Weak Kyle-Back equilibrium models for Max and ArgMax

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Abstract

The goal of this article is to introduce a new approach to model equilibrium in financial markets with an insider. We prove the existence and uniqueness in law of equilibrium for these markets. Our setting is weaker than Back's one and it can be interpreted as a first theoretical step towards developing statistical test procedures. Additionally, it allows various forms of insider information to be considered under the same framework and compared. As major examples, we consider the cases of the maximum of the demand and the time at which this maximum is taken, which have not been previously treated in the literature of equilibrium in financial markets with inside information. Simulations indicate that the expected wealth for the maximum is greater than the expected wealth for its argument.

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1 Introduction

In recent years, the study of mathematical models for financial markets with asymmetry of information has been gaining an increasing attention from mathematical finance researchers. In a seminal paper and from the market microstructure point of view, Kyle [16] introduced a model in which an insider, who knows the value of the stock at some future time, optimizes his wealth while the market-maker makes prices rational, that is, a rational expectations equilibrium model. The main features of Kyle's model are that it gives finite utilities and that it is a model of price formation. That is, the insider controls the price process through his demand of stock shares. Kyle's model has been extended by Back [2], Lasserre [17], Cho [5] and Campi and Çetin [4], among others.

We consider a continuous time market composed of one risk-free asset and one risky asset. We assume, without loss of generality, that the risk-free rate is zero. Trading in the risky asset is continuous in time and quantity. Furthermore, the market is order-driven, that is, prices are determined by the demand on the risky asset.

There is to be a public release of information at time $t = 1$. This information reveals the value of the risky asset, which we denote by ξ. As the market is order-driven, this entails that ξ will be the price at which the asset will be traded just after the release of information and, therefore, the final profit obtained through trading on this asset will depend on ξ.

There are three representative agents in the market: the market maker, the insider and the noise trader. The role of the market maker is to organize the market. That is, according to the asset's aggregate demand, the market maker sets the price of the asset and clears the market. The insider is assumed to

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know at the beginning of the trading period some strong information, say $\lambda = L(Y)$, not necessarily equal to ξ, which depends exclusively on the total demand Y. This agent uses this information in order to maximize his/her expected profit. The noise trader represents all the other participants in the market. Noise trader's orders are a consequence of liquidity or hedging issues and are assumed to be independent of λ, but not necessarily of ξ. Thanks to the demand of the noise trader, denoted by Z, the market maker cannot observe the demand of the insider.

Our formulation is weak in the sense that the vector (ξ,λ,Z) is not given beforehand, in contrast with the previous literature on this subject. The initial data in our formulation is (μ,L) where μ is the law of ξ and the other ingredients of an equilibrium are part of the problem. The mathematical motivation for using a weak set-up is due to the fact that in a strong formulation the relationship between λ,ξ and Z can not be simply stated in general. This relationship is not unique if one only wants to give as initial data the law of the final price. Furthermore, in general, ξ is not independent of λ or Z. However, it is assumed that ξ is made public at the end of the trading period. Hence, ξ is incorporated in the functional to be optimized in the equilibrium.

From the economic modelling point of view, the situation can be explained as follows. Suppose the existence of a financial controller, say a member of an exchange commission, which would like to test after the time interval has been totally observed (say $[0,1]$) the large trader/insider behavior in a sector of the market. By a sector, we understand a collection of homogeneous companies sharing a similar activity, for which one can assign a law for ξ, its value at $t=1$. The financial controller observes the data for different companies in the sector and after some renormalization we can regard the data as different realizations or sample points in his universe.

With the data, a law μ for ξ can be inferred and a functional L of the total demand is fixed for testing. The first step for the controller is to know if it is possible that there exists insiders trading in the stocks of this sector using the information $L(Y)$ and being in equilibrium. Our paper addresses this question. The next step would be to design a statistical test according to the probabilistic properties of the equilibrium, but we do not pursue this goal in this paper.

Now, we briefly discuss the concept of weak equilibrium used in this paper.

The difference between the classical notion of equilibrium used in Back [2], Section 1, and the one proposed here is that in Back the information is exogenously given while here is also part of the definition of weak equilibrium.

In particular, condition vii) in Definition 7 states that if we fix the noise trade process and the information, the strategy used by the insider maximizes his expected final wealth within a suitable admissible space. This condition can be also interpreted as a local equilibrium condition because the noise trade process and the information are fixed.

This interpretation is linked to the notion of partial (or local) equilibrium. If the insider finds himself/herself at such partial equilibrium point there is no particular reason to move from such a point.

