A MARKET MODEL WITH MEDIUM/LONG TERM EFFECTS DUE TO AN INSIDER

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Abstract. In this article, we consider a modification of the Karatzas-Pikovsky model of insider trading. Specifically, we suppose that the insider agent influences the long/medium term evolution of prices of the Black-Scholes type model through the drift of the model. We say that the insider agent is using a portfolio leading to a partial equilibrium if the following three properties are satisfied: a) the portfolio used by the insider leads to a stock price which is a semimartingale under its own filtration and its own filtration enlarged with the final price b) the portfolio used by the insider is optimal in the sense that it maximizes the logarithmic utility for the insider when its filtration is fixed and c) the optimal logarithmic utility in b) is finite. We give sufficient conditions for the existence of a partial equilibrium and show in some explicit models how to apply these general results.

1. Introduction

Lawful insider trading is a financial empirical fact which can be studied from various points of view. In the literature, one may find empirical studies, financial economic theoretic studies and recently studies in mathematical finance. These studies try to give an explanation to the basic puzzle of why there is trading in the stock market if there are traders that are better informed than others.

The Karatzas-Pikovsky model for insider trading considers the Black-Scholes model

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t),\ S(0) = S_0,$$

where $(\Omega, \mathcal{F}, P)$ is a standard Wiener space and W is the canonical Wiener process in this space. An insider trader is modeled as an investor which uses a bigger information stream for trading. That is, his/her portfolio policies are adapted to a filtration $\mathcal{G} \supseteq \mathcal{F}$. Usually $\mathcal{G} = \mathcal{F} \vee \sigma(S(T))$ where $\sigma(X)$ stands for the sigma algebra generated by the random variable X and $\vee$ stands for the minimal filtration satisfying the usual conditions which contains $\mathcal{F}$ and $\sigma(X)$.

In this setting, Karatzas-Pikovsky found the optimal utility for the insider, which blows up at time T and leads to the conclusion that the optimal utility of the insider is infinite. From the mathematical point of view is important to note that in order to make sense of the wealth process of the insider, one needs to have that W is a semimartingale in the filtration $\mathcal{G}$, which is the case in the Karatzas-Pikovsky model.

This model has been carefully studied from a mathematical point of view in recent years showing that this blow up effect is present in various other situations (see for example Imkeller [12], Imkeller et. al. [13]). From the financial economics point of view, regardless of the simplicity of the above set-up, this model has not attracted much attention essentially because the utility of the insider is infinite which is at contradiction with the reality of lawful insiders.

On the other hand, many generalizations of the Karatzas-Pikovsky model have been undertaken by Prof. Øksendal 's research group [1]. In particular, they have considered (jump type) models where the drift and diffusion coefficients may be anticipating and depend on the portfolio of the insider agent under a general filtration set-up. In particular, example 1 considered in section 5, was first considered in [6] where existence of the optimal portfolio was obtained for a fixed filtration $\mathcal{G}$.

Still, in the opinion of the authors, the domain lacks a greater variety of explicit examples where the optimal insider portfolio can be explicitly written, the filtration have a strong relationship with the information carried by stock prices and the utility of the insider agent is finite. The goal of the present article is to present a set-up where the insider has an effect on the price dynamics, its logarithmic utility is finite and a partial equilibrium condition is satisfied. In an accompanying paper, we have also considered another two examples which fall in the class introduced here (see [10]).

In this article, we study the following model. Let $(\Omega, \mathcal{F}, P; (\mathcal{F}_t)_{t\geq 0})$ to be a complete filtered probability space with the augmented Wiener filtration $(\mathcal{F}_t)_{t\geq 0}$ generated by the 1-dimensional Wiener process W.

On this probability space, we define a financial market with one risk-free asset with price S^0 given by

$$dS_t^0 = rS_t^0 dt, \quad S_0^0 > 0,$$

and one risky investment, whose price $S_t \equiv S_t^\pi$ is described by

$$dS_t = S_t \left\{ \left(\mu + a\pi_t - b\int_0^t H(t,u)\pi_u du \right) dt + \sigma dW_t \right\}, \ S_0 > 0. \tag{1.1}$$

Here $r > 0, \mu, a, \ b \in \mathbb{R}$, $H(t,u) > 0$ for all $t \in [0,T]$ and $\pi := (\pi_t)_{t\geq 0}$ is the proportion of the insider agent wealth invested in the risky asset, an element of the set of admissible strategies

$$\mathcal{A}_{\mathcal{S}^\pi} := \left\{ (\pi_t)_{t\geq 0}; (\pi_t)_{t\in[0,T]} \in \mathcal{L}_T^2 \right\}, \tag{1.2}$$

where $\mathcal{L}_T^2$ is the totality of $\mathcal{S}^\pi$-progressively measurable processes π on the time interval $[0,T]$ such that $E\left[\int_0^T \pi_t^2 dt\right] < \infty$. Here,

$$\mathcal{S}_t^\pi = \sigma(S^\pi(s); \ s \leq t) \vee \sigma\left(\ln(S^\pi(T)) + \sigma W'((T-s)^\theta); \ s \leq t\right), \tag{1.3}$$

where W' is a 1-dimensional Brownian motion independent of W and $\theta > 0$.

Clearly the proposed model can be considered as a perturbation of the Karatzas-Pikovsky model ($a = b = 0$).

Furthermore, note that in (1.1), if $a \neq 0$ or $b \neq 0$, then the insider influences the stock price dynamics through his/her strategy. The minus sign in front of b can be interpreted as a term which offsets the influence of the insider by considering variations of the strategy of the insider with respect to past time averages of its own strategy.

Now we explain the condition (1.3). W' is a dynamic perturbation of the information generated by the insider so as to assume that this agent does have some uncertainty on the information used for trading.

Furthermore, W' and θ, characterize the fact that the information of the insider is dynamic and changes through time. In Corcuera et al. [3], it was proven that if $\theta < 1$

[1]see e.g. http://folk.uio.no/oksendal/publications.html

then the insider has finite optimal utility. That is, if the rate at which the insider improves his views on future events is slow enough then the optimal utility is finite.

Clearly, the stochastic integral (1.1) is not well defined in general. Therefore we assume that this integral can be defined as a semimartingale integral in the filtration $\mathcal{S}^{\pi}$[2].

Finally, our goal is to prove that there exists a portfolio $\pi^* \in \mathcal{A}_{\mathcal{S}^{\pi^*}}$ such that W is a semimartingale in the filtration $\mathcal{S}^{\pi^*}$ and π^* is an optimal portfolio when the filtration is fixed to be $\mathcal{S}^{\pi^*}$. Furthermore its corresponding optimal logarithmic utility is finite.

The model presented here can be considered as a large trader model. These models have been studied for a long time and the current discussion in that area if much farther and complicated than the goal of this article. Within various models, the model closest to the one considered here is the model of Cuoco-Cvitanic [4].

In that model, only the drift of the model depends on the strategy of the large trader. This model has been extended to insider trading models by Grorud-Pontier [9]. In both cases the representative example is given by the "pressure on rates" example. That is, as the portfolio policy of the insider increases it creates a bounded reduction on the rate of growth. This case is clearly different from the situation we consider here.

This difference creates a qualitative mathematical difference in both models. While in the large trader model, concavity properties appear, in our case these properties are not satisfied. This in turn, introduces complications when one wants to use duality theory to solve the optimization problem.

Finally, some of the algebraic proofs and accessory results appear in the Appendices.

2. Existence and uniqueness of optimal portfolios for fixed enlarged filtrations

Our objective in this section is to consider a logarithmic utility optimization problem for the model (1.1), where $\pi := (\pi_t)_{t\geq 0}$ is the proportion of wealth that the insider invested in the risky asset. π an element of the set of admissible strategies

$$\mathcal{A}_{\mathcal{G}} := \left\{ (\pi_t)_{t\geq 0}; (\pi_t)_{t\in[0,T]} \in \mathcal{L}(\mathcal{G})^2_T \right\},$$

where $\mathcal{L}(\mathcal{G})^2_T$ is the totality of $(\mathcal{G}_t)$-progressively measurable processes on the time interval $[0,T]$ such that $E\left[\int_0^T \pi_t^2 dt\right] < \infty$. $\mathcal{G}$ is a fixed filtration satisfying $\mathcal{G} \supseteq \mathcal{F}$ and we assume through this section that on the filtration $\mathcal{G}$, W is a semi-martingale on $[0,T]$ with the decomposition

$$W(t) = \hat{W}(t) + \int_0^t g_1(s)\alpha(s)ds, \tag{2.1}$$

where $g \in L^2[0,T]$ is a deterministic strictly positive function, α is a $\mathcal{G}$-adapted integrable process and $\hat{W}$ is a Wiener process in the filtration $\mathcal{G}$[3].

We consider the maximization of the logarithmic expected utility function

$$\Psi(T) = \sup_{\pi\in\mathcal{A}_{\mathcal{G}}} J(\pi), \tag{2.2}$$

[2]One may also use the forward integral. Although this would give greater generality, it will also increase the technicality of the proofs.

[3]This particular type of compensator is used in order to allow easy application of Proposition 9.1

where $J(\pi)$ is defined by

$$J(\pi) := E\left[\log(\widehat{V}_T^\pi)\right].$$

Here $\widehat{V}^\pi$ is given by

$$d\widehat{V}_t^\pi = \pi_t \widehat{V}_t^\pi \widehat{S}_t^{-1} d\widehat{S}_t, \ \widehat{V}_0^\pi = 1. \tag{2.3}$$

Here $\widehat{S}$ denotes the discounted stock price defined by $\widehat{S}_t = e^{-rt}S_t$. Note that the explicit solution of equation (2.3) is given by

$$\widehat{V}_t^\pi = \exp\left[\int_0^t \left\{\pi_u\left(\mu - r + a\pi_u - b\int_0^u H(u,v)\pi_v dv\right) - \frac{1}{2}\sigma^2\pi_u^2\right\} du + \int_0^t \pi_u \sigma dW_u\right].$$

Theorem 2.1. *Consider a filtration $\mathcal{G} \supseteq \mathcal{F}$ such that $(W, \mathcal{G})$ is a semimartingale with the decomposition* (2.1)*, where α is a $\mathcal{G}$-adapted process and g_1 is a deterministic function such that $E\left[\int_0^T |g_1(s)\alpha(s)|^2 ds\right] < \infty$. Assume*

$$0 \le a < \frac{\sigma^2 - 2|b|K(T)}{2} \quad \text{for } K(T) := \left(\int_0^T \int_0^t H^2(t,u)dudt\right)^{1/2}. \tag{2.4}$$

If $\pi^* \in \mathcal{A}_\mathcal{G}$ satisfies the optimality equation

$$\begin{aligned}\mu - r + (2a - \sigma^2)\pi(t) - b\int_0^t H(t,u)\pi(u)du \\ + \sigma g_1(t)\alpha(t) - bE\left[\int_t^T H(u,t)\pi(u)du\Big|\mathcal{G}_t\right] = 0,\end{aligned} \tag{2.5}$$

then π^* is an optimal portfolio for the problem (2.2)*. Furthermore, there exists at most one solution for equation* (2.5) *in the space $L^2(\Omega \times [0,T])$.*

Proof. First, we rewrite $J(\pi)$ as

$$\begin{aligned}J(\pi) = E\Bigg[\int_0^T \Bigg\{\pi(t)\left(\mu - r + a\pi(t) - b\int_0^t H(t,u)\pi(u)du + \sigma g_1(t)\alpha(t)\right) \\ -\frac{\sigma^2}{2}\pi(t)^2\Bigg\} dt\Bigg],\end{aligned} \tag{2.6}$$

and notice that by the Cauchy-Schwarz inequality

$$\begin{aligned}&\left|\int_0^T \pi(t)\int_0^t H(t,u)\pi(u)dudt\right| \\ &\le \left(\int_0^T \pi(u)^2 du\right)^{1/2}\left(\int_0^T\left(\int_0^t H(t,u)\pi(u)du\right)^2 dt\right)^{1/2} \\ &\le K(T)\left(\int_0^T \pi(u)^2 du\right).\end{aligned} \tag{2.7}$$

Next we prove that $J(\pi)$ is concave. That is,

$$J(\alpha\pi_1 + (1-\alpha)\pi_2) \ge \alpha J(\pi_1) + (1-\alpha)J(\pi_2); \ \alpha \in [0,1], \pi_1, \pi_2 \in \mathcal{A}_\mathcal{G}.$$

