r(\delta) > 0$ (which depends on δ but not on t) such that $B_{r(\delta)}(f(t, x_n^1)) \subset U_t$. Then we define $\tau_n = \eta \wedge \inf\{t > t_n : |f(t, x_n^1) - z_t| \geq r(\delta)\}$. We have $z_{t_n} = b_2(x_n) = f(t_n, x_n^1)$ so $\tau_n > 0$. Furthermore, for every $t_n < t < \tau_n$ we have $z_t \in B_{r(\delta)}(f(t, x_n^1))$ so we may define $x_t^1 = g(t, z_t)$ where g is the inverse of f. Notice that $\tau_n - t_n$ does not depend on n. It depends on the modulus of continuity of z_t, on the Lipschitz continuity constant of $\partial_1 b_2$ and on $\sup_{t\leq\tau} |z_t|$ but not on n. So we have obtained an extension of x_t^1 on $(t_n, \tau_n]$. Since $\tau_n - t_n$ does not depend on n and $t_n \uparrow \tau$ we obtain an extension of x_t^1 beyond τ. And this is in contradiction with the definiton of τ. So $iii)$ is proved.

Suppose now that there exists $\overline{\tau}$ and $\overline{x}_t, t < \overline{\tau}$ which satisfy $i), ii), ii)$. By the uniqueness argument given in step 1, for every $t < \tau \wedge \overline{\tau}$ we have $\overline{x}_t = x_t^1$. In particular, $\lim_{t\uparrow\tau\wedge\overline{\tau}} |\partial_1 b_2(x_t^1)| = 0$. And then $ii)$ implies that $\tau = \overline{\tau}$. So we have uniqueness of the pair $(\tau, \{x_t^1, t < \tau\})$.

B. We recall that $f(t,a) = b_2(a, x_t^2(z))$ and $x_t^1(z) = g(t, z_t)$ where $g(t,\cdot)$ is the inverse of $f(t,\cdot)$. Fix $t_0 < \tau(z)$, and we know that $\left|\partial_a f(t_0, x_{t_0}^1(z))\right| = |\partial_1 b_2(x_{t_0}(z))| = \delta$ for some $\delta > 0$. So there exists $r > 0$ such that for every $t \in (t_0 - r, t_0 + r)$ the function $f(t,.) : B_r(x_{t_0}^1(z)) \to U_t = f(t, B_r(x_{t_0}^1(z)))$ is invertible and $g(t,.) : U_t \to B_r(x_{t_0}^1(z))$ is its inverse. So for every $t \in (t_0 - r, t_0 + r)$ and $a \in U_t$ we have $f(t, g(t,a)) = a$ and for every $a \in B_r(x_{t_0}^1(z))$ we have $g(t, f(t,a)) = a$. Finally, $\partial_a f(t,a) = \partial_1 b_2(a, x_t^2(z))$ and $\partial_t f(t,a) = \partial_2 b_2(a, x_t^2(z))\partial_t x_t^2(z) = \partial_2 b_2(a, x_t^2(z)) z_t$.

From here, the exact argument (in localized version) that was used in the proof of Lemma 15 applies. □

So far, we have constructed z and then defined x. We now prove that the reverse procedure is also possible. For this, we define the following sets. For $x, y \in \mathbb{R}^2$ and $T > 0$ and we define $\mathcal{C}_T(x,y)$ to be the class of the differentiable functions $z : [0,\infty)$ such that $\tau(z) > T$ and

$$z_0 = b_2(x), z_T = b_2(y) \text{ and } \int_0^T z_t dt = y^2 - x^2.$$

Let $\Lambda_T(x,y)$ be the set of derivatives $\partial_t x_t^2(\phi)$, $t \in [0,T]$ such that there exists $x_t^1(\phi)$ so that for $x(\phi) = (x^1(\phi), x^2(\phi))$ satisfies $x_0(\phi) = x$, $x_T(\phi) = y$ and it solves Eqs. (28).

Lemma 19 $\Lambda_T(x,y) = \mathcal{C}_T(x,y)$

Proof. Let $z \in \mathcal{C}_T(x,y)$. As in the previous Lemma define $x_t^2 = x^2 + \int_0^t z_s ds$ and using x_t^1 as in the statement of the previous lemma we obtain that $x_t = (x_t^1, x_t^2) \in \Lambda_T(x,y)$. Similarly, if $x \in \Lambda_T(x,y)$ then one defines $z_t = \partial_t x_t^2(\phi)$ and all the needed properties follow straightforwardly. □

Using the previous results we can study the following examples.

