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ESTIMATION OF DENSITIES FOR HESTON-TYPE MODELS THROUGH THE MALLIAVIN-THALMAIER METHOD AND ITS APPLICATION TO THE CALCULATION OF GREEKS

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Abstract. *In this paper, we estimate joint-density function of the stock price and its volatility for the Heston model and the double volatility Heston model through the Malliavin-Thalmaier formula. First, we give the representation of the joint density of the stock and its volatility. Next, we simulate these formulas on computer and compare them to other existing methods. We conclude from these experiments that our method has the smallest variance. Finally we apply the formula to the calculation of Greeks in finance.*
Keywords: Density estimation, Malliavin-Thalmaier formula, Greeks.

1. Introduction.

Financial systems require a careful continuous risk control in order to avoid undesirable results due to sudden large portfolio movements. One of the components in this problem is the risk control of option contracts.

From the mathematical point of view, the quantity that expresses this risk is the derivative of the price of the option price with respect to the parameters in the problem (these quantities are called "the Greeks" as many of them are denoted used capital Greek letters). In systems theory this problem falls in the area of sensitivity analysis.

In this article, we deal with such a problem for binary options which are certain particular type of option contracts for which is difficult to compute these sensitivity quantities. In a particular case, this problem is equivalent to the estimation of density functions for random variables arising from solutions of stochastic differential equations. For a detailed description, see Section 4.

In general, such differential equations are not explicitly solvable and therefore the need for Monte Carlo simulation arises. The theory in order to estimate density functions through the Malliavin calculus has been available since the eighties. See Nualart [11] (Proposition 2.1.5) or Sanz-Solé [12] (Proposition 5.4). These formulae were applied in finance in Fournié et. al. [4] for one dimensional financial models.

When this formula is used to simulate multidimensional density functions one finds a "curse of dimensionality" problem. In fact, the classical formulae that give simulatable expressions for the density involve multiple iterated Skorohod integrals. The Skorohod integral is an extension of the Ito's integral. Multiple stochastic integrals for fully multidimensional settings are costly to compute which lead to the above curse of dimensionality problem.

To solve this problem, the Malliavin-Thalmaier formula was introduced in Malliavin, Thalmaier [10] to simulate multidimensional density functions through the Monte-Carlo method. This method is based on an alternative expression for the delta (Schwarz) distribution function based on the Poisson kernel. Using this formula one can get away from the "curse of dimensionality", but the variance of this formula is infinite due to the degeneracy of the Poisson kernel.

The variance of the Malliavin-Thalmaier formula is infinite and therefore the formula is unstable for simulation. In order to be able to use this formula, Kohatsu-Higa, Yasuda [6] introduced an approximation of this formula in the spirit of the kernel density estimation method in non-parametric statistics.

In [6], we obtained the order of the bias, the L^2 error and a central limit theorem (CLT) for this approximation. And in Kohatsu-Higa, Yasuda [7], we gave a proposal of how to choose an approximate optimal size of the approximation parameter through the calculation of the leading constants in the error expansion. This is achieved using an appropriate CLT and a pilot simulation.

In the previous articles, "toy models" were used to demonstrate the method on a particular type of sensitivity quantity called Delta (the derivative with respect to the initial price). In this article, we give explicit formulas and carry out the experiments on models that are actually in use in the financial technology and we compute as well the formulae for other sensitivity quantities that are used in risk analysis. Furthermore we compare our methods with various other stochastic methods that could be used in order to carry the calculation of these sensitivities. We now a brief description of the contents of the article.

We estimate the joint-density function of the stock and volatility in the Heston model (see Heston [5]) and the double volatility Heston model (see Fonseca, Grasselli [3]), which are important stochastic volatility models currently in use as a financial model.

First we give an expression of these densities through the Malliavin-Thalmaier formula. In particular, we compute their Malliavin weights explicitly. Then we simulate the densities by using the optimal size of the approximation parameter and compare them to other methods. That is, we compare the Malliavin-Thalmaier formula without approximation, the kernel density estimation method (KDE) (see Scott [13]), which is a nonparametric method for density estimation in statistics using the Gaussian kernel, and the Laplacian of the Poisson kernel (see Kohatsu-Higa, Yasuda [8]).

Through our numerical results, we can confirm that our approximation formula has the smallest variance within these methods.

In [6], we also applied the Malliavin-Thalmaier formula to the calculation of Greeks and gave a new theoretical representation of Greeks, which are important risk indices in finance. Here we simulate Delta, Vega and Kappa of two-asset digital put option in the Black-Scholes model. We compare these results with the finite difference method and a method by Fournié et al. [4]. Then through numerical results, we find that our expression has the smallest variance in the three methods.

In physical systems, problems as the one described above are also common. These systems may be discrete or continuous, with a chaotic component described through an appropiate stochastic processes. In such a case density estimation is the central problem to be able to solve filtering problems and perform non-parametric estimation. For some examples, see e.g. [14], [15], [16].

In short, the article has the following sections: in Section 2, we will give the Malliavin-Thalmaier formula and its approximation. In Section 3, we will provide explicit expressions of the approximation to the Malliavin-Thalmaier formula in the case of the Heston and double volatility Heston model and also give their numerical results. In Section 4, we will deal with the calculation of Greeks.

2. **Malliavin-Thalmaier Representation of Multidimensional Density Functions and its Approximation.** Suppose that $F = (F_1, ..., F_d)$ is a nondegenerate random vector in $(\mathbb{D}^\infty)^d$ and G is a random variable in $\mathbb{D}^\infty$. Notations and terminologies can be found in Nualart [11].

Definition 2.1. *Given the $\mathbb{R}^d$-valued random vector F and the $\mathbb{R}$-valued random variable G, a multi-index $\alpha \in \bigcup_{j\in\mathbb{N}}\{1, ..., d^j$ and a power $p \geq 1$ we say that there is an integration by parts formula (IBP formula) in Malliavin sense if there exists a random variable $H_\alpha(F;G) \in L^p(\Omega)$ such that*

$$IP_{\alpha,p}(F,G): \ E\left[\frac{\partial^\alpha}{\partial \mathbf{x}^\alpha} f(F)G\right] = E\Big[f(F)H_\alpha(F;G)\Big] \ for \ all \ f \in C_0^{|\alpha|}(\mathbb{R}^d).$$

Here $|\alpha|$ denotes the length of the multi-index α.

Related to the Malliavin-Thalmaier formula [10], Bally and Caramellino [2], have obtained the following result.

