Two examples of an insider with medium/long term effects on the underlying*

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In a recent article [4], we have developed a market model where an insider trades using future information on the value of the underlying, through these trades it creates an effect on the drift of the underlying model. We find points of partial equilibria. That is, when the filtration is fixed the chosen portfolio is optimal, leads to finite utility of the insider and prices are semimartingales in their own filtration. In this article, we treat two explicit examples in detail. The first is an insider which has a medium term effect on the price. The second is an insider which has a long term effect on the price with memory effects. These examples were quoted in [4] but no details were given.

Key words: Insider trading, Enlargement of filtration, large trading models, equilibrium, long/medium term effects

1. Introduction

In this article, we consider the following set-up. Let $(\Omega, \mathcal{F}, P; (\mathcal{F}_t)_{t\geq 0})$ to be a complete filtered probability space with the augmented Brownian filtration $(\mathcal{F}_t)_{t\geq 0}$ generated by the 1-dimensional Brownian motion W. The risk-free investment, $S^0(t)$ is given by

$$dS^0(t) = rS^0(t)dt, \quad S^0(0) > 0$$

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and the risky investment, whose price $S_t \equiv S^{\pi}_t$ is described by

$$dS(t) = S(t)\left\{\left(\mu + a\pi(t) - b\int_0^t H(t,u)\pi(u)du\right)dt + \sigma dW(t)\right\},\ S_0 > 0. \tag{1.1}$$

Here $r > 0$ is the interest rate, μ is the return rate for the stock when there is no insider effect, $a,\ b \in \mathbb{R}$ and $H : [0,T]^2 \to \mathbb{R}_+$ are the parameters that characterize the effect of the insider on stock prices. $\pi := (\pi(t))_{t\geq 0}$ is the proportion of wealth that the insider has invested in the risky asset, an element of the set of admissible strategies

$$\mathcal{A}_{\mathcal{S}^\pi} := \left\{(\pi(t))_{t\geq 0}; (\pi(t))_{t\in[0,T]} \in \mathcal{L}^2_T\right\}, \tag{1.2}$$

where $\mathcal{L}^2_T$ is the totality of $\mathcal{S}^\pi$-progressively measurable processes π on the time interval $[0,T]$ such that $E\left[\int_0^T \pi(t)^2 dt\right] < \infty$. Here,

$$\mathcal{S}^\pi_t = \sigma(S^\pi(s);\ s \leq t) \vee \sigma\left(\ln(S^\pi(T)) + \sigma W'((T-s)^\theta);\ s \leq t\right), \tag{1.3}$$

where W' is a 1-dimensional Brownian motion independent of W and $\theta > 0$. W' is a dynamic perturbation of the information which states that the future information possessed by the insider is uncertain. Furthermore, $(T-s)^\theta$, characterizes this uncertainty. In fact, when θ is close to zero the information of the insider is more blurred by noise. This characterizes the fact that the information of the insider is dynamic and changes through time.

Several important comments are due at this point. First, it is natural to consider that proportion of the insider wealth should be the determinant variable and not money amount or number of stock shares held by the insider. Clearly, if an insider holds 70% of his wealth on a stock is much more important to know than if he holds 1 billion yen on the company stock shares. This is an important difference with large traders who by definition do not have inside information on the company. This is further enhanced by the empirical literature on the issue of insider effects. We give various references in the bibliography section.

a and b are parameters that express the strength of the influence of the insider on the underlying. H represents a weighting average function of the past wealth proportions of the agent. Taking the differences between the current wealth proportion with a weighted average of the past strategy allows to weaken the long term dependence of the price drift on the strategies of the insider.

The definition of long term and medium term is ad-hoc in this paper. We say that the effect is long term if $b = 0$ as small changes in the portfolio values will create a change on all values of the integrals $a\int S(s)\pi(s)ds$ for a fixed process S. The medium term effect is the fact that the change in the previous integrals is diminished by taking the difference with previous averaged values. The short term effect in this context would mean that the stock prices jumps immediately after a change in the investment portfolio of the insider and no other after effect is seen afterwards. Here also these effects are in concordance with what is seen in the empirical literature of insider trading. In fact, most of the insider trading is driven by long and medium term effects rather than short term effects. This is a clear difference between lawful and unlawful insiders.

A final mathematical remark is necessary here. It is clear that the set-up (1.1) is not well posed. In fact, we demand that portfolios π should be adapted to $\mathcal{S}^{\pi}$. These portfolios change also the values of the processes $S \equiv S^{\pi}$. Therefore part of the mathematical problem is to give meaning to the stochastic integrals appearing in (1.1).

Assuming that W is a semimartingale in the filtration $\mathcal{S}^{\pi}$ we consider the discounted wealth process $\widehat{V}^{\pi}(t)$ of the insider using a self-financing portfolio, which starts with the initial capital 1 and invests the proportion $\pi(t)$ of his wealth at time t in the risky asset, which is defined by

$$d\widehat{V}^{\pi}(t) = \pi(t)\widehat{V}^{\pi}(t)\widehat{S}^{-1}(t)d\widehat{S}(t), \ \widehat{V}^{\pi}(0) = 1. \tag{1.4}$$

Here $\widehat{S}$ denotes the discounted stock price defined by $\widehat{S}(t) = \mathrm{e}^{-rt}S(t)$. The explicit solution of equation (1.4) is given by

$$\widehat{V}^{\pi}(t) = \exp\left[\int_0^t \left\{\pi(u)\left(\mu - r + a\pi(u) - b\int_0^u H(u,v)\pi(v)dv\right)\right.\right. \\ \left.\left. -\frac{1}{2}\sigma^2\pi(u)^2\right\} du + \int_0^t \pi(u)\sigma dW(u)\right]. \tag{1.5}$$

With this setting, the main aim in [4] was to show that there exists a portfolio π^* satisfying the following properties. A portfolio π^* satisfying these properties is called a partial equilibrium.

1. The portfolio π^* leads to a filtration $\mathcal{S}_t^{\pi^*}$ under which W is a semimartingale and therefore the model (1.1) and the wealth equation, (1.4), makes sense as a linear stochastic differential equation driven by a $\mathcal{S}_t^{\pi^*}$-semimartingale.

2. If the filtration $\mathcal{S}_t^{\pi^*}$ is fixed (and therefore the resulting price process S is well defined) then $\pi = \pi^* \in \mathcal{A}_{\mathcal{S}^{\pi^*}}$ is the optimal portfolio of the maximization with the logarithmic expected utility function:

$$\sup_{\pi\in\mathcal{A}_{\mathcal{S}^{\pi^*}}} E\left[\log\left(\widehat{V}^{\pi}(T)\right)\right]. \tag{1.6}$$

3. The logarithmic utility obtained using π^* is finite.

Partial equilibrium means the following. Supposing that the insider has the above information and that there is an extra cost/risk for changing strategies (or that he is only able to carry out local optimization procedures), he will not have much of a technical reason to change strategies although there may be other portfolios which perform better than the one above.

With this setting, Hata-Kohatsu [4] proves general theorems about the existence and characterization of partial equilibria. In order to prove how to apply the general results we consider some particular examples. In particular, we make the following assumption

ASSUMPTION (A): $H(t,u) = h_1(u)h_2(t)^{-1}$, $h_1(t), h_2(t) > 0$ for all $t \in [0,T]$ and g_1, g_2, h_1 and $h_2 \in C^1([0,T))$.