Example 20 *1. The Asian case. Let $\sigma(x) = x_1, b_1(x) = x_1$ and $b_2(x) = 0$ for $x = (x_1, x_2)$. In this case, note that $\partial_1 b_2 = 1$ therefore $\tau = \infty$. As in the previous section we consider as z^*, a linear funcion on $[0, T/2]$ joining x^1 and q and then joining q and y^1 on $[T/2, T]$. Note that the path z^* stays away from zero. To make the arguments simple, assume that*

$$q = \frac{y^2 - x^2}{T/2} - \frac{x^1 + y^1}{2} > \max\{x^1, y^1\} > 0.$$

Then we have that

$$|\phi_t(z^*)| \leq \max\left\{\frac{q - x^1}{x^1 T/2}, \frac{q - y^1}{y^1 T/2}\right\} \equiv \bar{\phi}.$$

Therefore in order to apply Theorem 14, note that $\varepsilon_\phi(t) = \min\{x^1, y^1\}$, $C_* = 1$, $C_\phi(t) = 2q$ so that we obtain

$$p_T(x,y) \geq K_1 \left(\frac{2q}{\min\{x^1, y^1\}}\right)^{220} \exp\left(-K_2 T \left(1 + \overline{\phi}^2\right) \left(\frac{2q}{\min\{x^1, y^1\}}\right)^{150}\right).$$

where

$$K_1 = \frac{\sqrt{3}}{4\pi},$$
$$K_2 = \frac{2}{T \wedge C^*(1)} \left(\ln(2^9 \times 21\pi^{1/2} (T \wedge C^*(1))^{-1}) + 199\right)$$
$$C^* \leq \min\left\{ \left(CC_d 196^{\frac{145}{2}}\right)^{-1} e^{-49(C^*)^2}, \sqrt{\frac{2}{9}} \right\}.$$

From here one can obtain the following result

$$\lim_{y^2 \to \infty} \inf \frac{\ln\left((y^2)^{-220} p_T(x,y) \right)}{(y^2)^{152}} \geq -C(x^1, y^1, x^2, T)$$

where $C(x^1, y^1, x^2, T)$ *is a positive constant. Similar results can be easily obtained. For example, the above result is also valid for the case* $y^2 = \alpha y^1$ *for* $\alpha > T$.□

Example 21 *Consider the case* $\sigma = 1, b_1 = 0$ *and* $b_2(x)$ *is a smooth function such that* $b_2(x) = \sqrt{x_1}$ *for* $x_1 \geq 1$. *In this case,* $\partial_1 b_2(x) = \frac{1}{2} x_1^{-1/2}$. *As before we consider the path* $z : [0,T] \to (1, +\infty)$ *to be a piecewise linear function as in the proof of Lemma 16 such that* $z_0 = \sqrt{x^1}$, $z_T = \sqrt{y^1}$, $z_{T/2} = q > 1$ *with*

$$\int_0^T z_t dt = y^2 - x^2.$$

Therefore $\tau = \infty$. *For the sake of the argument we assume that* $\min\{x^1, y^1\} > 1$ *and* $q = \frac{1}{T/2}(y^2 - x^2) - \frac{\sqrt{x^1} + \sqrt{y^1}}{2} > 1$. *In this case, we have*

$$p_T(x,y) \geq K_1(\mu) \left(2^{-1} z_T^{-2}\right)^{220} \exp\left(-K_2(\mu) \int_0^T \left(1 + (2 z_t \partial_t z_t)^2\right) \left(2^{-1} z_t^{-2}\right)^{150} dt \right).$$

where

$$K_1(\mu) = \frac{\sqrt{3}}{4\pi\mu^{446}},$$
$$K_2(\mu) = \frac{2\mu^{296}}{T \wedge C^*(\mu)} \left(\ln(2^9 \times 21\pi^{1/2} \mu^{744} \bar{\mu}^6 (T \wedge C^*(\mu))^{-1}) + 150 + 49\mu^4\right).$$

$$\bar{\mu} = \mu \frac{\max\left\{\left|\frac{q-\sqrt{x^1}}{T/2}\right|, \left|\frac{q-\sqrt{y^1}}{T/2}\right|\right\}}{\min\left\{\left|\frac{q-\sqrt{x^1}}{T/2}\right|, \left|\frac{q-\sqrt{y^1}}{T/2}\right|\right\}}$$
$$\mu = \frac{\max\{\sqrt{x^1}, \sqrt{y^1}, q\}}{\min\{\sqrt{x^1}, \sqrt{y^1}, q\}}.$$

