Stochastics and Dynamics

MODELLING OF FINANCIAL MARKETS WITH INSIDE INFORMATION IN CONTINUOUS TIME

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We give a survey about the different approaches to model financial markets with inside information in continuous time. In particular, we consider the Karatzas-Pikovsky, Kyle-Back and the weak Kyle-Back approach. These three types of modelling are based on the enlargement of filtration problem, which we explain with some examples and use it for these three modelling approaches.

Keywords: enlargement of filtrations; continuous time market; insider modelling.

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1. Introduction

This paper is devoted to survey some probabilistic techniques and approaches in the modelling of financial markets with asymmetric information. These markets are characterized by the presence of at least two types of agents with different information flows. Although some of such models have been developed for general semimartingale price processes, we will focus our attention in models based on the Brownian motion, as the paradigm of a continuous local martingale. In particular, we are interested in the modelling of privileged information, also known as insider trading. We begin in Section 2 reviewing the main probabilistic techniques used in modelling such markets. We will recall some basic ideas and results on the theory of initial and progressive enlargement of filtrations. These techniques will be critical to make sense of the stochastic integrals involved in the different models and also to find optimal strategies for the insider. Section 3 aims to present the two most popular approaches to insider trading. In Section 3.1 we discuss the basic model of portfolio optimization with privileged information introduced by Karatzas and

Pikovsky in [35]. This model has the nice feature that easily allows to consider different kinds of information but the drawback of allowing infinite expected utilities. In Section 3.2 we introduce the Kyle-Back's model of market equilibrium. This model combines price formation, portfolio optimization and privileged information. It has the advantage that yields finite expected utilities, but the kind of additional information held by the insider is somewhat restrictive. Finally, in Section 3.3 we present a new model of equilibrium with insider trading. In this model we have finite expected utilities and flexibility in the modelling of insider's additional information.

2. Probabilistic techniques

2.1. *Enlargement of filtrations*

The enlargement of filtrations is an important subject in the general theory of stochastic processes, see [31], and its study was initiated by Itô [21] trying to give a meaning to $\int_0^t W_1 dW_s$, where W_t is a Brownian motion. This topic was further studied in the fundamental works of Jeulin [22], Jacod and Yor [24] among others. For up to date references see Yor [37] and Mansuy and Yor [30]. An increasing interest to this question has risen recently from asset pricing models and portfolio optimization problems in stochastic finance. In this area, the enlargement of filtrations theory is an important tool in the modelling of asymmetric information between different agents and the possible additional gain due to this information, as we will see below.

Let us state the problem in a quite general setting. Let $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, P)$ be a filtered probability space satisfying the usual hypotheses. Let $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$ be another filtration satisfying the usual hypotheses and such that $\mathcal{F}_t \subseteq \mathcal{G}_t$, for all $t \geq 0$. Some natural questions are the following: what happen with the $\mathbb{F}$-semimartingales when considered as stochastic processes in the larger filtration $\mathbb{G}$? Do they remain $\mathbb{G}$-semimartingales? If this is the case, we say that $(\mathbb{F}, \mathbb{G})$ satisfy the (H') hypothesis. Moreover, how are their $\mathbb{G}$-semimartingale decompositions? The theory of enlargement of filtrations tries to solve the previous problems. Due to its generality, these questions are only partially answered. The problem, as it is stated above, it is too general to get any result and, therefore, we need to assume some structure to the filtration $\mathbb{G}$. There are, essentially, two ways of enlarging filtrations: *initially and progressively.*

2.1.1. *Initial enlargement of filtrations*

In an initial enlargement, the additional information is a σ-algebra $\mathcal{H}$ and is added to $\mathbb{F}$ at the beginning of the the time interval, i.e., $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}, t \geq 0$.

One of the most important results on initial enlargements comes from Jacod, see pages 15-35 in [24], and deals with the particular case in which the enlarged filtration is of the following form $\mathcal{G}_t \triangleq \bigcap_{\varepsilon>0} (\mathcal{F}_{t+\varepsilon} \vee \sigma(L))$, where L is any $\mathcal{F}$-measurable random variable.

Theorem 2.1 (Jacod's Criterion). *Let L be an $\mathcal{F}$-measurable random variable with values in a standard Borel space*[a] $(E, \mathcal{E})$ *and let* $P_t^L(dx)$ *denote the regular conditional distribution of* L *given* $\mathcal{F}_t, t \geq 0$. *Assume that for each* $t \geq 0$, *there exists a positive* σ*-finite measure* $\nu_t(dx)$ *on* $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ *such that* $P_t^L(dx) \ll \nu_t(dx), P$*-a.s.. Then* $(\mathbb{F}, \mathbb{G})$ *satisfy the* (H') *hypothesis.*

It can be proved that the existence of a positive σ-finite measure $\nu_t(dx)$ is equivalent to the existence of one positive measure $\nu(dx)$ such that $P_t^L(dx) \ll \nu(dx), P$-a.s. and, in this case, it can be taken to be the distribution of L, see for instance Theorem 11, chapter 6 in [36].

Along the paper, $\phi(x,t)$ will denote the density of a centered Gaussian random variable with variance t, the law is denoted by $\mathcal{N}(0,t)$ and $\delta_S(dx)$ will denote the Dirac delta measure on S, where S will be a random variable.

Example 2.1 (Gaussian expansion). Let $\mathbb{F}$ be the standard Brownian filtration satisfying the usual hypotheses, with W a standard Brownian motion. Let $L \triangleq \int_0^\infty g(s)\, dW_s$, where $g \in L^2([0,\infty))$ and let h be a bounded Borel measurable function, then

$$\begin{aligned}\mathbb{E}[h(L)\,|\mathcal{F}_t] &= \mathbb{E}\left[h\left(\int_0^t g(s)\,dW_s + \int_t^\infty g(s)\,dW_s\right)|\mathcal{F}_t\right] \\ &= \int_{\mathbb{R}} h(x)\,\phi\left(x - \int_0^t g(s)\,dW_s, \int_t^\infty g^2(s)\,ds\right)dx.\end{aligned}$$

The previous equation yields that

$$P_t^L(dx) = \phi\left(x - \int_0^t g(s)\,dW_s, \int_t^\infty g^2(s)\,ds\right)dx.$$

Therefore, L conditioned to $\mathcal{F}_t$ is distributed as a $\mathcal{N}\left(\int_0^t g(s)\,dW_s, \int_t^\infty g^2(s)\,ds\right)$. Define $a \triangleq \inf\{t > 0; \int_t^\infty g^2(s)\,ds = 0\}$. If $a = \infty$ one has that

$$P_t^L(dx) \ll \phi\left(x, \int_t^\infty g^2(s)\,ds\right)dx, \quad P\text{-a.s.},$$

and the Jacod's criterion applies. Note, that, actually,

$$P_t^L(dx) \ll \phi\left(x, \int_0^\infty g^2(s)\,ds\right)dx, \quad P\text{-a.s.},$$

which is the law of L. Note that if $L \in \mathcal{F}_1$ then $a = 1$ and $\mathcal{G}_t = \mathcal{F}_t$ for $t \geq 1$. A relevant example of this type of Gaussian expansion is $L = W_1$, which corresponds to the function $g(x) = \mathbf{1}_{[0,1]}(x)$. In this example, Jacod's criterion applies for all $t < 1$, but not for $t \geq 1$, because $P_t^{W_1}(dx) = \delta_{W_1}(dx)$, for $t \geq 1$, which is P-a.s. singular with respect to the Lebesgue measure.

[a] $(E, \mathcal{E})$ is standard Borel space if there is a set $\Gamma \in \mathcal{B}(\mathbb{R})$, where $\mathcal{B}(\mathbb{R})$ is the Borel σ-algebra of $\mathbb{R}$, and an injective mapping $\phi : E \to \Gamma$ such that ϕ is $\mathcal{E}$-measurable and ϕ^{-1} is $\mathcal{B}(\mathbb{R})$-measurable.

Although Jacod's criterion provides a sufficient condition on the initial enlargement to assure that semimartingales remain semimartingales in the new filtration, it does not provide their semimartingale decomposition.

From now on, it will be assumed that $\mathbb{F}$ is the natural filtration generated by a real-valued Brownian motion $W = (W_t)_{t\geq 0}$ and L is a $\mathcal{F}_\infty$-measurable random variable. For any bounded Borel function f, denote by $\{\lambda_t^L(f)\}_{t\geq 0}$ the continuous version of the martingale $\{\mathbb{E}[f(L)|\mathcal{F}_t]\}_{t\geq 0}$. Note that $\lambda_t^L(f) = \int_{\mathbb{R}} f(x) P_t^L(dx)$, where $P_t^L(dx)$ is the regular conditional distribution of L given $\mathcal{F}_t, t \geq 0$. From the representation property of Brownian continuous martingales as stochastic integrals with respect W, one obtains the existence of a predictable process $\{\dot{\lambda}_t^L(f)\}_{t\geq 0}$ such that

$$\lambda_t^L(f) = \mathbb{E}[f(L)] + \int_0^t \dot{\lambda}_s^L(f)\, dW_s, \quad t \geq 0.$$

The next result provides a way to compute the semimartingale decomposition in the enlarged filtration.