Moreover [4] treats the following explicit examples:

(a) Market model with insider long term effects on the price:

$$a > 0, \quad b = 0, \quad h_1(t) \equiv 1, \quad \text{and,} \quad h_2(t) \equiv 1. \tag{1.7}$$

(b) Market model with insider medium term effects on the price (weighted average):

$$a > 0, \quad b > 0, \quad h_1(t) = \mathrm{e}^{\frac{b}{a}t}, and \quad h_2(t) = \mathrm{e}^{\frac{b}{a}t}.$$

In this paper, we treat the following examples, which are related to (a) and (b):

(c) Market model with insider medium term effects on the price (unweighted average):

$$a > 0, \quad b > 0, \quad h_1(t) \equiv 1, \quad \text{and,} \quad h_2(t) \equiv 1. \tag{1.8}$$

(d) Market model with insider long term effects on the price (with memory):

$$a = 0, \quad b < 0, \quad h_1(t) \equiv 1, \quad \text{and,} \quad h_2(t) \equiv 1. \tag{1.9}$$

2. Optimal portfolios with fixed enlarged filtration

2.1 Enlargement of filtrations

In order to prove the three properties of partial equilibrium, [4] concentrates in a particular type of enlargement of filtration problem and treats the case where the enlarged filtration $\mathcal{G}$ is defined by

$$\begin{aligned} \mathcal{G}_t &= \mathcal{F}_t \vee \sigma\left(W'(u) - W'\left((T-t)^\theta\right); u \in [(T-t)^\theta, T^\theta]\right) \\ &\vee \sigma\left(\int_t^T g_1(s)dW(s) + \int_0^{(T-t)^\theta} g_2(s)dW'(s)\right) \\ &= \mathcal{F}_t \vee \sigma(I(s);\ s \le t), \end{aligned} \tag{2.1}$$

where

$$I(t) = \int_0^T g_1(s)dW(s) + \int_0^{(T-t)^\theta} g_2(s)dW'(s),$$

for two deterministic positive functions g_1, $g_2 \in L^2[0,T]$ and W' another one dimensional Brownian motion independent of W. We define $G(t) := \int_t^T g_1(s)^2 ds +$

$\int_0^{(T-t)^\theta} g_2(s)^2 ds > 0$ for all $t \in [0,T]$. **This hypothesis is in force for all the article**. Note that $\mathcal{G}$ will eventually become the filtration of the insider (i.e. (1.3)) when the partial equlibrium is achieved. This construction has a specific financial meaning which can be understood by reading the proof of Theorem 2.4 in [4].

In particular, we can say that g_1 models the volatility of the final value of the stock price and g_2 models the noise in the insider information.

The compensator of W under $\mathcal{G}$ is obtained in [4].

Proposition 2.1. *([4] Proposition* 2.1*) Suppose that* $\int_0^T \frac{|g_1(s)|}{\sqrt{G(s)}} ds < \infty$*, then* $\{W(t); t \in [0,T]\}$ *is a semimartingale in the filtration* $\mathcal{G}$ *and the semimartingale decomposition of* W *is given by*

$$W(t) = \hat{W}(t) + \int_0^t g_1(s)\alpha(s)ds, \tag{2.2}$$

$$\alpha(s) = \frac{1}{G(s)}\left(\int_s^T g_1(u)dW(u) + \int_0^{(T-s)^\theta} g_2(u)dW'(u)\right); \; s < T, \tag{2.3}$$

where $\{\hat{W}(t); t \in [0,T]\}$ *is a* $\mathcal{G}$*-Wiener process and* α *is a Gaussian* $\mathcal{G}$*-martingale in* $[0,T)$ *with quadratic variation given by*

$$\langle\alpha\rangle_t = \int_0^t \left(\frac{g_1(s)}{G(s)}\right)^2 ds + \int_{(T-t)^\theta}^{T^\theta} \left(\frac{g_2(s)}{G(s)}\right)^2 ds.$$

2.2 Existence and uniqueness of optimal portfolios for fixed enlarged filtrations

Our objective in this subsection is to consider the model

$$dS(t) = S(t)\left\{\left(\mu + a\pi(t) - b\int_0^t H(t,u)\pi(u)du\right)dt + \sigma dW(t)\right\}, \; S_0 > 0.$$

Here $\pi := (\pi(t))_{t\geq 0}$ is the proportion of wealth invested in the risky asset, an element of the set of admissible strategies

$$\mathcal{A}_{\mathcal{G}} := \left\{(\pi(t))_{t\geq 0}; (\pi(t))_{t\in[0,T]} \in \mathcal{L}(\mathcal{G})_T^2\right\},$$

where $\mathcal{L}(\mathcal{G})_T^2$ is the totality of $(\mathcal{G}_t)$-progressively measurable processes π on the time interval $[0,T]$ such that $E\left[\int_0^T \pi(t)^2 dt\right] < \infty$. Throughout this subsection $\mathcal{G}$ is a fixed filtration satisfying $\mathcal{G} \supseteq \mathcal{F}$.

We consider the maximization of the logarithmic expected utility function

$$\Psi(T) = \sup_{\pi\in\mathcal{A}_{\mathcal{G}}} J(\pi), \tag{2.4}$$

$$J(\pi) := E\left[\log\left(\widehat{V}^\pi(T)\right)\right].$$

where $\widehat{V}^{\pi}$ satisfies (1.4) and is explicitly given by (1.5). Then the optimal equation for the fixed filtration $\mathcal{G}$ is known.

Theorem 2.1. ([4] *Theorem* 3.1) *Consider, in general, a filtration* $\mathcal{G} \supseteq \mathcal{F}$ *such that* $(W, \mathcal{G})$ *is a semimartingale with the decomposition* (2.2)*, where* α *is a* $\mathcal{G}$ *adapted process and* g_1 *is a deterministic function such that* $E\left[\int_0^T |g_1(s)\alpha(s)|^2 ds\right] < \infty$*. Assume that*

$$(2.5) \qquad 0 \le a < \frac{\sigma^2 - 2|b|K(T)}{2} \text{ for } K(T) := \left(\int_0^T \int_0^t H^2(t,u)dudt\right)^{1/2}.$$

If $\pi^* \in \mathcal{A}_{\mathcal{G}}$ *satisfies the optimality equation*

$$(2.6) \qquad \mu - r + (2a - \sigma^2)\pi(t) - b\int_0^t H(t,u)\pi(u)du + \sigma g_1(t)\alpha(t) - bE\left[\int_t^T H(u,t)\pi(u)du \Big| \mathcal{G}_t\right] = 0,$$

then π^* *is an optimal portfolio for the problem* (2.4)*. Furthermore, there exists at most one solution for equation* (2.6) *in the space* $\mathcal{L}(\mathcal{G})_T^2$*.*

2.3 Characterization of optimal portfolios and optimal utility function

In Theorem 2.1 we have given general conditions that an optimal portfolio should satisfy for a general fixed enlarged filtration. In this subsection, we use the particular filtration of the type (2.1) to characterize the solutions of the optimality equation (2.6) using classical stochastic control methods. In order to do this, we assume **ASSUMPTION (A)**. This hypothesis becomes important in order to determine the explicit form of the optimal portfolio π^*. From now on, this assumption is in force. To characterize the optimal portfolio satisfying (2.6), we need to introduce a Ricatti equation and a linear ordinary differential equations. As this procedure will also be needed to characterize the value function, we will use the following general structural notation. We say that Z is a solution of a (Z_1, Z_2, Z_3)-Ricatti equation if it is a solution of the following ordinary differential equation:

$$\begin{cases} \dot{Z}(t) + Z_1(t)Z^2(t) + Z_2(t)Z(t) + Z_3(t) = 0, \ t \in [0,T], \\ Z(T) = 0, \end{cases}$$

Similarly we say that Z is a solution of a (Z_1, Z_2)-linear ODE if it is a solution of the linear ordinary differential equation

$$\begin{cases} \dot{Z}(t) + Z_1(t)Z(t) + Z_2(t) = 0, \ t \in [0,T], \\ Z(T) = 0. \end{cases}$$

With this notation we now set P to be the solution of a (Z_1^P, Z_2^P, Z_3^P)-Ricatti equation with

$$Z_1^P(t) = b\frac{h_1(t)}{h_2(t)}, \quad Z_2^P(t) = \frac{2b}{2a-\sigma^2}\frac{h_1(t)}{h_2(t)} - \frac{\dot h_1(t)}{h_1(t)} - \frac{\dot h_2(t)}{h_2(t)}, \quad Z_3^P(t) = \frac{b}{(2a-\sigma^2)^2}\frac{h_1(t)}{h_2(t)}. \tag{2.7}$$

and Q to be a solution of a (Z_1^Q, Z_2^Q)−linear ODE with

$$Z_1^Q(t) = b\left(\frac{1}{2a-\sigma^2} + P(t)\right)\frac{h_1(t)}{h_2(t)} - \frac{\dot h_1(t)}{h_1(t)}, \quad Z_2^Q(t) = -\sigma\left\{\frac{\dot h_2(t)}{h_2(t)}g_1(t) + \dot g_1(t)\right\}P(t). \tag{2.8}$$

Similarly L is a solution of a (Z_1^L, Z_2^L)−linear ODE with

$$Z_1^L(t) = b\left(\frac{1}{2a-\sigma^2} + P(t)\right)\frac{h_1(t)}{h_2(t)} - \frac{\dot h_1(t)}{h_1(t)}, \quad Z_2^L(t) = -b\frac{\mu-r}{2a-\sigma^2}\left(\frac{1}{2a-\sigma^2} + P(t)\right)\frac{h_1(t)}{h_2(t)}. \tag{2.9}$$