Note that for $|x_1| > 1$, we have $\partial_\sigma b_2(x) = \left(2\sqrt{|x|}\right)^{-1}$ so we have a degeneracy at infinity. Therefore $x_t^1 = (z_t)^2$, $\phi_t = 2z_t\partial_t z_t$ and $x_2(t) = x^2 + \int_0^t z_s ds$. We then have that

$$\begin{aligned}
\varepsilon_\phi(t) &= (2z_t)^{-1}, \\
C_\phi(t) &= z_t, \\
h_\phi(t) &= \min\left\{h_*, C^*(\mu)\left(2^{-1}z_t^{-2}\right)^{148}\right\}
\end{aligned}$$

where we have that $z_t, z_t^{-2} \in \Lambda_{\mu,h_*}$ for $h_* = T$. Furthermore we also set $|\phi_t| = |2z_t\partial_t z_t| = \bar{\phi}_t \in \Lambda_{\bar{\mu},h_*}$. As before, one can use the lower bound for the density to prove that

$$\lim \inf_{y^2\to\infty} \frac{\ln((y^2)^{446} p_t(x,y))}{(y^2)^4} \geq -C(x^1, y^1, x^2, T).$$

□

Remark: A variety of similar situations can be treated with the above arguments. For example, in the case that $\sigma = 1, b_1 = 0$ and $b_2(x)$ is a smooth function such that for $|x_1|$ big enough $b_2(x) = e^{-|x_1|}$. It is clear from the above argument that if we assume that $\min\{x^1, y^1\} > 1$, a similar argument as the one above can be used to obtain a lower bound for the density in this case.

We now give a lower bound in optimal form. For $z \in C_T(x, y)$ we denote

$$\begin{aligned}
\varepsilon_z(t) &= |\partial_1 b_2(x_t(z))| \wedge |\sigma(x_t(z))| \wedge 1 > 0, \\
C_z(t) &= (1 + |\sigma| + |b| + |\partial_\sigma\sigma| + |\partial_\sigma b| + |Lb|)(x_t(z)),
\end{aligned}$$

We also fix $\mu > 1 > h$ and we define the class

$$\mathcal{C}_{T,\mu,h}(x, y) = \{z \in \mathcal{C}_T(x, y); \varepsilon_z, C_z, \varepsilon_z C_z^{-1} \in \Lambda_{\mu,h}\}.$$

Following (24), we define

$$h_z(t) := \min\left\{h_*, C^*(\mu)\left(\frac{\varepsilon_z(t)}{C_z(t)}\right)^{148}\right\}$$
$$\phi_t(z) = \frac{\partial_t z_t - \partial_b b_2(x_t(z))}{\partial_\sigma b_2(x_t(z))}.$$

So, for $z \in C_{T,\mu,h}(x,y)$, the hypothesis $\mathbf{H}_2(\phi)i)$ holds with $\phi = \phi(z), \mathbf{H}_1(\phi)$ holds with $\varepsilon_\phi = \varepsilon(z)$ and $\mathbf{H}_3(\phi)$ holds with $\bar{\phi} = \phi$, μ and $h_* = 1$.

Then as an immediate consequence of Theorem 14 we obtain:

Theorem 22 *Suppose that the* $(\mathbf{H}_2(\phi))$ *ii) (see (9)) is satisfied and* $C_T(x,y) \neq \emptyset$ *for some* $\mu > 1 > h$*. Then*

$$p_T(x,y) \geq \sup_{\mu>1>h} \sup_{z\in\mathcal{C}_T(x,y)} K_1(\mu)\left(\frac{C_z(T)}{\varepsilon_z(T)}\right)^{220} \times$$
$$\times \exp\left(-K_2(\mu)\int_0^T \left(1+\left(\frac{\partial_t z_t - \partial_b b_2(x_t(z))}{\partial_\sigma b_2(x_t(z))}\right)^2\right)\left(\frac{C_z(t)}{\varepsilon_z(t)}\right)^{150} dt\right).$$

Here,

$$K_1(\mu) = \frac{\sqrt{3}}{4\pi\mu^{448}},$$
$$K_2(\mu) = \frac{2\mu^{296}}{C^*(\mu)}\left(\ln(2^9 \times 21\pi^{1/2}\bar{\mu}^6\mu^{744}C^*(\mu)^{-1}) + 150 + 49\mu^4\right).$$

In this Theorem one may use $\bar{\mu} = \max_{t\in[0,T]}|\phi_t(z)|$.