Proposition 2.1. *(Bally, Caramellino [2]) Suppose that for some $p > 1$*

$$\sup_{|\mathbf{a}|\leq R} E\left[\left|\frac{\partial}{\partial x_i} Q_d(F-\mathbf{a})\right|^{\frac{p}{p-1}} + \left|Q_d(F-\mathbf{a})\right|^{\frac{p}{p-1}}\right] < \infty \ for \ all \ R > 0, \ \mathbf{a} \in \mathbb{R}^d.$$

If $IP_{i,p}(F;1)$ $(i = 1, ..., d)$ holds then the law of F is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^d$ and the density p_F is represented as

$$p_F(\hat{\mathbf{x}}) = E\left[\sum_{i=1}^d \frac{\partial}{\partial x_i} Q_d(F-\hat{\mathbf{x}})\, H_{(i)}(F;1)\right] \ for \ \hat{\mathbf{x}} \in \mathbb{R}^d.$$

Although the above formula makes sense as a duality in L^p, $p > 2$, one obtains that the variance of the proposed estimator is infinite. In fact, one can easily prove that

$$E\left[\left(\frac{\partial}{\partial x_i} Q_d(F-\hat{\mathbf{x}})\right)^2\right] = \infty.$$

In order to avoid the explosion of the variance of the Malliavin-Thalmaier estimator, we have proposed the use of a kernel density type alternative to this estimator. For this reason, we define

$$\frac{\partial}{\partial x_i} Q_d^h(\mathbf{x}) := A_d \frac{x_i}{|\mathbf{x}|_h^d}.$$

where $|\cdot|_h$ is defined as

$$|\mathbf{x}|_h := \sqrt{\sum_{i=1}^d x_i^2 + h} \qquad (h > 0, \ \mathbf{x} \in \mathbb{R}^d).$$

Then we define the approximation to the density function of F as; for $\hat{\mathbf{x}} \in \mathbb{R}^d$,

$$p_F^h(\hat{\mathbf{x}}) := E\left[\sum_{i=1}^d \frac{\partial}{\partial x_i} Q_d^h(F-\hat{\mathbf{x}})\, H_{(i)}(F;1)\right].$$

Note that clearly, $Q_d = Q_d^0$. The bias and L^2 errors together with the central limit theorem are given in [6].

The advantage of the above formulation in comparison with the classical results in Nualart [11] (Proposition 2.1.5) or Sanz-Solé [12] (Proposition 5.4) is that the above formulas require only one integration by parts. That is $H_{(i)}$ is only required while in the classical formulas H_α is required with $|\alpha| = d$. One could also avoid the approximation above by using the integration by parts formula twice which will lead to $H_{(i,j)}$. But as the objective here is to reduce the quantity of integration by parts, we find that this approach performs better.

3. **Numerical Results for the Heston-type Density.** In this section, we simulate the joint-density of the Heston model and the double volatility Heston model through the Malliavin-Thalmaier formula. In the following sections, we give expressions of Malliavin weights $H_{(i)}(F;G)$ without proof. These weights are calculated using arguments of Malliavin calculus. Note that one has to be careful about the non-differentiability of the square root function at zero. As these arguments are quite long to write in the case of the Heston model and the double volatility Heston model, we do not include them here. Nevertheless the computational complexity is unaffected by the length of these equations.

3.1. **The Heston Model.** In this section, we consider the joint-density function of the Heston model, which is the most popular stochastic volatility model in finance. The model is given as follows;

$$
\begin{aligned}
dS_t &= \mu S_t dt + \sqrt{v_t} S_t dW_t^{(1)}, \\
dv_t &= -\gamma(v_t - \theta)dt + \kappa\sqrt{v_t} dW_t^{(2)},
\end{aligned}
$$

where $\mu,\ \gamma,\ \theta,\ \kappa$ are constants with $2\gamma\theta \geq \kappa^2$ (see Lamberton, Lapeyre [9]) and $W_t^{(1)},\ W_t^{(2)}$ are Wiener processes with $E[W_t^{(1)} W_t^{(2)}] = \rho t$.

We introduce a new Wiener process Z_t, which is independent of $W_t^{(2)}$ and $W_t^{(1)} = \rho W_t^{(2)} + \sqrt{1-\rho^2} Z_t$. We also perform the following change of variables. Set $X_t := \ln(S_t/S_0) - \mu t$ and $u_t := a v_t$. Then from Itô's formula, we have the following dynamics;

$$
\begin{aligned}
dX_t &= -\frac{u_t}{2a}dt + \sqrt{\frac{u_t}{a}}\left\{\rho dW_t^{(2)} + \sqrt{1-\rho^2} dZ_t\right\}, \\
du_t &= -\gamma(u_t - a\theta)dt + \sqrt{a}\kappa\sqrt{u_t} dW_t^{(2)}.
\end{aligned}
\tag{1}
$$

As the exact value of the joint density value of (X_t, u_t) is unknown we will use the following Monte Carlo methods to estimate this value. We estimate this value using the Malliavin-Thalmaier formula (2) whose variance explodes, then its approximated version (3) and finally two KDE methods ((4) and (6)).

First we give the Malliavin-Thalmaier formula for the transformed Heston model (1).

Theorem 3.1. *Set $F := (F_1, F_2) := (X_t, u_t)$ for fixed $t > 0$. Assume that $E[\int_0^t u_s^{-3} ds] < \infty$. For $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$, we have*

$$
p_F(\mathbf{x}) = \frac{1}{2\pi} E\left[\sum_{i=1}^{2} \frac{F_i - x_i}{|F - \mathbf{x}|^2} H_{(i)}(F;1)\right], \tag{2}
$$

where

$$
\begin{aligned}
H_{(1)}(F;1) &:= \frac{\sqrt{a}}{\sqrt{1-\rho^2}t}\int_0^t \frac{1}{\sqrt{u_s}}dZ_s,\\
H_{(2)}(F;1) &:= \frac{1}{t}\left\{A-B\right\},\\
A &:= \frac{1}{\sqrt{a}\kappa e(t)}\int_0^t \frac{e(s)}{\sqrt{u_s}}dW_s^{(2)} + \frac{1}{2e(t)}\int_0^t \frac{e(s)}{u_s}ds + \frac{a\kappa^2}{8e(t)}\int_0^t s\frac{e(s)}{u_s^2}ds - \frac{\sqrt{a}\kappa}{4e(t)}\int_0^t s\frac{e(s)}{u_s^{\frac{3}{2}}}dW_s^{(2)},\\
B &:= \frac{\rho}{\kappa\sqrt{a(1-\rho^2)}e(t)}\int_0^t \frac{e(s)}{\sqrt{u_s}}dZ_s - \frac{1}{2\sqrt{a(1-\rho^2)}e(t)}\int_0^t e(r)\int_0^r \frac{1}{\sqrt{u_s}}dZ_s dr\\
&\quad + \frac{\rho}{2\sqrt{1-\rho^2}e(t)}\int_0^t \frac{e(r)}{\sqrt{u_r}}\int_0^r \frac{1}{\sqrt{u_s}}dZ_s dW_r^{(2)} + \frac{1}{2e(t)}\int_0^t \frac{e(r)}{\sqrt{u_r}}\int_0^r \frac{1}{\sqrt{u_s}}dZ_s dZ_r,\\
e(t) &:= \exp\left(-\gamma t - \frac{a\kappa^2}{8}\int_0^t \frac{1}{u_r}dr + \frac{\sqrt{a}\kappa}{2}\int_0^t \frac{1}{\sqrt{u_r}}dW_r^{(2)}\right).
\end{aligned}
$$

Sufficient conditions in order to obtain that $E[\int_0^t u_s^{-3}ds] < \infty$ are $2\gamma\theta > 3\kappa^2$, see e.g. Alós, Ewald [1].