From the point of view of a financial controller, the procedure is carried out after all the data is available. That is, the final price has already been announced and the controller wants to test the existence of some insiders in the market. Once the type of information is selected, one can statistically check if the strategy used by the insider(s) is locally optimal.

It is important to point out that this optimal strategy has the same functional form as the compensator of the Brownian motion W with respect to its natural filtration enlarged with the random variable $L(W)$.

Besides the weak equilibrium feature, there are various delicate mathematical points where our results and techniques differ from previously mentioned research. Briefly summarizing, we mention:

1) Due to the generality of the functional L, we use variational calculus (or dynamic principle) and we do not obtain an HJB equation formulation. In particular, optimal strategies do not depend only on the insider's additional information and the value process, the admissible strategies do not form a linear space and the expected profit depends on ξ which it is not measurable with respect to the insider's filtration. These features introduce some difficulties in order to obtain the optimality results.

2) One of the conditions of admissibility require that the optimal strategy has to be adapted to the filtration generated by noise traders process and the insider information. This result which was

easy to obtain in previous articles (in fact, this was just a property of Brownian bridges) becomes extremely difficult in the generality presented here. In fact, we consider as examples the case where $L(Y) = \max_t Y_t$ corresponds to the maximum of the demand and to the argument of this maximum $L(Y) = \arg\max_t Y_t$. This leads to the study of existence and uniqueness of solutions of stochastic differential equations with path dependent coefficients which degenerate at random times.

Finally, we compare the expected wealth obtained by the large trader/insider in the two main examples considered. The simulations indicate that knowing τ, the time at which the maximum of the total demand is achieved, gives less expected wealth than knowing M, the maximum of the total demand.

As a final remark, we want to state that one of the main goals of the article is to raise/contribute to the discussion on the issue of the equilibrium concept for insider-large trader for general information as it is explained in the article. We do not pretend that this is the unique way to solve the problem. We hope that other researchers will also present alternative proposals and comments on this model.

The paper is organized as follows. In Section 2 we give some basic definitions and introduce our weak formulation of equilibrium. Section 3 contains the discussion of the optimization problem for the insider and a optimality equation is deduced. In Section 4 we relate the properties of the solutions of the optimality equation with the rationality of prices. Section 5 is devoted to state the main results on the existence and uniqueness in law of a weak equilibrium. In Section 6 we deal with two basic examples, previously treated in the literature, the Back's example and an example on binary information. Section 7 aims to introduce two new examples in the literature of equilibrium for asymmetric markets. First, in the case that the additional information held by the insider is the maximum of the total demand and second, in the case of the time at which this maximum is attained. We state and prove the existence and uniqueness in law of a weak equilibrium in both cases. Finally, we numerically compare the expected wealth obtained by the insider in these two examples. Section 8 is dedicated to the conclusions. Finally, Section 9 contains an appendix devoted to prove some technical results.

Throughout the article C will denote a constant that may change from line to line. $\mathscr{L}(\mathscr{X})$ denotes the law of the random element $\mathscr{X}$.

2 Weak formulation of equilibrium

In this section we introduce the concept of weak equilibrium. First we define the class of pricing rules and admissible strategies.

Definition 1 *We say that a function $F : [0,1] \times \mathbb{R} \to \mathbb{R}$ satisfies an exponential growth condition if there exist positive constants A, B such that $|F(t,y)| \leq Ae^{B|y|}$, for all $(t,y) \in [0,1] \times \mathbb{R}$.*

Definition 2 *A pricing rule is a function $H \in \mathscr{C}^{1,3}((0,1) \times \mathbb{R})$, such that $H_y(t,y) > 0, \forall t \in [0,1]$ and H, H_t, H_y, H_{yy} satisfy an exponential growth condition. We denote by $\mathscr{H}$ the set of functions H satisfying these properties.*

In the previous definition we require the pricing rules to satisfy some regularity and growth conditions for technical reasons. From a modeling point of view, the important assumption is the requirement that $H_y(t,y) > 0, \forall t \in [0,1]$. This implies that the insider can invert the price process to obtain the total demand and, hence, the noise trader demand, see Remark 8 c) below.