A straightforward calculation shows that

$$J(\alpha\pi_1 + (1-\alpha)\pi_2) - \alpha J(\pi_1) - (1-\alpha)J(\pi_2) = \alpha(1-\alpha)N(\pi_1 - \pi_2),$$

where

$$N(\pi) := E\left[\int_0^T \left\{\frac{\sigma^2 - 2a}{2}\pi^2(t) + b\pi(t)\int_0^t H(t,u)\pi(u)du\right\} dt\right].$$

From (2.4) and (2.7), we see that $N(\pi_1 - \pi_2) \geq 0$ and therefore $J(\pi)$ is concave. Now to prove (2.5), consider the directional derivative

$$D_v J(\pi) := \lim_{\epsilon \to 0} \frac{J(\pi + \epsilon v) - J(\pi)}{\epsilon}, \;\; v, \pi \in \mathcal{A}_{\mathcal{G}}.$$

Using Fubini's theorem, one obtains that

$$\begin{aligned}
&J(\pi + \epsilon v) - J(\pi) \\
=&\epsilon E\left[\int_0^T \left\{\mu - r + (2a - \sigma^2)\pi(t) - b\int_0^t H(t,u)\pi(u)du + \sigma g_1(t)\alpha(t)\right.\right. \\
&\left.\left. -bE\left[\int_t^T H(u,t)\pi(u)du \Big| \mathcal{G}_t\right]\right\} v(t)dt\right] - \epsilon^2 N(v).
\end{aligned}$$

Therefore, we obtain that

$$\begin{aligned}
D_v J(\pi) =&E\left[\int_0^T \left\{\mu - r + (2a - \sigma^2)\pi(t) - b\int_0^t H(t,u)\pi(u)du + \sigma g_1(t)\alpha(t)\right.\right. \\
&\left.\left. -bE\left[\int_t^T H(u,t)\pi(u)du \Big| \mathcal{G}_t\right]\right\} v(t)dt\right],
\end{aligned}$$

and that $D_v J(\pi^*) = 0$. Noting that $J(\pi)$ is concave, for all $v, \pi \in \mathcal{A}_{\mathcal{G}}$ and $\epsilon \in (0,1)$, we have

$$\begin{aligned}
J(\pi + \epsilon v) - J(\pi) &\geq (1-\epsilon)J\left(\frac{\pi}{1-\epsilon}\right) + \epsilon J(v) - J(\pi) \\
&= J\left(\frac{\pi}{1-\epsilon}\right) - J(\pi) + \epsilon\left\{J(v) - J\left(\frac{\pi}{1-\epsilon}\right)\right\}. \qquad (2.8)
\end{aligned}$$

Now, with $\frac{1}{1-\epsilon} = 1 + \eta$ we have

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon}\left\{J\left(\frac{\pi}{1-\epsilon}\right) - J(\pi)\right\} = \lim_{\eta \to 0} \frac{1+\eta}{\eta}\left\{J(\pi + \eta\pi) - J(\pi)\right\} = D_\pi J(\pi).$$

Combining this with (2.8), we get for $v, \pi \in \mathcal{A}_{\mathcal{G}}$

$$D_v J(\pi) = \lim_{\epsilon \to 0} \frac{J(\pi + \epsilon v) - J(\pi)}{\epsilon} \geq D_\pi J(\pi) + J(v) - J(\pi).$$

In particular, applying this to $\pi = \pi^*$ and using that $D_v J(\pi^*) = 0$, we get

$$J(v) - J(\pi^*) \leq 0 \quad \text{for all } v \in \mathcal{A}_{\mathcal{G}},$$

which proves that π^* is optimal. For the uniqueness is enough to note that in fact for $\pi \neq 0$ one has that $N(\pi) > 0$ therefore J is strictly concave which gives the uniqueness of solutions.

Note also that (2.6) and the conditions $\pi^* \in \mathcal{A}_{\mathcal{G}}$ and $E\Big[\int_0^T |g_1(s)\alpha(s)|^2 ds\Big] < \infty$ imply that $J(\pi^*) < \infty$. $\square$

We remark here that the condition (2.4) is essential in order to interpret the model correctly. In particular, this condition states that the volatility has to be big enough (or the coefficients a, b and the function H small enough) so as to allow for the concavity of J. In strategical terms, this means that volatility has to be big enough to allow the insider to "hide" his behavior within the movements of the volatility. Also the fact that T can not be too big means that this type of insider behavior can not continue for long periods of time (for $b \neq 0$). From now on, condition (2.4) will be assumed.

3. Characterization of optimal portfolios and optimal utility function under an explicit filtration

In Theorem 2.1 we have given general conditions that an optimal portfolio should satisfy in order to maximize the logarithmic utility for a general fixed enlarged filtration. In order to obtain optimal portfolios as explicitly as possible, we use a particular filtration of the type

$$\begin{aligned} \mathcal{G}_t = \mathcal{F}_t \vee \sigma\left(W'(u) - W'\left((T-t)^\theta\right); u \in [(T-t)^\theta, T^\theta]\right) \\ \vee\, \sigma\left(\int_t^T g_1(s)dW(s) + \int_0^{(T-t)^\theta} g_2(s)dW'(s)\right) \quad (3.1)\\ = \mathcal{F}_t \vee \sigma(I(s);\ s \leq t), \quad (3.2) \end{aligned}$$

where

$$I(t) = \int_0^T g_1(s)dW(s) + \int_0^{(T-t)^\theta} g_2(s)dW'(s),$$

for two fixed deterministic strictly positive functions g_1, $g_2 \in L^2[0,T]$ and W' another one dimensional Wiener process independent of W.

In fact, the equality (3.2) is satisfied because g_2 is strictly positive and therefore

$$\sigma\left(W'(u) - W'\left((T-t)^\theta\right); u \in [(T-t)^\theta, T^\theta]\right) = \sigma\left(\int_{(T-t)^\theta}^{(T-s)^\theta} g_2(u)dW'(u);\ s \leq t\right).$$

We obtain the explicit semimartingale decomposition of the Wiener process W in this filtration as well as some related properties of its compensator process in Proposition 9.1. In particular, this filtration satisfies the hypotheses in the previous section. Also, the property that the Radon-Nikodym derivative of the compensator is square integrable is important to obtain the finiteness of the utility as we have seen in Theorem 2.1.

Our objective in this section is to characterize the solution of the optimality equation (2.5) and the corresponding optimal value $\Psi(T)$ of the optimization problem (2.2) using classical stochastic control methods.

In order to obtain an optimal portfolio, π^*, as explicitly as possible we concentrate on the decomposable case $H(t,u) = h_1(u)h_2(t)^{-1}$. This allows the characterization of optimal strategies through the use of techniques of stochastic control. This explicit representation is obtained in Theorem 3.1 which then requires the solution of certain Ricatti type equations.

ASSUMPTION (A): $H(t,u) = h_1(u)h_2(t)^{-1}$, $h_1(t), h_2(t) > 0$ for all $t \in [0,T]$ and g_1, g_2, h_1 and $h_2 \in C^1([0,T))$.

From now on, this assumption is in force. To characterize the optimal portfolio satisfying (2.5), we need to introduce the following Ricatti and linear ordinary differential equations. This procedure will also be needed to characterize the optimal logarithmic utility. We will use the following general structural notation. We say that Z is a solution of a (Z_1, Z_2, Z_3)-Ricatti equation if it is a solution of the following ordinary differential equation :

$$\begin{cases} \dot{Z}(t) + Z_1(t)Z^2(t) + Z_2(t)Z(t) + Z_3(t) = 0, \\ Z(T) = 0. \end{cases}$$

Similarly we say that Z is a solution of a (Z_1, Z_2)-linear ODE if it is a solution of the linear ordinary differential equation

$$\begin{cases} \dot{Z}(t) + Z_1(t)Z(t) + Z_2(t) = 0, \\ Z(T) = 0. \end{cases}$$

With this notation we now set P to be the solution of a (Z_1^P, Z_2^P, Z_3^P)-Ricatti equation with

$$\begin{aligned} Z_1^P(t) &= b\frac{h_1(t)}{h_2(t)}, \\ Z_2^P(t) &= \frac{2b}{2a-\sigma^2}\frac{h_1(t)}{h_2(t)} - \frac{\dot{h}_1(t)}{h_1(t)} - \frac{\dot{h}_2(t)}{h_2(t)}, \\ Z_3^P(t) &= \frac{b}{(2a-\sigma^2)^2}\frac{h_1(t)}{h_2(t)}, \end{aligned} \tag{3.3}$$

and Q to be a solution of a $(Z_1^Q, Z_2^Q)-$linear ODE with

$$\begin{aligned} Z_1^Q(t) &= b\left(\frac{1}{2a-\sigma^2} + P(t)\right)\frac{h_1(t)}{h_2(t)} - \frac{\dot{h}_1(t)}{h_1(t)}, \\ Z_2^Q(t) &= -\sigma\left\{\frac{\dot{h}_2(t)}{h_2(t)}g_1(t) + \dot{g}_1(t)\right\}P(t). \end{aligned} \tag{3.4}$$

Similarly L is a solution of a $(Z_1^L, Z_2^L)-$linear ODE with $Z_1^L = Z_1^Q$ and

$$Z_2^L(t) = -b\frac{\mu - r}{2a-\sigma^2}\left(\frac{1}{2a-\sigma^2} + P(t)\right)\frac{h_1(t)}{h_2(t)}. \tag{3.5}$$

Furthermore, we define $X \in L^2(\Omega \times [0,T])$ by

(3.6)
$$\begin{aligned} X_t := \eta_t \Bigg[\sigma g_1(0)\alpha(0) + \int_0^t \eta_u^{-1}\Bigg[\left\{-b\frac{h_1(u)}{h_2(u)}Q(u) + \sigma\left(\dot{g}_1(u) + g_1(u)\frac{\dot{h}_2(u)}{h_2(u)}\right)\right\}\alpha(u) \\ -b\frac{h_1(u)}{h_2(u)}\left(L(u) - \frac{\mu - r}{2a-\sigma^2}\right)\Bigg]du + \sigma\int_0^t \eta_u^{-1}g_1(u)d\alpha(u)\Bigg], \end{aligned}$$

where $\dot{\eta}_t\eta_t^{-1} = Z_1^Q(t)$ or more explicitly

$$\eta_t := \exp\left[\int_0^t \left\{b\left(\frac{1}{2a-\sigma^2} + P(u)\right)\frac{h_1(u)}{h_2(u)} - \frac{\dot{h}_2(u)}{h_2(u)}\right\}du\right]. \tag{3.7}$$

Theorem 3.1. *Assume* **(A)**, (2.4) *and that the* (Z_1^P, Z_2^P, Z_3^P)*-Ricatti equation has a solution. Then,* $\pi^* \in L^2(\Omega \times [0,T])$ *defined by*

$$\pi^*(t) = -\left(\frac{1}{2a-\sigma^2} + P(t)\right) X_t + Q(t)\alpha(t) + \left(L(t) - \frac{\mu - r}{2a-\sigma^2}\right), \tag{3.8}$$

is the unique solution of the optimal equation (2.5).

In fact, in order to obtain the above formulas one plugs (3.8) into the optimality equation (2.5) obtaining the corresponding equations for P, Q and L as the deterministic coefficients corresponding to X, α and the non-random term. The explicit steps of this proof are carried out in Appendix 10.

The above theorem characterizes the optimal portfolio for an insider where the enlarged filtration is fixed to be (3.1). This result also shows that under such a filtration, α retains some Markovian properties which are used in the proof of Theorem 3.1. Therefore, classical stochastic control techniques can still be applied. This also leads to the definition of a value function associated to this problem which will be denoted by $v(t,x,\alpha)$. This is the maximal logarithmic utility value when the problem is considered in the interval $[t,T]$ and the driving process $(X(s),\alpha(s))_{s\in[t,T]}$ departs from $(X(t),\alpha(t)) = (x,\alpha)$.