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6 Appendix A: Proof of Theorem 2

We recall the reader that the notation appearing in this proof corresponds to the one in [2]. We use that framework with $\tilde{M}_k = \frac{1}{a_k} M_k$, $\tilde{H}_k = \frac{a_k^{1/2} H_k}{a_{k-1}^{1/2}}$, $\tilde{a}_k = a_k^2$ with the set $\tilde{A}_k$ redefined as

$$\tilde{A}_k = \{\omega \in \Omega : |F_{i-1} - y_i|_{\tilde{i}} \leq \frac{\tilde{\rho}_i}{2}, i = 1, ..., k\}.$$

Here the norm $|x|_{\tilde{i}} = \left\langle \tilde{M}_i^{-1} x, x \right\rangle^{1/2}$ and $\tilde{\rho}_i = a_i^{1/2} \rho_i$. Then the proofs in [2] apply exactly changing the constant 1 which appears in the definition of the hypothesis $(H_1, \tilde{a}, \tilde{A}, z)$ by $\tilde{\rho}$ and $\eta \in (0, \tilde{c}\sqrt{\Delta_{\tilde{M}}})$ which gives the same result except that e^2 is changed by $e^{\frac{(\tilde{\rho}+\tilde{c})^2}{2}}$. This factor is not considered in the definition of C_d above as it was the case in [2]. Therefore in this setting the constant e^2 in [2] starts to depend on $\tilde{\rho}$ and $\tilde{c}$. Similarly, the condition $(H_2, \tilde{a}, \tilde{A}, z)$ is changed to

$$\left\| \tilde{M}^{-1/2} R \right\|_{t,\delta,d+2,p_d} \leq \frac{1}{\tilde{a}^{4(d+1)^2} C_d e^{\frac{(\tilde{\rho}+\tilde{c})^2}{2}}}.$$

Proposition 8 becomes

$$p_\eta(z)(\omega) \geq \frac{1}{4 e^{\frac{(\tilde{\rho}+\tilde{c})^2}{2}} (2\pi\tilde{a})^{d/2} \sqrt{\det \tilde{M}}}$$

for $\omega \in A \subset \{\omega : \|V(\omega) - z\|_{M^{-1}} \leq \tilde{\rho}\}$ and some $\eta \in (0, \tilde{c}\sqrt{\Delta_{\tilde{M}}})$. Then section 2.3 applies exactly with the following changes:

1. The estimate in Corollary 9 becomes (where $\tilde{c}_k = \frac{\tilde{\rho}_k}{8\tilde{H}_k}$)

$$P(\tilde{A}_k) \geq \frac{P(\tilde{A}_{k-1})\tilde{\rho}_k^d}{8^{d+1}\tilde{H}_k^d e^{\frac{(\tilde{c}_k+\tilde{\rho}_{k-1}/2)^2}{2}}(2da_{k-1}\pi)^{d/2}}$$

under the condition that

$$\left\|\tilde{M}_k^{-1/2}R_k\right\|_{t_{k-1},\delta_k,d+2,p_d} \leq \frac{1}{C_d\tilde{a}_k^{4(d+1)^2}e^{\frac{(\tilde{c}_k+\tilde{\rho}_{k-1}/2)^2}{2}}}$$

2. This estimate naturally leads to the estimate in Theorem 15 with

$$p_{F_N}(x_N) \geq \frac{e^{-Nd\theta}}{4(2\pi)^{d/2}\sqrt{\det \tilde{M}_N}}$$

with

$$\theta = \ln(8^2(2\pi d)^{1/2}) + \frac{1}{2N}\sum_{k=1}^{N}\ln(\tilde{a}_k) + \frac{1}{N}\sum_{k=2}^{N}\ln(\frac{\tilde{H}_k}{\tilde{\rho}_k}) + \frac{1}{8dN}\sum_{k=2}^{N}(\tilde{c}_k + \frac{\tilde{\rho}_{k-1}}{2})^2.$$

Finally replacing all the above parameters changes and the inequality

$$\left|\tilde{c}_k + \frac{\tilde{\rho}_{k-1}}{2}\right| = a_{k-1}^{1/2}\left|\frac{\rho_{k-1}}{2} + \frac{\rho_k}{8H_k}\right| \leq a_{k-1}^{1/2}\left|\frac{\rho_{k-1}+\rho_k}{2}\right|$$

one obtains the result. □