Theorem 2.2 (Enlargement formula). *Assume that there exists a predictable family $\{\dot{P}_t^L(dx)\}_{t\geq 0}$ of measures such that $\dot{\lambda}_t^L(f) = \int_{\mathbb{R}} f(x)\dot{P}_t^L(dx)$, $dt \times dP$-a.e. In addition, assume that $dt \times dP$-a.e., the measure $\dot{P}_t^L(dx)$ is absolutely continuous with respect to $P_t^L(dx)$ and denote by $\alpha_t(x)$ its Radon-Nikodym derivative. Then, for any continuous $\mathbb{F}$-martingale (which can be taken of the form $\{M_t = \int_0^t m_s dW_s\}_{t\geq 0}$) there exists a continuous local $\mathbb{G}$-martingale $\{\widetilde{M}_t\}_{t\geq 0}$, such that*

$$M_t = \int_0^t \alpha_s(L)\, d\langle M, W\rangle_s + \widetilde{M}_t = \int_0^t \alpha_s(L)\, m_s ds + \widetilde{M}_t$$

provided that

$$\int_0^t |\alpha_s(L)|\, d|\langle M, W\rangle_s| < \infty, \quad P\text{-a.s.} \tag{2.1}$$

In particular, if $\int_0^t |\alpha_s(L)|\, ds < \infty, P$-a.s., then $\{W\}_{t\geq 0}$ decomposes as $W_t = \int_0^t \alpha_s(L)\, ds + \widetilde{W}_t$, with $\{\widetilde{W}\}_{t\geq 0}$ a $\mathbb{G}$-Brownian motion.

Let us briefly sketch the proof of the previous theorem. Let f be a bounded Borel measurable function, $A \in \mathcal{F}_s, s < t$, then

$$\begin{aligned}
\mathbb{E}[\mathbf{1}_A f(L)(M_t - M_s)] &= \mathbb{E}[\mathbf{1}_A \mathbb{E}[f(L)(M_t - M_s)|\mathcal{F}_s]] \\
&= \mathbb{E}[\mathbf{1}_A \left(\lambda_t^L(f) M_t - \lambda_s^L(f) M_s\right)] \\
&= \mathbb{E}[\mathbf{1}_A \int_s^t m_u \dot{\lambda}_u^L(f)\, du] \\
&= \mathbb{E}[\mathbf{1}_A \int_s^t m_u \left(\int_{\mathbb{R}} f(x)\dot{P}_u^L(dx)\right) du] \\
&= \mathbb{E}[\mathbf{1}_A f(L) \int_s^t m_u \alpha_u(L)\, du].
\end{aligned}$$

By a monotone class argument, the last equality yields that $M_t - \int_0^t \alpha_s(L)\, m_s ds$ is a continuous $\mathbb{G}$-martingale. Then, the process $\{\alpha_t(L)\, m_t\}_{t\geq 0}$ is the *compensator* of M with respect to the filtration $\mathbb{G}$. This terminology is intuitively clear as $\int_0^t \alpha_s(L)\, m_s ds$ compensates the $\mathbb{G}$-semimartingale M to obtain a $\mathbb{G}$-martingale. To find the compensator of a martingale with respect to a initially enlarged filtration, assuming it exists, one usually tries to reproduce the type of computations done in the proof of Theorem 2.2. A difficulty appears when the enlarging random variable is a random time. The strategy then is to compute separately the compensator *before* and *after* the random time and finally check that they paste well. This strategy is carried on for the random time at which the Brownian motion in a finite interval $[0,T]$ achieves its maximum in [27].

The previous theorem is useful, for instance, to compute the semimartingale decomposition in the Gaussian expansion case.

Example 2.2 (Gaussian expansion formulae). For the Gaussian expansion case, $L \triangleq \int_0^\infty g(s)\, dW_s$, with $g \in L^2([0,\infty))$, we have that

$$P_t^L(dx) = \phi\left(x - \int_0^t g(s)\, dW_s, \int_t^\infty g^2(s)\, ds\right) dx$$

and

$$\dot{P}_t^L(dx) = g(t)\, \frac{x - \int_0^t g(s)\, dW_s}{\int_t^\infty g^2(s)\, ds}\phi\left(x - \int_0^t g(s)\, dW_s, \int_t^\infty g^2(s)\, ds\right) dx.$$

Hence, $\alpha_t(x) = g(t)\, \frac{x - \int_0^t g(s)dW_s}{\int_t^\infty g^2(s)ds}$. If we use the Enlargement Formula (Theorem 2.2) with $g(t) = \mathbf{1}_{[0,1]}(t)$ and $M = W$ we obtain that

$$W_t = \int_0^t \alpha_s(L)\, ds + \widetilde{W}_t = \int_0^{t\wedge 1} \frac{W_1 - W_s}{1-s} ds + \widetilde{W}_t, \quad t \geq 0,$$

with $\{\widetilde{W}\}_{t\geq 0}$ a $\mathbb{G}$-Brownian motion. Note that condition (2.1) translates to $\int_0^{t\wedge 1} \left|\frac{W_1 - W_s}{1-s}\right| ds < \infty, P$-a.s., for all $t \geq 0$, which is easy to check. Notice that, for $t \geq 1$, Jacod's condition does not apply and, actually, there exist $\mathbb{F}$-martingales which do not remain semimartingales in the enlarged filtration $\mathbb{G}$, see for instance Proposition 1.7 in [30].

Another example in which Theorem 2.2 can be applied is the Brownian filtration enlarged with the maximum of the Brownian motion in the interval $[0,1]$. In this case, Jacod's criterion does not apply for any $t > 0$, as $P_t^L(dx)$ is not absolutely continuous with respect to the Lebesgue measure due to the appearance of the point mass $\delta_{M_t}(dx)$ in the semimartingale decomposition as we see in the following example.

Example 2.3 (The maximum of Brownian motion). Let $\mathbb{F}$ be the standard Brownian filtration satisfying the usual hypotheses, with W a standard Brownian motion. Let $L = \max_{0 \le t \le 1} W_t$ and $M_t = \max_{0 \le s \le t} W_s$, then it can be proved (see Example 1.7 in [30]) that

$$P_t^L(dx) = R_{1-t}(M_t - W_t)\,\delta_{M_t}(dx) + r_{1-t}(x - W_t)\,\mathbf{1}_{(M_t,\infty)}(x)\,dx,$$
$$\dot{P}_t^L(dx) = -r_{1-t}(M_t - W_t)\,\delta_{M_t}(dx) + \frac{x - W_t}{1-t} r_{1-t}(x - W_t)\,\mathbf{1}_{(M_t,\infty)}(x)\,dx,$$

where

$$r_t(x) = 2\phi(x,t)\,\mathbf{1}_{(0,\infty)}(x), \quad R_t(x) = \int_0^x r_t(y)\,dy, \quad x \ge 0.$$

Therefore, $\alpha_t(x) = \frac{x - W_t}{1-t}\mathbf{1}_{(M_t,\infty)}(x) - \varphi(M_t - W_t, 1-t)\,\mathbf{1}_{\{M_t\}}(x)$, where $\varphi(x,t) \triangleq \frac{r_t(x)}{R_t(x)}$, and

$$W_t = \widetilde{W}_t + \int_0^t \frac{L - W_t}{1-t}\mathbf{1}_{\{M_t < L\}} - \varphi(M_t - W_t, 1-t)\,\mathbf{1}_{\{M_t = L\}} ds, \quad t \ge 0,$$

with $\{\widetilde{W}\}_{t \ge 0}$ a $\mathbb{G}$-Brownian motion.

Imkeller [17] studied the initial enlargement of filtrations with a smooth random variable L in the Malliavin sense, see Nualart [34], section 1.2.. He finds a sufficient condition in terms of the Malliavin derivative DL for Jacod's condition to apply.

Imkeller, Pontier and Weisz [20] develop a Malliavin calculus for measure valued random variables. With this new tool they are able to replace the Jacod's criterion by the following more general and natural one in the setting of Wiener space:

$$(\mathbf{AC}) \qquad k_t(dx) \ll P_t^L(dx), \quad P\text{-a.s., } t \in [0,1],$$

where $k_t(dx) = \lim_{s \downarrow t} D_t P_s^L(dx)$ in the sense of the weak* convergence in $L^2(\Omega \times [0,1])$. Their main findings are summarized in the following theorem.

Theorem 2.3. *Let $\{W_t\}_{t\in[0,1]}$ be a standard Brownian motion and L a $\mathcal{F}_1$-measurable random variable. Assume that (**AC**) is satisfied and denote by $\alpha_t(x)$ a measurable version of the density of $k_t(dx)$ with respect to $P_t^L(dx)$.*

(1) If $\int_0^t |\alpha_s(L)|ds < \infty, P$-a.s. for any $0 \le t < 1$, then, W is a $\mathbb{G}$-semimartingale with the following decomposition $W_t = \widetilde{W}_t + \int_0^t \alpha_s(L)\,ds$.