In order to proceed with the characterization of the optimal logarithmic utility $\Psi(T) = J(\pi^*)$ we define the following ordinary differential equations. R is a solution of a (Z_1^R, Z_2^R)-linear ODE with:

$$\begin{aligned} Z_1^R(t) &= \frac{\dot h_2(t)}{h_2(t)} - \frac{\dot h_1(t)}{h_1(t)}, \\ Z_2^R(t) &= -b\frac{h_1(t)}{h_2(t)}Q^2(t) + 2\sigma\left\{\frac{\dot h_2(t)}{h_2(t)}g_1(t) + \dot g_1(t)\right\}Q(t). \end{aligned} \tag{3.9}$$

M is a solution of a (Z_1^M, Z_2^M)-linear ODE with:

$$\begin{aligned} Z_1^M(t) &= \frac{\dot h_2(t)}{h_2(t)} - \frac{\dot h_1(t)}{h_1(t)}, \\ Z_2^M(t) &= -b\frac{h_1(t)}{h_2(t)}\left(L(t) - \frac{\mu-r}{2a-\sigma^2}\right)Q(t) + \sigma\left\{\frac{\dot h_2(t)}{h_2(t)}g_1(t) + \dot g_1(t)\right\}L(t), \end{aligned} \tag{3.10}$$

and N is a solution of a (Z_1^N, Z_2^N)-linear ODE with:

$$\begin{aligned} Z_1^N(t) &= \frac{\dot h_2(t)}{h_2(t)} - \frac{\dot h_1(t)}{h_1(t)}, \\ Z_2^N(t) &= -\frac{b}{2}\frac{h_1(t)}{h_2(t)}\left(L(t) - \frac{\mu-r}{2a-\sigma^2}\right)^2 \\ &\quad + \frac{g_1^2(t) + \theta(T-t)^{\theta-1}g_2^2((T-t)^\theta)}{G^2(t)}\left\{-\sigma^2 g_1^2(t)P(t) + \sigma g_1(t)Q(t) + R(t)\right\}. \end{aligned} \tag{3.11}$$

Then, we obtain the following result which characterizes the optimal logarithmic utility of the insider.

Theorem 3.2. *Assume* **(A)**, $b \neq 0$ *and* $E\left[\int_0^T |g_1(t)\alpha(t)|^2 dt\right] < \infty$. *Define*

$$v(t,x,\alpha) = \frac{2a-\sigma^2}{b}\frac{h_2(t)}{h_1(t)}\left\{-\frac{1}{2}P(t)x^2 + Q(t)x\alpha + \frac{1}{2}R(t)\alpha^2 + L(t)x + M(t)\alpha + N(t)\right\},$$

where Q, R, L, M *and* N *are solutions of the respective* (Z_1^F, Z_2^F)*-linear ODE's for* $F = Q, R, L, M, N$ *equations and* P *is a solution of the* (Z_1^P, Z_2^P, Z_3^P)*-Ricatti equation. If* g_1 *is a continuous positive function and the conditions of Theorem 3.1 are satisfied and if*

$$E\left[v(t, X_t, \alpha(t))\right] \to 0 \text{ as } t \to T,$$

then the optimal value of the problem (2.2) *is finite and*

$$\Psi(T) = J(\pi^*) = E\left[v(0, X_0, \alpha(0))\right].$$

Proof. From the proof of Theorem 2.1, we have that for any admissible portfolio $\pi \in \mathcal{A}_{\mathcal{G}}$,

$$\begin{aligned} J(\pi) &\leq E\left[\int_0^T \left(\pi(t)(\mu - r + \sigma g_1(t)\alpha(t)) - \frac{1}{2}(\sigma^2 - 2a - 2|b|K(T))\pi(t)^2\right) dt\right] \\ &\leq \frac{1}{2(\sigma^2 - 2a - 2|b|K(T))} E\left[\int_0^T (\mu - r + \sigma g_1(t)\alpha(t))^2 dt\right] < \infty. \end{aligned}$$

Recall that from (3.6), using the Itô formula for the product $h_2(t)X_t$, we have that X_t satisfies

$$\begin{aligned} &dX_t + \frac{\dot{h}_2(t)}{h_2(t)} X_t dt \\ &= b\frac{h_1(t)}{h_2(t)}\left\{\left(\frac{1}{2a-\sigma^2} + P(t)\right)X_t - Q(t)\alpha(t) - \left(L(t) - \frac{\mu - r}{2a-\sigma^2}\right)\right\} dt \\ &\quad + \sigma\left(\dot{g}_1(t) + g_1(t)\frac{\dot{h}_2(t)}{h_2(t)}\right)\alpha(t)dt + \sigma g_1(t) d\alpha(t). \end{aligned} \tag{3.12}$$

Using Itô's formula, Proposition 9.1 and (3.12), we have for $s < T$

$$\begin{aligned} &v(s, X_s, \alpha(s)) - v(0, X_0, \alpha(0)) \\ &= \int_0^s \left(\frac{\partial v}{\partial t} dt + v_x dX_t + v_\alpha d\alpha(t) + \frac{v_{xx}}{2} d\langle X\rangle_t + v_{x\alpha} d\langle X, \alpha\rangle_t + \frac{v_{\alpha\alpha}}{2} d\langle \alpha\rangle_t\right) \\ &= m(s) + \frac{2a-\sigma^2}{b}\int_0^s \frac{h_2(t)}{h_1(t)} F_1(t, X_t, \alpha(t)) dt, \end{aligned} \tag{3.13}$$

where

$$\begin{aligned} m(s) = &\frac{2a-\sigma^2}{b}\int_0^s \frac{h_2(t)}{h_1(t)}\left\{Q(t)X(t) + R(t)\alpha(t) + M(t)\right. \\ &\left. + \sigma g_1(t)\left(-P(t)X(t) + Q(t)\alpha(t) + L(t)\right)\right\} d\alpha(t), \end{aligned}$$

and

$$F_1(t, x, \alpha) = \sum_{0 \leq i+j \leq 2} A_{ij}(t) x^i \alpha^j.$$

Here, the coefficients A_{ij} have explicit expressions (see Section 11) depending on P, Q, L, M, R, g_1, h_1 and h_2. Furthermore from Proposition 9.1 and (3.6) the continuity of Q, R, M, g_1, P and L it follows that m is a $\mathcal{G}$ martingale in $[0, T)$.

On the other hand, using (3.8)

$$
\begin{aligned}
J(\pi^*) &= E\left[\int_0^T \left\{\frac{2a-\sigma^2}{2}(\pi^*(t))^2 + (\mu - r + X_t)\pi^*(t)\right\} dt\right] \\
&= E\left[\int_0^T \left\{\frac{2a-\sigma^2}{b}\frac{h_2(t)}{h_1(t)}F_2(t, X_t, \alpha(t))\right\} dt\right],
\end{aligned}
\tag{3.14}
$$

where

$$
F_2(t, x, \alpha) := \sum_{0 \le i+j \le 2} B_{ij}(t) x^i \alpha^j.
$$

Here B_{ij} can also be written explicitly as A_{ij} (see Section 11). Then if we consider $F_1(t,x,\alpha) + F_2(t,x,\alpha)$ and in particular $A_{ij} + B_{ij}$ we see that these coefficients corresponds to each of the ODE's defining P, Q, R, M, N and L. Therefore $F_1(t,x,\alpha) + F_2(t,x,\alpha) = 0$ (see Section 11). Next, as $E[v(s, X_s, \alpha(s))] \to 0$ as $s \to T$ then applying these results into (3.13), we obtain that $E\left[v(0, X_0, \alpha(0))\right] = J(\pi^*)$. □

The following section describes the sufficient conditions so that the filtration $\mathcal{G}$ becomes the insider's filtration and therefore the optimal portfolio obtained in the previous theorem corresponds to the optimal portfolio for the insider. Later we prove that such a portfolio leads to a finite utility.

4. The partial equilibrium equation

Up until this point the filtration is fixed to be (3.1). In order to make it coincide with (1.3) we introduce the optimal portfolio characterization obtained in Theorem 3.1 into (1.3) and equal this result to (3.1). This gives an equation which may be rightly called a "partial equilibrium equation" which is obtained in Theorem 4.1.

We will show that there exists a portfolio π^* satisfying the following properties.

Definition 4.1. *A portfolio π^* is called a* **partial equilibrium** *if it satisfies the following three properties*

1. The portfolio π^* leads to a filtration $\mathcal{S}^{\pi^*}$ where W is a semimartingale[4] *the model* (1.1) *and the wealth equation (see* (2.3)*) makes sense as a linear stochastic differential equation driven by a semimartingale.*

2. If the filtration $\mathcal{S}^{\pi^*}_t$ is fixed (and therefore the resulting price process S is well defined by condition 1.) then $\pi = \pi^*$ is the optimal portfolio for the logarithmic expected utility function within all portfolios in the class $\mathcal{S}^{\pi^*}_t$. That is,

$$
\pi^* = argmax\{E[\log(\widehat{V}^\pi_T)];\ \pi \in \mathcal{A}_{\mathcal{S}^{\pi^*}}\},
\tag{4.1}
$$

where $\widehat{V}^\pi$ is the discounted wealth process defined in (2.3)*.*

3. The logarithmic utility obtained using π^* is finite.

Heuristically speaking, partial equilibrium means the following. Supposing that the insider has the above information and that there is an extra cost/risk for changing strategies (or that he is only able to carry out local optimization procedures), he will not have much of a technical reason to change strategies although there may be other portfolios which perform better than the one above.

[4]Therefore by Stricker's Theorem, S^{π^*} is also a semimartingale under its own filtration. See Theorem 4, page 57 in Protter [19]

In this section, we will establish our main theoretical result which states that the optimal portfolio π^* obtained in Theorem 3.1 can be characterized as a policy obtained from the filtration $\mathcal{S}^{\pi^*}$ for some particular type of function g_1. Finally we characterize the optimal value function for the problem in Theorem 3.2.

In order to introduce our main result, we first define the following processes:

$$\begin{aligned}
U_t^1 &= \int_0^t g_1(s)F(t,s)dW(s),\\
U_t^2 &= \int_0^T g_1(s)dW(s) + \int_0^{(T-t)^\theta} g_2(s)dW'(s),\\
k_s(u) &= -\int_u^s \left\{-a\left(\frac{1}{2a-\sigma^2}+P(v)\right)+1\right\}\eta_v dv \eta_u^{-1} b\frac{h_1(u)}{h_2(u)} + a,\\
A_s(u) &:= k_s(u)\left\{Q(u) - \sigma\left(\frac{1}{2a-\sigma^2}+P(u)\right)g_1(u)\right\},\ u\in[0,s],\\
F(t,s) &= 1 - \int_s^t \frac{A_t(u)}{G(u)}du - \int_0^s \frac{A_T(u)}{G(u)}du.
\end{aligned}$$

Theorem 4.1. *Assume* **(A)**, (2.4) *and that Q and L are solutions of the respective (Z_1^F, Z_2^F)-linear ODE's for $F = Q, L$ equations and P is a solution of the (Z_1^P, Z_2^P, Z_3^P)-Ricatti equation. Let $g_1(t) = g_2((T-t)^\theta)$ be a strictly positive solution of*

$$g_1(t) = g_1(t)\int_0^t \frac{A_T(s)}{G(s)}ds + \sigma,\ t\in[0,T). \tag{4.2}$$

Moreover assume that $\int_0^T \frac{|g_1(s)|}{\sqrt{G(s)}}ds < \infty$, $\inf_{t\in[0,T]}|F(t,t)| \geq c_0 > 0$ and $\sup_{0\le s\le t\le T}|\partial_t F(t,s)| < \infty$. Then

$$\mathcal{G}_t = \mathcal{S}_t^{\pi^*},$$

where π^* is given by (3.8).

Note that the partial equilibrium equation (4.2) of Volterra type is non-linear in g_1 as P and Q depend on g_1 as well, through their characterizing equations. It is also clear from the above equation that $A_s(u)$ is a continuous function for $u \in [0,s]$ and therefore $g_1 \in C^1([0,T))$.

Note that as $G(T) = 0$, solving equation (4.2) appears to be difficult in general and this degeneracy problem will appear in some of the examples where we have been able to prove existence of solutions for (4.2) using relative compactness arguments. As equation (4.2) means that there is a stability in the flow of information, we call it the partial equilibrium equation in what follows.

Proof. In the proof we use the symbol σ for two different purposes, one for $\sigma-$fields and the other for the volatility coefficient. As their roles are clearly differentiated there should be no confusion.