Next, we describe the approximated version of the Malliavin-Thalmaier formula in the transformed Heston model (1).

Corollary 3.1. *Set $F := (F_1, F_2) := (X_t, u_t)$ for fixed $t > 0$. Assume that $E[\int_0^t u_s^{-3}ds] < \infty$. For $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$, we have*

$$p_F^h(\mathbf{x}) = \frac{1}{2\pi}E\left[\sum_{i=1}^2 \frac{F_i - x_i}{|F-\mathbf{x}|_h^2}H_{(i)}(F;1)\right], \tag{3}$$

where $H_{(i)}(F;1)$, $i = 1,2$ is the same as Theorem 3.1.

We compare the density value obtained from the above calculation with the KDE method. To implement this method we use the multidimensional Gaussian kernel with equal bandwidth sizes. That is, for $F := (F_1, \cdots, F_d)$ and $\mathbf{x} = (x_1, ..., x_d) \in \mathbb{R}^d$,

$$p_F(\mathbf{x}) \approx \frac{1}{N}\sum_{j=1}^N \frac{1}{h^d}\prod_{i=1}^d \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{(F_i^{(j)} - x_i)^2}{2h^2}\right), \tag{4}$$

where $F_i^{(j)}$, $i = 1, ..., d$, $j = 1, ..., N$ is a sequence of r.v.'s, independent copies of F_i.

We also can estimate the density function through the Laplacian of the Poisson kernel. That is, for $\hat{\mathbf{x}} \in \mathbb{R}^d$,

$$p_F(\hat{\mathbf{x}}) = E\left[\delta_{\mathbf{0}}(F - \hat{\mathbf{x}})\right] = E\left[\sum_{i=1}^d \frac{\partial^2}{\partial x_i^2}Q(F - \hat{\mathbf{x}})\right]. \tag{5}$$

If we simulate (5) directly, it is clear that the simulation will return either zero or an error. Therefore we introduce the following approximation of (5); for $h > 0$,

$$p_{Poi}^h(\hat{\mathbf{x}}) := E\left[\sum_{i=1}^d \frac{\partial^2}{\partial x_i^2}Q_d^h(F - \hat{\mathbf{x}})\right]. \tag{6}$$

TABLE 1. Parameters of the Heston model

parameter	notation	value
initial log stock price	S_0	100
(initial volatility)^2	v_0	0.1
scale parameter	a	3
expected return	μ	0.1
speed of mean reversion	γ	2
long term mean	θ	0.1
volatility of volatility process	κ	0.2
correlation	ρ	-0.8
maturity	t	1
time step size	Δt	1/50=0.02

3.2. **Numerical Results of the Heston Model.** In the numerical simulations, we have used the parameters for the Heston model that appear in Table 1. Next, in order to determine the value of h in (3) we use an optimal approximation-parameter size through a pilot simulation for the approximated Malliavin-Thalmaier formula. For more details, see Kohatsu-Higa, Yasuda [7]. Similarly, we use the same method to deduce the optimal bandwidth sizes for the KDE method (see Kohatsu-Higa, Yasuda [8]).

We simulate the following quantities; (i). the approximated Malliavin-Thalmaier formula with the optimal parameter size, (ii). the Malliavin-Thalmaier formula without approximation, (iii). the KDE method with the Gaussian kernel and the optimal bandwidth size, (iv). the KDE method on the Laplacian of the Poisson kernel with the optimal bandwidth size and (v,vi). results of 10^8-times Monte-Carlo simulation through the Malliavin-Thalmaier formula with and without approximation.

In Figure 1, we give the simulation results where the X-axis denotes the number of the Monte-Carlo simulation and the Y-axis its density value. We can observe that the KDE method with the Gaussian kernel swings wildly and the KDE method using the Laplacian of the Poisson kernel has large bias-error. The approximated Malliavin-Thalmaier formula is slightly stable, compared to the one without approximation.

Now we discuss the variance of the above methods in Figure 2. Obviously the KDE method has quite large variance (most of values are out of the frame). The KDE method using the Laplacian of the Poisson kernel and the Malliavin-Thalmaier formula without approximation have similar variance values, which are more than twice of the value of the associated variance for the approximated Malliavin-Thalmaier formula.

Note that the Malliavin-Thalmaier formula without approximation has some singular points even if the number of the Monte-Carlo simulation is big enough (around 8.5×10^5 times). So we can conclude that our proposal of approximation is better than the other proposals.

At first, it may seem odd that the KDE method does so poorly in this case, as its optimality is a well-known property in statistics. The main reason for this, is that the application of an integration by parts formula changes the order of degeneracy of the variance and therefore the KDE method becomes clearly suboptimal.

Remark 3.1. *Here we did not simulate the density through the classical method in the Malliavin calculus (see* [11] *or* [12]*). In Kohatsu-Higa, Yasuda* [7]*, we gave a numerical result of the joint-density in the case of two assets of the Black-Scholes model. Then we obtained that the variance of the classical method was much larger than one of the*