Furthermore, we define X by

$$X_t := \eta_t\left[\sigma g_1(0)\alpha(0) + \int_0^t \eta_u^{-1}\left[\left\{-b\frac{h_1(u)}{h_2(u)}Q(u) + \sigma\left(\dot g_1(u) + g_1(u)\frac{\dot h_2(u)}{h_2(u)}\right)\right\}\alpha(u) - b\frac{h_1(u)}{h_2(u)}\left(L(u) - \frac{\mu-r}{2a-\sigma^2}\right)\right]du + \sigma\int_0^t \eta_u^{-1}g_1(u)d\alpha(u)\right], \tag{2.10}$$

where

$$\eta_t := \exp\left[\int_0^t\left\{b\left(\frac{1}{2a-\sigma^2} + P(u)\right)\frac{h_1(u)}{h_2(u)} - \frac{\dot h_2(u)}{h_2(u)}\right\}du\right]. \tag{2.11}$$

Theorem 2.2. *Assume* **(A)**, (2.5) *and that the* (Z_1^P, Z_2^P, Z_3^P)*-Ricatti equation has a solution. Then,* $\pi^*(t)$ *defined by*

$$\pi^*(t) = -\left(\frac{1}{2a-\sigma^2} + P(t)\right)X_t + Q(t)\alpha(t) + \left(L(t) - \frac{\mu-r}{2a-\sigma^2}\right) \tag{2.12}$$

is the unique solution of the optimal equation (2.6).

In fact, in order to obtain the above formulas one plugs (2.12) into the optimality equation (2.6) obtaining the corresponding equations for P, Q and L as the deterministic coefficients corresponding to X, α and the non-random term.

The above theorem characterizes the optimal portfolio for an insider where the enlarged filtration is fixed to be $\mathcal{G}$.

In order to obtain the optimal value $\Psi(T) = J(\pi^*)$ we define the following ordinary differential equations. R is a solution of a (Z_1^R, Z_2^R)-linear ODE with:

$$\begin{aligned} Z_1^R(t) &= \frac{\dot{h}_2(t)}{h_2(t)} - \frac{\dot{h}_1(t)}{h_1(t)}, \\ Z_2^R(t) &= -b\frac{h_1(t)}{h_2(t)}Q^2(t) + 2\sigma\left\{\frac{\dot{h}_2(t)}{h_2(t)}g_1(t) + \dot{g}_1(t)\right\}Q(t). \end{aligned} \tag{2.13}$$

M is a solution of a (Z_1^M, Z_2^M)-linear ODE with:

$$\begin{aligned} Z_1^M(t) &= \frac{\dot{h}_2(t)}{h_2(t)} - \frac{\dot{h}_1(t)}{h_1(t)}, \\ Z_2^M(t) &= -b\frac{h_1(t)}{h_2(t)}\left(L(t) - \frac{\mu - r}{2a - \sigma^2}\right)Q(t) + \sigma\left\{\frac{\dot{h}_2(t)}{h_2(t)}g_1(t) + \dot{g}_1(t)\right\}L(t) \end{aligned} \tag{2.14}$$

and N is a solution of a (Z_1^N, Z_2^N)-linear ODE with:

$$\begin{aligned} Z_1^N(t) &= \frac{\dot{h}_2(t)}{h_2(t)} - \frac{\dot{h}_1(t)}{h_1(t)}, \\ Z_2^N(t) &= -\frac{b}{2}\frac{h_1(t)}{h_2(t)}\left(L(t) - \frac{\mu - r}{2a - \sigma^2}\right)^2 \\ &\quad + \frac{g_1^2(t) + \theta(T-t)^{\theta-1}g_2^2((T-t)^\theta)}{G^2(t)}\left\{-\sigma^2 g_1^2(t)P(t) + \sigma g_1(t)Q(t) + R(t)\right\}. \end{aligned} \tag{2.15}$$

Then, we obtain the following result which characterizes the value function associated to the optimization problem for the insider.

Theorem 2.3. *Assume* **(A)** *and let* $b \neq 0$ *and* $E\left[\int_0^T |g_1(t)\alpha(t)|^2 dt\right] < \infty$. *Define*

$$v(t,x,\alpha) = \frac{2a-\sigma^2}{b}\frac{h_2(t)}{h_1(t)}\left\{-\frac{1}{2}P(t)x^2 + Q(t)x\alpha + \frac{1}{2}R(t)\alpha^2 + L(t)x + M(t)\alpha + N(t)\right\}.$$

where Q, R, L, M *and* N *are solutions of the respective* (Z_1^F, Z_2^F)*-linear ODE's for* $F = Q, R, L, M, N$ *equations and* P *is a solution of the* (Z_1^P, Z_2^P, Z_3^P)*-Ricatti equation. If* g_1 *is a continuous positive function and the conditions of Theorem 2.2 are satisfied and if*

$$E\left[v(t, X_t, \alpha(t))\right] \to 0 \text{ as } t \to T,$$

then the value function is finite and

$$\Psi(T) = J(\pi^*) = E\left[v(0, X_0, \alpha(0))\right].$$

The following section describes the sufficient conditions so that the filtration $\mathcal{G}$ becomes the insider's filtration and therefore the optimal portfolio obtained in the previous theorem corresponds to the optimal portfolio for the insider. Later we will see that such a portfolio leads to a finite utility.

2.4 The partial equilibrium equation

In this section, we establish our main theoretical result which states that the optimal portfolio π^* obtained in Theorem 2.2 can be characterized as the policy which determines the filtration $\mathcal{S}^{\pi^*}$ for some particular type of function g_1. Finally we characterize the optimal value function for the problem in Theorem 2.3.

In order to introduce our main result, we first define the following processes:

$$\begin{aligned}
U_t^1 &= \int_0^t g_1(s)F(t,s)dW(s),\\
U_t^2 &= \int_0^T g_1(s)dW(s) + \int_0^{(T-t)^\theta} g_2(s)dW'(s),\\
k_s(u) &= -\int_u^s \left\{-a\left(\frac{1}{2a-\sigma^2} + P(v)\right) + 1\right\}\eta_v dv \eta_u^{-1} b\frac{h_1(u)}{h_2(u)} + a,\ u \in [0,s],\\
A_s(u) &:= k_s(u)\left\{Q(u) - \sigma\left(\frac{1}{2a-\sigma^2} + P(u)\right)g_1(u)\right\},\ u \in [0,s],\\
F(t,s) &= 1 - \int_s^t \frac{A_t(u)}{G(u)}du - \int_0^t \frac{A_T(u)}{G(u)}du.
\end{aligned}$$

Theorem 2.4. *Assume* **(A)** *and that* (2.7) *has a unique solution. Let* g_1 *satisfy* $\int_0^T \frac{|g_1(s)|}{\sqrt{G(s)}}ds < \infty$ *and* $g_1(t) = g_2((T-t)^\theta)$ *is a strictly positive solution of*

$$g_1(t) = g_1(t)\int_0^t \frac{A_T(s)}{G(s)}ds + \sigma. \tag{2.16}$$

Moreover assume that $|F(t,t)| \geq c_0 > 0$ *and* $\sup_{0\leq s\leq t\leq T}|\partial_t F(t,s)| < \infty$. *Then*

$$\mathcal{G}_t = \mathcal{S}_t^{\pi^*},$$

where π^* *is given by* (2.12).

Note that the Volterra type equation (2.16) is non-linear in g_1 as P and Q depend on g_1 as well, through their characterizing equations. It is also clear from the above equation that $A_s(u)$ is a continuous function for $u \in [0,s]$ and therefore $g_1 \in C^1([0,T))$.

Note that as $G(T) = 0$, solving equation (2.16) appears to be difficult in general and this degeneracy problem will appear in some of the examples where we have been able to prove existence of solutions for (2.16) using relative compactness arguments. As equation (2.16) means that there is a stability in the flow of information, we call it the equilibrium equation in what follows.

The left side of the equation (2.16) appears due to the type of information set up in (2.1). The first term on the right side of (2.16) appears due to the large insider effects in the model (1.1) expressed through the optimal portfolios and finally the second term on the right side is the volatility of the model. Therefore this equation states that the insider information structure, the large trader effect and the volatility in the model have to combine in order to achieve partial equilibrium.