First, note that X_t, defined in (3.6) satisfies

$$X_t = -\frac{b}{h_2(t)}\int_0^t h_1(u)\pi^*(u)du + \sigma g_1(t)\alpha(t). \tag{4.3}$$

If we consider the solution of (1.1) in the filtration $\mathcal{G}$ with the portfolio process π^* defined in (3.8) and using (4.3) one has that

$$\sigma(S^{\pi^*}(t)) = \sigma\left(\int_0^t \left(-a\left(\frac{1}{2a-\sigma^2} + P(s)\right) + 1\right) X(s) + (aQ(s) - \sigma g_1(s))\alpha(s)ds + \sigma W(t)\right). \tag{4.4}$$

On the right hand side of the above equality, we have deleted the deterministic terms, as it clearly does not change the $\sigma-$field. This property will be used again without further mention.

Now, we replace X given by (3.6) in (4.4) and applying the integration by parts formula, we have

$$\begin{aligned}
&\int_0^s \eta_u^{-1} g_1(u)d\alpha(u) = \eta_s^{-1} g_1(s)\alpha(s) - g_1(0)\alpha(0) \\
&- \int_0^s \left[-\left\{b\frac{h_1}{h_2}(u)\left(\frac{1}{2a-\sigma^2} + P(u)\right) - \frac{\dot{h}_2}{h_2}(u)\right\} g_1(u) + \dot{g}_1(u)\right]\alpha(u)\eta_u^{-1}du,
\end{aligned}$$

together with Fubini's theorem, we obtain that

$$\sigma(S^{\pi^*}(t)) = \sigma\left(\int_0^t A_t(u)\alpha(u)du + \sigma W(t)\right).$$

Using (9.2) and the stochastic Fubini theorem, we rewrite $\int_0^t A_t(u)\alpha(u)du$ so that

$$\begin{aligned}
\sigma(S^{\pi^*}(t)) = \sigma\Bigg(&\int_0^t \left(g_1(s)\int_0^s \frac{A_t(u)}{G(u)}du + \sigma\right) dW(s) \\
&+ \int_0^t \frac{A_t(u)}{G(u)}du\left(U_t^2 - \int_0^t g_1(s)dW(s)\right) \\
&+ \int_{(T-t)^\theta}^{T^\theta} g_2(s)\int_0^{T-s^{1/\theta}} \frac{A_t(u)}{G(u)}dudW'(s)\Bigg).
\end{aligned}$$

Therefore as g_1 is a solution of (4.2), we have that

$$\begin{aligned}
\sigma(S^{\pi^*}(t)) = \sigma\Bigg(&\int_0^t g_1(s)dW(s) + \int_0^t g_1(s)\int_0^s \frac{A_t(u) - A_T(u)}{G(u)}dudW(s) \\
&+ \int_0^t \frac{A_t(u)}{G(u)}du\left(U_t^2 - \int_0^t g_1(s)dW(s)\right) \\
&+ \int_{(T-t)^\theta}^{T^\theta} g_2(s)\int_0^{T-s^{1/\theta}} \frac{A_t(u)}{G(u)}dudW'(s)\Bigg).
\end{aligned} \tag{4.5}$$

Similarly, as $g_2((T-t)^\theta)$ solves (4.2), we obtain that

$$\begin{aligned}
&\sigma\left(\log\left(S^{\pi^*}(T)\right) + \sigma W'((T-t)^\theta)\right) \\
&= \sigma\left(U_t^2 + \int_{(T-t)^\theta}^{T^\theta} g_2(s)\int_0^{T-s^{1/\theta}} \frac{A_T(u)}{G(u)}dudW'(s)\right).
\end{aligned} \tag{4.6}$$

Finally,

$$\begin{aligned}
\text{(4.7)} \qquad \mathcal{S}_t^{\pi^*} &= \sigma(S^{\pi^*}(s); s \le t) \vee \sigma\left(\log\left(S^{\pi^*}(T)\right) + \sigma W'((T-s)^\theta); s \le t\right) \\
&= \sigma(S^{\pi^*}(s); s \le t) \vee \sigma\left(\log\left(S^{\pi^*}(T)\right) + \sigma W'((T-t)^\theta)\right) \\
&\quad \vee \sigma\left(W'(u) - W'((T-t)^\theta); u \ge (T-t)^\theta\right).
\end{aligned}$$

Using (4.5) and (4.6) and after some linear operations on the processes in (4.7) one obtains that

$$\mathcal{S}_t^{\pi^*} = \sigma\left(U_s^1; s \le t\right) \vee \sigma(U_t^2) \vee \sigma\left(W'(s) - W'((T-t)^\theta); s \ge (T-t)^\theta\right).$$

Now we concentrate on the process U^1. Applying Theorem 4.4 in Hida-Hitsuda [11] page 69, we will prove that $\sigma\left(U_s^1; s \le t\right) = \mathcal{F}_t$. According to that theorem, the result will hold if the trivial solution $x \equiv 0$ is the only solution to the characteristic equation

$$\int_0^t F(t,s)x(s)ds = 0.$$

In fact, first differentiate the equation with respect to t to obtain

$$x(t)F(t,t) + \int_0^t \partial_t F(t,s)x(s)ds = 0.$$

As $|F(t,t)| = \left|1 - \int_0^t \frac{A_T(u)}{G(u)}du\right| \ge c_0 > 0$ then x is a continuous function and and if $\sup_{0 \le s \le t \le T} |\partial_t F(t,s)| < \infty$ then by Gronwall inequality $x \equiv 0$. From here the result follows. □

Remark: 1. From the proof of Theorem 4.1, one can give a financial interpretation of $\mathcal{G}$ in (3.1). In fact, $\sigma\left(W'(u) - W'((T-t)^\theta); (T-t)^\theta \le u \le T^\theta\right)$ is generated by the insider signal noise in the interval $[0,t]$. Next, given this σ-field one sees that $\sigma\left(\log\left(S^{\pi^*}(T)\right) + \sigma W'((T-t)^\theta)\right)$ is the sigma field $\sigma(I(0))$. Finally, given these two σ-fields then $\sigma(S^{\pi^*}(s); s \le t)$ reduces to $\mathcal{F}_t$. Again, given these σ-fields one also obtains that $\sigma(I(0))$ becomes $\sigma(I(s);\ s \le t)$.

Therefore, loosely speaking, $\sigma\left(\int_t^T g_1(s)dW(s) + \int_0^{(T-t)^\theta} g_2(s)dW'(s)\right)$, represents the insider information. $\mathcal{F}_t$ represents the price information and

$$\sigma\left(W'(u) - W'\left((T-t)^\theta\right); u \in [(T-t)^\theta, T^\theta]\right)$$

represents the insider information noise. But this interpretation is only valid when all these σ fields are considered together.

2. Considering the elements in the proof above, one may interpret the elements of the partial equilibrium equation (4.2) as follows. The left side of the equation (4.2) appears due to the type of information set up in (3.2). The first term on the right side of (4.2) appears due to the insider effects in the price model (1.1) (in fact, if $a = b = 0$ this term vanishes) and finally, the second term on the right side is the volatility of the model. Therefore this equation states that the insider information structure, its leverage on the price and the volatility in the model have to combine in order to achieve partial equilibrium.

3. Observing the previous proof, one understands that there may be other portfolios satisfying the conditions in the definition of partial equilibrium.

The following lemma provides an easy way to check the conditions required on F in the previous theorem.

Lemma 4.1. *Assume that P, Q, h_1, h_2 and g_1 are bounded functions, $\theta \le 1$ and that there exists a positive constant c such that $\inf_{t\in[0,T]} \min\{h_2(t), g_2(t)\} > c$. Then there exists a constant c_0 such that $\inf_{t\in[0,T]} |F(t,t)| \ge c_0 > 0$ and $\sup_{0\le s\le t\le T} |\partial_t F(t,s)| < \infty$.*

Proof. If $g_1(t) \le C$ then we have that

$$F(t,t) = \frac{\sigma}{g_1(t)} \ge \frac{\sigma}{C},$$

$$\partial_t F(t,s) = \left\{-a\left(\frac{1}{2a-\sigma^2} + P(t)\right) + 1\right\} \eta_t \int_s^t \frac{\eta_u^{-1} b h_1(u)}{G(u) h_2(u)} \left\{Q(u) - \sigma\left(\frac{1}{2a-\sigma^2} + P(u)\right) g_1(u)\right\} du$$
$$+ G(t)^{-1} \int_t^T \left\{-a\left(\frac{1}{2a-\sigma^2} + P(u)\right) + 1\right\} \eta_u du \eta_t^{-1} b \frac{h_1(t)}{h_2(t)} \left\{Q(t) - \sigma\left(\frac{1}{2a-\sigma^2} + P(t)\right) g_1(t)\right\}.$$

The proof finishes by noting that $G(t) \ge c^2 (T-t)^\theta$. □

In the remaining sections we consider some representative examples.

5. Example 1: An insider with long term effects

Before we start studying in detail the examples, we give a detailed description of the model interpretation.

5.1. **Model interpretation.** Consider $Y_t^\pi = \int_0^t \frac{dS_s^\pi}{S_s^\pi} - \mu t - \sigma W_t$. Then if δ is a perturbation of the portfolio process π, we have that $Y_T^{\pi+\delta} - Y_T^\pi = \int_0^T (a - b\int_s^T H(u,s)du)\delta_s ds$. Y measures the change from a Black-Scholes type structure. The above calculation allows us to give some interpretation on the influence of perturbations in the insider portfolio composition in the model. Let's start discussing some particular examples.

The case $a > 0$, $b = 0$ stands for an insider with long term effects on the stock price dynamics. In fact, then in the previous difference taken at time T, we see that the same weight is given to a change in portfolio composition at time T and at time 0.

Another way of explaining this effect is as follows. Suppose that the insider invests his money on the stock during a time interval $[d, e]$ and then takes his money out of the market. Then the stock price for large time T has a drift $\mu T + a(e-d)$. Then if T is large then the effect of $a(e-d)$ can be neglected. That is, the asymptotic rate of return is $\mu + a(e-d)/T$. So when T approaches infinite the effect of the large trader dissipates.

Similarly, the case $a > 0$, $b > 0$, stands for an insider with medium term effects on the stock price dynamics. In fact, in that case we see that through an appropriate determination of H and b the effects of change of portfolios near time 0 can be weakened. In fact, H represents a weighting average function of the past wealth proportions of the agent. Taking the differences between the current wealth proportion with a weighted average of the past strategy allows to weaken the long term dependence of the price drift on the strategies of the insider. Note also that in general, this should allow more flexibility on insider strategies as his/her portfolio strategies will have less effects on stock prices.

5.2. **Model specification, optimal portfolios and optimal utilities.** In this section, we assume

$$a>0,\quad b=0,\quad h_1(t)\equiv 1,\quad \text{and}\quad h_2(t)\equiv 1. \tag{5.1}$$

Namely, we consider the following insider model:

$$dS_t = S_t\left\{(\mu + a\pi_t)\,dt + \sigma dW_t\right\},\ S_0>0.$$

Moreover, we assume that

$$0<a<\frac{\sigma^2}{2}. \tag{5.2}$$

Therefore hypotheses (4.2) and (**A**) are immediately satisfied.

Although this model has very simplistic features it shows various aspects that we can expect in the next example to follow. First, we give a Lemma whose proof is straightforward.

Lemma 5.1. *Assume* (5.1) *and* (5.2). *Then,* (2.4) *is satisfied and*

$$P(t)\equiv 0,\quad Q(t)\equiv 0,\quad L(t)\equiv 0,\quad \eta_t\equiv 1, \tag{5.3}$$

$$A_s(t) = \frac{a\sigma}{\sigma^2-2a}g_1(t),\ t\in[0,s].$$

From the above lemma, we can rewrite the equilibrium equation (4.2) as

$$g_1(t) = g_1(t)\int_0^t \frac{a\sigma g_1(s)}{(\sigma^2-2a)\,G(s)}ds + \sigma. \tag{5.4}$$

In the next theorem, we study the existence of solutions for the partial equilibrium equation (5.4). In particular, we note that here the value of θ becomes of importance to determine if the utility is finite or not. As it is discussed in [3] this is related with the speed of transmission of information into the market.

Theorem 5.1. *Assume* (5.1), (5.2), *and that*

$$\theta<1. \tag{5.5}$$

Then

$$g_1(t) = \sigma\exp\left(\int_0^t\left(\frac{\sigma^2}{a}(T-s)+\frac{(\sigma^2-2a)}{\sigma^2}(T-s)^\theta\right)^{-1}ds\right)$$

is the unique solution of the equilibrium equation (5.4) *which is bounded and strictly positive. Furthermore* $\int_0^T \frac{|g_1(s)|^2}{G(s)}ds<\infty$ *if and only if* $\theta<1$.