(2) If $\mathbb{E}[\int_0^1 \alpha_s^2(L)\,ds] < \infty$, then the relative entropy of the conditional law $P_t^L(dx)$ with respect to $P^L(dx)$ (the law of L) is finite. In particular, $P_t^L(dx) \ll P^L(dx), P$-a.s., $t \in [0,1]$.

(3) If $\mathbb{E}[\exp(\int_0^1 \alpha_s^2(L)\,ds)] < \infty$, then $P_t^L(dx)$ is equivalent to $P^L(dx), P$-a.s., $t \in [0,1]$.

2.1.2. *Progressive enlargement of filtrations*

In a progressive enlargement, the additional information is a filtration $\mathbb{H} = (\mathcal{H}_t)_{t\ge0}$ and is added to $\mathbb{F}$ progressively, i.e., $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t, t \ge 0$. The most well studied case in the literature is when the enlarged filtration is of the following form $\mathcal{G}_t :=$

$\bigcap_{\varepsilon>0} \left(\mathcal{F}_{t+\varepsilon} \vee \sigma\left(\Lambda \wedge (t+\varepsilon)\right)\right)$, where Λ is a given random time, that is, a random variable taking values in $[0, \infty]$. Note that this filtration is the smallest filtration (satisfying the usual conditions) that contains $\mathbb{F}$ and makes Λ a stopping time. From now on, we will assume that all $\mathbb{F}$-martingales are continuous, which is sometimes known as condition **(C)** in the literature. Also note that the Brownian filtration satisfies condition **(C)**. Moreover, we will focus our attention to a particular class of random times, the honest times.

Definition 2.1. A random time Λ is said to be honest if Λ is the end of an $\mathbb{F}$-optional set Γ, that is, $\Lambda = \sup\{t \leq \infty : (t, \omega) \in \Gamma\}$.

Honest times constitute a subclass of random times which is large enough to include interesting examples for financial modelling, see [32]. Note that stopping times belong to this class, but this class is strictly larger it contains, for example, the argument of the maximum of a continuous local martingale.

Barlow [7], discovered that $(\mathbb{F}, \mathbb{G})$ automatically satisfy the (H') hypothesis. Therefore, no result such as Jacod's Criterion is needed when dealing with progressive enlargements with honest times. This intuitively means that initial enlargements can add more information than progressive ones, just preventing the (H') hypothesis to hold. The main result in this theory is the following one, due to Jeulin and Yor [23].

Theorem 2.4. *Let $\{M_t\}_{t\geq 0}$ be a $\mathbb{F}$-local martingale, then there exists a $\mathbb{G}$-local martingale $\{\widetilde{M}_t\}_{t\geq 0}$ such that*

$$M_t = \widetilde{M}_t + \int_0^{t\wedge\Lambda} \frac{d\left\langle M, Z^\Lambda \right\rangle_s}{Z^\Lambda_{s-}} - \int_\Lambda^{t\vee\Lambda} \frac{d\left\langle M, Z^\Lambda \right\rangle_s}{1 - Z^\Lambda_{s-}}, \tag{2.2}$$

where $\{Z^\Lambda_t\}_{t\geq 0}$ is the Azema's supermartingale, that is, $Z^\Lambda_t = P(\Lambda > t|\mathcal{F}_t)$. In particular, $(\mathbb{F}, \mathbb{G})$ satisfy the (H') hypothesis.

The explicit computation of Azema's supermartingale $\{Z^\Lambda_t\}_{t\geq 0}$ is, in general, difficult to perform and almost all examples are given in the Brownian motion case, for a list of examples see [30]. Recently, Nikeghbali and Yor [33] have used a multiplicative decomposition of $\{Z^\Lambda_t\}_{t\geq 0}$ to provide further examples of progressive enlargements beyond the Brownian setting.

Example 2.4 (The argument of the maximum of the Brownian motion). Let $\{W_t\}_{t\geq 0}$ be a $\mathbb{F}$-Brownian motion and $M_{s,t} \triangleq \max_{s\leq u\leq t} W_u$, $M_t \triangleq M_{0,t}$ and $\tau \triangleq \arg M_1$. We have

$$\begin{aligned} Z^\tau_t &\triangleq P\left(\tau > t|\mathcal{F}_t\right) = P\left(M_{t,1} \geq M_t|\mathcal{F}_t\right) = P\left(M_{t,1} - W_t \geq M_t - W_t|\mathcal{F}_t\right) \\ &= \int_{M_t - W_t}^{+\infty} 2\phi\left(y, 1-t\right) dy = 2(1 - \Phi\left(M_t - W_t, 1-t\right)), \end{aligned}$$

where $\phi\left(y, t\right)$ and $\Phi\left(y, t\right)$ are the density and the distribution, respectively, of a $\mathcal{N}(0, t)$ random variable. Using that $M_t - W_t$ and $|W_t|$ have the same law, Itô and

Tanaka's formula one obtains the following enlargement formula

$$W_t = \widetilde{W}_t + \int_0^{t\wedge\tau} \frac{\phi\left(M_s - W_s, 1-s\right)}{1-\Phi\left(M_s - W_s, 1-s\right)} ds - \int_\tau^{t\vee\tau} \frac{\phi\left(M_s - W_s, 1-s\right)}{\Phi\left(M_t - W_t, 1-t\right)) - 1/2} ds.$$

Remark 2.1. For arbitrary random times, the (H') hypothesis is not satisfied in general. However, every $\mathbb{F}$-local martingale **stopped** at a random time Λ is a semimartingale with respect to the progressively enlarged filtration with Λ.

3. Modelling approaches to insider trading

3.1. *Karatzas-Pikovsky approach to insider trading*

In the seminal paper [35], Karatzas and Pikovsky used the theory of initial enlargement of filtrations to model financial markets with asymmetric information. Their model is, essentially, the following. Assume $(\Omega, \mathcal{F}, P)$ is a complete probability space equipped with $\mathbb{F} \triangleq \{\mathcal{F}_t^W\}_{t\in[0,1]}$, the natural filtration generated by a Brownian motion W augmented so as to satisfy the usual conditions of completeness and right-continuity. The dynamics of the prices are described by the following stochastic differential equations

$$\begin{aligned} dS_t &= S_t\left(\mu_t dt + \sigma_t dW_t\right), \quad S_0 = s > 0, \\ dB_t &= \rho_t B_t dt, \quad B_0 = 1, \end{aligned}$$

where B_t is the price of the bond and S_t is the price of the risky asset at time t, on the finite horizon $t \in [0,1]$. The following hypotheses are imposed in order to put this model in the framework of the classical model for utility optimization problems, see [25]:

- The coefficients ρ, μ and σ are assumed to be $\mathbb{F}$-progressively measurable processes. Moreover, ρ and μ are assumed to be bounded.
- $\sigma^2 \in L^1\left([0,1]\times\Omega\right).$
- σ is strictly positive for every $(t,\omega) \in [0,1]\times\Omega$.
- $$\mathbb{E}[\int_0^1 \left(\frac{\mu_t - \rho_t}{\sigma_t}\right)^2 dt] < \infty \tag{3.1}$$

Let V_t denote the wealth of the investor at time t and π_t the fraction of the total amount that he invests in the risky asset. A measurable process π which satisfies $\int_0^1 \left(\sigma_t\pi_t\right)^2 dt < \infty$, a.s., will be called a portfolio process.

3.1.1. *Optimization problem*

The objective of the investor is to maximize the expected logarithmic utility from the terminal wealth, by means of choosing an appropriate portfolio process for a fixed initial wealth. Thus, the objective is to find an optimal portfolio $\pi^* \triangleq \arg\max \mathbb{E}[\log V_1^\pi]$, where the maximum is taken over the set of all "admissible" portfolios, and V_t^π represents the wealth, at time t, corresponding to the portfolio process π. Obviously, the question that naturally arises is what is the appropriate class of admissible portfolios. When considering $\mathbb{F}$-adapted portfolios the problem is well known, see for instance [25] and [26]. One has that V^π satisfies the following s.d.e.

$$\begin{aligned} \frac{dV_t^\pi}{V_t^\pi} &= (1-\pi_t)\frac{dB_t}{B_t} + \pi_t \frac{dS_t}{S_t} \\ &= (\rho_t + \pi_t(\mu_t - \rho_t))dt + \pi_t\sigma_t dW_t, \quad V_0^\pi = x > 0. \end{aligned} \tag{3.2}$$

Furthermore, the solution to the previous s.d.e. is

$$V_t^\pi = x\exp\left\{\int_0^t (\rho_u + \pi_u(\mu_u-\rho_u) - \frac{1}{2}\sigma_u^2\pi_u^2)du + \int_0^t \sigma_u\pi_u dW_u\right\}$$

and the optimum is provided by $\pi_t^* = \frac{\mu_t-\rho_t}{\sigma_t^2}, 0 \le t \le 1$, with value

$$\begin{aligned} \mathcal{V}_1^{\mathbb{F}} &\triangleq \mathbb{E}[\log V_1^{\pi^*}] \\ &= \log x + \mathbb{E}[\int_0^1 \rho_u du] + \frac{1}{2}\mathbb{E}[\int_0^1 \left(\frac{\mu_u-\rho_u}{\sigma_u}\right)^2 du]. \end{aligned}$$

Karatzas and Pikovsky study the case in which the portfolios are $\mathbb{G}$-adapted, where $\mathbb{G} = \{\mathcal{G}_t\}_{t\in[0,1]}$ is given by $\mathcal{G}_t \triangleq \mathcal{F}_t^W \vee \sigma(L), 0 \le t \le 1$, where L is an $\mathcal{F}_1^W$-measurable random variable. It is worth noticing that the portfolio π can not be allowed to depend on the whole future at time t. If this were the case, then the value of the problem would clearly become infinite, since an investor could exploit all the fluctuations of the market.