The following lemma provides an easy way to check the conditions required on F in the previous theorem.

Lemma 2.1. *Assume that P, Q, h_1, h_2 and g_1 are bounded functions, $\theta \leq 1$ and that there exists a positive constant c such that $\min\{h_2(t), g_2(t)\} > c$. Then there exists a constant c_0 such that $|F(t,t)| \geq c_0 > 0$ and $\sup_{0\leq s\leq t\leq T} |\partial_t F(t,s)| < \infty$.*

In the remaining sections we consider some representative examples. We will always assume that $g_1(t) = g_2((T-t)^\theta)$ is satisfied.

3. Example (c): A market with insider medium term effects on the price

In this section we assume that

$$a > 0, \quad b > 0, \quad h_1(t) \equiv 1, \quad \text{and} \quad h_2(t) \equiv 1. \tag{3.1}$$

$$dS^\pi(t) = S^\pi(t)\left\{\left(\mu + a\pi(t) - b\int_0^t \pi(u)du\right)dt + \sigma dW(t)\right\},\ S(0) > 0.$$

That is, the insider creates an effect on the drift through the variations of his strategies with respect to a past time average. This model is somewhat more realistic than the example (a) introduced in [4] but this model also allows the insider to hide his trades easier than in the model (a).

A mathematical way to explain the medium term effects is as follows. Consider $Y^\pi(t) = \int_0^t \frac{dS^\pi(s)}{S^\pi(s)} - \mu t - \sigma W(t)$. Then if δ is a perturbation of the portfolio process π, we have that $Y^{\pi+\delta}(T) - Y^\pi(T) = \int_0^T \{a - b(T-s)\}\,\delta(s)ds$. Therefore if we take $b = aT^{-1}$ then the effects of trade perturbations at the beginning of the time interval $[0,T]$ are small. These perturbation effects slowly increase towards the same effect of example (a) as time approaches T. In this sense, we say that the insider has medium term effects on the price.

Lemma 3.1. *Besides* (3.1)*, we assume that*

$$\lim_{t\to T}\{(T-t)g_1(t)\} = 0. \tag{3.2}$$

Then, we obtain the following explicit formulas for $P(t), Q(t), L(t), \eta_t, A_s(t)$ *and* X_t*:*

$$P(t) = \frac{b(T-t)}{(\sigma^2-2a)\{\sigma^2-2a+b(T-t)\}}, \tag{3.3}$$

$$Q(t) = -\frac{b\sigma}{(\sigma^2-2a)\{\sigma^2-2a+b(T-t)\}}\left\{-(T-t)g_1(t) + \int_t^T g_1(u)du\right\}, \tag{3.4}$$

$$L(t) = -\frac{b(\mu-r)(T-t)}{(\sigma^2-2a)\{\sigma^2-2a+b(T-t)\}}, \tag{3.5}$$

$$\eta_t = \frac{\sigma^2-2a+b(T-t)}{\sigma^2-2a+bT}, \tag{3.6}$$

$$A_s(t) = \frac{\sigma k_s(t)}{\sigma^2-2a}\left\{g_1(t) - \frac{b}{\sigma^2-2a+b(T-t)}\int_t^T g_1(u)du\right\}, \ s \geq t, \tag{3.7}$$

$$k_s(t) = -\frac{b}{\sigma^2-2a+b(T-t)}\left\{(\sigma^2-a)(s-t) + \frac{b}{2}\left\{(T-t)^2-(T-s)^2\right\}\right\} + a, \tag{3.8}$$

$$X_t = \sigma g_1(t)\alpha(t) - \int_0^t c(t,u)\alpha(u)du - \frac{b(\mu-r)t}{\sigma^2-2a+bT}, \tag{3.9}$$

where $c(t,u)$ *is defined by*

$$c(t,u) := \frac{b\sigma\{\sigma^2-2a+b(T-t)\}}{(\sigma^2-2a)\{\sigma^2-2a+b(T-u)\}} \times\left\{g_1(u) - \frac{b}{\sigma^2-2a+b(T-u)}\int_u^T g_1(v)dv\right\}.$$

For the proof, see Appendix 6. From these results, we have that the equilibrium equation (2.16) can be rewritten as

$$\begin{aligned} g_1(t) &= g_1(t)\int_0^t \frac{A_T(s)}{G(s)}ds + \sigma, \\ A_T(t) &= \frac{\sigma}{\sigma^2-2a}k_T(t)B(t), \\ B(t) &:= g_1(t) - \frac{b}{\sigma^2-2a+b(T-t)}\int_t^T g_1(u)du. \end{aligned} \tag{3.10}$$

Here we note that $g_1(t) = g_2((T-t)^\theta)$ and that k_T is defined in (3.8).

Lemma 3.2. *Assume that*

$$a < \frac{\sigma^2 - \sqrt{2}bT}{2}, \tag{3.11}$$

$$Tb < \sqrt{\sigma^2 - 2a}\left(\sigma - \sqrt{\sigma^2 - 2a}\right). \tag{3.12}$$

Then, $0 < k_T(u) \leq a$ *for* $u \in [0, T]$ *holds.*

For the proof, see Appendix 6. The above condition on $k_T(u)$ will be used when studying existence of solutions for the equilibrium equation. In particular, (3.12) is used in order to obtain the existence of solutions and it seems to be a technical condition which therefore does not have a direct interpretation.

Remark 3.1. If we assume (3.1) and (3.11), the condition (2.5) is satisfied. Therefore, we can use the results in Sections 2.2–2.4.

Theorem 3.1. *Assume* (3.1), (3.11) *and* (3.12). *Assume also that*

$$\theta < 1, \tag{3.13}$$

and

$$T^{1-\theta} < \frac{(1-\theta)(\sigma^2 - 2a)}{2a}. \tag{3.14}$$

Then the following assertions are satisfied.
1. *The equilibrium equation* (3.10) *has a strictly positive bounded solution such that* (3.2) *is satisfied,* $g_1(0) = \sigma$, $\dot{g}_1(t) > 0$ *for* $t \in [0, T]$.
2. *Define the portfolio*

$$\hat{\pi}(t) = -\left(\frac{1}{2a - \sigma^2} + P(t)\right)X_t + Q(t)\alpha(t) + \left(L(t) - \frac{\mu - r}{2a - \sigma^2}\right). \tag{3.15}$$

Then $\hat{\pi} \in \mathcal{A}_{\mathcal{S}^{\hat{\pi}}}$. *Furthermore,*

$$\mathcal{G}_t = \mathcal{S}_t^{\hat{\pi}}.$$

3. *The strategy given by* (3.15) *is a solution for the problem* (1.6) *leading to a finite utility and its value is* $E\left[v(0, X_0, \alpha(0))\right]$, *where*

$$v(t, x, \alpha) = \frac{2a - \sigma^2}{b}\left\{-\frac{1}{2}P(t)x^2 + Q(t)x\alpha + \frac{1}{2}R(t)\alpha^2 + L(t)x + M(t)\alpha + N(t)\right\}.$$

Here P, Q *and* L *are given by* (3.3), (3.4) *and* (3.5) *respectively. Moreover,* R, M *and* N *are the solutions of the following* $(0, Z_2^F)$, $F = R, M, N$ *linear ODE's with*

$$\begin{cases} Z_2^R(t) = -bQ^2(t) + 2\sigma\dot{g}_1(t)Q(t), \\ Z_2^M(t) = -b\left(L(t) - \dfrac{\mu - r}{2a - \sigma^2}\right)Q(t) + \sigma g_1(t)L(t), \\ Z_2^N(t) = -\dfrac{b}{2}\left(L(t) - \dfrac{\mu - r}{2a - \sigma^2}\right)^2 + \dfrac{g_1^2(t) + \theta(T-t)^{\theta-1}g_2^2((T-t)^\theta)}{G^2(t)} \\ \qquad \times\left\{-\sigma^2 g_1^2(t)P(t) + \sigma g_1(t)Q(t) + R(t)\right\}. \end{cases} \tag{3.16}$$

For the proof, see Appendix 6. Note that condition (3.13) is a condition on the speed of transmission of information. Condition (3.14) is a condition that ties the investment time horizon, the speed of transmission and the effect of the insider on the prices. A condition of this type is natural although it should also involve b which appears in condition (3.12). Still these conditions are not optimal.