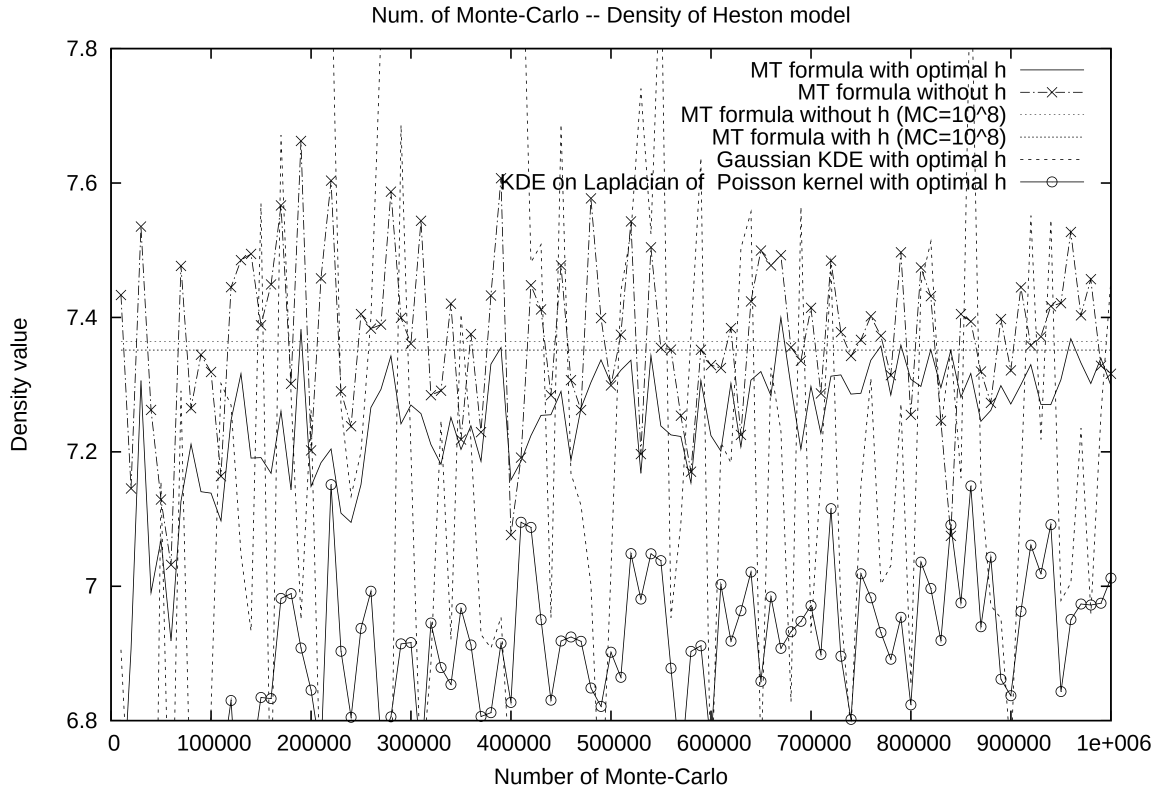

Figure 1. Density estimates for the Heston model

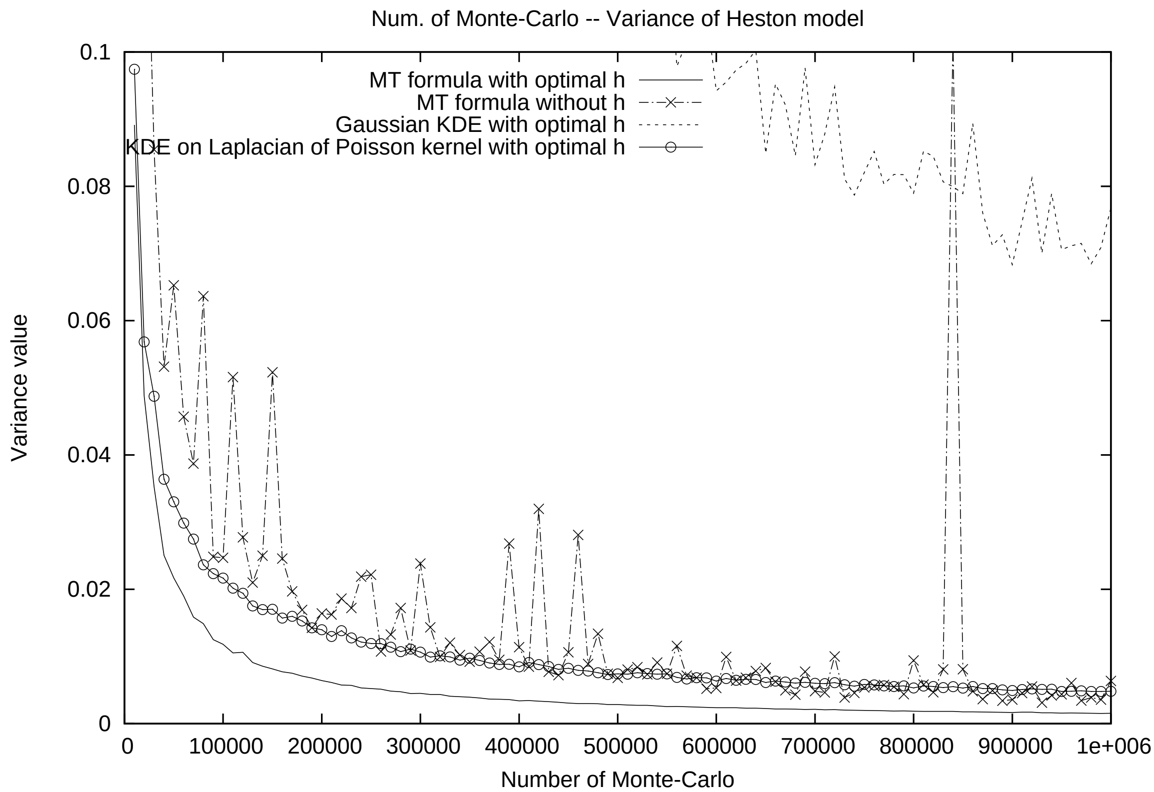

Figure 2. Variance of the density estimates for the Heston model

Malliavin-Thalmaier formula. The Heston model has higher complexity than the Black-Scholes model, so one can easily guess that the calculation will become extremely long and that same conclusion will probably follow. Furthermore, the classical formula suffers from the curse of dimensionality problem and therefore its use is limited. A similar comment can be also applied to the PDE method.

3.3. **The Double Volatility Heston Model.** First we define the double volatility Heston Model [3] as follows;

$$\begin{aligned} dS_t &= \mu S_t dt + \sqrt{v_t} S_t dW_t^1 + \sqrt{u_t} S_t dW_t^2, \\ dv_t &= \gamma(\theta - v_t)dt + \kappa\sqrt{v_t} dB_t^1, \\ du_t &= \alpha(\beta - u_t)dt + \tau\sqrt{u_t} dB_t^2, \end{aligned}$$

where $W_t^1, W_t^2, B_t^1, B_t^2$ are standard Wiener processes with $E[W_t^1 B_t^1] = \rho_1 t,\ E[W_t^2 B_t^2] = \rho_2 t\ (-1 \leq \rho_1, \rho_2 \leq 1)$ and $W^1 \sqcup W^2,\ W^1 \sqcup B^2,\ W^2 \sqcup B^1,\ B^1 \sqcup B^2$ and $\mu, \gamma, \theta, \kappa, \alpha, \beta, \tau$ are constants satisfying $\gamma\theta \geq \frac{\kappa^2}{2}$ and $\alpha\beta \geq \frac{\tau^2}{2}$ (see [9]). Here $\sqcup$ denotes independence.