Definition 3 *Given a process Z and a random variable M, we define $\Theta_{\sup}(M,Z)$ as the class of $\mathbb{F}^I = \mathbb{F}^Z \vee \sigma(M)$-adapted càglàd processes in $[0,1)$ which satisfy*

$$\int_0^1 |\theta_s|\, ds \in L^1(\Omega), \tag{1}$$

$$\sup_{0\leq t\leq 1} \left| \int_0^t H\left(s, \int_0^s \theta_u du + Z_s\right) \theta_s ds \right| \in L^1(\Omega), \tag{2}$$

$$\sup_{0\leq t\leq 1}\left|\int_0^t H_y\left(s,\int_0^s \theta_u du+Z_s\right)\theta_s ds\right| \in L^{1+\varepsilon}(\Omega)\,,\ \textit{for some } \varepsilon>0 \tag{3}$$

and

$$\exp\left(C\sup_{0\leq t\leq 1}\left|\int_0^t \theta_s ds\right|\right)\in L^1(\Omega),\quad \forall C>0 \tag{4}$$

for all $H\in\mathscr{H}$.

Remark 4 *We could replace the technical conditions* $(1),(2),(3)$ *and* (4) *in the definition of* $\Theta_{\sup}(M,Z)$ *by the stronger ones*

$$\int_0^1 |\theta_s|^{1+\varepsilon} ds\in L^1(\Omega),\ \textit{for some } \varepsilon>0 \tag{5}$$

and

$$\exp\left(C\sup_{0\leq t\leq 1}\left|\int_0^t \theta_s ds\right|\right)\in L^1(\Omega),\quad \forall C>0. \tag{6}$$

The advantage is that conditions (5) *and* (6) *define a linear space. On the other hand, condition* (5) *is difficult to verify in specific examples.*

Definition 5 *Given a process Z and a random variable M, independent of Z, we say that a process X is a (M,Z)-strategy process if there exists $\theta\in\Theta_{\sup}(M,Z)$ such that $X_t=\int_0^t\theta_s ds, t\in[0,1]$.*

Definition 6 *Given a final price $\xi\in L^2(\Omega)$, a stochastic process Z, a random variable M, independent of Z, a price semimartingale process $P\equiv(P_t)_{t\in[0,1]}$ with respect to $\mathbb{F}^Z\vee\sigma(M)$ and a (M,Z)-strategy process $X=(X_t)_{t\in[0,1]}$, we denote by $V=V(X,P,\xi)$ the agent final wealth defined by*

$$V(X,P,\xi)=V_0+\int_0^1 X_s dP_s+(\xi-P_1)X_1,$$

whenever the above stochastic integral is well defined. Here V_0 is a constant.

Definition 7 (Weak Equilibrium) *Let $L:C[0,1]\to\mathbb{R}^k$ be a measurable functional on the canonical Wiener space and μ be a probability measure on $\mathbb{R}$ with $\int_{\mathbb{R}}x^2\mu(dx)<\infty$. We say that there exists a (L,μ)-weak equilibrium if there exists some probability space $(\Omega,\mathscr{F},P)$ where there exists three processes Y^*, θ^* and Z^*, a random variable ξ^*, a random vector λ^* and a function $H^*\in\mathscr{H}$ such that*

i) $Y_t^*=X_t^*+Z_t^*$, where $X_t^*=\int_0^t\theta_s^* ds$ for $t\in[0,1]$.

ii) $\lambda^*=L(Y^*)$ is independent of the process Z^*.

iii) Z^* is a Brownian motion.

iv) ξ^* has the law μ.

v) $\theta^*\in\Theta_{\sup}(\lambda^*,Z^*)$.

vi) Prices are rational. That is, $P_t^*\triangleq H^*(t,Y_t^*)=\mathbb{E}[\xi^*|\mathscr{F}_t^{Y^*}]$ for $t\in[0,1]$.

vii) For all $\theta\in\Theta_{\sup}(\lambda^*,Z^*)$, one has

$$\mathbb{E}[V(X,P,\xi^*)]\leq\mathbb{E}[V(X^*,P^*,\xi^*)],$$

where $X_\cdot=\int_0^\cdot\theta_s ds$, $Y^\theta=X+Z^*$ and $P_\cdot=H^*(\cdot,Y_\cdot^\theta)$.

Now we give a series of remarks related to this definition.