4. Example (d): A market with an insider with memory effects on the price

In this section we assume

$$a = 0, \quad b < 0, \quad h_1(t) \equiv 1, \text{ and } h_2(t) \equiv 1. \tag{4.1}$$

Moreover, we set

$$\overline{b} := -b > 0.$$

$$dS(t) = \left(\mu + \overline{b}\int_0^t \pi(s)ds\right) S(t)dt + \sigma S(t)dW(t).$$

In comparison with example (a), the effect of the insider is not as strong. In this case the impact of the trades of the large trader-insider is smoothed through the integral term in the drift.

Following similar arguments as in Section 3, we can derive the results of this section, therefore we only give the main results here and sketch the proofs in Appendix 7.

Lemma 4.1. *In addition to* (4.1)*, we assume that*

$$\lim_{t\to T}\{(T-t)g_1(t)\} = 0. \tag{4.2}$$

Then, we obtain the following explicit formulas for P, Q, L, η, A *and* X*:*

$$P(t) = -\frac{\overline{b}(T-t)}{\sigma^2\{\sigma^2 - \overline{b}(T-t)\}}, \tag{4.3}$$

$$Q(t) = \frac{\overline{b}}{\sigma\{\sigma^2 - \overline{b}(T-t)\}}\left\{-(T-t)g_1(t) + \int_t^T g_1(u)du\right\}, \tag{4.4}$$

$$L(t) = \frac{\overline{b}(\mu - r)(T-t)}{\sigma^2\{\sigma^2 - \overline{b}(T-t)\}}, \tag{4.5}$$

$$\eta_t = \frac{\sigma^2 - \overline{b}(T-t)}{\sigma^2 - \overline{b}T}, \tag{4.6}$$

$$k_s(t) = \frac{\overline{b}}{\sigma^2 - \overline{b}(T-t)}\left[\sigma^2(s-t) + \frac{\overline{b}}{2}\left\{(T-s)^2 - (T-t)^2\right\}\right], \tag{4.7}$$

$$B(t) = g_1(t) + \frac{\overline{b}}{\sigma^2 - \overline{b}(T-t)} \int_t^T g_1(u)du,$$

(4.8) $$A_s(t) = \frac{1}{\sigma} k_s(t) B(t), \ s \le t,$$

$$X_t = \frac{\overline{b}}{\sigma} \int_0^t \frac{\sigma^2 - \overline{b}(T-t)}{\sigma^2 - \overline{b}(T-u)} \left\{ g_1(u) + \frac{\overline{b}}{\sigma^2 - \overline{b}(T-u)} \int_u^T g_1(v)dv \right\} \alpha(u)du + \sigma g_1(t)\alpha(t) + \frac{\overline{b}(\mu - r)t}{\sigma^2 - \overline{b}T}.$$

Therefore, the equilibrium equation (2.16) can be rewritten as

(4.9) $$g_1(t) = g_1(t) \int_0^t \frac{A_T(s)}{G(s)} ds + \sigma,$$
$$A_T(t) = \frac{1}{\sigma} k_T(t) B(t),$$
$$k_T(t) := \overline{b}(T-t) \frac{\sigma^2 - \overline{b}/2 \cdot (T-t)}{\sigma^2 - \overline{b}(T-t)}.$$

Here we note that $g_1(t) = g_2((T-t)^\theta)$. The proof of Lemma 4.1 is obtained as the proof of Lemma 3.1 by setting $a = 0$ and $b = -\bar{b}$.

Theorem 4.1. *Assume* (4.1). *Assume also that*

(4.10) $$2\overline{b}T < \sigma^2,$$

and

(4.11) $$\theta < 1.$$

Then, the following assertions are true.
1. *The equilibrium equation* (4.9) *has a strictly positive bounded solution such that* (4.2) *is satisfied,* $g_1(0) = \sigma$, $\dot{g}_1(t) > 0$ *for* $t \in [0, T]$.
2. *Define*

(4.12) $$\hat{\pi}(t) = -\left(-\frac{1}{\sigma^2} + P(t)\right) X_t + Q(t)\alpha(t) + \left(L(t) + \frac{\mu - r}{\sigma^2}\right).$$

Then $\hat{\pi} \in \mathcal{A}_{\mathcal{S}^{\hat{\pi}}}$ *and furthermore*

$$\mathcal{G}_t = \mathcal{S}_t^{\hat{\pi}}.$$

3. *The strategy given by* (4.12) *is a strategy that solves problem* (1.6) *with finite utility and its value is* $E\left[v(0, X_0, \alpha(0))\right]$, *where*

$$v(t, x, \alpha) = -\frac{\sigma^2}{b} \left\{ -\frac{1}{2} P(t)x^2 + Q(t)x\alpha + \frac{1}{2} R(t)\alpha^2 + L(t)x + M(t)\alpha + N(t) \right\}.$$

Here P, Q and L are given by (4.3)*,* (4.4) *and* (4.5) *respectively. Moreover, R, M and N are the solutions of the linear ODE's* (3.16)*.*

For the proof, see Appendix 7.

5. Concluding remarks

In this article, we consider two examples of insider agents which have medium/long term effects on the price through the proportion of their wealth invested in the stock. We prove that under certain conditions one can achieve partial equilibrium therefore pointing to the possibility of a stable market.

Certainly, the modeling used in the present article is a modification of the usual price function in economics which is due to the fact that insider effect on the price of the underlying is better measured by the proportion of wealth invested in the underlying rather than money value as it is used in the large investor literature. Obviously, one could also say that in order to change the units into money invested in the underlying it would just be a matter of multiplying the $\frac{dS(t)}{S(t)}$ by the total money value of the investment. So one may consider that a renormalization procedure has been performed.

In [4] is also proved that if the effect of the insider is measured using the total value of investment then the model explodes. Clearly, we only present here a proposal given the current empirical facts and the current mathematical possibilities for modeling this problem.

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6. Appendix 1: Proofs of Section 3

Proof of Lemma 3.1

If $h_1(t) \equiv 1$ and $h_2(t) \equiv 1$ holds, the coefficients (2.7), (2.8) and (2.9) give that:

$$\begin{cases} \dot{P}(t) + bP^2(t) + \dfrac{2b}{2a-\sigma^2}P(t) + \dfrac{b}{(2a-\sigma^2)^2} = 0, \\ P(T) = 0, \end{cases} \tag{6.1}$$

$$\begin{cases} \dot{Q}(t) + b\left(\dfrac{1}{2a-\sigma^2} + P(t)\right)Q(t) - \sigma\dot{g}_1(t)P(t) = 0, \\ Q(T) = 0, \end{cases} \tag{6.2}$$

$$\begin{cases} \dot{L}(t) + b\left(\dfrac{1}{2a-\sigma^2} + P(t)\right)L(t) - \dfrac{b(\mu-r)}{2a-\sigma^2}\left(\dfrac{1}{2a-\sigma^2} + P(t)\right) = 0, \\ L(T) = 0. \end{cases}$$

Supposing that $P(t) \neq \frac{1}{\sigma^2-2a}$, we have

$$\frac{\dot{P}(t)}{\left\{P(t) - \frac{1}{\sigma^2-2a}\right\}^2} = -b.$$

When we integrate both sides of this expression and use the terminal condition, we obtain (3.3). As for (3.6): from (2.11) and (3.3), we have

$$\eta_t = \exp\left[\int_0^t b\left(\frac{1}{2a-\sigma^2} + P(u)\right)du\right] = \frac{\sigma^2 - 2a + b(T-t)}{\sigma^2 - 2a + bT}.$$

For $Q(t)$, by using (3.3) we rewrite (6.2) as

$$\dot{Q}(t) - \frac{b}{\sigma^2 - 2a + b(T-t)}Q(t) = \frac{b\sigma(T-t)}{(\sigma^2-2a)\{\sigma^2 - 2a + b(T-t)\}}\dot{g}_1(t).$$

Therefore, using the variations of constants method we have that

$$\begin{aligned} Q(t) &= -\frac{b\sigma}{\sigma^2-2a}\int_t^T e^{-\int_t^u \frac{b}{\sigma^2-2a+b(T-v)}dv}\frac{T-u}{\sigma^2-2a+b(T-u)}\dot{g}_1(u)du \\ &= -\frac{b\sigma}{(\sigma^2-2a)\{\sigma^2-2a+b(T-t)\}}\left\{-(T-t)g_1(t) + \int_t^T g_1(u)du\right\}, \end{aligned}$$

where we have used (3.2). Next, we use the variation of constants method in order to compute L in (3.5).