Set Wiener processes Z_t^1 and Z_t^2; $W_t^1 = \rho_1 B_t^1 + \sqrt{1-\rho_1^2} Z_t^1,\ W_t^2 = \rho_2 B_t^2 + \sqrt{1-\rho_2^2} Z_t^2$, where $B^1 \sqcup Z^1,\ B^2 \sqcup Z^2,\ Z^1 \sqcup Z^2$. And set $X_t := \ln(S_t/S_0) - \mu t,\ V_t := a_1 v_t,\ U_t := a_2 u_t$, where a_1, a_2 are positive constants. Then we have

$$\begin{aligned} dX_t &= -\frac{1}{2}\left(\frac{V_t}{a_1} + \frac{U_t}{a_2}\right) dt + \frac{\rho_1}{\sqrt{a_1}}\sqrt{V_t} dB_t^1 + \frac{\sqrt{1-\rho_1^2}}{\sqrt{a_1}}\sqrt{V_t} dZ_t^1 \\ &\quad + \frac{\rho_2}{\sqrt{a_2}}\sqrt{U_t} dB_t^2 + \frac{\sqrt{1-\rho_2^2}}{\sqrt{a_2}}\sqrt{U_t} dZ_t^2, \\ dV_t &= \gamma\left(a_1\theta - V_t\right) dt + \sqrt{a_1}\kappa\sqrt{V_t} dB_t^1, \\ dU_t &= \alpha\left(a_2\beta - U_t\right) dt + \sqrt{a_2}\tau\sqrt{U_t} dB_t^2. \end{aligned}$$

Finally, we have the following joint-density expression of the transformed double volatility Heston model through the Malliavin-Thalmaier formula.

Theorem 3.2. *Set* $F := (F_1, F_2, F_3) := (X_t, V_t, U_t)$ *for fixed* $t > 0$. *Assume that* $E\left[\int_0^t U_s^{-3} + V_s^{-3} ds\right] < \infty$. *Then, for* $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$, *we have that*

$$p_F(\mathbf{x}) := \frac{1}{4\pi} E\left[\sum_{i=1}^{3} \frac{F_i - x_i}{|F - \mathbf{x}|^3} H_{(i)}(F; 1)\right],$$

where expressions for $H_{(i)}(F;1)$, $i = 1, 2, 3$, *are given in the Appendix.*

As in the Heston model, a sufficient condition of $E[\int_0^t U_s^{-3} + V_s^{-3} ds] < \infty$ is $2\gamma\theta > 3\kappa^2$ and $2\alpha\beta > 3\tau^2$.

Then the approximated version is as follows;

Corollary 3.2. *Set* $F := (F_1, F_2, F_3) := (X_t, V_t, U_t)$ *for fixed* $t > 0$. *Assume that* $E\left[\int_0^t U_s^{-3} + V_s^{-3} ds\right] < \infty$. *For* $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$,

$$p_F^h(\mathbf{x}) = \frac{1}{4\pi} E\left[\sum_{i=1}^{3} \frac{F_i - x_i}{|F - \mathbf{x}|_h^3} H_{(i)}(F; 1)\right],$$

where $H_{(i)}(F;1)$, $i = 1, 2, 3$, *is the same as Theorem 3.2.*

Table 2. Parameters of the double volatility Heston model

parameter	notation	value
correlations	(ρ_1, ρ_2)	(0.2,-0.15)
scale parameters	(a_1, a_2)	(1,1)
speed of mean reversion	(γ, α)	(2,1.5)
long term mean	(θ, β)	(0.2,0.15)
volatility of volatility process	(κ, τ)	(0.2,0.15)
initial value of volatility process	(V_0, U_0)	(0.2,0.15)
initial log stock price	X_0	100
maturity	t	1
time step size	Δt	1/200=0.005

3.4. **Numerical Results of the Double Volatility Heston Model.** We use the parameters for the double volatility Heston model in Table 2.

Here we simulate the double volatility Heston model by using the same methods as in the Heston model. Furthermore, we obtain the optimal approximation-parameter and bandwidth size through a pilot simulation.

Figure 3 gives a the simulation results. The KDE method with the Gaussian kernel moves wildly and the KDE method based on the Laplacian of the Poisson kernel has a large bias-error. The approximated Malliavin-Thalmaier formula is slightly more stable than the one without approximation.

Figure 4 shows the variance of the methods used in the simulation. The KDE method with the Gaussian kernel has a comparatively large variance and is mainly out of the frame of the graph. The KDE method on the Laplacian of the Poisson kernel and the Malliavin-Thalmaier formula without approximation have larger variances than one with approximation.

We also observe some points where the Malliavin-Thalmaier formula without approximation spikes due to the large variance. Clearly, the approximated version improves this aspect and has stability.

4. **Greeks Calculation.** In this section, we apply the Malliavin-Thalmaier formula to calculate Greeks in finance. A Greek is a sensitivity of an option price and an important risk index for derivative traders in banks and security companies.

4.1. **Settings and Expressions of Greeks.** First we give the 2-dimensional Black-Scholes model,

$$\begin{aligned} dS_t^{(1)} &= \mu_1 S_t^{(1)} dt + v_1 S_t^{(1)} dW_t^{(1)}, \\ dS_t^{(2)} &= \mu_2 S_t^{(2)} dt + v_2 S_t^{(2)} \left(\rho dW_t^{(1)} + \sqrt{1-\rho^2} dW_t^{(2)} \right), \end{aligned}$$

where $W_t^{(1)}$ and $W_t^{(2)}$ are independent Wiener processes, and $\mu_1, \mu_2, v_1, v_2, \rho$ are constants.

Now we consider the following Digital put option;

$$p\left(S_0^{(1)}, S_0^{(2)}, v_1, v_2, \rho\right) := E^Q\left[e^{-rt} \mathbf{1}\left(S_t^{(1)} \le K_1\right) \mathbf{1}\left(S_t^{(2)} \le K_2\right)\right], \tag{7}$$

where K_1, K_2, r are positive constants and E^Q is an expectation with respect to a risk neutral measure Q. Without loss of generality, we assume $r = 0$. We calculate Delta which is a sensitivity w.r.t. $S_t^{(1)}$, Vega which is a sensitivity w.r.t. v_1 and Kappa which is