Remark 8 *a)It is clear that Z^* is a $\mathbb{F}^{Z^*} \vee \sigma(\lambda^*)$-Brownian motion, as Z^* is adapted to this filtration and is independent of λ^*.*
b)The price of the asset will be equal to ξ^*, just after the release of information at time $t = 1$. This price has to have the pre-specified law μ. Furthermore the relationship between ξ^* and λ^* is specified through the rationality of prices (property vi) above) and in general ξ^* is not independent from Z^*.
c)The natural definition of the insider filtration is $\mathbb{F}^* = \mathbb{F}^{Z^*} \vee \sigma(\lambda^*)$. This is due to the monotonicity of the pricing rule and the fact that the insider observes the prices, one has that at time t the insider can infer Z_t^*. Note that in general $\mathbb{F}^{Y^*}$ is not necessarily included in $\mathbb{F}^{Z^*}$.
d)The set of (λ^*, Z^*)-strategies is usually non empty. Furthermore, in the optimization problem vii), one may think that is more natural to restrict the opportunity set of strategies to the ones satisfying $L(Y^\theta) = \lambda^*$. This means that the insider would realize the giving strong information on the total demand. In the next section, we will see that the optimum, with or without this restriction, it is the same given condition ii) in the Definition 7.
e)From now on we will always assume that μ is a probability measure on $\mathbb{R}$ satisfying $\int_{\mathbb{R}} x^2 \mu(dx) < \infty$ without any further mention.

Note that in the above (partial) equilibrium set up the insider optimizes his expected profit given the information λ^* and Z^*. In this aspect, the above equilibrium is a partial one. In other words, if the agent uses the strategy θ^*, there is no (local) reason to change strategy. It can also be considered as an stable point where the insider can actually realize all the conditions for a stable market. The above set-up and the subsequent proofs to follow are not constructive.

3 Optimization problem for the insider

In this section we give necessary conditions for a process to solve the optimization problem stated in property vii)**.** Given a Wiener process Z and a fixed random variable M, which is independent of Z, we define $\mathbb{F}^I = \mathbb{F}^Z \vee \sigma(M)$. As pointed out in the introduction, we use the classical approach of variational calculus.

From now on, we denote by a super-index θ on Y_t the dependence of the total demand on the strategy of the insider. Then, $Y_t^\theta = \int_0^t \theta_s ds + Z_t$. Before studying the optimization problem we remark the following property for the portfolio process θ.

Lemma 9 *If $\theta \in \Theta_{\sup}(M,Z)$ then the price process $P_t^\theta = H\left(t, Y_t^\theta\right)$ is a $\mathbb{F}^I$-semimartingale and its decomposition is given by*

$$P_t^\theta = P_0^\theta + \int_0^t \{H_t(s,Y_s^\theta) + \frac{1}{2}H_{yy}(s,Y_s^\theta) + H_y(s,Y_s^\theta)\theta_s\}ds + \int_0^t H_y(s,Y_s^\theta)dZ_s.$$

The proof is a straightforward application of Itô's formula, as $H \in C^{1,2}((0,1)\times\mathbb{R})$ and Y^θ is a semimartingale in the filtration $\mathbb{F}^I = \mathbb{F}^Z \vee \sigma(M)$. The next lemma is obtained using the integration by parts formula.

Lemma 10 *Let θ be any $\mathbb{F}^I$-adapted process such that $\int_0^1 |\theta_s| ds < \infty$ a.s. Then, we have that*

$$V \triangleq V(X, P^\theta, \xi) = V_0 + \int_0^1 (\xi - H(t, Y_t^\theta))\theta_t dt.$$

Without loss of generality we assume from now on that $V_0 = 0$. The optimization problem we consider in this section is $\max_{\theta \in \Theta_{\sup}(M,Z)} J(\theta)$, where

$$J(\theta) \triangleq \mathbb{E}\left[\int_0^1 (\xi - H(t, Y_t^\theta))\theta_t dt\right], \quad \theta \in \Theta_{\sup}(M,Z). \tag{7}$$

We also denote by $\theta^* \triangleq \arg \max_{\theta \in \Theta_{\sup}(M,Z)} J(\theta)$ when this process exists. The difficulties to solve this problem are due to the nonlinearity of the functional J and the fact that $\Theta_{\sup}(M,Z)$ is not a linear space.

Remark 11 *Note that, for* $\theta \in \Theta_{\sup}(M,Z)$ *, we have that*

$$|J(\theta)| \leq \mathbb{E}\left[|\xi|\left|\int_0^1 \theta_t dt\right|\right] + \mathbb{E}\left[\left|\int_0^1 H(t,Y_t^\theta)\theta_t dt\right|\right] < \infty,$$

due to $\xi \in L^2(\Omega)$ *and that* θ *satisfies conditions* (4) *and* (2)*. Furthermore, if* θ *is* $\mathbb{F}^I$*-adapted and satisfies the integrability conditions that define* $\Theta_{\sup}(M,Z)$*, but is not necessarily càglàd, then we also have that* $|J(\theta)| < \infty$*.*