From (3.3) and (3.4) we have

$$Q(t) - \sigma\left(\frac{1}{2a-\sigma^2} + P(t)\right) g_1(t) = \frac{\sigma}{\sigma^2 - 2a}\left\{g_1(t) - \frac{b}{\sigma^2 - 2a + b(T-t)}\int_t^T g_1(u)du\right\}. \tag{6.3}$$

Using (3.3) and (3.6), we have

$$\begin{aligned} k_s(t) &= -\int_t^s \left\{-a\left(\frac{1}{2a-\sigma^2} + P(u)\right) + 1\right\}\eta_u du \eta_t^{-1} b + a \\ &= -\frac{b}{\sigma^2 - 2a + b(T-t)}\left[(\sigma^2 - a)(s-t) + \frac{b}{2}\left\{(T-t)^2 - (T-s)^2\right\}\right] + a. \end{aligned}$$

This result together with (6.3), gives (3.7).

Finally, we note that if we perform an integration by parts on the stochastic integral in (2.10) we have the following alternative expression for X

$$X_t = \sigma g_1(t)\alpha(t) - \eta_t b\int_0^t \eta_u^{-1}\left[\left\{Q(u) - \sigma g_1(u)\left(\frac{1}{2a-\sigma^2} + P(u)\right)\right\}\alpha(u) + L(u) - \frac{\mu - r}{2a-\sigma^2}\right]du, \tag{6.4}$$

then the result follows from (3.5), (3.6) and (6.3). □

Proof of Lemma 3.2

A straightforward calculation shows that

$$\frac{dk_s(t)}{ds} = \frac{-b(\sigma^2 - a + b(T-s))}{\sigma^2 - 2a + b(T-t)} < 0.$$

Therefore, $k_s(t)$ is decreasing for $s \in [t, T]$ and

$$a = k_t(t) \geq k_s(t) \geq k_T(t) = -b\frac{(\sigma^2 - a)(T-t) + \frac{b}{2}(T-t)^2}{\sigma^2 - 2a + b(T-t)} + a.$$

Similarly, one obtains that

$$k_T(t) \geq k_T(0) = \frac{-\frac{b^2}{2}T^2 + b(2a - \sigma^2)T + a(\sigma^2 - 2a)}{\sigma^2 - 2a + bT} + a.$$

Analyzing the quadratic polynomial in the numerator, one obtains that it is positive if and only if

$$bT < \sqrt{\sigma^2 - 2a}(\sigma - \sqrt{\sigma^2 - 2a}).$$

□

Proof of Theorem 3.1

1. To study the existence of solutions for the equation (3.10), we introduce the following approximating equation for $n \geq 3$:

$$
\begin{aligned}
(6.5) \qquad g_n(t) &= g_n(t) \int_0^t \frac{A_T^n(s)}{G_n(s)} ds + \sigma, \\
G_n(t) : &= \int_t^T \left\{ 1 + \theta \left(T - u + \frac{1}{n} \right)^{\theta - 1} \right\} g_n^2(u) du + \frac{g_n^2(T)}{n^\theta}, \\
A_T^n(t) &:= \frac{\sigma}{\sigma^2 - 2a} k_T(t) B_n(t), \\
B_n(t) : &= g_n(t) - \frac{b}{\sigma^2 - 2a + b(T-t)} \int_t^T g_n(u) du.
\end{aligned}
$$

First, we assume $g_n(T) = x_0 > 0$ and we shall prove that there exists $x_0 > 0$ such that $g_n(0) = \sigma$. Note that differentiating both sides of the equation (6.5) we have

$$
(6.6) \qquad \dot{g}_n(t) = \frac{g_n^2(t) A_T^n(t)}{\sigma G_n(t)}.
$$

It is clear that $B_n(t) > 0$ for t close to T by continuity. Therefore, $\dot{g}_n(t) > 0$ and $B_n(t) > 0$ in a maximal neighborhood represented as $I_0 = (t_0, T]$, and therefore g_n is increasing on I_0. From this we have that as g_n is increasing then $G_n(t) \geq g_n(t)^2 (T-t)^\theta$. Then using Lemma 3.2 and $\theta < 1$, we have that for $t \in I_0$

$$
\begin{aligned}
(6.7) \qquad g_n(T) - g_n(t) &\leq \frac{a}{\sigma^2 - 2a} g_n(T) \int_t^T \frac{du}{(T-u)^\theta} \\
&\leq \frac{a}{(\sigma^2 - 2a)(1-\theta)} T^{1-\theta} g_n(T).
\end{aligned}
$$

From (3.14) we see that $g_n(t) > \frac{g_n(T)}{2} = \frac{x_0}{2} > 0$, for all $t \in I_0$.

By contradiction, assume that $B_n(t) > 0$ for $t \in (t_0, T]$ and $B_n(t_0) = 0$. Then, since g_n is continuous and increasing on $[t_0, T]$ and we have due to (3.11)

$$
\begin{aligned}
(6.8) \qquad 0 = B_n(t_0) &\geq \frac{g_n(T)}{2} - \frac{b(T-t_0)}{\sigma^2 - 2a + b(T-t_0)} g_n(T) \\
&= \frac{\sigma^2 - 2a - b(T-t_0)}{2\{\sigma^2 - 2a + b(T-t_0)\}} x_0 \\
&> 0.
\end{aligned}
$$

This gives a contradiction. Therefore, we see that $B_n(t) > 0$ and $\dot{g}_n(t) > 0$ for $t \in [0, T]$. From here it also follows that

(6.9)

$$
\frac{x_0}{2} < g_n(t) \leq x_0, \ 0 < A_T^n(t) \leq \frac{a \sigma x_0}{\sigma^2 - 2a} \text{and } 0 < \dot{g}_n(t) \leq \frac{a n^\theta x_0}{\sigma^2 - 2a} \quad \text{for } t \in [0, T].
$$

To prove the third inequality, we observe that (6.6), $G_n(t) \geq x_0^2 n^{-\theta}$ and $\dot{g}_n(t) > 0$ for $t \in [0, T]$ hold, therefore we see that

$$\dot{g}_n(t) \leq \frac{an^\theta x_0}{\sigma^2 - 2a},$$

for $t \in [0, T]$ and (6.9) follows.

From (6.5), we will construct a system of ordinary differential equations. For this, we introduce the following auxiliary function

$$M_n(t) := \frac{\dot{g}_n(t)}{g_n^2(t)} G_n(t). \tag{6.10}$$

From (6.9) and (6.5), we note that

$$0 < M_n(t) \leq \frac{4an^\theta x_0}{\sigma^2 - 2a} \left\{ T + \left(T + \frac{1}{n} \right)^\theta \right\} \quad \text{for } t \in [0, T]. \tag{6.11}$$

Then, using (6.6), we have

$$M_n(t) = \frac{A_T^n(t)}{\sigma} = \frac{k_T(t)}{\sigma^2 - 2a} \left\{ g_n(t) - \frac{b}{\sigma^2 - 2a + b(T - t)} \int_t^T g_n(u) du \right\}. \tag{6.12}$$

Then, by direct calculations, from (6.10) and (6.12) we have

$$2\frac{\dot{g}_n(t)}{g_n(t)} + \frac{\dot{M}_n(t)}{M_n(t)} - \frac{\ddot{g}_n(t)}{\dot{g}_n(t)} = -\left\{ 1 + \theta \left(T - t + \frac{1}{n} \right)^{\theta - 1} \right\} \frac{\dot{g}_n(t)}{M_n(t)}, \tag{6.13}$$

and

$$-\frac{b}{\sigma^2 - 2a + b(T - t)} + \frac{\dot{M}_n(t)}{M_n(t)} - \frac{\dot{k}_T(t)}{k_T(t)} - \frac{k_T(t)}{\sigma^2 - 2a} \frac{\dot{g}_n(t)}{M_n(t)} = 0. \tag{6.14}$$