The toy example we consider is $L = W_1$. Then, the first problem is to give sense to the s.d.e. (3.2) for V^π as the coefficients, which depend on π, are non adapted to the Brownian filtration $\mathbb{F}$. To solve this problem, the idea is to use the semimartingale decomposition of W with respect to $\mathbb{G}$, that is, the process

$$\widetilde{W}_t \triangleq W_t - \int_0^t \frac{W_1 - W_u}{1-u}du, \quad 0 \le t \le 1, \tag{3.3}$$

is a $\mathbb{G}$-Brownian motion, see Example 2.2. Therefore,

$$\begin{aligned} \frac{dV_t^\pi}{V_t^\pi} &= (1-\pi_t)\frac{dB_t}{B_t} + \pi_t \frac{dS_t}{S_t} \\ &= (\rho_t + \pi_t(\mu_t - \rho_t))dt + \pi_t\sigma_t dW_t \\ &= (\rho_t + \pi_t(\mu_t - \rho_t + \sigma_t\alpha_t))dt + \pi_t\sigma_t d\widetilde{W}_t \\ V_0^\pi &= x > 0, \end{aligned}$$

where $\alpha_t \triangleq \frac{W_1 - W_t}{1-t}$. Note that, for any $T < 1$, the following bound is satisfied

$$\mathbb{E}[\int_0^T \alpha_u^2 du] = \log\left(\frac{1}{1-T}\right) < \infty. \tag{3.4}$$

Assumptions (3.1) and (3.4) put the model within the framework of the classical theory, as long as $T<1$. Then, for any $\mathbb{G}$-progressively measurable portfolio process π, the solution to the wealth equation is

$$V_T^\pi = x\exp\left\{\int_0^T \{\rho_u + \pi_u(\mu_u - \rho_u + \sigma_u\alpha_u) - \frac{1}{2}\sigma_u^2\pi_u^2\}du + \int_0^T \sigma_u\pi_u dW_u\right\}.$$

Let $\mathcal{A}(\mathbb{G},T)$ the class of $\mathbb{G}$-adapted portfolios on the subinterval $[0,T]$ and $V_T^{\mathbb{G}} \triangleq \max_{\pi\in\mathcal{A}(\mathbb{G},T)} \mathbb{E}[\log V_T^\pi]$. It can be proved that for this problem the optimal portfolio in $\mathcal{A}(\mathbb{G},T)$ has the form $\pi_t^* = \frac{\mu_t-\rho_t}{\sigma_t^2} + \frac{\alpha_t}{\sigma_t}$ and

$$\mathcal{V}_T^{\mathbb{G}} = \mathbb{E}[\log X_T^{\pi^*}] = \log x + \mathbb{E}[\int_0^T \rho_u du] + \frac{1}{2}\mathbb{E}[\int_0^T \left(\frac{\mu_u-\rho_u}{\sigma_u} + \alpha_u\right)^2 du].$$

Notice that,

$$\mathcal{V}_T^{\mathbb{G}} = \mathcal{V}_T^{\mathbb{F}} + \frac{1}{2}\mathbb{E}[\int_0^T \alpha_u^2 du] + \mathbb{E}[\int_0^T \frac{\mu_u-\rho_u}{\sigma_u}\alpha_u du].$$

Furthermore, due to condition (3.1) and decomposition (3.3) we have that

$$\int_0^T \frac{\mu_u-\rho_u}{\sigma_u}\alpha_u du = \int_0^T \frac{\mu_u-\rho_u}{\sigma_u} dW_u + \int_0^T \frac{\mu_u-\rho_u}{\sigma_u} d\widetilde{W}_u,$$

where the last stochastic integral is well defined and, hence, $\mathbb{E}[\int_0^T \frac{\mu_u-\rho_u}{\sigma_u}\alpha_u du] = 0$.

A natural definition for the fair price for the insider additional information $L = W_1$, on the interval $[0,T]$, is

$$\Delta\mathcal{V}_T \triangleq \mathcal{V}_T^{\mathbb{G}} - \mathcal{V}_T^{\mathbb{F}} = \frac{1}{2}\mathbb{E}[\int_0^T \alpha_u^2 du] = \frac{1}{2}\log\left(\frac{1}{1-T}\right), \tag{3.5}$$

which explodes as T tends to 1. Now, one can consider, for every $T<1$, the portfolio $\pi_t^T := \pi_t^* 1_{[0,T]}(t)$, which corresponds to optimal investment up to time T and then to move all the investment to the riskless asset and keep it there until terminal time 1. If $\{T_n\}_{n\geq 1}$ is a sequence strictly increasing to 1, then $\{\pi_t^{T_n}\}_{n\geq 1}$ is a sequence of portfolios which satisfy $\lim_{n\to\infty}\mathbb{E}[\log V_1^{\pi^{T_n}}] = \infty$, and it can be concluded that $\mathcal{V}_1^{\mathbb{G}} = \sup_{\pi\in\mathcal{A}(\mathbb{G},1)}\mathbb{E}[\log V_1^\pi] = \infty$.

We have sketched the proof of the main result in [35], which is the following lemma. Here we present the one dimensional version.

Lemma 3.1 (Characterization lemma). *Assume that for the given $\mathcal{F}_1$-measurable random variable L, we can find a measurable process $\alpha(L) : [0,1)\times\Omega \longrightarrow \mathbb{R}$, such that:*

(1) $\alpha(L)$ is adapted to $\mathbb{G} = \{\mathcal{G}_t\}_{0\leq t\leq 1}$, with $\mathcal{G}_t \triangleq \mathcal{F}_t^W \vee \sigma(L)$ and W a Brownian motion on $[0,1]$;

(2) the process $\widetilde{W}_t \triangleq W_t - \int_0^t \alpha_u(L)\,du$ is a $\mathbb{G}$-Brownian motion on $[0,1)$;

(3) $\mathbb{E}[\int_0^T \alpha_u^2(L)\,du] < \infty$ for any $T<1$.

For every $T \in (0,1]$, let $\mathcal{A}(\mathbb{G},T)$ (resp. $\mathcal{A}(\mathbb{F},T)$) be the class of $\mathbb{G}$ (resp. $\mathbb{F}$)-adapted processes $\pi : [0,T] \times \Omega \longrightarrow \mathbb{R}^d$ with $\int_0^T (\sigma_u \pi_u)^2 du < \infty$, almost surely, and let

$$\mathcal{V}_T^{\mathbb{G}} \triangleq \sup_{\pi \in \mathcal{A}(\mathbb{G},T)} \mathbb{E}[\log V_T^{\pi}], \quad \mathcal{V}_T^{\mathbb{F}} \triangleq \sup_{\pi \in \mathcal{A}(\mathbb{F},T)} \mathbb{E}[\log V_T^{\pi}],$$

denote the values of the portfolio optimization problem over these two respective classes. Then

$$\mathcal{V}_T^{\mathbb{F}} = \log x + \mathbb{E}[\int_0^T (\rho_u + \frac{1}{2}\left(\frac{\mu_u - \rho_u}{\sigma_u^2}\right)^2)du],$$
$$\mathcal{V}_T^{\mathbb{G}} = \mathcal{V}_T^{\mathbb{F}} + \frac{1}{2}\mathbb{E}[\int_0^T \alpha_u^2\,(L)\,du], \quad 0 < T \leq 1,$$

and thus $\mathcal{V}_1^{\mathbb{G}} < \infty \iff \mathbb{E}[\int_0^1 \alpha_u^2\,(L)\,du] < \infty$. When this latter condition is satisfied, an optimal portfolio is given by $\pi_t^* = \frac{(\mu_t - \rho_u)}{\sigma_t^2} + \frac{\alpha_t}{\sigma_t}$.

Remark 3.1. If $L = \max_{0 \leq t \leq 1} W_t$, Example 2.3 yields that there exists a process $\alpha_t\,(L)$ satisfying assumptions (1) and (2) in the previous theorem. However, one can check that for any $T < 1$, $\mathbb{E}[\int_0^T \alpha_u^2\,(L)\,du] = +\infty$. This means that the initial additional information provided by $L = \max_{0 \leq t \leq 1} W_t$ is too informative to yield a realistic model of a financial market.