Proof. First we note that from (5.4) one obtains that $g_1(0)=\sigma$, $g_1(t)\neq 0$ and that $g_1'(t)\neq 0$ for all $t\in[0,T]$. Therefore we have that dividing (5.4) by $g_1(t)$ and differentiating we obtain

$$\frac{ag_1(s)^3}{(\sigma^2-2a)\,g_1'(s)} = G(s). \tag{5.6}$$

If we differentiate again the above equation, we obtain

$$\frac{a}{(\sigma^2-2a)}\left(3-\frac{g_1(s)g_1''(s)}{g_1'(s)^2}\right) = -1-\theta(T-s)^{\theta-1}.$$

Now, we perform the change of variables $G_1(s) = g_1(s)(g_1'(s))^{-1}$. Then the above ordinary differential equation becomes

$$\frac{a}{(\sigma^2-2a)}\left(2+(G_1(s))'\right) = -1-\theta(T-s)^{\theta-1}.$$

From here, one obtains that for a fixed $t_0 \in [0,T]$ and some suitable constants C and C_1, the solution is

$$g_1(t) = C_1 \exp\left(\int_{t_0}^t \left(C - \frac{\sigma^2 s}{a} + \frac{(\sigma^2-2a)}{a}\left((T-s)^\theta - T^\theta\right)\right)^{-1} ds\right).$$

In order to determine the constants we take $t_0 = 0$, then one obtains that $C_1 = \sigma$ due to the initial condition $g_1(0) = \sigma$. Next, suppose that the above integral exists and

$$C \neq \frac{\sigma^2 T}{a} + \frac{(\sigma^2-2a)}{a}T^\theta,$$

then $g_1(T) \neq 0$ and $g_1'(T) \neq 0$ but this is a contradiction with (5.6) as $G(T) = 0$.

Therefore $C = \frac{\sigma^2 T}{a} + \frac{(\sigma^2-2a)}{a}T^\theta$. From here, we obtain that

$$\begin{aligned} g_1(t) &= \sigma \exp\left(\int_0^t \left(\frac{\sigma^2}{a}(T-s) + \frac{(\sigma^2-2a)}{a}(T-s)^\theta\right)^{-1} ds\right) \\ G(t) &= \frac{a g_1(t)^2}{\sigma^2-2a}\left(\frac{\sigma^2}{a}(T-t) + \frac{(\sigma^2-2a)}{a}(T-t)^\theta\right). \end{aligned}$$

Then one has that for $\theta < 1$

$$\lim_{s\uparrow T} \frac{G(s)}{g_1(s)^2\,(T-s)^\theta} = 1.$$

Similarly, for $\theta \geq 1$ we have that

$$\lim_{s\uparrow T} \frac{G(s)}{g_1(s)^2\,(T-s)} = \frac{\sigma^2}{\sigma^2-2a}.$$

and therefore one easily concludes that $\int_0^T \frac{|g_1(s)|^2}{G(s)}ds < \infty$ if and only if $\theta < 1$. □

From the characterization of g_1 obtained in the previous Theorem we obtain the optimal portfolio.

Theorem 5.2. *Assume that* (5.1)*,* (5.2) *and* (5.5) *are satisfied. Then,* (3.8) *can be written as*

$$\hat{\pi}(s) = \frac{\mu - r + \sigma g_1(s)\alpha(s)}{\sigma^2-2a}. \tag{5.7}$$

Moreover, we have

$$\mathcal{G}_t = \mathcal{S}_t^{\hat{\pi}}. \tag{5.8}$$

Proof. Using (5.3), we obtain from (3.6) that $X(t) = \sigma g_1(t)\alpha(t)$. Applying this to (3.8) then (5.7) follows at once. We note that clearly the conditions of Lemma 4.1 are satisfied therefore we obtain (5.8) from Theorem 4.1. □

Note that as $a \to 0$ the model converges to the Black-Scholes model and the portfolio tends to the solution of the Karatzas-Pikovsky problem.

The portfolio (5.7) can be fully interpreted. The first part, $\frac{\mu-r}{\sigma^2-2a}$, corresponds to the Merton optimal portfolio where the volatility is modified from the usual σ^2 into σ^2-2a. That is, the effective volatility for the insider is reduced due to his effect on the market. The second term corresponds to the insider effect (information advantage) in the market which is characterized by the Radon-Nikodym derivative of the compensator of the semimartingale decomposition of W in the enlarged filtration.

The variance of the second term in the expression (5.7) is of the order $(T-t)^{-\theta}$ therefore the requirement that $\theta < 1$ becomes important in order to obtain that $E[\int_0^T \pi(s)^2 ds] < \infty$ which leads to finite utility for this strategy.

Theorem 5.3. *Assume that* (5.1), (5.2) *and* (5.5) *are satisfied. Then, the policy given by* (5.7) *is a solution for the problem* (4.1) *and the maximal expected log utility of the insider in partial equilibrium is given by*

$$\Psi(T) = \frac{1}{2(\sigma^2-2a)} \left\{ (\mu-r)^2 T + \int_0^T \frac{\sigma^2 du}{\frac{\sigma^2}{\sigma^2-2a}(T-u) + (T-u)^\theta} \right\}.$$

Furthermore, the above utility is finite if and only if $\theta < 1$.

Proof. Using (2.6) and Proposition 9.1 gives that

$$\begin{aligned} J(\hat{\pi}) &= E\left[\int_0^T \left\{ \hat{\pi}_u \left(\mu - r + a\hat{\pi}_u + \sigma g_1(u)\alpha_u\right) - \frac{1}{2}\sigma^2 \hat{\pi}_u^2 \right\} du \right] \\ &= \frac{1}{2(\sigma^2-2a)} \left\{ (\mu-r)^2 T + \sigma^2 \int_0^T \frac{g_1^2(u)}{G(u)} du \right\}. \end{aligned}$$

Therefore, the result clearly follows from Theorem 5.2. □

Note that the above maximal expected log utility is increasing with respect to a. That is, that the higher the price impact the higher the maximal expected log utility becomes. Furthermore, the case $a = 0$ corresponds to the Karatzas-Pikovsky model[5].

6. Example 2: An insider with medium term effects and a weighted average

In order to attenuate the influence of the agent we consider in this section a model which satisfies

$$a > 0, \quad b > 0, \quad h_1(t) = e^{\frac{b}{a}t}, \quad \text{and} \quad h_2(t) = e^{\frac{b}{a}t}. \tag{6.1}$$

$$dS_t = S_t \left\{ \left(\mu + a\pi_t - b\int_0^t e^{\frac{b}{a}(u-t)} \pi_u du \right) dt + \sigma dW_t \right\}, \ S_0 > 0.$$

That is, the insider creates an effect on the drift through the variations of his strategies with respect to a past time weighted average. Values closer to present time have a bigger weight. And the past time weighted average seems to allow the insider to hide his trades.

We only state the main results here and give the proofs in Appendix 12. First, by direct calculations we obtain the following preliminary result.

[5]We thank a referee for this remark

Lemma 6.1. *Besides* (6.1), *we assume that*

$$\sigma^2 - 2a > 0,$$

and

$$\lim_{t\to T}\left\{\left(\mathrm{e}^{D(T-t)} - \mathrm{e}^{-D(T-t)}\right) g_1(t)\right\} = 0, \tag{6.2}$$

where D is defined by

$$D := \frac{b\sigma}{a\sqrt{\sigma^2 - 2a}} > 0.$$

Then, we obtain the following explicit formulas for P, Q, L, η, A_s and X:

$$P(t) = \frac{r_+ r_- \left(\mathrm{e}^{D(T-t)} - \mathrm{e}^{-D(T-t)}\right)}{r_+\mathrm{e}^{D(T-t)} - r_-\mathrm{e}^{-D(T-t)}} < r_-, \text{ where} \tag{6.3}$$

$$r_\pm := \frac{1}{a}\left\{\frac{\sigma^2 - a}{\sigma^2 - 2a} \pm \frac{\sigma}{\sqrt{\sigma^2 - 2a}}\right\} > 0,$$

$$\eta_t = \frac{r_+\mathrm{e}^{D(T-t)} - r_-\mathrm{e}^{-D(T-t)}}{r_+\mathrm{e}^{DT} - r_-\mathrm{e}^{-DT}} > 0, \text{ and } \eta \text{ is decreasing,} \tag{6.4}$$

(6.5)

$$\begin{aligned} Q(t) = &-\frac{\sigma r_+ r_-}{r_+\mathrm{e}^{D(T-t)} - r_-\mathrm{e}^{-D(T-t)}}\left[-\left(\mathrm{e}^{D(T-t)} - \mathrm{e}^{-D(T-t)}\right) g_1(t)\right. \\ &\left.+\frac{b}{a}\int_t^T \left(\mathrm{e}^{D(T-u)} - \mathrm{e}^{-D(T-u)}\right) g_1(u)du + D\int_t^T \left(\mathrm{e}^{D(T-u)} + \mathrm{e}^{-D(T-u)}\right) g_1(u)du\right], \end{aligned}$$

$$\begin{aligned} L(t) = &\frac{\mu - r}{\sigma(\sigma^2 - 2a)^2\left(r_+\mathrm{e}^{D(T-t)} - r_-\mathrm{e}^{-D(T-t)}\right)} \\ &\times\left\{-\sqrt{\sigma^2 - 2a}\left(\mathrm{e}^{\frac{D}{2}(T-t)} - \mathrm{e}^{-\frac{D}{2}(T-t)}\right)^2 - \sigma\left(\mathrm{e}^{D(T-t)} - \mathrm{e}^{-D(T-t)}\right)\right\}, \end{aligned} \tag{6.6}$$

$$A_s(t) = \frac{\sigma k_s(t)}{\sigma^2 - 2a}B(t), \tag{6.7}$$

where

$$\begin{aligned} B(t) = g_1(t) - &\frac{1}{(\sigma^2 - 2a)\left(r_+\mathrm{e}^{D(T-t)} - r_-\mathrm{e}^{-D(T-t)}\right)} \\ &\times\left\{\frac{b}{a}\int_t^T \left(\mathrm{e}^{D(T-u)} - \mathrm{e}^{-D(T-u)}\right) g_1(u)du + D\int_t^T \left(\mathrm{e}^{D(T-u)} + \mathrm{e}^{-D(T-u)}\right) g_1(u)du\right\}, \end{aligned}$$

$$k_s(t) = \frac{a\left(r_+\mathrm{e}^{D(T-s)} - r_-\mathrm{e}^{-D(T-s)}\right)}{r_+\mathrm{e}^{D(T-t)} - r_-\mathrm{e}^{-D(T-t)}},$$

and $X_t = \sigma g_1(t)\alpha(t) - \int_0^t c_1(t,u)\alpha(u)du - \int_0^t c_2(t,u)du,$

where $c_1(t,u)$ and $c_2(t,u)$ are defined by

$$c_1(t,u) := \frac{b\sigma\left(r_+\mathrm{e}^{D(T-t)} - r_-\mathrm{e}^{-D(T-t)}\right)}{(\sigma^2 - 2a)\left(r_+\mathrm{e}^{D(T-u)} - r_-\mathrm{e}^{-D(T-u)}\right)}B(u),$$

and

$$c_2(t,u) := \frac{b(\mu-r)\left(r_+ \mathrm{e}^{D(T-t)} - r_- \mathrm{e}^{-D(T-t)}\right)}{a\sigma(\sigma^2-2a)^{\frac{3}{2}}\left(r_+ \mathrm{e}^{D(T-u)} - r_- \mathrm{e}^{-D(T-u)}\right)^2} \times \left\{2a + (\sigma\sqrt{\sigma^2-2a}+\sigma^2-a)\mathrm{e}^{D(T-u)} - (\sigma\sqrt{\sigma^2-2a}-\sigma^2+a)\mathrm{e}^{-D(T-u)}\right\}$$

respectively.