The first step in our strategy to solve the problem is to study the properties of $J(\theta)$ in the following linear subset of $\Theta_{\sup}(M,Z)$:

$$\Theta_b(M,Z) = \{\theta \in \Theta_{\sup}(M,Z) : \text{there exists } K > 0, \text{ s.t. } \forall \omega, \ |\theta_s(\omega)| \leq K\}.$$

Lemma 12 *If* $v, \theta \in \Theta_b(M,Z)$*, then*

$$\begin{aligned} D_v J(\theta) &\triangleq \frac{d}{d\varepsilon} J(\theta + \varepsilon v)|_{\varepsilon=0} \\ &= \mathbb{E}\left[\int_0^1 v_t(\xi - H(t,Y_t^\theta))dt\right] - \mathbb{E}\left[\int_0^1 \left(\int_0^t v_s ds\right) H_y(t,Y_t^\theta)\theta_t dt\right]. \end{aligned} \tag{8}$$

Furthermore, the operator $D_\cdot J(\theta) : v \longrightarrow D_v J(\theta)$ *is linear.*

Proof. First note that for any θ, $v \in \Theta_b(M,Z)$ and $\varepsilon > 0$ one has that $\theta + \varepsilon v \in \Theta_b(M,Z)$ and differentiating under the integral sign (see Lemma 46) and applying Fubini's theorem, one obtains

$$\begin{aligned} \frac{d}{d\varepsilon} J(\theta + \varepsilon v)|_{\varepsilon=0} &= \mathbb{E}\left[\int_0^1 v_t\{(\xi - H(t,Y_t^\theta)) - \int_t^1 H_y(s,Y_s^\theta)\theta_s ds\} dt\right] \\ &= \mathbb{E}\left[\int_0^1 v_t(\xi - H(t,Y_t^\theta))dt\right] - \mathbb{E}\left[\int_0^1 \left(\int_0^t v_s ds\right) H_y(t,Y_t^\theta)\theta_t dt\right]. \end{aligned}$$

■

Remark 13 *If* $\theta \in \Theta_{\sup}(M,Z)$ *is such that*

$$\mathbb{E}\left[(\xi - H(t,Y_t^\theta)) - \int_t^1 H_y(s,Y_s^\theta)\theta_s ds \,|\mathscr{F}_t^I\right] = 0, \tag{9}$$

then for any $v \in \Theta_b(M,Z)$ *we have*

$$\mathbb{E}\left[\int_0^1 v_t(\xi - H(t,Y_t^\theta))dt\right] - \mathbb{E}\left[\int_0^1 \left(\int_0^t v_s ds\right) H_y(t,Y_t^\theta)\theta_t dt\right] = 0.$$

From now on, we refer to equation (9) *as the optimality equation.*

The next step is to prove the concavity of $J(\theta)$ in $\Theta_b(M,Z)$. This is done in the following proposition, which makes use of a general result on convex analysis, see Proposition 45.

Proposition 14 *If* $H \in \mathscr{H}$ *satisfies*

$$H_{ty}(t,y) + \frac{1}{2} H_{yyy}(t,y) \leq 0, \tag{10}$$

then $J(\theta)$ *is concave in* $\Theta_b(M,Z)$*.*

Proof. We will show that for every $\theta,\eta\in\Theta_b(M,Z)$, we have $D_{\eta-\theta}J(\theta)\geq J(\eta)-J(\theta)$, which thanks to Proposition 45 is equivalent to $J(\theta)$ being concave. Given $\theta,\eta\in\Theta_b(M,Z),\alpha\in[0,1]$, define $\delta\triangleq\eta-\theta$ and $\Psi^{\alpha}\triangleq\theta+\alpha\delta$. For $\alpha\in[0,1]$, define $\varphi(\alpha)\triangleq D_{\eta-\theta}J(\Psi^{\alpha})=\frac{d}{d\alpha}J(\Psi^{\alpha})$. We will show that $\varphi'(\alpha)=\frac{d^2}{d\alpha^2}J(\Psi^{\alpha})\leq 0$. First, by Lemma 46 we have that

$$\varphi'(\alpha)=-\mathbb{E}\left[\int_0^1\left(\int_0^t\delta_s ds\right)^2 H_{yy}(t,Y_t^{\alpha})\Psi_t^{\alpha}dt]-\mathbb{E}[2\left(\int_0^t\delta_s ds\right)H_y(t,Y_t^{\alpha})\delta_t dt\right].$$