From (6.9) and (6.11), we can define $w(t) := \log g_n(t)$, $v(t) := \log \dot{g}_n(t)$, and $m(t) := \log M_n(t)$. Furthermore, due to (6.13) and (6.14) we obtain the following system:

$$\dot{\begin{pmatrix} w(t) \\ v(t) \\ m(t) \end{pmatrix}} = f(t, w(t), v(t), m(t)), \quad \begin{pmatrix} w(T) \\ v(T) \\ m(T) \end{pmatrix} = \begin{pmatrix} \log x_0 \\ \log \left(\frac{n^\theta a x_0}{\sigma^2 - 2a} \right) \\ \log \left(\frac{a x_0}{\sigma^2 - 2a} \right) \end{pmatrix},$$

where

$$f(t, w(t), v(t), m(t)) :=$$

$$\begin{pmatrix} e^{v(t) - w(t)} \\ 2e^{v(t) - w(t)} + \left\{ 1 + \frac{k_T(t)}{\sigma^2 - 2a} + \theta \left(T - t + \frac{1}{n} \right)^{\theta - 1} \right\} e^{v(t) - m(t)} + \frac{b}{\sigma^2 - 2a + b(T - t)} + \frac{\dot{k}_T(t)}{k_T(t)} \\ \frac{k_T(t)}{\sigma^2 - 2a} e^{v(t) - m(t)} + \frac{b}{\sigma^2 - 2a + b(T - t)} + \frac{\dot{k}_T(t)}{k_T(t)} \end{pmatrix}.$$

As for this system, the existence and uniqueness of a local solution is given by Picard's theorem as the function f is locally Lipschitz. Noticing (6.9), (6.11) and (6.8), one realizes that the system can not explode in finite time for fixed n. Therefore repeating the application of Picard's theorem, we see that this system has a unique solution for $t \in [0,T]$ and fixed n. Moreover, the solution of this system depends continuously on the terminal value. Therefore, we see that for $t \in [0,T]$, $g_n(t)$ is continuous on x_0. From (6.9) we see that $g_n(t) \to \infty$ as $x_0 \to \infty$ and $g_n(t) \to 0$ as $x_0 \to 0$. Therefore, by the intermediate value theorem, there exists $x_0 > 0$ such that $g_n(0) = \sigma$.

Next, we shall show the uniform boundedness of g_n. For that, note that as g_n is increasing, $A^n_T(t) \le \frac{\sigma a}{\sigma^2 - 2a} g_n(t)$, (6.6) and $G_n(t) \ge g_n^2(t)(T-t)^\theta$ then we have

$$\frac{\dot{g}_n(t)}{g_n(t)} = \frac{A^n_T(t) g_n(t)}{\sigma G_n(t)} \le \frac{a}{(\sigma^2 - 2a)(T-t)^\theta},$$

where we have used that g_n is increasing for $t \in [0,T]$. By integrating the above inequality, we obtain

$$\sigma \le g_n(t) \le \sigma e^{\frac{a}{(\sigma^2-2a)(1-\theta)} T^{1-\theta}} \text{ for } t \in [0,T].$$

Here we have used that $g_n(0) = \sigma$ and g_n is increasing for $t \in [0,T]$. In particular note that (3.2) is satisfied for g_n.

Moreover, in a similar way to (6.7) we have

$$\begin{aligned}|g_n(t) - g_n(s)| &\le \frac{a}{\sigma^2 - 2a} g_n(T) \int_s^t \frac{du}{(T-u)^\theta} \\ &\le \frac{a\sigma}{(\sigma^2 - 2a)(1-\theta)} e^{\frac{a\sigma}{(\sigma^2-2a)(1-\theta)} T^{1-\theta}} |(T-s)^{1-\theta} - (T-t)^{1-\theta}| \text{ for } t, s \in [0,T].\end{aligned}$$

Since, for all n, g_n is uniformly continuous, we see that $\{g_n(t)\}_n$ is equicontinuous. Therefore, there is a uniformly convergent subsequence $\{g_{n_k}(t)\}_k$. Setting $\bar{g}(t) := \lim_{k\to\infty} g_{n_k}(t)$ for $t \in [0,T]$ we see that $\bar{g}(t)$ solves the equation (6.5) by using the bounded convergence theorem.

2. All the necessary properties are stated in Theorem 3.1 therefore the result follows from Theorem 2.4. Also note that as g_1 is bounded and $G(t) \ge (T-t)^\theta g_1(t)^2$ then the condition $\int_0^T \frac{|g_1(s)|}{\sqrt{G(s)}} ds < \infty$ as well as the other conditions in Lemma 2.1.

3. From Theorems 2.4 and 2.3, it is sufficient to check that

$$E[|v(t, X_t, \alpha(t))|] \to 0 \text{ for } t \to T.$$

First, we note that

$$\begin{aligned}|v(t, X_t, \alpha(t))| \le K_T \Big\{ &|P(t)| X_t^2 + |Q(t)|(X_t^2 + \alpha^2(t)) + |R(t)|\alpha^2(t) \\ &+ |L(t)|(X_t^2 + 1) + |M(t)|(\alpha^2(t) + 1) + |N(t)| \Big\}.\end{aligned}$$

Next, as $\theta < 1$ we see that

$$E\left[\int_0^t \alpha^2(u)du\right] = \int_0^t G(u)^{-1}du \leq \frac{(T-t)^{1-\theta}}{\sigma^2(1-\theta)}. \tag{6.15}$$

Moreover, we have from (3.9) that

$$X_t^2 \leq K_T\left\{1 + \alpha^2(t) + \int_0^t \alpha^2(u)du\right\}.$$

Hence, from (6.15) we obtain that

$$\begin{aligned} E\left[X_t^2\right] &\leq E\left[K_T\left\{1 + \alpha^2(t) + \int_0^t \alpha^2(u)du\right\}\right] \\ &\leq K_T\left\{1 + \frac{1}{(T-t)^\theta} + (T-t)^{1-\theta}\right\}. \end{aligned} \tag{6.16}$$

Then, we observe the following:

$$|A_T(t)| \leq K_T,\ |P(t)| \leq K_T(T-t),\ |Q(t)| \leq K_T(T-t),\ \text{and}\ |L(t)| \leq K_T(T-t). \tag{6.17}$$

Furthermore, noting that

$$\begin{aligned} |\dot{g}_1(t)| &\leq K_T\left|\frac{g_1^2(t)}{G(t)}\right| \leq \frac{K_T}{(T-t)^\theta}, \\ |R(t)| &\leq K_T\int_t^T \left\{|Q(u)|^2 + |\dot{g}_1(u)||Q(u)|\right\}du, \\ |M(t)| &\leq K_T\int_t^T \{|QL(u)| + |Q(u)| + |\dot{g}_1(u)||L(u)|\}\,du, \\ |N(t)| &\leq K_T\int_t^T \left\{|L(u)|^2 + \left|\frac{\{1+\theta(T-u)^{\theta-1}\}g_1^2(u)}{G^2(u)}\right|\left(|g_1^2P(u)| + |g_1Q(u)| + |R(u)|\right)\right\}du. \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} &|R(t)| \leq K_T(T-t)^{2-\theta},\ |M(t)| \leq K_T(T-t)^{2-\theta}, \\ \text{and}\quad &|N(t)| \leq K_T(T-t)^{2-2\theta}. \end{aligned} \tag{6.18}$$

By using (6.15), (6.16), (6.17) and (6.18), we see that

$$\lim_{t\to T} E[|v(t, X_t, \alpha(t))|] \to 0 \text{ as } t \to T.$$

Therefore, we conclude this theorem by using Theorems 2.4 and 2.3. The utility is finite because $E\left[\int_0^T |g_1(t)\alpha(t)|^2dt\right] < \infty$. □

7. Appendix 2: Proofs of Section 4

Proof of Theorem 4.1

1. We apply the guidelines set by the proof of Theorem 3.1. Still the proof is slightly different due to the fact that here k_T is not bounded away from zero. To study the existence of solutions to the equation (4.9), we introduce the following equation:

$$
\begin{aligned}
(7.1) \qquad g_n(t) &= g_n(t)\int_0^t \frac{A_T^n(s)}{G_n(s)}ds + \sigma,\\
G_n(t) &:= \int_t^T \left\{1+\theta\left(T-u+\frac{1}{n}\right)^{\theta-1}\right\} g_n^2(u)du + \frac{g_n^2(t)}{n},\\
A_T^n(t) &:= \frac{1}{\sigma}k_T^n(t)B_n(t),\\
k_T^n(t) &:= \overline{b}\left(T-t+\frac{1}{n}\right)\frac{\sigma^2-\overline{b}/2\cdot(T-t)}{\sigma^2-\overline{b}(T-t)},\\
B_n(t) &:= g_n(t) + \frac{\overline{b}}{\sigma^2-\overline{b}(T-t)}\int_t^T g_n(u)du.
\end{aligned}
$$