Therefore, the equilibrium equation (4.2) can be rewritten as

$$\begin{aligned} g_1(t) &= g_1(t)\int_0^t \frac{A_T(s)}{G(s)}ds + \sigma, \\ A_T(t) &= \frac{\sigma}{\sigma^2-2a}k_T(t)B(t), \\ k_T(t) &:= \frac{2\sigma}{\sqrt{\sigma^2-2a}}\left(r_+ \mathrm{e}^{D(T-t)} - r_- \mathrm{e}^{-D(T-t)}\right)^{-1}. \end{aligned} \tag{6.8}$$

Here we note that $g_1(t) = g_2((T-t)^\theta)$.

Theorem 6.1. *Assume* (6.1) *and the following conditions*

$$a < \min\left\{\frac{1}{2}\left[\sigma^2 - \sqrt{2ab\left(T + \frac{a}{2b}(\mathrm{e}^{-\frac{2bT}{a}}-1)\right)}\right], \frac{\sigma^2}{3}\right\}, \tag{6.9}$$

$$\theta < 1, \tag{6.10}$$

$$T^{1-\theta} < \frac{(1-\theta)(\sigma^2-2a)}{2a}. \tag{6.11}$$

Then the following assertions are satisfied.
1. $k_T(t) \in (0,a]$ *and the equation* (6.8) *has a strictly positive solution in such that* (6.2) *is satisfied,* $g_1(0)=\sigma$, $\dot{g}_1(t) > 0$ *and is bounded for* $t \in [0,T]$.
2. *Define*

$$\hat{\pi}(t) = -\left(\frac{1}{2a-\sigma^2} + P(t)\right)X_t + Q(t)\alpha(t) + \left(L(t) - \frac{\mu-r}{2a-\sigma^2}\right). \tag{6.12}$$

Then $\hat{\pi} \in \mathcal{A}_{\mathcal{S}^{\hat{\pi}}}$ *and furthermore*

$$\mathcal{G}_t = \mathcal{S}_t^{\hat{\pi}}.$$

3. *The strategy given by* (6.12) *is a strategy satisfying* (4.1) *with finite utility and its value is*

$$E\left[v(0, X_0, \alpha(0))\right],$$

where

$$v(t,x,\alpha) = \frac{2a-\sigma^2}{b}\left\{-\frac{1}{2}P(t)x^2 + Q(t)x\alpha + \frac{1}{2}R(t)\alpha^2 + L(t)x + M(t)\alpha + N(t)\right\}.$$

Here P,Q and L are given by (6.3), (6.5) *and* (6.6) *respectively. Moreover, R,M and N are the solutions of the following $(0,Z_2^F)$-linear ODEs, $F=R,M,N$*

$$
\left\{
\begin{aligned}
&Z_2^R(t) = -bQ^2(t) + 2\sigma\left(\frac{b}{a}g_1(t)+\dot{g}_1(t)\right)Q(t),\\
&Z_2^M(t) = -b\left(L(t)-\frac{\mu-r}{2a-\sigma^2}\right)Q(t) + \sigma\left(\frac{b}{a}g_1(t)+\dot{g}_1(t)\right)L(t),\\
&Z_2^N(t) = -\frac{b}{2}\left(L(t)-\frac{\mu-r}{2a-\sigma^2}\right)^2\\
&\quad + \frac{g_1^2(t)\left\{1+\theta(T-t)^{\theta-1}\right\}}{G^2(t)}\left\{-\sigma^2 g_1^2(t)P(t)+\sigma g_1(t)Q(t)+R(t)\right\}.
\end{aligned}
\right.
\tag{6.13}
$$

For the proof, see Appendix 12. Note that if we assume (6.9), (2.4) is satisfied. Hence, we can use the results in Sections $2-4$.

Although the restriction (6.11) may not be optimal, it seems reasonable that some kind of restriction of this type should appear. In fact, if the effect of the agent on the prices dissipates quickly in the dynamics of the underlying the possibility of creating arbitrage increases. In short, this restriction states that this kind of insider effect can not happen over large periods of time.

7. Conclusions

The method to prove the existence of solutions to the partial equilibrium equation in Example 2 is quite intricate as the non-linear ordinary differential equation characterizing the partial equilibrium, (4.2), degenerates at time T. Therefore we have resorted to an approximative argument in order to study the relative compactness of the family of solutions with respect to a small parameter that makes the system non-degenerate. We then characterize the strategy for the insider and the utility which is finite as $\theta<1$ is assumed.

We have also treated two additional examples ($a>0,\ b>0$ and $a=0,\ b<0$ with $h_1=h_2=1$) in a related article [10]. The techniques used in those cases resemble the ones used in Example 2.

There are many obvious criticisms to the present modeling, such as why the influence of the insider does not appear in an non-bijective way in the volatility. Another interesting aspect is the relation of the insider investor with the small investor. In our setting this relation is only established under the restriction that markets do not blow up. That is, the optimal portfolios π^* lead to a finite utility and therefore optimization and non-existence of arbitrage for the small trader can be carried out. In Appendix 8, we also prove that a model where the effect of the insider is measured through the amount of money invested in the stock will explode for all times with positive probability.

Undoubtedly this is a first attempt and we hope that many other articles discussing other approaches appear in the future.

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8. Appendix: Investor's optimal policy explodes for drifts depending on the total amount of money invested

Here we consider the situation where the stock price is influenced through a power function of the amount of money invested by an agent in the underlying. That is, our model is

$$dS_t = S_t \left\{ (\mu + a\,(p_t S_t)^{\alpha})dt + \sigma dW_t \right\},\ S_0 > 0,$$

where p denotes the number of shares that the agent invests in the underlying and $\alpha, a > 0$.

In this model there is no insider characteristics and for simplicity we assume that the interest rate $r \equiv 0$. We furthermore simplify the setting considering the particular case that the investor invests all his wealth in the underlying. In such a case the discounted wealth process satisfies the equation

$$d\widehat{V}_t = \widehat{V}_t \left[\left\{ \mu + a \left(\widehat{V}_t \right)^{\alpha} \right\} \right] dt + \sigma \widehat{V}_t dW_t.$$

If we transform the above equation using $Y_t = \log(\widehat{V}_t)$ we obtain that Y_t satisfies the equation

$$dY_t = \left(\mu_0 + a\mathrm{e}^{\alpha Y_t}\right) dt + \sigma W_t,$$

where we assume that $\mu_0 = \mu - r - \frac{\sigma^2}{2} < 0$. This equation can be analyzed using the methodology of Engelbert-Schmidt (see Karatzas-Shreve [16] Section 5.5). In that case the scale function is given by

$$p(x) = \int_c^x \exp\left\{-2\sigma^{-2}\int_c^\xi (\mu_0 + a\mathrm{e}^{\alpha\zeta})d\zeta\right\} d\xi.$$

Here, c is any positive constant. From the properties of the exponential function we have that $p(+\infty) < \infty$ and $p(-\infty) > -\infty$ as $\mu_0 < 0$. As p is a strictly increasing function, its inverse q exists on $(p(-\infty), p(+\infty))$. According to Proposition 5.5.13 in Karatzas-Shreve [16] we have that $Y = p^{-1}(Z)$ where Z is the solution of the equation

$$dZ_t = \tilde{\sigma}(Z_t)dW_t,$$

where

$$\tilde{\sigma}(y) = 1(y < p(+\infty))p'(q(y))\sigma.$$

Note that there exists two positive constants c_0 and c_1 such that $0 < c_0 < p'(q(y))\sigma < c_1$. Therefore the function $\tilde{\sigma}$ has a jump at $p(+\infty)$.

The Engelbert-Schmidt result also provides through a time change of the Brownian motion, an almost explicit weak construction method for Z. In fact, let B be a Brownian motion, define

$$T_s = \int_0^{s+} \frac{du}{\tilde{\sigma}(Z_0 + B_u)^2},$$

$$A_t = \inf\{s \geq 0; T_s > t\},$$

then $Z_t = Z_0 + B_{A_t}$. Clearly, as $P(T_s = +\infty) > 0$ for all $s > 0$ then $P(A_s = A_\infty) > 0$ for all $s > 0$. This is equivalent to $P(Z_s > p(+\infty)) > 0$ for all $s > 0$. This clearly implies that $P(Y_s = +\infty) > 0$ for all $s > 0$ and therefore in such a model the agent's best policy implies that the wealth process (and therefore the price process) is infinite in finite time with positive probability.

This shows that the setting in Cuoco-Cvitanic may lead to infinite utilities and price explosion if the drift is not decreasing in p.

9. Appendix: An enlargement of filtration problem

In this section we treat an enlargement of filtration problem which will be used in the calculations. It is related to the insider problem with a fixed enlarged filtration. The goal is to obtain the compensator of W under the filtration $\mathcal{G}$ defined by (3.1). Here, for $g_1,\ g_2 \in L^2[0,T]$ strictly positive functions, we define $G(t) := \int_t^T g_1(s)^2 ds + \int_0^{(T-t)^\theta} g_2(s)^2 ds > 0$ for all $t \in [0,T)$.

Proposition 9.1. *Suppose that* $\int_0^T \frac{|g_1(s)|}{\sqrt{G(s)}} ds < \infty$, *then* $\{W(t); t \in [0,T]\}$ *is a semimartingale in the filtration* $\mathcal{G}$ *and the semimartingale decomposition of* W *is given by*

$$W_t = \hat{W}_t + \int_0^t g_1(s)\alpha(s)ds, \tag{9.1}$$

$$\alpha(s) = \frac{1}{G(s)}\left(\int_s^T g_1(u)dW(u) + \int_0^{(T-s)^\theta} g_2(u)dW'(u)\right); \; s < T, \tag{9.2}$$

where $\{\hat{W}(t); t \in [0,T]\}$ *is a* $\mathcal{G}$*-Wiener process and* α *is a Gaussian* $\mathcal{G}$*-martingale in* $[0,T)$ *with quadratic variation given by*

$$\langle \alpha \rangle_t = \int_0^t \left(\frac{g_1(s)}{G(s)}\right)^2 ds + \int_{(T-t)^\theta}^{T^\theta} \left(\frac{g_2(s)}{G(s)}\right)^2 ds.$$

Proof. First, note that by the joint law of increments for the Brownian motion, we have that for $s < u < t$

$$\begin{aligned} E\left[W(t) - W(u)\right| \mathcal{G}_s] &= E\left[W(t) - W(u)\left| \int_s^T g_1(r)dW(r) + \int_0^{(T-s)^\theta} g_2(r)dW'(r)\right.\right] \\ &= \int_u^t g_1(r)dr \frac{1}{G(s)}\left(\int_s^T g_1(r)dW(r) + \int_0^{(T-s)^\theta} g_2(r)dW'(r)\right). \end{aligned}$$

Here we have used that for a Gaussian random vector (X,Y), we have that $E[X|Y] = \frac{E[XY]}{E[Y^2]}Y$. Then (9.1) follows from a direct calculation. In fact,

$$\begin{aligned} &E\left[W(t) - W(s) - \int_s^t \frac{g_1(r)}{G(r)}\left(\int_r^T g_1(u)dW(u) + \int_0^{(T-r)^\theta} g_2(u)dW'(u)\right) dr \right| \mathcal{G}_s\Bigg] \\ &= \int_s^t g_1(r)dr \frac{1}{G(s)}\left(\int_s^T g_1(r)dW(r) + \int_0^{(T-s)^\theta} g_2(r)dW'(r)\right) \\ &- \int_s^t \frac{g_1(r)}{G(r)} E\left[\int_r^T g_1(u)dW(u) + \int_0^{(T-r)^\theta} g_2(u)dW'(u) \right| \mathcal{G}_s\Bigg] dr. \end{aligned}$$

The proof finishes because we have that for $r > s$

$$\begin{aligned} &E\left[\int_r^T g_1(u)dW(u) + \int_0^{(T-r)^\theta} g_2(u)dW'(u)\right| \mathcal{G}_s\Bigg] \\ &= E\left[\int_r^T g_1(u)dW(u) + \int_0^{(T-r)^\theta} g_2(u)dW'(u)\right| \int_s^T g_1(u)dW(u) + \int_0^{(T-s)^\theta} g_2(u)dW'(u)\Bigg] \\ &= \frac{G(r)}{G(s)}\left(\int_s^T g_1(u)dW(u) + \int_0^{(T-s)^\theta} g_2(u)dW'(u)\right). \end{aligned}$$

Therefore the process α is a $\mathcal{G}$-martingale and $\hat{W}$ defined by (9.1) is a $\mathcal{G}$ continuous martingale and therefore by Levy's characterization of Brownian motion one obtains that $\hat{W}$ is a Brownian motion for any interval $[0,t]$ for $t < T$. Note that the condition $\int_0^T \frac{|g_1(s)|}{\sqrt{G(s)}} ds < \infty$ is used to guarantee the integrability of the compensator. That is, using the Gaussian property for the sum of two stochastic integrals with deterministic integrands, we have for some positive constant C

$$E\left|\int_0^T \frac{g_1(s)}{G(s)}\left(\int_s^T g_1(u)dW(u)+\int_0^{(T-s)^\theta} g_2(u)dW'(u)\right)ds\right| \le C\int_0^T \frac{|g_1(s)|}{\sqrt{G(s)}}ds < \infty.$$

Therefore $\hat{W}$ is well defined in $[0,T]$. □

10. Appendix: Proof of Theorem 3.1

Proof of Theorem 3.1

Proof. Using (3.7), (3.8) and (3.12), we obtain an expression

$$d(h_2(t)X_t) = -bh_1(t)\pi^*(t)dt + \sigma d(g_1(t)h_2(t)\alpha(t)).$$

Noting that $X_0 = \sigma g_1(0)\alpha(0)$, we obtain (4.3).