Then we apply integration by parts in the second expectation to obtain that

$$\begin{aligned}\mathbb{E}\left[\int_0^1 2\left(\int_0^t\delta_s ds\right)H_y(t,Y_t^{\Psi^{\alpha}})\delta_t dt\right]&=\mathbb{E}\left[\left(\int_0^1\delta_s ds\right)^2 H_y(1,Y_1^{\Psi^{\alpha}})\right]\\ -\mathbb{E}\left[\int_0^1\left(\int_0^t\delta_s ds\right)^2\{H_{ty}(t,Y_t^{\Psi^{\alpha}})+\frac{1}{2}H_{yyy}(t,Y_t^{\Psi^{\alpha}})+H_{yy}(t,Y_t^{\Psi^{\alpha}})\Psi_t^{\alpha}\}dt\right].\end{aligned}$$

Hence,

$$\begin{aligned}\varphi'(\alpha)=&-\mathbb{E}\left[\left(\int_0^1\delta_s ds\right)^2 H_y(1,Y_1^{\Psi^{\alpha}})\right]\\ &+\mathbb{E}\left[\int_0^1\left(\int_0^t\delta_s ds\right)^2\{H_{ty}(t,Y_t^{\Psi^{\alpha}})+\frac{1}{2}H_{yyy}(t,Y_t^{\Psi^{\alpha}})\}dt\right].\end{aligned}$$

Using that $H_y>0$ and equation (10) we conclude that $\varphi'(\alpha)\leq 0$ and therefore φ is a decreasing function. On the other hand, an application of the mean value theorem gives that $J(\eta)-J(\theta)=D_{\eta-\theta}J(\Psi^{\alpha^*})=\varphi(\alpha^*)$, for some $\alpha^*\in[0,1]$. Therefore $J(\eta)-J(\theta)\leq\varphi(0)=D_{\eta-\theta}J(\theta)$. ■

The following theorem gives a sufficient condition to find the optimal process in $\Theta_{\sup}(M,Z)$.

Theorem 15 *Let $H\in\mathscr{H}$ satisfy (10). If $\theta^*\in\Theta_{\sup}(M,Z)$ is such that*

$$\mathbb{E}\left[(\xi-H(t,Y_t^{\theta^*}))-\int_t^1 H_y(s,Y_s^{\theta^*})\theta_s^* ds\,|\mathscr{F}_t^I\right]=0,$$

then $J(\theta)\leq J(\theta^*),\theta\in\Theta_{\sup}(M,Z)$.

Proof. According to Proposition 48 and Proposition 49, there exists a sequence $\{\theta^{*,n}\}_{n\in\mathbb{N}}\subseteq\Theta_b(M,Z)$ such that $\lim_{n\to\infty}J(\theta^{*,n})=J(\theta^*)$ and $\lim_{n\to\infty}D_{\theta-\theta^{*,n}}J(\theta^{*,n})=0,\forall\theta\in\Theta_b(M,Z)$. Using Proposition 14, we obtain that J is concave in $\Theta_b(M,Z)$ and therefore we have that $\theta\in\Theta_b(M,Z)$

$$J(\theta)\leq J(\theta^{*,n})+D_{\theta-\theta^{*,n}}J(\theta^{*,n}).$$

Therefore, taking limits one gets that $J(\theta)\leq J(\theta^*)$ for all $\theta\in\Theta_b(M,Z)$. Using Proposition 48 again, we have that for all $\theta\in\Theta_{\sup}(M,Z)$ there exists a sequence $\{\theta^n\}_{n\in\mathbb{N}}\subseteq\Theta_b(M,Z)$ such that $\lim_{n\to\infty}J(\theta^n)=J(\theta)\leq J(\theta^*)$. ■

4 Properties of the solutions to the optimality equation

The following proposition is important to find a strategy θ^* satisfying the optimality equation and yielding a rational price. It tell us that given an insider's strategy satisfying the optimality equation, then the price process associated to this strategy is rational if and only if the market maker sees the associated total demand as a Brownian motion. In other words, the associated price process is rational if and only if the market maker sees the total demand as if only the noise trader was buying or selling stocks. Moreover this suggests the connection between the optimal insider's demand and the compensator of a Brownian motion with respect to a enlarged filtration, see Remark 19 below.