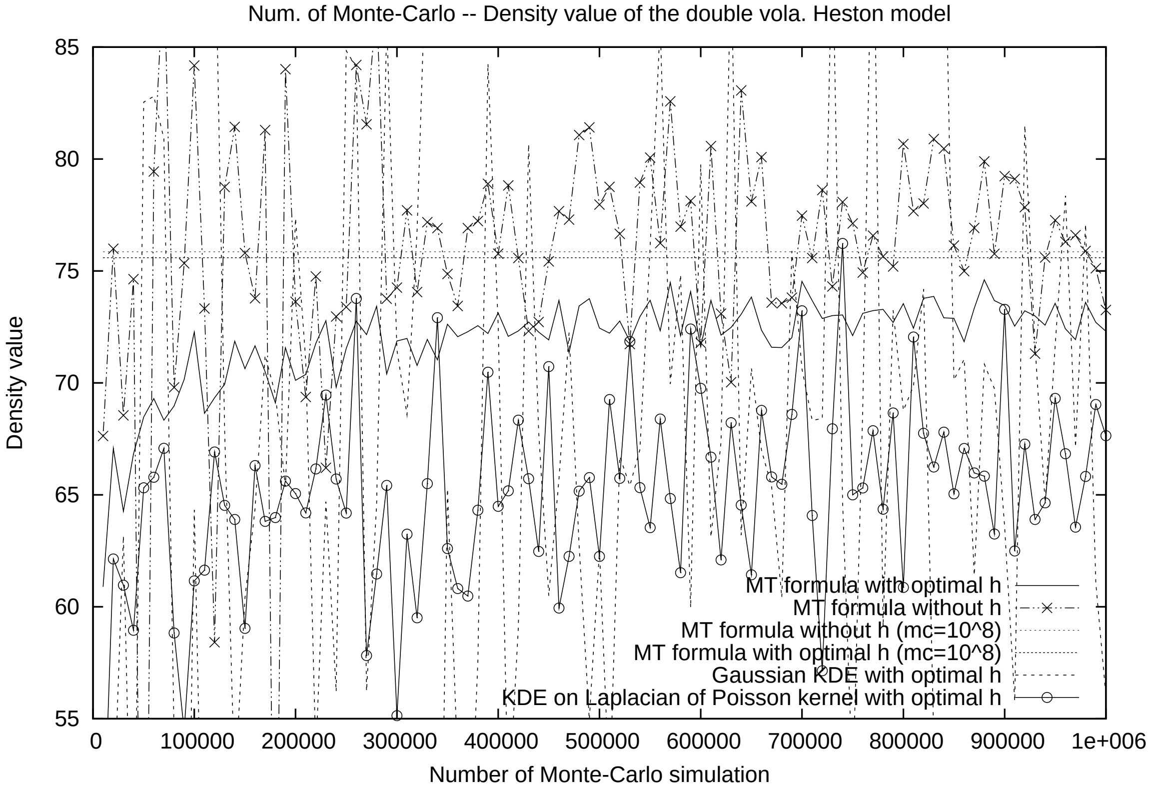

FIGURE 3. Density estimates for double volatility Heston model

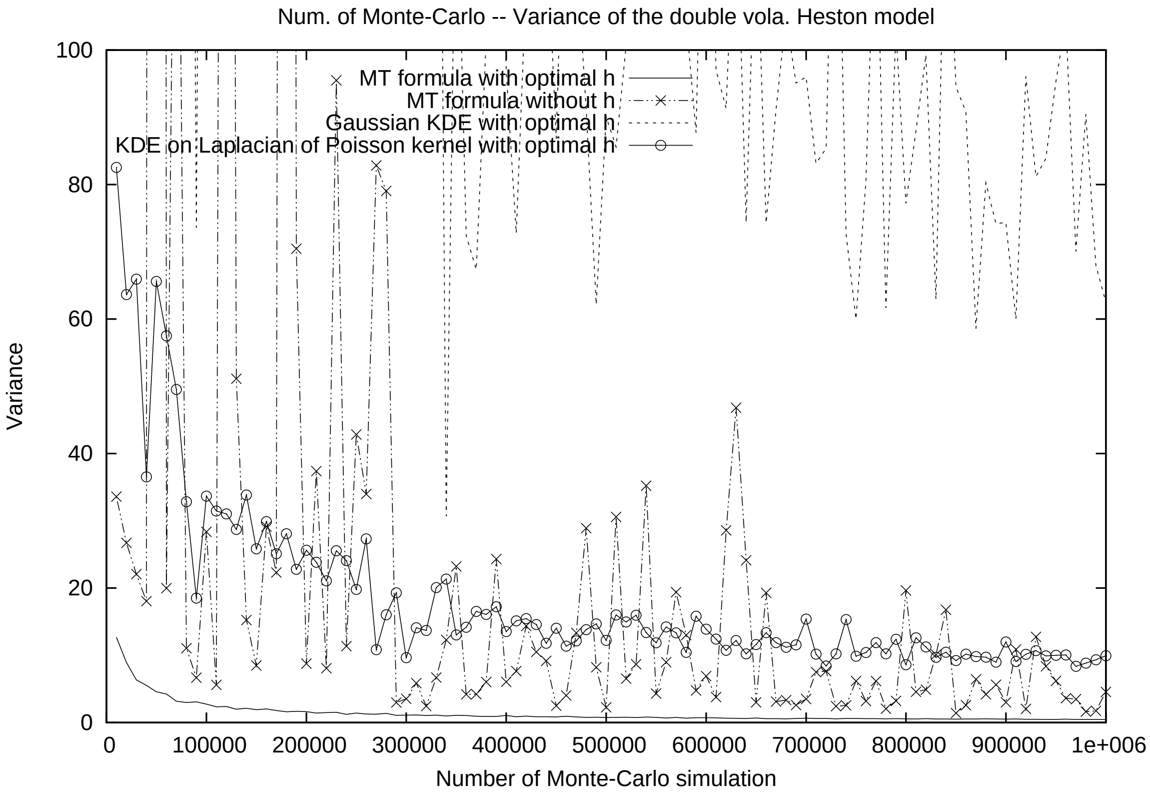

FIGURE 4. Variance of the density estimates for double volatility Heston model

a sensitivity w.r.t. ρ, of (7), that is,

$$\frac{dp}{dS_0^{(1)}},\ \frac{dp}{dv_1} \text{ and } \frac{dp}{d\rho}.$$

Kohatsu-Higa, Yasuda [6] gave a new expression of Greeks by using the Malliavin-Thalmaier formula. In the present case, (7), we have the following expressions

Theorem 4.1. *For $x, y \in \mathbb{R}$, set*

$$g_{1,1}(x,y) := \frac{1}{2\pi}\left\{\arctan\frac{y}{x} - \arctan\frac{y-K_2}{x} - \arctan\frac{y}{x-K_1} + \arctan\frac{y-K_2}{x-K_1}\right\},$$

$$g_{1,2}(x,y) = g_{2,1}(x,y) := \frac{1}{4\pi}\ln\left(\frac{(x^2+y^2)((x-K_1)^2+(y-K_2)^2)}{((x-K_1)^2+y^2)(x^2+(y-K_2)^2)}\right),$$

$$g_{2,2}(x,y) := \frac{1}{2\pi}\left\{\arctan\frac{x}{y} - \arctan\frac{x-K_1}{y} - \arctan\frac{x}{y-K_2} + \arctan\frac{x-K_1}{y-K_2}\right\}.$$

(i). *The Delta in the first stock $S_t^{(1)}$ is given as follows;*

$$\begin{aligned}\frac{dp}{dS_0^{(1)}} &= E^Q\left[g_{1,1}\left(S_t^{(1)}, S_t^{(2)}\right) H_{(1)}\left(S_t^{(1)}, S_t^{(2)}; \frac{\partial S_t^{(1)}}{\partial S_0^{(1)}}\right)\right] \\ &\quad + E^Q\left[g_{2,1}\left(S_t^{(1)}, S_t^{(2)}\right) H_{(2)}\left(S_t^{(1)}, S_t^{(2)}; \frac{\partial S_t^{(1)}}{\partial S_0^{(1)}}\right)\right],\end{aligned}$$

where set

$$\begin{aligned}H_{(1)}\left(S_t^{(1)}, S_t^{(2)}; \frac{\partial S_t^{(1)}}{\partial S_0^{(1)}}\right) &:= \frac{W_t^{(1)}}{v_1 t S_0^{(1)}} - \frac{\rho W_t^{(2)}}{\sqrt{1-\rho^2} v_1 t S_0^{(1)}}, \\ H_{(2)}\left(S_t^{(1)}, S_t^{(2)}; \frac{\partial S_t^{(1)}}{\partial S_0^{(1)}}\right) &:= \frac{S_t^{(1)}}{S_0^{(1)} S_t^{(2)}}\left(\frac{W_t^{(2)}}{\sqrt{1-\rho^2} v_2 t} + 1\right).\end{aligned}$$