Remark 3.2. Ankirchner et al.[5], in the continuous semimartingale setting, have proved a kind of reverse of Lemma 3.1. They set a similar optimization problem for an insider and deduce the existence of a semimartingale decomposition of the price process with respect to the insiders filtration whenever the additional expected logarithmic utility of the insider is finite.

3.1.2. *Additional logarithmic utility and entropy*

Karatzas and Pikovsky [35] also noticed the following relationship between the additional utility of the insider and the relative entropy of probability measures. If we assume that the exponential $(P,\mathbb{G})$-local martingale

$$\zeta_t \triangleq \exp\left\{-\int_0^t \alpha_s\,(L)\,d\widetilde{W}_s - \frac{1}{2}\int_0^t \alpha_u^2\,(L)\,du\right\}, \quad 0 \leq t < 1,$$

is actually a martingale, then for every $T \in (0,1), Q_T \triangleq \mathbb{E}[\zeta_T \mathbf{1}_A]$ defines a probability measure on $\mathcal{G}_T$, under which W is a $\mathbb{G}$-Brownian motion. The relative entropy (or Fisher information) of the measure P with respect to Q_T, is defined as

$$\mathcal{H}(P|Q_T) \triangleq \mathbb{E}[\log \frac{dP}{dQ_T}] = \mathbb{E}[\log \zeta_T^{-1}] = \mathbb{E}[\int_0^t \alpha_s\,(L)\,d\widetilde{W}_s + \frac{1}{2}\int_0^t \alpha_u^2\,(L)\,du]$$
$$= \frac{1}{2}\mathbb{E}[\int_0^T \alpha_u^2\,(L)\,du].$$

Moreover, as the family of probability measures $\{Q_T\}_{0\leq T<1}$ is consistent, we can define the relative entropy of the probability measure P with respect to $\{Q_T\}_{0\leq T<1}$ as $\mathcal{H} \triangleq \lim_{T\to 1} \mathcal{H}(P|Q_T)$, and conclude that $\mathcal{H} = \frac{1}{2}\mathbb{E}[\int_0^T \alpha_u^2 du] = \mathcal{V}_T^{\mathbb{G}} - \mathcal{V}_T^{\mathbb{F}}$.

This connection between the additional utility and the entropy was also established in the continuous semimartingale setting by Amendinger et al. [3]. Recently, Ankirchner et al. [4], also in the continuous semimartingale setting but with more general insiders' filtrations (not necessarily obtained by initial or progressive enlargements), have established similar relationships between the insider's additional logarithmic expected utility and several concepts in information theory, such as the Shannon entropy of a filtration.

3.1.3. *Free lunches and arbitrage opportunities*

Imkeller et al. [20] and Imkeller [19] studied the existence of arbitrage and free lunches in the presence of insiders. Their model is essentially the one dimensional Karatzas-Pikovsky with $\rho = 0$ and a slightly more general assumption on the pair of processes (μ_t, σ_t) which determines the risky asset dynamics. They take as portfolio processes the progressively measurable processes π such that $\int_0^1 |\mu_t \pi_t| dt < \infty$ and $\int |\sigma_t \pi_t|^2 dt < \infty, P$-a.s. They consider the excess yield process R and wealth process V^π given by $dR_t = dS_t/S_t$ and $V_t^\pi = \int_0^t \pi_s dR_s, 0 \leq t \leq 1$. Let us recall the classic notions on arbitrage and free lunches, see Delbaen and Schachermayer [14]. A portfolio process is tame if there exists some constant $c \in \mathbb{R}$ such that $V_t^\pi \geq c$ for all $0 \leq t \leq 1$. Let

$$K_0 \triangleq \{V_1^\pi = \int_0^1 \pi_s dR_s : \pi \text{ is tame}\}$$

and let C_0 denote the cone of functions dominated by elements of K_0, that is, $C_0 = K_0 - L_+^0$. Set $C = C_0 \cap L^\infty$.

Condition 3.1 (NA). The semimartingale R is said to satisfy the condition of no arbitrage if $C \cap L_+^\infty = \{0\}$.

Condition 3.2 (NFLVR). The semimartingale R is said to satisfy the condition of no free lunch with vanishing risk if for the closure $\overline{C}$ of C in L^∞ we have $\overline{C} \cap L_+^\infty = \{0\}$.

We assume that the additional information of the insider is given by $L = \sup_{0\leq t\leq 1} W_t$. Hence, for the insider, the risky asset is driven by $(\widetilde{\mu}, \widetilde{\sigma}) = (\mu_t + \sigma_t \alpha_t(L), \sigma_t)$, where

$$\widetilde{W}_t = W_t - \int_0^t \alpha_s(L) ds$$

is a Brownian motion in the insider's filtration. Recall that, given a semimartingale R with respect to a filtration and a probability P, a probability measure Q is called an equivalent martingale measure, if P and Q are equivalent and R is a local martingale with respect to the same filtration and Q.

Theorem 3.3. *R does not satisfy the condition (NFLVR).*

The proof of this result is based in a general classic result of Delbaen and Schachermayer [15] that states the equivalence between R satisfying (NFLVR) and the existence of an equivalent martingale measure that makes R a local martingale. In the Brownian case, if this martingale measure exists, it must have the form

$$\frac{dQ}{dP} = \exp\{-\int_0^1 \alpha_t dM_t - \frac{1}{2}\int_0^1 \alpha_t^2 d\langle M\rangle_t\},$$

if R has the Doob-Meyer decomposition $R_t = M_t + \int_0^t \alpha_s d\langle M\rangle_s$. Therefore, if R satisfies the condition (NFLVR) we have that $M_t = \int_0^t \sigma_s d\widetilde{W}_s$ and $\alpha_t = \frac{\widetilde{\mu}_t}{\sigma_t^2}$. If we define θ_t as the progressively measurable process that satisfies $\theta_t \sigma_t = \widetilde{\mu}_t$, we obtain that

$$\frac{dQ}{dP} = \exp\{-\int_0^1 \theta_t d\widetilde{W}_t - \frac{1}{2}\int_0^1 \theta_t^2 d\langle M\rangle_t\}.$$

Lemma 4.1 in [20] shows that $\int_0^1 \theta_t^2 dt = \infty$ on a set of positive probability. Hence, $\frac{dQ}{dP} = 0$ in a set of positive probability and P and Q can not be equivalent therefore finding a contradiction. In [20] the authors also construct explicit free lunch possibilities, showing that in these cases even the (NA) condition is violated.

The previously cited Lemma 4.1 is of technical nature and makes use of the particular form of L. In [18], Imkeller improved the previous result. He assumes that the insider's filtration is the progressive enlargement of the Brownian filtration with a honest time Λ. Using the general formula (2.2), he is able to prove that the (NFLVR) is not satisfied if $\int_0^t \alpha_s^2(\Lambda) ds = \infty$ in a set of positive probability.

3.1.4. *Imperfect dynamical information*

Corcuera et al. [12] studied a market driven by a Wiener process in which the insider has a privileged information which has been deformed by a independent noise vanishing as the revelation time approaches. Let $\{W_t\}_{t\in[0,T]}$ be a standard Brownian motion defined in a complete probability space $(\Omega, \mathcal{F}, P)$ and denote by $\mathbb{F}^W = \{\mathcal{F}_t^W\}_{t\in[0,T]}$ the P-completed natural filtration generated by W. The additional information until time t, is given by $\mathcal{H}_t \triangleq \sigma\{L_s, s \le t\}$ where the family of random variables $\{L_s, s \le t\}$ has the following structure: $L_t = G(L, Y_t)$, where L is a $\mathcal{F}_T$-measurable random variable (not necessarily a random time), the process $\{Y_t\}_{t\in[0,T]}$ is independent of $\mathcal{F}_T$ and $G : \mathbb{R}^2 \to \mathbb{R}$ is a measurable function. Denote by $\mathbb{G} = \{\mathcal{G}_t\}_{t\in[0,T]}$ the usual augmentation of $\{\mathcal{F}_t^W \vee \mathcal{H}_t\}_{t\in[0,T]}$. Hence, $\mathbb{G}$ is a particular example of progressive enlargement which is different from the progressive enlargement with honest times explained in Section 2.1.2. It turns out that if one has the semimartingale decomposition of W with respect to $\mathbb{G}^L := \{\bigcap_{\varepsilon>0} (\mathcal{F}^W_{(t+\varepsilon)\wedge T} \vee \sigma(L))\}_{t\in[0,T]}$ (that is, $\mathbb{F}$ is initially enlarged with L), then one can obtain the semimartingale decomposition with respect to $\mathbb{G}$. The main result in Corcuera et al. [12] is the following proposition.