Proposition 16 *Assume there exists a process* $\theta^* \in \Theta_{\sup}(M,Z)$ *satisfying the optimality equation* (9)*. Then* $H(\cdot, Y_\cdot^{\theta^*})$ *is a* $\mathbb{F}^{Y^{\theta^*}}$*-martingale if and only if* Y^{θ^*} *is a* $\mathbb{F}^{Y^{\theta^*}}$*-Brownian motion.*

Proof. Assume that θ^* and $Y_t^{\theta^*} = \int_0^t \theta_s^* ds + Z_t$ satisfy the optimality equation. Note that this equation is equivalent to

$$H(t, Y_t^{\theta^*}) - \int_0^t H_y(s, Y_s^{\theta^*})\theta_s^* ds = \mathbb{E}[\xi | \mathscr{F}_t^I] - M_t \tag{11}$$

where $M_t = \mathbb{E}[\int_0^1 H_y(s, Y_s^{\theta^*})\theta_s^* ds | \mathscr{F}_t^I]$. Making $t = 0$ in (11), we obtain $H(0,0) + M_0 = \mathbb{E}[\xi | \mathscr{F}_0^I]$. Applying Itô's formula to $H(t, Y_t^{\theta^*})$ in equation (11), we get

$$\begin{aligned} &\int_0^t \{H_t(s, Y_s^{\theta^*}) + \frac{1}{2} H_{yy}(s, Y_s^{\theta^*})\} ds \\ &= -\int_0^t H_y(s, Y_s^{\theta^*}) dZ_s + \mathbb{E}[\xi | \mathscr{F}_t^I] - \mathbb{E}[\xi | \mathscr{F}_0^I] - (M_t - M_0), \end{aligned} \tag{12}$$

for all $t \in [0,1]$. The r.h.s of equation (12) is a continuous $\mathbb{F}^I$- local martingale with initial value 0 and the l.h.s. is a finite variation process with continuous paths. Therefore, both processes must be identically zero. Therefore, we have that

$$H_t(t, Y_t^{\theta^*}) + \frac{1}{2} H_{yy}(t, Y_t^{\theta^*}) = 0, \quad t \in [0,1]. \tag{13}$$

Combining the above equation with Itô's formula, we have

$$H(t, Y_t^{\theta^*}) = H(0,0) + \int_0^t H_y(s, Y_s^{\theta^*}) dY_s^{\theta^*}. \tag{14}$$

If Y^{θ^*} is a $\mathbb{F}^{Y^{\theta^*}}$-Brownian motion then the stochastic integral $\int_0^t H_y(s, Y_s^{\theta^*}) dY_s^{\theta^*}$ is a martingale due to Lemma 44. Therefore $H(t, Y_t^{\theta^*})$ is a $\mathbb{F}^{Y^{\theta^*}}$-martingale. Conversely, note that as $H_y > 0$, we can write $Y_t^{\theta^*} = \int_0^t \frac{dH(s, Y_s^{\theta^*})}{H_y(s, Y_s^{\theta^*})}$. Hence, if we assume that $H(t, Y_t^{\theta^*})$ is a $\mathbb{F}^{Y^{\theta^*}}$-martingale, then Y^{θ^*} is a $\mathbb{F}^{Y^{\theta^*}}$-local martingale. As Y^{θ^*} has the same quadratic variation as Z we obtain that Y^{θ^*} is actually a Brownian motion with respect to its own filtration. ■

Corollary 17 *If there exists a process* $\theta^* \in \Theta_{\sup}(M,Z)$ *satisfying equation* (9) *and* $H(\cdot, Y_\cdot^{\theta^*})$ *is an* $\mathbb{F}^{Y^{\theta^*}}$*-martingale, then* H *and* ξ *must satisfy*

$$H_t(t,y) + \frac{1}{2} H_{yy}(t,y) = 0 \tag{15}$$

and

$$H(1, Y_1^{\theta^*}) = \mathbb{E}[\xi | \mathscr{F}_1^I]. \tag{16}$$

Proof. Equation (13) and the fact that Y^{θ^*} is a Brownian motion in its own filtration leads to equation (15). Making $t = 1$ in the optimality equation (9), one obtains (16). ■