(ii). *The Vega in the first volatility v_1 is given as follows;*

$$\begin{aligned}\frac{dp}{dv_1} &= E^Q\left[g_{1,1}\left(S_t^{(1)}, S_t^{(2)}\right) H_{(1)}\left(S_t^{(1)}, S_t^{(2)}; \frac{\partial S_t^{(1)}}{\partial v_1}\right)\right] \\ &\quad + E^Q\left[g_{2,1}\left(S_t^{(1)}, S_t^{(2)}\right) H_{(2)}\left(S_t^{(1)}, S_t^{(2)}; \frac{\partial S_t^{(1)}}{\partial v_1}\right)\right],\end{aligned}$$

where

$$\begin{aligned}H_{(1)}\left(S_t^{(1)}, S_t^{(2)}; \frac{\partial S_t^{(1)}}{\partial v_1}\right) &:= \frac{(W_t^{(1)})^2}{v_1 t} - \frac{1}{v_1} - W_t^{(1)} - \frac{\rho}{\sqrt{1-\rho^2} v_1 t}\left(W_t^{(1)} - v_1 t\right) W_t^{(2)}, \\ H_{(2)}\left(S_t^{(1)}, S_t^{(2)}; \frac{\partial S_t^{(1)}}{\partial v_1}\right) &:= \frac{S_t^{(1)}}{\sqrt{1-\rho^2} v_2 t S_t^{(2)}}\left(W_t^{(1)} - v_1 t\right)\left(W_t^{(2)} + \sqrt{1-\rho^2} v_2 t\right).\end{aligned}$$

(iii). *The Kappa is given as follows;*

$$\begin{aligned}\frac{dp}{d\rho} &= E^Q\left[g_{1,2}\left(S_t^{(1)}, S_t^{(2)}\right) H_{(1)}\left(S_t^{(1)}, S_t^{(2)}; \frac{\partial S_t^{(1)}}{\partial \rho}\right)\right] \\ &\quad + E^Q\left[g_{2,2}\left(S_t^{(1)}, S_t^{(2)}\right) H_{(1)}\left(S_t^{(1)}, S_t^{(2)}; \frac{\partial S_t^{(1)}}{\partial \rho}\right)\right],\end{aligned}$$

where

$$H_{(1)}\left(S_t^{(1)}, S_t^{(2)}; \frac{\partial S_t^{(1)}}{\partial \rho}\right)$$

$$:= \frac{v_2 S_t^{(2)}}{v_1 t S_t^{(1)}} \left\{ \left(W_t^{(1)}\right)^2 - \rho v_2 t W_t^{(1)} - t + v_1 t W_t^{(1)} - \frac{\rho W_t^{(2)}}{\sqrt{1-\rho^2}} \left[W_t^{(1)} - \rho v_2 t + v_1 t\right] \right.$$

$$\left. - \frac{\rho}{\sqrt{1-\rho^2}} \left(W_t^{(1)} W_t^{(2)} - \sqrt{1-\rho^2} v_2 t W_t^{(1)} - \frac{\rho (W_t^{(2)})^2}{\sqrt{1-\rho^2}} + \rho v_2 t W_t^{(2)} + \frac{\rho t}{\sqrt{1-\rho^2}} \right) \right\},$$

$$H_{(2)}\left(S_t^{(1)}, S_t^{(2)}; \frac{\partial S_t^{(1)}}{\partial \rho}\right) := \frac{W_t^{(1)} W_t^{(2)}}{\sqrt{1-\rho^2} t} - \frac{\rho}{t(1-\rho^2)} \left(\left(W_t^{(2)}\right)^2 - t \right).$$

Proof: We can derive these Malliavin weights through arguments using Malliavin calculus. □

4.2. **Numerical Results of Calculation of Greeks.** We use the parameters that appear in Figure 5. Here we compare the results of Theorem 4.1 with the finite central difference (FD) method and the classical method by Forunié et al. [4]. Then we give simulation results of Delta w.r.t. the first stock $S_t^{(1)}$, Vega w.r.t. the first volatility v_1 and Kappa of the above option.

In Figure 6, we compare Delta and Variance computed using these three methods. The Delta value computed with any of the above three methods look similar at a first glance. But once we look at the Variance, we find that variance of the FD method is much larger than the other methods and the variance of the method by Fournié et al. is more than triple the one by the Malliavin-Thalmaier formula method.

We compare Vega and Variance in Figure 7 and Kappa and Variance in Figure 8 using the above three methods. Then their Delta results are close to each other. In Figure 8, variance of the FD method is extremely large and completely out of the frame.

5. **Conclusions.** In this paper, we gave expressions that allow Monte Carlo simulation of the joint-density of the stock price and its volatility in the Heston and the double volatility Heston model by using the Malliavin-Thalmaier formula. Then through numerical experiments, we found that the approximated Malliavin-Thalmaier formula, which had been introduced in Kohatsu-Higa, Yasuda [6], have the smallest variance in comparison with other methods applied in the financial industry. Next we applied the Malliavin-Thalmaier formula for the calculation of Greeks. We found that this new method has smaller variance than the method by Fournié et al. [4].

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Figure 5. Parameters of the Greeks calculation

parameter	value
$S_0^{(2)}$	100
(μ_1, μ_2)	(0,0)
v_2	0.2
(K_1, K_2)	(100,100)
t	1
Δt	1/8=0.125
N	10^6

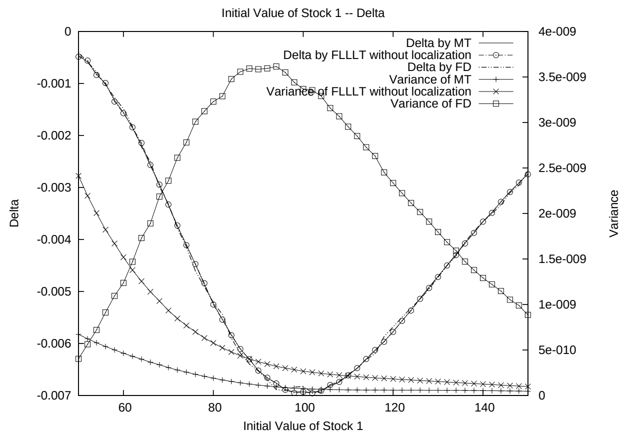

Figure 6. Initial Value of Stock 1 – Delta value (left-axis) and Variance (right-axis) ($S_0^{(1)} \in [50, 150]$, $v_1 = 0.3$, $\rho = 0.2$; parameter size of FD is $0.01 \times S_0^{(1)}$)