Proposition 3.1. *Let L be an $\mathcal{F}_T^W$-measurable random variable and assume that there exists an integrable,$\mathbb{G}^L$-progressively measurable process $\alpha = \{\alpha_t(L)\}_{t\in[0,T)}$, such that $W - \int_0^\cdot \alpha_s(L)\,ds$ is an $\mathbb{G}^L$-Brownian motion. Then, $\widehat{W} \triangleq W - \int_0^\cdot \beta_s ds$ is a $\mathbb{G}$-Brownian motion, where β_t an appropriate version of $\mathbb{E}[\alpha_t(L)|\mathcal{G}_t]$.*

In the previously mentioned article is also proven the following general formula for the compensator in the case of an additive noise.

Proposition 3.2. *Suppose that the assumptions of the previous proposition are fulfilled. Let $t \in [0,T]$, the random variable L_t be given by $L_t = L + Y_t, Y_t = Z_{T-t}$, where $\{Z_t\}_{t\in[0,T]}$ is a continuous process with independent increments, independent of $\mathcal{F}_T^W$ and whose marginal Z_t has density q_t. Then, we have for $t \in [0,T]$*

$$\beta_t = \frac{\int_{\mathbb{R}} \alpha_t(x) q_{T-t}(L_t - x) P_t^L(dx)}{\int_{\mathbb{R}} q_{T-t}(L_t - x) P_t^L(dx)},$$

where we denote by $P_t^L(dx)$ a regular version of the conditional law of L given $\mathcal{F}_t^W$.

In [12], it is proposed the Karatzas-Pikovsky model but with the insider's filtration given by $\mathbb{G}$. Hence, under the hypothesis of Proposition 3.1, one has that the dynamics of the price process is given by

$$\frac{dS_t}{S_t} = (\mu_t + \sigma_t\beta_t)dt + \sigma_t d\widehat{W}_t, \quad S_0 = s > 0.$$

Moreover, the optimization problem for the insider can be solved analogously to the Karatzas-Pikovsky model and yields that the additional expected utility of the insider is given by $\frac{1}{2}\mathbb{E}[\int_0^T \beta_t^2 dt]$.

Example 3.1. Let $L = \max_{0\le t\le T} W_t, Y_t = \overline{W}_{g(T-t)}$ and $L_t = L + Y_t, t \in [0,T]$, where $\overline{W}$ is a Brownian motion independent of $\mathcal{F}_T^W$ and $g : [0,T] \to [0,+\infty)$ is a strictly increasing bounded function with $g(0) = 0$. By Example 2.3, we are under the hypothesis of Proposition 3.1 and, by Proposition 3.2, we obtain

$$\beta_t = \frac{-r_{T-t}(M_t - W_t)\, q_{T-t}(L_t - M_t) + \int_{M_t}^\infty r_{T-t}(x - W_t)\frac{x-W_t}{T-t} q_{T-t}(L_t - x)dx}{R_{T-t}(M_t - W_t)\, q_{T-t}(L_t - M_t) + \int_{M_t}^\infty r_{T-t}(x - W_t) q_{T-t}(L_t - x)dx}$$

for $t \in [0,T]$. Integrating by parts the second expression in the numerator of β_t one obtains that

$$\beta_t = \frac{1}{g(T-t)} \frac{\int_{M_t}^\infty r_{T-t}(x - W_t)(L_t - x) q_{T-t}(L_t - x)dx}{R_{T-t}(M_t - W_t)\, q_{T-t}(L_t - M_t) + \int_{M_t}^\infty r_{T-t}(x - W_t) q_{T-t}(L_t - x)dx}.$$

Then, noting that $P(L \in A|\mathcal{H}_t) = \frac{\int_A q_{T-t}(L_t-x)P_t^L(dx)}{\int_{\mathbb{R}} q_{T-t}(L_t-x)P_t^L(dx)}, A \in \mathcal{B}(\mathbb{R})$, one can show that

$$\beta_t = \frac{1}{g(T-t)}\mathbb{E}[Y_t \mathbf{1}_{\{L>M_t\}}|\mathcal{H}_t].$$

Finally, by applying Cauchy-Schwarz's inequality we have

$$\mathbb{E}[\beta_t^2] \leq \frac{1}{g(T-t)^2}\mathbb{E}[Y_t^2 \mathbf{1}_{\{L>M_t\}}|\mathcal{H}_t] \leq \frac{1}{g(T-t)^2}\mathbb{E}[Y_t^2] = g\,(T-t)^{-1}\,.$$

Therefore, $\mathbb{E}[\int_0^T \beta_t^2 dt] < \infty$ if $\int_0^T g(T-t)^{-1} dt < \infty$. So, if we set $g\,(t) = Kt^p, K > 0, 0 < p < 1$, we obtain that the additional expected utility for $L = \max_{0\leq t\leq T} W_t$ is finite.

To sum up, the approach of imperfect dynamical information allows the construction of better models that the Karatzas-Pikovsky's approach (see Remark 3.1). Moreover, under stronger conditions on g this model, one can also ensure the absence of arbitrage, see page 447 in [12].

The main ideas in [35] have been extended to various frameworks by many authors. Amendinger [1], Grorud and Pontier [16], Amendinger et al. [2] and Amendinger et al. [3] extended these results to the semimartingale setting. Campi [9] studied the problem of quadratic hedging in a semimartingale market with an insider.

It is remarkable that almost all the salient features of insider trading models with strong initial information are already present in [35]. For instance, they studied the case in which the information is distorted by noise and also mention the connection of the additional expected logarithmic utility with the Fisher information. The following is an interesting quote from the authors:

> *Even a small amount of information about the future blows up the value of the problem, provided that this information is exact (i.e. not distorted by noise), and the investor is not bound by constraints.*

3.2. *Kyle-Back approach to insider trading*

In [28], Kyle studied the problem of insider trading from a market microstructure point of view. He proposes a rational equilibrium setup with three different representative agents interacting in the market: an insider, a market maker and a noise trader. He shows how the insider can profit from his private information by rationally anticipating how his orders will influence the market price. Furthermore, the insider hides his trades behind the orders from the noise traders so that the market maker cannot determine his orders and, hence, cannot infer his signal. Kyle's model is discrete in time and assumes normality of all relevant random variables. This assumption makes possible the existence of a unique equilibrium with a linear pricing rule and linear optimal strategy. Kyle also extends his model *heuristically* to continuous time. The continuous-time version of Kyle's model was formalized and extended by Back in [6]. Back solves in closed form the equilibrium pricing rule of market makers, for a rather general distribution of the asset value, not necessarily normal. He also gives an explicit strategy for the insider who maximizes the expected utility of his final wealth. What is more, he finds that, in equilibrium, the price of the risky asset is a geometric Brownian motion, when it is assumed lognormality for the distribution of the insider's information. Here is a description of Back's set up and his paper's main results.

There is to be a public release of information, which reveals the value of a financial asset. The value of the asset is denoted by ξ and the scheduled date of release is normalized to be 1. It is assumed that ξ is the price at which the asset will be traded after the information release. It is also assumed that the market is continuous (in time and in trading quantity) and order driven (this means that the price is set after clearing the orders in the market). Moreover, the risk-free rate is taken to be zero for the sake of simplicity. In this market there are three representative agents:

- *The market maker*: Represents the competitive market making firms, which set the price and clear the market. This agent is assumed to be risk-neutral.
- *The insider or informed agent*: Represents the market participants that know the asset value ξ at the beginning of the trading interval and maximize the expected utility of their wealth.
- *The noise trader*: Represents all the other participants in the market. These participants trade for liquidity or hedging reasons and trade independently of the asset liquidation value.

The previous setting translates into the following mathematical model. Let $(\Omega, \mathcal{F}, P)$ be a probability space in which are defined a random variable ξ and a Brownian motion $Z = \{Z_t\}_{t\in[0,1]}$, independent of ξ, and with variance $\sigma^2 t$. It is assumed that the random variable ξ is square integrable, has convex support and continuous distribution function F_ξ, so that the inverse distribution function F_ξ^{-1} is well defined on the interval $(0,1)$. Let Φ be the distribution function of a standard normal random variable, then the function $h \triangleq F_\xi^{-1} \circ \Phi$ is well defined on $\mathbb{R}$, strictly increasing and $h^{-1}(\xi)$ is a standard normal random variable. The cumulative orders of the noise trader are modelled by the process Z and the cumulative orders of the insider trading are denoted by X. The market maker observes the total cumulative orders $Y_t = X_t + Z_t$, so his filtration is $\mathbb{F}^Y \triangleq \{\mathcal{F}_t^Y\}_{t\in[0,1]}$ and he cannot distinguish between the insider and the noise trader. Let S_t denote the price of the asset at any time $t \in [0,1]$. In this model, S_t will only depend on Y_t, the cumulative orders at time t , and not on the history of orders until time t. Hence, it will be assumed that $S_t = H(t, Y_t)$, for some function H, which is called a pricing rule.