First, we assume $g_n(T) = x_0 > 0$ and we shall prove that there exists $x_0 > 0$ such that $g_n(0) = \sigma$. By differentiating both sides of the equation (7.1) we have

$$
(7.2) \qquad \dot{g}_n(t) = \frac{g_n^2(t)A_T^n(t)}{\sigma G_n(t)}.
$$

It is clear that $B_n(t) > 0$ for t close to T by continuity. Therefore, $\dot{g}_n(t) > 0$ and $B_n(t) > 0$ in a maximal neighborhood represented as $I_0 = (t_0, T]$, and therefore g_n is increasing on I_0. Then, for $t \in I_0$ we have

$$
\begin{aligned}
g_n(T) - g_n(t) &= \int_t^T \frac{g_n^2(u)A_T^n(u)}{\sigma G_n(u)}du\\
&\leq \bar{b}g_n(T)\int_t^T \frac{(T-u+\frac{1}{n})\{\sigma^2-\bar{b}/2\cdot(T-u)\}}{\{(T-u+\frac{1}{n})+(T-u)^{\theta}\}\{\sigma^2-\bar{b}(T-u)\}^2}du\\
&\leq \frac{\overline{b}(T-t)}{\sigma^2-\overline{b}(T-t)}g_n(T)\\
&\leq \frac{\overline{b}T}{\sigma^2-\overline{b}T}g_n(T).
\end{aligned}
$$

Hence we see that

$$
(7.3) \qquad g_n(t) > \frac{\sigma^2-2\overline{b}T}{\sigma^2-\overline{b}T}x_0 > 0
$$

for $t \in I_0$. Here we use (4.10).

By contradiction, suppose that $B_n(t) > 0$ for $t \in I_0$ and $B_n(t_0) = 0$. Since g_n is continuous and increasing on I_0 and we have using (7.3) that

$$\begin{aligned} 0 = B_n(t_0) &\geq \frac{\sigma^2 - 2\bar{b}T}{\sigma^2 - \bar{b}T} \left\{ g_n(T) + \frac{\bar{b}(T - t_0)}{\sigma^2 - \bar{b}(T - t_0)} g_n(T) \right\} \\ &= \frac{\sigma^2(\sigma^2 - 2\bar{b}T)}{(\sigma^2 - \bar{b}T)(\sigma^2 - \bar{b}(T - t_0))} g_n(T) > 0. \end{aligned}$$

This gives a contradiction. Therefore for $t \in [0, T]$, $B_n(t) > 0$ and $\dot{g}_n(t) > 0$ hold. Hence we have

$$\frac{\sigma^2 - 2\bar{b}T}{\sigma^2 - \bar{b}T} x_0 < g_n(t) < x_0 \text{ and } 0 < \dot{g}_n(t) \leq \frac{\bar{b}\sigma^2 x_0}{(\sigma^2 - \bar{b}T)^2}, \tag{7.4}$$

for $t \in [0, T]$. From (7.1) we shall construct a system of ordinary differential equations. As in the proof of Theorem 3.1, we introduce the following auxiliary function

$$M_n(t) := \frac{\dot{g}_n(t)}{g_n^2(t)} G_n(t). \tag{7.5}$$

Here note that due to (7.4), we have

$$0 < M_n(t) = \sigma^{-1} A_T^n(t) \leq \frac{\bar{b}(T + n^{-1})x_0}{(\sigma^2 - \bar{b}T)^2}. \tag{7.6}$$

Then, using (7.2), we have

$$M_n(t) = \frac{k_T^n(t)}{\sigma^2} \left\{ g_n(t) + \frac{\bar{b}}{\sigma^2 - \bar{b}(T - t)} \int_t^T g_n(u)du \right\}. \tag{7.7}$$

Note that from (7.4) and (7.6), we can define $w(t) := \log g_n(t)$, $v(t) := \log \dot{g}_n(t)$ and $m(t) := \log M_n(t)$. From (7.4), (7.5) and (7.7), we obtain the following system:

$$\dot{\begin{pmatrix} w(t) \\ v(t) \\ m(t) \end{pmatrix}} = f(t, w(t), v(t), m(t)), \quad \begin{pmatrix} w(T) \\ v(T) \\ m(T) \end{pmatrix} = \begin{pmatrix} \log x_0 \\ \log\left(\frac{\bar{b}x_0}{\sigma^2}\right) \\ \log\left(\frac{\bar{b}x_0}{\sigma^2 n}\right) \end{pmatrix},$$

where

$$f(t, w(t), v(t), m(t)) :=$$

$$\begin{pmatrix} e^{v(t)-w(t)} \\ 2e^{v(t)-w(t)} + \left\{1 + \frac{k_T^n(t)}{\sigma^2} + \theta\left(T - t + \frac{1}{n}\right)^{\theta-1}\right\} e^{v(t)-m(t)} + \frac{2}{n} e^{2v(t)-m(t)-w(t)} - \frac{\bar{b}}{\sigma^2 - \bar{b}(T-t)} + \frac{\dot{k}_T^n(t)}{k_T^n(t)} \\ \frac{k_T^n(t)}{\sigma^2} e^{v(t)-m(t)} - \frac{\bar{b}}{\sigma^2 - \bar{b}(T-t)} + \frac{\dot{k}_T^n(t)}{k_T^n(t)} \end{pmatrix}.$$

As before, this system has a choice of x_0 such that $g_n(0) = \sigma$. Next, we shall show the uniform boundedness of g_n. For that, from (7.1) and (7.4) we have

$$\frac{\dot{g}_n(t)}{g_n(t)} = \frac{A_T^n(t)g_n(t)}{\sigma G_n(t)} \le \frac{\overline{b}\sigma^2}{(\sigma^2 - \overline{b}T)(\sigma^2 - 2\overline{b}T)}. \tag{7.8}$$

By integrating both sides of (7.8), we obtain

$$\sigma \le g_n(t) \le \sigma e^{\frac{\overline{b}\sigma^2}{(\sigma^2 - \overline{b}T)(\sigma^2 - 2\overline{b}T)}} \text{ for } t \in [0, T].$$

Here we have used that $g_n(0) = \sigma$. Moreover, from (7.8) we have

$$|g_n(t) - g_n(s)| \le \frac{\overline{b}\sigma^2}{(\sigma^2 - \overline{b}T)^2} e^{\frac{\overline{b}\sigma^2}{(\sigma^2 - \overline{b}T)(\sigma^2 - 2\overline{b}T)}} |t - s|.$$

Since, for all n, g_n is uniformly continuous, we see that $\{g_n(t)\}_n$ is equicontinuous. Therefore, there is a uniformly convergent subsequence $\{g_{n_k}(t)\}_k$. Setting $\bar{g}(t) := \lim_{k\to\infty} g_{n_k}(t)$ for $t \in [0, T]$ we see that $\bar{g}(t)$ solves the equation (7.1) by using the bounded convergence theorem.
2. All the necessary properties are stated in Theorem 4.1 therefore the result follows from Theorem 2.4.
3. We note the result of Theorem 2.3. Then, as in the proof of Theorem 3.1 we can observe the following:

$$E[|v(t, X_t, \alpha(t))|] \le K_T \Big[\{|P(t)| + |Q(t)| + |L(t)|\} E[X_t^2] + \{|Q(t)| + |R(t)| + |M(t)|\} E[\alpha^2(t)] + |L(t)| + |M(t)| + |N(t)|\Big], \tag{7.9}$$

$$E[\alpha^2(t)] \le \frac{1}{\sigma^2 (T - t)^\theta}, \tag{7.10}$$

$$E[X_t^2] \le K_T \left\{1 + \frac{1}{(T - t)^\theta} + (T - t)^{1-\theta}\right\}. \tag{7.11}$$

Moreover, we can observe the following:

$$\begin{gathered} |P(t)| \le K_T(T - t),\ |Q(t)| \le K_T(T - t),\ |R(t)| \le K_T(T - t), \\ |L(t)| \le K_T(T - t),\ |M(t)| \le K_T(T - t)^2,\ |N(t)| \le K_T(T - t)^{1-\theta}. \end{gathered} \tag{7.12}$$

Here we use $|\dot{g}_1(t)| \le K_T$ to obtain the estimate of $|R(t)|$ and $|M(t)|$. From (7.9), (7.10), (7.11) and (7.12) we can get $E[|v(t, X_t, \alpha(t))|] \to 0$ as $t \to T$.

Similarly one obtains that $J(\hat{\pi}) < \infty$. □