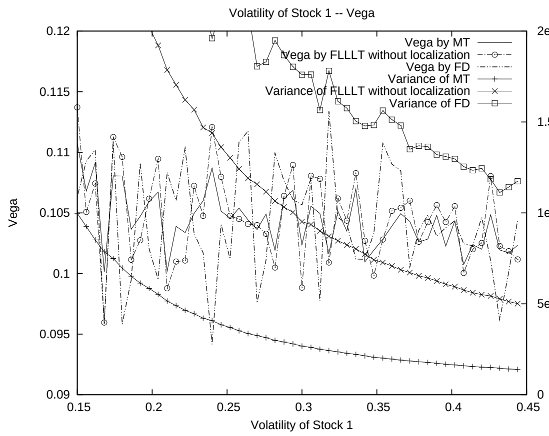

Figure 7. Volatility of Stock 1 – Vega value (left-axis) and Variance (right-axis) ($S_0^{(1)} = 100$, $v_1 \in [0.15, 0.45]$, $\rho = 0.2$; parameter size of FD is $0.01 \times v_1$)

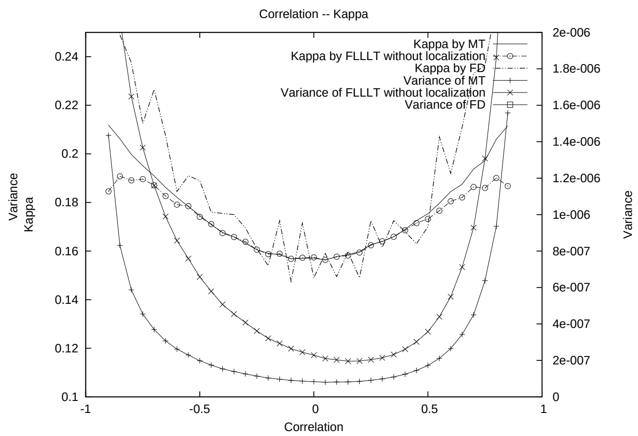

Figure 8. Correlation – Kappa value (left-axis) and Variance (right-axis) ($S_0^{(1)} = 100$, $v_1 = 0.3$, $\rho \in [-0.9, 0.9]$; parameter size of FD is 0.001)

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Appendix. Here we give explicit expressions for $H_{(i)}(F;1)$, $i = 1,2,3$, in the double volatility Heston model. For $j = 1,2$,

$$
\begin{aligned}
H_{(1)}(F;1) &:= \frac{\sqrt{a_1 a_2}}{t}\int_0^t \frac{dZ_s^1 + dZ_s^2}{\sqrt{a_2(1-\rho_1^2)V_s} + \sqrt{a_1(1-\rho_2^2)U_s}},\\
H_{(2)}(F;1) &:= A - B_1 - B_2,\\
A &:= \frac{1}{e_V(t)t}\left\{\frac{1}{\sqrt{a_1}\kappa}\int_0^t \frac{e_V(s)}{\sqrt{V_s}}dB_s^1 + \frac{1}{2}\int_0^t \frac{e_V(s)}{V_s}ds\right.\\
&\qquad \left. + \frac{a_1\kappa^2}{8}\int_0^t s\frac{e_V(s)}{V_s^2}ds - \frac{\sqrt{a_1}\kappa}{4}\int_0^t s\frac{e_V(s)}{V_s^{\frac{3}{2}}}dB_s^1\right\},\\
B_j &:= \frac{\sqrt{a_2}}{e_V(t)t}\left\{\frac{\rho_1}{\sqrt{a_1}\kappa}\int_0^t \frac{e_V(s)dZ_s^j}{\sqrt{a_2(1-\rho_1^2)V_s} + \sqrt{a_1(1-\rho_2^2)U_s}}\right.\\
&\qquad - \frac{1}{2\sqrt{a_1}}\int_0^t e_V(r)\int_0^r \frac{dZ_s^j dr}{\sqrt{a_2(1-\rho_1^2)V_s} + \sqrt{a_1(1-\rho_2^2)U_s}}\\
&\qquad \left. + \frac{1}{2}\int_0^t \frac{e_V(r)}{\sqrt{V_r}}\int_0^r \frac{dZ_s^j(\rho_1 dB_r^1 + \sqrt{1-\rho_1^2}dZ_r^1)}{\sqrt{a_2(1-\rho_1^2)V_s} + \sqrt{a_1(1-\rho_2^2)U_s}}\right\},\\
e_V(t) &:= \exp\left(-\gamma t - \frac{a_1\kappa^2}{8}\int_0^t \frac{1}{V_s}ds + \frac{\sqrt{a_1}\kappa}{2}\int_0^t \frac{1}{\sqrt{V_s}}dB_s^1\right),\\
H_{(3)}(F;1) &:= C - D_1 - D_2,\\
C &:= \frac{1}{e_U(t)t}\left\{\frac{1}{\sqrt{a_2}\tau}\int_0^t \frac{e_U(s)}{\sqrt{U_s}}dB_s^2 + \frac{1}{2}\int_0^t \frac{e_U(s)}{U_s}ds\right.\\
&\qquad \left. + \frac{a_2\tau^2}{8}\int_0^t s\frac{e_U(s)}{U_s^2}ds - \frac{\sqrt{a_2}\tau}{4}\int_0^t s\frac{e_U(s)}{U_2^{\frac{3}{2}}}dB_s^2\right\},\\
D_j &:= \frac{\sqrt{a_1}}{e_U(t)t}\left\{\frac{\rho_2}{\sqrt{a_2}}\int_0^t \frac{e_U(s)dZ_s^j}{\sqrt{a_2(1-\rho_1^2)V_s} + \sqrt{a_1(1-\rho_2^2)U_s}}\right.\\
&\qquad - \frac{1}{2\sqrt{a_2}}\int_0^t e_U(r)\int_0^r \frac{dZ_s^j dr}{\sqrt{a_2(1-\rho_1^2)V_s} + \sqrt{a_1(1-\rho_2^2)U_s}}\\
&\qquad \left. + \frac{1}{2}\int_0^t \frac{e_U(r)}{\sqrt{U_r}}\int_0^r \frac{dZ_s^j(\rho_2 dB_r^2 + dZ_r^2)}{\sqrt{a_2(1-\rho_1^2)V_s} + \sqrt{a_1(1-\rho_2^2)U_s}}\right\},\\
e_U(t) &:= \exp\left(-\alpha t - \frac{a_2\tau^2}{8}\int_0^t \frac{1}{U_s}ds + \frac{\sqrt{a_2}\tau}{2}\int_0^t \frac{1}{\sqrt{U_s}}dB_s^2\right).
\end{aligned}
$$