Definition 3.1. Denote by $\mathcal{H}$ the set of functions H satisfying $H \in C^{1,2}((0,1)\times\mathbb{R})$, $H_y(t,y) > 0, \forall t \in [0,1]$ and

$$\mathbb{E}[(H(1,Z_1))^2] < \infty \quad \text{and} \quad \mathbb{E}[\int_0^1 (H(t,Z_t))^2 dt] < \infty.$$

All the pricing rules are assumed to belong to $\mathcal{H}$. The fact that $H \in \mathcal{H}$ is monotone increasing in y for each t yields that the insider can invert H to compute

Y_t at each time t. Furthermore, thanks to the continuity of Z the insider can infer $\{\mathcal{F}_s^Z\}_{0\le s\le t}$ at each time t and hence his filtration is $\mathbb{F}^I := \{\mathcal{F}_t^Z \vee \sigma(\xi)\}_{0\le t\le 1}$.

Definition 3.2. Denote by $\mathcal{X}$ the class of semimartingales X adapted to $\mathbb{F}^I$ such that

$$\mathbb{E}[\int_0^1 H(t, X_{t-} + Z_t))^2 dt] < \infty, \quad \forall H \in \mathcal{H},$$

where the symbol X_{t-} denotes the left limit $\lim_{s\uparrow t} X_s$. A trading strategy for the insider is an element of $\mathcal{X}$.

Let $V = \{V_t\}_{t\in[0,1]}$ denote the wealth of the insider. The dynamics of V is given by $dV_t = X_{t-}dS_t, t \in [0,1]$. Note that, by Itô's formula, the process $S_t = H(t, Y_t)$ will be a semimartingale. Furthermore, the left continuous process X_{t-} is predictable and locally bounded, so the stochastic integral $\int_0^t X_{s-}dS_s$ is well defined. This model allows for the possibility that there will be a jump in the price process after the release of the information at terminal time. Therefore, including the capital gain from such a jump, the final wealth of the informed trader is

$$V_{1+} = (\xi - S_1) X_1 + \int_0^{1-} X_{t-} dS_t,$$

where it is assumed, without loss of generality, that $V_0 = 0$. Integrating by parts, the following alternative expression for V_{1+} is obtained.

$$V_{1+} = \int_0^1 (\xi - S_{t-}) dX_t - [S, X]_1,$$

where $[S, X]$ is the quadratic covariation. Sometimes, we write $V_{1+}(H, X)$ to stress the dependence of the final wealth V_{1+} on the insider's trading strategy X and the pricing rule H.

In order to define an equilibrium, it is needed to specify first what is a rational price and an optimal strategy.

Definition 3.3. Given a trading strategy $X \in \mathcal{X}$, a pricing rule $H \in \mathcal{H}$ is said to be X-rational if $H(t, Y_t) = \mathbb{E}[\xi|\mathcal{F}_t^Y], 0 \le t \le 1$, where $Y_t = X_t + Z_t$.

Definition 3.4. Given a pricing rule $H \in \mathcal{H}$, a trading strategy X^* is said to be H-optimal if $\mathbb{E}[V_{1+}(H, X)] \le \mathbb{E}[V_{1+}(H, X^*)]$, for all $X \in \mathcal{X}$.

Definition 3.5. A couple $(H^*, X^*) \in (\mathcal{H}, \mathcal{X})$ is termed an equilibrium if it verifies

(1) The market efficiency condition: H^* is X^*-rational.
(2) The optimality condition: X^* is H^*-optimal.

The main results in [6] are the following.

Theorem 3.4. *Define*

$$H(t, y) \triangleq \mathbb{E}[h(y + Z_1 - Z_t)], \tag{3.6}$$

where $h \triangleq F_{\xi}^{-1} \circ \Phi$ *and* $X_t \triangleq (1-t)\int_0^t \frac{h^{-1}(\xi)-Z_s}{(1-s)^2}ds$. *Then,* (H, X) *is an equilibrium.*

Theorem 3.5. *The pricing rule (3.6) is the unique equilibrium pricing rule* H *for which there exists a nonnegative, smooth function* $J(v,t,y)$ *on* $supp(\xi) \times \mathbb{R} \times [0,1]$ *satisfying the Bellman equation*

$$\max_{\alpha\in\mathbb{R}}\{J_t(v,t,y) + \frac{1}{2}\sigma^2 J_{yy}(v,t,y) + \alpha(J_y(v,t,y) + (v - H(t,y)))\} = 0 \qquad (3.7)$$

and boundary condition

$$J(v,1,y) > J(v,t,h^{-1}(v)) = 0, \forall v \in supp(\xi), \forall y \neq h^{-1}(v), \qquad (3.8)$$

where $h(\cdot) \triangleq H(1,\cdot)$.

Theorem 3.6. *Let* (H, X) *be an equilibrium. Suppose* H *is such that there exists a smooth solution* J *to the Bellman equation (3.7) and boundary condition (3.8). Then,* $dS_t = H_y(t,Y_t)\,dY_t$, *and the process* Y *is distributed as a Brownian motion with zero drift and variance* σ^2, *given the market maker's filtration. The process* $H(t,Z_t)$ *is a martingale with respect to the insider's filtration. If* F_ξ *has a density function and* $\mathbb{E}[H_y(1,Z_1)] < \infty$, *then the process* $H_y(t,Z_t)$ *is a martingale with respect to the insider's filtration and the process* $H_y(t,Y_t)$ *is a martingale with respect to the market's maker filtration.*

By Theorem 3.4 and the fact that $Z_t = Y_t - X_t$, it is easy to see that the optimal strategy for the insider is given by

$$dX_t = \frac{h^{-1}(\xi) - Y_t}{1-t}dt.$$

Furthermore, by the definition of h one has that $h^{-1}(\xi)$ is normally distributed and independent of Z. Combining these facts, one obtains that Y satisfies

$$dY_t = \frac{h^{-1}(\xi) - Y_t}{1-t}dt + dZ_t,$$

and, hence, from the insider point of view Y is a Brownian bridge, beginning at zero and ending at $h^{-1}(\xi)$. However, by Theorem 3.6 the process Y is a Brownian motion with respect to the market's maker filtration.

The next two examples show the application of the previous results and the type of price processes generated in equilibrium.

Example 3.2 (Normal prices). Assume that ξ has distribution $\mathcal{N}(\mu,\delta^2)$. Then, it is easy to check that $h(y) = \mu + \frac{\delta}{\sigma}y$ and $H(t,y) = \mu + \lambda y$, where $\lambda = \delta/\sigma$. From Theorem 3.6, one obtains that $dS_t = \lambda dY_t$ and Y is viewed as a Brownian motion by the market maker, which yield that the prices follow a Brownian motion. This is a drawback from the modelling point of view, because implies that the prices can be negative.

Example 3.3 (Lognormal prices). Assume that $\xi \sim \text{LogNormal}(\mu,\delta^2)$, that is $\log\xi \sim \mathcal{N}(\mu,\delta^2)$. Then, $h(y) = \exp(\mu+\lambda y)$, where as before $\lambda = \delta/\sigma$ and

$H(t, y) = \exp\left(\mu + \lambda y + \delta^2 (1-t)/2\right)$. Note that $H_y(t, Y_t) = \lambda H(t, Y_t)$ so, from Theorem 3.6, one obtains that $dS_t = \lambda S_t dY_t$ and Y is viewed as a Brownian motion by the market maker, which yield that the prices follow a geometric Brownian motion. The Black-Scholes model of option pricing assumes that the prices of the assets follow geometric Brownian motions. Hence, this model is interesting because we can recover the Black-Scholes model from a market microstructure point of view.

The results of Kyle and Back have been extended in various directions. In [11], Cho allows the pricing rules to take into account the history of cumulative market orders. Furthermore, he also studies the case in which the insider is risk averse and solves the optimization problem using different utility functions. Lasserre [29] extends this model to the multivariate case and also allows the insider to be risk averse. Campi et al. [10] study an equilibrium model, in the Back's sense, for the pricing of a defaultable zero coupon bond issued by a company. They find that in equilibrium the pricing model becomes structural, otherwise being of reduced form, see for instance [8]. Recently, Danilova [13] studies an equilibrium model for asset prices with imperfect dynamic information. She assumes that the insider knows the value of the asset perturbed by the stochastic integral of a deterministic process with respect to a Brownian motion, which is taken independent of the noise trader's demand and the value of the asset.

3.3. *Weak Kyle-Back approach to insider trading*

In [27], the authors have introduced a new approach to model equilibria in financial markets with an insider. Their model is a weaker version of the Kyle-Back's one but it allows to deal with different kinds of insider's information. Moreover, it has the advantage, from the modelling point of view, that the embedded optimization problem has a finite expected utility solution.

As in the Back's model, there is to be a public release of information at time $t = 1$. This information reveals the value of the risky asset, which we denote by ξ. There are also three representative agents in the market: the market maker, the insider and the noise trader. The role of the market maker is to organize the market. That is, according to the asset's aggregate demand, the market maker sets the price of the asset and clears the market. The insider is assumed to know at the beginning of the trading period some additional strong information, say $\lambda = L(Y)$, not necessarily equal to ξ, which depends exclusively on the total demand Y. This agent uses this information in order to maximize his/her expected profit. The noise trader represents all the other participants in the market. Noise trader's orders are a consequence of liquidity or hedging issues and are assumed to be independent of λ, but not necessarily of ξ.