Define

$$\begin{aligned} F(\pi) :=& \mu - r + (2a-\sigma^2)\pi(t) - \frac{b}{h_2(t)}\int_0^t h_1(u)\pi(u)du \\ &+ \sigma g_1(t)\alpha(t) - bh_1(t)E\left[\int_t^T \frac{\pi(u)}{h_2(u)}du\Big|\mathcal{G}_t\right]. \end{aligned}$$

Using (3.8) and (4.3), we have

$$F(\pi^*) = (2a-\sigma^2)\left\{-P(t)X_t + Q(t)\alpha(t) + L(t)\right\} - bh_1(t)E\left[\int_t^T \frac{\pi^*(u)}{h_2(u)}du\Big|\mathcal{G}_t\right].$$

Since α is a $\mathcal{G}$-martingale, we have that

$$\begin{aligned} E\left[\int_t^T \frac{\pi^*(u)}{h_2(u)}du\Big|\mathcal{G}_t\right] =& -\int_t^T \frac{1}{h_2(u)}\left(\frac{1}{2a-\sigma^2}+P(u)\right)E[X_u|\mathcal{G}_t]du \\ &+ \alpha(t)\int_t^T \frac{Q(u)}{h_2(u)}du + \int_t^T \frac{1}{h_2(u)}\left(L(u) - \frac{\mu-r}{2a-\sigma^2}\right)du. \end{aligned}$$

Furthermore, using (3.6) for $u > t$,

$$\begin{aligned} E[X_u|\mathcal{G}_t] =& \eta_u\eta_t^{-1}X_t \\ &+ \eta_u\left(\int_t^u \eta_v^{-1}\left\{-b\frac{h_1(v)}{h_2(v)}Q(v) + \sigma\left(\dot{g}_1(v) + g_1(v)\frac{\dot{h}_2(v)}{h_2(v)}\right)\right\}dv\right)\alpha(t) \\ &- b\eta_u\int_t^u \frac{h_1(v)}{h_2(v)}\eta_v^{-1}\left(L(v) - \frac{\mu-r}{2a-\sigma^2}\right)dv. \end{aligned}$$

Therefore, we rewrite $F(\pi^*)$ as

$$F(\pi^*) = \left\{-\eta_t^{-1}M_X(t)X(t) + M_\alpha(t)\alpha(t) + M(t)\right\}h_1(t),$$

where

$$
\begin{aligned}
M_X(t) &:= (2a-\sigma^2)P(t)\frac{\eta_t}{h_1(t)} - b\int_t^T \frac{1}{h_2(u)}\left(\frac{1}{2a-\sigma^2}+P(u)\right)\eta_u du,\\
M_\alpha(t) &:= (2a-\sigma^2)\frac{Q(t)}{h_1(t)} + b\int_t^T \frac{1}{h_2(u)}\left(\frac{1}{2a-\sigma^2}+P(u)\right)\eta_u\\
&\cdot \left(\int_t^u \eta_v^{-1}\left\{-b\frac{h_1(v)}{h_2(v)}Q(v)+\sigma\left(\dot{g}_1(v)+g_1(v)\frac{\dot{h}_2(v)}{h_2(v)}\right)\right\}dv\right)du - b\int_t^T \frac{Q(u)}{h_2(u)}du,\\
M(t) &:= (2a-\sigma^2)\frac{L(t)}{h_1(t)} - b^2\int_t^T \frac{1}{h_2(u)}\left(\frac{1}{2a-\sigma^2}+P(u)\right)\eta_u\\
&\cdot\left\{\int_t^u \frac{h_1(v)}{h_2(v)}\eta_v^{-1}\left(L(v)-\frac{\mu-r}{2a-\sigma^2}\right)dv\right\}du - b\int_t^T \frac{1}{h_2(u)}\left(L(u)-\frac{\mu-r}{2a-\sigma^2}\right)du.
\end{aligned}
$$

Note that as P is a solution of the (Z_1^P, Z_2^P, Z_3^P)-Ricatti equation with coefficients given in (3.3), we see that M_X satisfies

$$
\begin{aligned}
\dot{M}_X(t) = \frac{(2a-\sigma^2)\eta_t}{h_1(t)}\Bigg[&\dot{P}(t) + b\frac{h_1(t)}{h_2(t)}P^2(t)\\
&+\left\{\frac{2b}{2a-\sigma^2}\frac{h_1(t)}{h_2(t)} - \frac{\dot{h}_1(t)}{h_1(t)} - \frac{\dot{h}_2(t)}{h_2(t)}\right\}P(t) + \frac{b}{(2a-\sigma^2)^2}\frac{h_1(t)}{h_2(t)}\Bigg]\\
&= 0
\end{aligned}
$$

for all $t \in [0,T]$. Noting that $M_X(T) = 0$, we see that $M_X(t) = 0$ for all $t \in [0,T]$. In a similar fashion and using that $M_X(t) = 0$ for all $t \in [0,T]$ we obtain that $\dot{M}_\alpha(t) = 0$ and $\dot{M}(t) = 0$ for all $t \in [0,T]$. Therefore, since $M_\alpha(t) = 0$ and $M(t) = 0$ hold for all $t \in [0,T]$, we conclude the proof as $F(\pi^*) = 0$ implies that π^* is the optimal portfolio by Theorem 2.1. □

11. Appendix: Proof of Theorem 3.2

Proof. Here we prove that

$$F_1(t,x,\alpha) + F_2(t,x,\alpha) = 0, \quad \text{for } t \in [0,T].$$

By direct calculations we can observe that

$$
\begin{aligned}
F_2(t,x,\alpha) := \frac{bh_1(t)}{2h_2(t)}\Bigg[&\left\{P^2(t) - \frac{1}{(2a-\sigma^2)^2}\right\}x^2 - 2P(t)Q(t)x\alpha + Q^2(t)\alpha^2\\
&-2\left\{\frac{\mu-r}{(2a-\sigma^2)^2} + P(t)L(t)\right\}x + 2Q(t)L(t)\alpha + L^2(t) - \left(\frac{\mu-r}{2a-\sigma^2}\right)^2\Bigg],
\end{aligned}
$$

and

$$
\begin{aligned}
F_1(t,x,\alpha) &= -\frac{1}{2}\left[\dot{P}(t)+\left\{2b\left(\frac{1}{2a-\sigma^2}+P(t)\right)\frac{h_1(t)}{h_2(t)}-\frac{\dot{h}_2(t)}{h_2(t)}-\frac{\dot{h}_1(t)}{h_1(t)}\right\}P(t)\right]x^2\\
&+\left[\dot{Q}(t)+\left\{b\left(\frac{1}{2a-\sigma^2}+2P(t)\right)\frac{h_1(t)}{h_2(t)}-\frac{\dot{h}_1(t)}{h_1(t)}\right\}Q(t)-\sigma\left(\dot{g}_1(t)+\frac{\dot{h}_2(t)}{h_2(t)}g_1(t)\right)P(t)\right]x\alpha\\
&+\frac{1}{2}\left[\dot{R}(t)+\left(\frac{\dot{h}_2(t)}{h_2(t)}-\frac{\dot{h}_1(t)}{h_1(t)}\right)R(t)+2\left\{-b\frac{h_2(t)}{h_1(t)}Q(t)+\sigma\left(\dot{g}_1(t)+\frac{\dot{h}_2(t)}{h_2(t)}g_1(t)\right)\right\}Q(t)\right]\alpha^2\\
&+\left[\dot{L}(t)+\left\{b\left(\frac{1}{2a-\sigma^2}+2P(t)\right)\frac{h_1(t)}{h_2(t)}-\frac{\dot{h}_1(t)}{h_1(t)}\right\}L(t)-\frac{b(\mu-r)}{2a-\sigma^2}\frac{h_1(t)}{h_2(t)}P(t)\right]x\\
&+\left[\dot{M}(t)+\left(\frac{\dot{h}_2(t)}{h_2(t)}-\frac{\dot{h}_1(t)}{h_1(t)}\right)M(t)\right.\\
&\qquad\left.+\left\{-2b\frac{h_2(t)}{h_1(t)}Q(t)+\sigma\left(\dot{g}_1(t)+\frac{\dot{h}_2(t)}{h_2(t)}g_1(t)\right)\right\}L(t)+\frac{b(\mu-r)}{2a-\sigma^2}\frac{h_1(t)}{h_2(t)}Q(t)\right]\alpha\\
&+\left[\dot{N}(t)+\left(\frac{\dot{h}_2(t)}{h_2(t)}-\frac{\dot{h}_1(t)}{h_1(t)}\right)N(t)-b\frac{h_2(t)}{h_1(t)}\left(L(t)-\frac{\mu-r}{2a-\sigma^2}\right)L(t)\right.\\
&\left.+\frac{g_1^2(t)+\theta(T-t)^{\theta-1}g_2^2((T-t)^\theta)}{G^2(t)}\left\{-\sigma^2 g_1^2(t)P(t)+\sigma g_1(t)Q(t)+R(t)\right\}\right].
\end{aligned}
$$

Using the ODE equations satisfied by P, Q, L, R, M and N, we see that $F_1(t,x,\alpha)+F_2(t,x,\alpha)=0$. □

12. Appendix: Proofs of Section 6

Proof of Lemma 6.1

If $h_1(t)=\mathrm{e}^{\frac{b}{a}t}$ and $h_2(t)=\mathrm{e}^{\frac{b}{a}t}$ holds, the ODE's for P, Q and L become:

$$
\begin{cases}
\dot{P}(t)+bP^2(t)-2\dfrac{b(\sigma^2-a)}{a(\sigma^2-2a)}P(t)+\dfrac{b}{(\sigma^2-2a)^2}=0,\\
P(T)=0,
\end{cases}
\tag{12.1}
$$

$$
\begin{cases}
\dot{Q}(t)+b\left\{-\dfrac{\sigma^2-a}{a(\sigma^2-2a)}+P(t)\right\}Q(t)-\sigma\left\{\dfrac{b}{a}g_1(t)+\dot{g}_1(t)\right\}P(t)=0,\\
Q(T)=0,
\end{cases}
\tag{12.2}
$$

$$
\begin{cases}
\dot{L}(t)+b\left\{-\dfrac{\sigma^2-a}{a(\sigma^2-2a)}+P(t)\right\}L(t)-b\dfrac{(\mu-r)}{2a-\sigma^2}\left(\dfrac{1}{2a-\sigma^2}+P(t)\right)=0,\\
L(T)=0.
\end{cases}
\tag{12.3}
$$

Suppose that $P(t)\neq r_\pm$. Then we can rewrite (12.1) as

$$
\frac{\dot{P}(t)}{P(t)-r_+}-\frac{\dot{P}(t)}{P(t)-r_-}=-2D. \tag{12.4}
$$

Integrate (12.4) on $[t,T]$. Then we get

$$\frac{P(t)-r_-}{P(t)-r_+} = \frac{r_-}{r_+}\mathrm{e}^{-2D(T-t)},$$

which leads to (6.3). Moreover, from (3.7) and after some calculations one obtains (6.3). Furthermore, we have

$$\frac{\dot{\eta}_t}{\eta_t} = -D\frac{r_+\mathrm{e}^{D(T-t)}+r_-\mathrm{e}^{-D(T-t)}}{r_+\mathrm{e}^{D(T-t)}-r_-\mathrm{e}^{-D(T-t)}}.$$

Integrating the above equality on $[0,t]$, we get (6.4).