The goal of the article is to construct a probability sample space where such a market can be realized.

Given a class of pricing rules $\mathcal{H}$ and a suitable space of $\mathbb{F}^Z \vee \sigma(\lambda)$- adapted strategies $\Theta_{\sup}(\lambda, Z)$, where λ is a random variable and Z is a process (see Definitions 2.2. and 2.3. in [27] for further details), the definition of a weak equilibrium is the following.

Definition 3.6 (Weak Equilibrium). Let $L : C[0,1] \to \mathbb{R}^k$ be a measurable functional on the canonical Wiener space and μ be a probability measure on $\mathbb{R}$ with $\int_{\mathbb{R}} x^2 \mu(dx) < \infty$. We say that there exists a (L, μ)-weak equilibrium if there exists some probability space $(\Omega, \mathcal{F}, P)$ where there exists three processes Y^*, θ^* and Z^*, a random variable ξ^*, a random vector λ^* and a function $H^* \in \mathcal{H}$ such that

i) $Y_t^* = X_t^* + Z_t^*$, where $X_t^* = \int_0^t \theta_s^* ds$ for $t \in [0,1]$.
ii) $\lambda^* = L(Y^*)$ is independent of the process Z^*.
iii) Z^* is a Brownian motion.
iv) ξ^* has the law μ.
v) $\theta^* \in \Theta_{\sup}(\lambda^*, Z^*)$.
vi) Prices are rational. That is, $S_t^* \triangleq H^*(t, Y_t^*) = \mathbb{E}[\xi^* | \mathcal{F}_t^{Y^*}]$ for $t \in [0,1]$.
vii) For all $\theta \in \Theta_{\sup}(\lambda^*, Z^*)$, one has

$$\mathbb{E}[V(X, S, \xi^*)] \leq \mathbb{E}[V(X^*, S^*, \xi^*)],$$

where $X_\cdot = \int_0^\cdot \theta_s ds$, $Y^\theta = X + Z^*$, $S_\cdot = H^*(\cdot, Y_\cdot^\theta)$ and

$$V(X, S, \xi) = V_0 + \int_0^1 X_s dS_s + (\xi - S_1) X_1.$$

This formulation is weak in the sense that the vector (ξ, λ, Z) is not given beforehand, in contrast with the previous literature on this subject. The initial data in this formulation is (μ, L) where μ is the law of ξ and the other ingredients of an equilibrium are part of the problem. The mathematical motivation for using a weak set-up is due to the fact that in a strong formulation the relationship between λ, ξ and Z can not be simply stated in general. This relationship is not unique if one only wants to give as initial data the law of the final price. Furthermore, in general, ξ is not independent of λ or Z. However, it is assumed that ξ is made public at the end of the trading period. Hence, ξ is incorporated in the functional to be optimized in the equilibrium.

The difference between the classical notion of equilibrium in insider trading (see Section 3.2, Definition 3.5), and the one proposed here is that in Back the information is exogenously given while here is also part of the definition of weak equilibrium.

In particular, condition vii) in Definition 3.6 states that if we fix the noise trade process and the information, the strategy used by the insider maximizes his expected final wealth within a suitable admissible space. This condition can be also interpreted as a local equilibrium condition because the noise trade process and the information are fixed.

For a process W and a random variable λ, we shall denote by $\mathbb{F}^W \vee \sigma(\lambda)$, the initial enlargement of $\mathbb{F}^W = \{\sigma(\{W_s\}_{0 \leq s \leq t})\}_{0 \leq t \leq 1}$ with λ. The main result in [27] is the following. There is also a uniqueness in law result for weak equilibriums, see Theorem 5.3. in [27].

Theorem 3.7 (Existence). *Given a measurable functional* $L : \mathcal{C}[0,1] \to \mathbb{R}^k$ *and* μ *a probability measure on* $\mathbb{R}$ *satisfying* $\int_{\mathbb{R}} x^2 \mu(dx) < \infty$. *Assume:*

1) *There exists* $H \in \mathcal{H}$ *such that it satisfies*

$$H_t(t,y) + \frac{1}{2} H_{yy}(t,y) = 0,$$

and

$$\mu(A) = \frac{1}{\sqrt{2\pi}} \int_{H(1,\cdot)^{-1}(A)} e^{-x^2/2} dx, \quad \forall A \in \mathcal{B}(\mathbb{R}).$$

2) *There exists a probability space* $(\Omega, \mathcal{F}, P)$ *supporting a Brownian motion* W *which is a semimartingale in the filtration* $\mathbb{F}^W \vee \sigma(\lambda)$, $\lambda \triangleq L(W)$, *with semimartingale decomposition* $W_t = \int_0^t \alpha_s(\lambda) ds + W_t^\lambda$, *where* W^λ *is a* $\mathbb{F}^W \vee \sigma(\lambda)$*-Brownian motion.*

3) $\alpha \in \Theta_{\sup}(\lambda, W^\lambda)$.

Then

$$(Y^*, \theta^*, Z^*, H^*, \xi^*, \lambda^*) = (W, \alpha(\cdot, \lambda), W^\lambda, H, H(1, W_1), \lambda)$$

is a (L,μ)*-weak equilibrium.*

Let us discuss a little bit the assumptions in the previous result. Assumption 1), essentially, tells us that the law μ must be a smooth transformation of a $\mathcal{N}(0,1)$ law. Assumption 2), states that the functional L, when applied to a Brownian motion W, must give a random variable λ for which W remains a semimartingale with respect to the initially enlarged filtration $\mathbb{F}^W \vee \sigma(\lambda)$. Furthermore, for this theorem to be useful, we need to know the compensator $\alpha_t(\lambda)$. Finally, Assumption 3) is the hardest hypothesis to check in particular examples. Regarding the integrability conditions which define $\Theta_{\sup}(\lambda, W^\lambda)$, Proposition 5.3. in [27] yields that if W is a semimartingale in the enlarged filtration we only need to check that the compensator belongs to $L^1(\Omega \times [0,1])$. This can be difficult to check in particular examples and it does not need to be true in general. On the other hand, the fact that $\alpha(\lambda)$ must be $\mathbb{F}^{W^\lambda} \vee \sigma(\lambda)$-adapted or, equivalently, the fact that $\mathbb{F}^W \vee \sigma(\lambda) = \mathbb{F}^{W^\lambda} \vee \sigma(\lambda)$ is even harder to check. That $\mathbb{F}^W \vee \sigma(\lambda) \supseteq \mathbb{F}^{W^\lambda} \vee \sigma(\lambda)$ is obvious because W^λ is a $\mathbb{F}^W \vee \sigma(\lambda)$-Brownian motion, but for the reverse inclusion we can not say anything in general. The approach to solve the problem is to show the existence and uniqueness of strong solutions of s.d.e.'s of the form

$$X_t = \int_0^t \alpha_s(G) ds + U_t, \tag{3.9}$$

where U is a Brownian motion and G is a random variable independent of U and with the appropriate support. This entails the $\mathbb{F}^U \vee \sigma(G)$ adaptedness of X. Substituting U by W^λ and G by λ one obtains that $\mathbb{F}^W \vee \sigma(\lambda) \subseteq \mathbb{F}^{W^\lambda} \vee \sigma(\lambda)$. The classical results on the existence and uniqueness of s.d.e.'s do not directly apply to s.d.e.'s of the form (3.9), because the drift α usually degenerates at a random point of the interval $[0,1]$. Note also that, in general, $\alpha_t(G)$ will depend on $X_{[0,t]}$, the path of X until t, not just X_t.

It should be noted that, contrasting with the previous literature on the subject, the portfolio optimization of the insider is not solved with Hamilton-Jacobi-Bellman techniques but using variational calculus, see Section 3 in [27].

The main examples studied in [27] are $L(Y) = \max_{0\leq t\leq 1} Y_t$ and $L(Y) = \arg\max_{0\leq t\leq 1} Y_t$, for both it is proved the existence of a weak equilibrium. It is important to notice again that the insider's optimization problem associated to these examples give finite expected utilities, contrasting with the same examples in the Karatzas-Pikovsky setting. As a by-product of the study of $L(Y) = \arg\max_{0\leq t\leq 1} Y_t$ it is obtained the semimartingale decomposition of a Brownian motion with respect to its natural filtration enlarged with the argument of the maximum.

Theorem 3.8. *Let $W = \{W_t\}_{0\leq t\leq 1}$ be a Brownian motion and $\tau \triangleq \arg\max_{0\leq t\leq 1} W_t$. Then W is a $\mathbb{F}^W \vee \sigma(\tau)$-semimartingale with the following decomposition $W_t = \int_0^t \alpha_s(\tau)\, ds + W_t^\tau$, where*

$$\alpha_s(\tau) = \frac{M_s - W_s}{\tau - s}\mathbf{1}_{[0,\tau)}(s) - \varphi(M - W_s, 1-s)\,\mathbf{1}_{[\tau,1]}(s),$$

W^τ is a $\mathbb{F}^W \vee \sigma(\tau)$-Brownian motion and φ is defined in Example 2.3.

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