Note that from (3.4), (3.5) and (3.7) and as $h_1 \equiv h_2$ then $Z_1^Q = Z_1^L = \dot{\eta}_t\eta_t^{-1}$ and therefore

$$(\eta_t Q(t))' = \sigma\left\{\frac{b}{a}g_1(t)+\dot{g}_1(t)\right\}P(t)\eta_t.$$

We integrate the above equality on $[t,T]$ and use (6.2), (6.3) and (6.4). Then we obtain (6.5). For L, we similarly have

$$(\eta_t L(t))' = -\frac{b(\mu-r)}{\sigma^2-2a}\left(P(t)-\frac{1}{\sigma^2-2a}\right)\eta_t.$$

Therefore, we obtain (6.6).

From (6.3) and (6.5) we have

$$Q(t)-\sigma\left(P(t)-\frac{1}{\sigma^2-2a}\right)g_1(t) = \frac{\sigma}{\sigma^2-2a}B(t). \tag{12.5}$$

Using (12.5) and (6.4), we have (6.7).

Finally, if we note that if we perform an integration by parts on the stochastic integral in (3.6) we have the following alternative expression for X

$$\begin{aligned} X_t = \sigma g_1(t)\alpha(t) - \eta_t b\int_0^t \eta_u^{-1}\frac{h_1(u)}{h_2(u)}\left[\left\{Q(u)-\sigma g_1(u)\left(\frac{1}{2a-\sigma^2}+P(u)\right)\right\}\alpha(u)\right. \\ \left. +L(u)-\frac{\mu-r}{2a-\sigma^2}\right]du, \end{aligned} \tag{12.6}$$

then the expression for X in the statement of Lemma 6.1 follows from (6.4), (6.6), (12.5), (6.3) and (6.5).

Proof of Theorem 6.1: Proof of 1. The proof that $k_T(t)\in(0,a]$ is straightforward. Also note that $k_T(T)=a$. To study the existence of solution to the partial equilibrium equation (6.8), we introduce the following approximating equations for $n\geq 2$, $t\in[0,T]$:

$$g_n(t) = g_n(t)\int_0^t \frac{A_T^n(s)}{G_n(s)}ds+\sigma, \tag{12.7}$$

$$G_n(t) := \int_t^T\left\{1+\theta\left(T-u+\frac{1}{n}\right)^{\theta-1}\right\}g_n^2(u)du+\frac{g_n^2(T)}{n^\theta},$$

$$A_T^n(t) = \frac{\sigma}{\sigma^2-2a}k_T(t)B_n(t), \tag{12.8}$$

$$B_n(t) := g_n(t) - \frac{1}{(\sigma^2-2a)\left(r_+\mathrm{e}^{D(T-t)} - r_-\mathrm{e}^{-D(T-t)}\right)}$$
$$\times \left\{ \frac{b}{a}\int_t^T \left(\mathrm{e}^{D(T-u)} - \mathrm{e}^{-D(T-u)}\right) g_n(u)du + D\int_t^T \left(\mathrm{e}^{D(T-u)} + \mathrm{e}^{-D(T-u)}\right) g_n(u)du \right\}.$$

First, assuming $g_n(T) = x_0 > 0$, we shall show that there exists $x_0 > 0$ such that $g_n(0) = \sigma$. Then, if g_n satisfies (12.7) then we have that $g_n(t) > 0$. Next, we divide the equation (12.7) by $g_n(t)$ and differentiating both sides of the equation we have

$$\dot{g}_n(t) = \frac{g_n(t)^2 A_T^n(t)}{\sigma G_n(t)}. \tag{12.9}$$

Now we will use (12.9) in order to find a bound for $g_n(T) - g_n(t)$. In fact, $k_T(t) \le a$ and $B_n(t) \le g_n(t)$ give that $A_T^n(t) \le \frac{\sigma a g_n(t)}{\sigma^2-2a}$. To this fact we add that $G_n(t) \ge g_n(t)^2(T-t)^\theta$ and the temporary assumption that g_n is increasing in an interval $I_0 = (t_0, T]$ to obtain that for $t \in I_0$

$$\begin{aligned} g_n(T) - g_n(t) &\le \frac{a g_n(T)}{\sigma^2-2a}\int_t^T \frac{du}{(T-u)^\theta} \\ &\le \frac{aT^{1-\theta} g_n(T)}{(\sigma^2-2a)(1-\theta)}. \end{aligned} \tag{12.10}$$

Therefore from the assumption (6.11), we see that $g_n(t) > \frac{g_n(T)}{2} > 0$ for $t \in I_0$.

By contradiction, assume that $B_n(t) > 0$ for $t \in (t_0, T]$ and $B_n(t_0) = 0$. Then, since g_n is continuous and increasing on $[t_0, T]$, we have from (12.7) and (6.9)

$$\begin{aligned} 0 = B_n(t_0) &\ge g_n(T)\left[\frac{1}{2} - \frac{1}{(\sigma^2-2a)(r_+\mathrm{e}^{D(T-t_0)} - r_-\mathrm{e}^{-D(T-t_0)})}\left\{\frac{b}{a}\int_{t_0}^T (\mathrm{e}^{D(T-u)} - \mathrm{e}^{-D(T-u)})du \right.\right. \\ &\left.\left. + D\int_{t_0}^T (\mathrm{e}^{D(T-u)} + \mathrm{e}^{-D(T-u)})du\right\}\right] \\ &= \frac{g_n(T)}{2a\sigma(\sigma^2-2a)\left(r_+\mathrm{e}^{D(T-t_0)} - r_-\mathrm{e}^{-D(T-t_0)}\right)}\left[4a\sqrt{\sigma^2-2a}\right. \\ &\left. +\sigma(\sigma^2-3a)\left(\mathrm{e}^{D(T-t_0)} - \mathrm{e}^{-D(T-t_0)}\right) + (\sigma^2-2a)^{\frac{3}{2}}\left(\mathrm{e}^{D(T-t_0)} + \mathrm{e}^{-D(T-t_0)}\right)\right] \\ &> 0. \end{aligned}$$

This is a contradiction. Repeating similar arguments, we see that $B_n(t) > 0$ and $\dot{g}_n(t) > 0$ for $t \in [0, T]$. From here it also follows that

$$\frac{x_0}{2} < g_n(t) < x_0, \text{ and } 0 < \dot{g}_n(t) \le \frac{a x_0 n^\theta}{\sigma^2-2a} \tag{12.11}$$

for $t \in [0, T]$. From (12.7) we shall construct a system of ordinary differential equations. For this, we introduce the following auxiliary function

$$M_n(t) := \frac{\dot{g}_n(t)}{g_n^2(t)} G_n(t), \tag{12.12}$$

and note that

$$0 < M_n(t) \le \frac{4a x_0 n^\theta}{\sigma^2-2a}\left\{T + \left(T + \frac{1}{n}\right)^\theta\right\}. \tag{12.13}$$

Then, using (12.9), we have

$$M_n(t) = \frac{A_T^n(t)}{\sigma}. \tag{12.14}$$

Setting $w(t) := \log g_n(t), v(t) := \log \dot{g}_n(t)$ and $m(t) := \log M_n(t)$. From (12.12) and (12.14), we obtain the following system:

$$\dot{\begin{pmatrix} w(t) \\ v(t) \\ m(t) \end{pmatrix}} = f(t, w(t), v(t), m(t)), \quad \begin{pmatrix} w(T) \\ v(T) \\ m(T) \end{pmatrix} = \begin{pmatrix} \log x_0 \\ \log\left(\frac{n^\theta a x_0}{\sigma^2 - 2a}\right) \\ \log\left(\frac{a x_0}{\sigma^2 - 2a}\right) \end{pmatrix},$$

where

(12.15)

$$f(t, w(t), v(t), m(t)) := \begin{pmatrix} e^{v(t)-w(t)} \\ \tilde{c} + 2e^{v(t)-w(t)} + \left\{1 + \frac{k_T(t)}{\sigma^2 - 2a} + \theta\left(T - t + \frac{1}{n}\right)^{\theta-1}\right\} e^{v(t)-m(t)} - \frac{b k_T(t)}{a(\sigma^2 - 2a)} e^{w(t)-m(t)} \\ \tilde{c} + \frac{k_T(t)}{\sigma^2 - 2a} e^{v(t)-m(t)} - \frac{b k_T(t)}{a(\sigma^2 - 2a)} e^{w(t)-m(t)} \end{pmatrix},$$

$$\tilde{c} = \tilde{c}_T(t) := \frac{\dot{k}_T(t)}{k_T(t)} + D \frac{r_+ e^{D(T-t)} + r_- e^{-D(T-t)}}{r_+ e^{D(T-t)} - r_- e^{-D(T-t)}}.$$

Noting that (12.11) and (12.13) and repeating the application of Picard's theorem, we see that this system has a unique solution for $t \in [0, T]$. Moreover, since for $t \in [0, T]$ $g_n(t)$ is continuous on x_0 and $\frac{x_0}{2} < g_n(t) < x_0$ for $t \in [0, T]$, by the intermediate value theorem, there exists $x_0 > 0$ such that $g_n(0) = \sigma$. Next, we shall prove the boundedness of g_n. For that, from (12.9), we have

$$\sigma \le g_n(t) \le \sigma e^{\frac{a}{(\sigma^2 - 2a)(1-\theta)} T^{1-\theta}} \quad \text{for } t \in [0, T]. \tag{12.16}$$

Here we have used that $g_n(0) = \sigma$ and g_n is increasing for $t \in [0, T]$. Furthermore, we see that $\{g_n(t)\}_n$ is equicontinuous. Indeed, in a similar way to (12.10) we have, for all n

$$\begin{aligned} |g_n(t) - g_n(s)| &\le \frac{a}{\sigma^2 - 2a} g_n(T) \int_s^t \frac{du}{(T - u)^\theta} \\ &\le \frac{a\sigma}{(\sigma^2 - 2a)(1 - \theta)} e^{\frac{a}{(\sigma^2 - 2a)(1-\theta)} T^{1-\theta}} \left| (T - s)^{1-\theta} - (T - t)^{1-\theta} \right|. \end{aligned}$$

Therefore, there exists a uniformly convergent subsequence $\{g_{n_k}(t)\}_k$, and $\bar{g}(t) := \lim_{k\to\infty} g_{n_k}(t)$ solves the equation (12.7) by using bounded convergence theorem. Furthermore $\bar{g}$ satisfies (12.16) and $\dot{\bar{g}}(t) \le \frac{a\bar{g}(t)}{(\sigma^2 - 2a)(T-t)^\theta}$ follows from the same arguments used in obtaining (12.10).

Proof of 2. All the necessary properties are stated in Theorem 6.1 therefore the result follows from Theorem 4.1.

Proof of 3. Note that we want to apply the result of Theorem 3.2. Therefore we obtain the following estimate.

$$E[|v(t,X_t,\alpha(t))|] \le K_T\left[\{|P(t)|+|Q(t)|+|L(t)|\}E[X_t^2] + \{|Q(t)|+|R(t)|+|M(t)|\}E[\alpha^2(t)]+|L(t)|+|M(t)|+|N(t)|\right], \tag{12.17}$$

$$E[\alpha^2(t)] \le \frac{1}{\sigma^2(T-t)^\theta}, \tag{12.18}$$

$$E[X_t^2] \le K_T\left\{1+\frac{1}{(T-t)^\theta}+(T-t)^{1-\theta}\right\}. \tag{12.19}$$

Moreover using Lemma 6.1, (6.13) and $|\dot{g}_1(t)| \le \frac{K_T}{(T-t)^\theta}$, we can observe the following:

$$|P(t)| \le K_T(T-t),\ |Q(t)| \le K_T(T-t),\ |L(t)| \le K_T(T-t), \tag{12.20}$$
$$|R(t)| \le K_T(T-t)^{2-\theta},\ |M(t)| \le K_T(T-t)^{2-\theta},\ |N(t)| \le K_T(T-t)^{1-\theta}.$$

From (12.17), (12.18), (12.19) and (12.20) we can get $E[|v(t,X_t,\alpha(t))|] \to 0$ as $t \to T$. Therefore the proof finishes using Theorem 3.2. □