5 Existence and uniqueness in law of weak equilibrium

We start this section with a result giving sufficient conditions to obtain a (L, μ)-weak equilibrium. The first condition in Theorem 18 essentially says that the law μ of the asset value ξ must be a smooth transformation of a standard normal random variable. Actually, in the examples of the following sections we do not specify the law μ but the pricing rule H, which gives this smooth transformation. We will consider the pricing rules studied previously in the literature, which are $H(t,y) = y$ and $H(t,y) = e^{y+(1-t)/2}$, see [2]. Notice that the exponential pricing rule has much more economical interpretation as it implies that prices are lognormally distributed. The second condition says that there exists a Brownian motion W such that W is a semimartingale with respect to enlarged filtration $\mathbb{F}^W \vee \sigma(L(W))$. According to Proposition 16, if the insider wants to obtain a rational price process then the total demand Y must be a Brownian motion with respect to its natural filtration. Therefore, it is natural to impose that the additional information of the insider, given by $L(Y)$, is such that the total demand remains a Brownian motion with respect to the enlarged filtration $\mathbb{F}^Y \vee \sigma(L(Y))$ and then use the compensator as the insider's strategy. The technical condition in order to carry out this argument is $\mathbb{F}^Y \vee \sigma(L(Y)) = \mathbb{F}^I$ plus some integrability conditions, which is the third condition in the theorem.

From the economic point of view, it seems reasonable to expect that the insider can not held "too much information" for an equilibrium to hold. In our framework this is reflected in the semimartingale property of W. In fact, if $L(Y)$ gives too much information to the insider, then W will not be a a semimartingale with respect to the enlarged filtration and, therefore, prices will not be rational. Although there exists a general criterion to ensure that a given functional satisfies this semimartingale condition, known as Jacod's criterion, see for instance [19], this criterion does not apply to our main examples $L(W) = \max_{0\leq t\leq 1} W_t$ or $L(W) = \arg\max_{0\leq t\leq 1} W_t$. In other models of insider trading, where the rationality of prices is not taken into account, this condition is not sufficient to provide realistic models with finite expected wealth for the insider optimization problem, see [12]. Usually the information held by the insider has to be perturbed by some noise, see [12] and [6].

Theorem 18 (Existence) *Given a measurable functional $L : \mathscr{C}[0,1] \to \mathbb{R}^k$ and μ a probability measure on $\mathbb{R}$ satisfying $\int_{\mathbb{R}} x^2 \mu(dx) < \infty$. Assume:*

1) *There exists $H \in \mathscr{H}$ such that it satisfies (15) and*

$$\mu(A) = \frac{1}{\sqrt{2\pi}} \int_{H(1,\cdot)^{-1}(A)} e^{-x^2/2} dx, \quad \forall A \in \mathscr{B}(\mathbb{R}).$$

2) *There exists a probability space $(\Omega, \mathscr{F}, P)$ supporting a Brownian motion W which is a semimartingale in the filtration $\mathbb{F}^W \vee \sigma(\lambda)$, $\lambda \triangleq L(W)$, with semimartingale decomposition $W_t = \int_0^t \alpha(s,\lambda)\, ds + W_t^\lambda$, where W^λ is a $\mathbb{F}^W \vee \sigma(\lambda)$-Brownian motion.*

3) $\alpha \in \Theta_{\sup}(\lambda, W^\lambda)$.

Then

$$(Y^*, \theta^*, Z^*, H^*, \xi^*, \lambda^*) = (W, \alpha(\cdot,\lambda), W^\lambda, H, H(1,W_1), \lambda)$$

is a (L,μ)-weak equilibrium.

Proof. Verification of properties i), iii), iv) and v) in the definition of weak equilibrium is straightforward. Property ii) follows from the fact that W_t^λ is an $\mathbb{F}^W \vee \sigma(\lambda)$-Brownian motion and, hence, independent from $\mathscr{F}_0^W \vee \sigma(\lambda) = \sigma(\lambda)$. From hypothesis 1) together with equation (14), we have that $H(\cdot, W_\cdot)$ is a $\mathbb{F}^W$-martingale. As $H(t,W_t) = \mathbb{E}[H(1,W_1) | \mathscr{F}_t^W]$, property vi) follows. To check property vii), we apply Itô's formula to $H(\cdot, W_\cdot)$ in the l.h.s. of the optimality equation (9) with $\theta = \alpha$. Due to hypothesis 1) we obtain that it is equal to

$$\mathbb{E}\left[\int_t^1 H_y(s,W_s)\, dW_s^\lambda | \mathscr{F}_t^{W^\lambda} \vee \sigma(\lambda)\right].$$

On the other hand, hypothesis 3) implies that $\alpha(\cdot,\lambda)$ is $\mathbb{F}^{W^\lambda} \vee \sigma(\lambda)$-adapted, which entails that W is $\mathbb{F}^{W^\lambda} \vee \sigma(\lambda)$-adapted and one can conclude that $\mathbb{F}^W \vee \sigma(\lambda) = \mathbb{F}^{W^\lambda} \vee \sigma(\lambda)$. Hence, due to Lemma 44, the above conditional expectation equals to zero and the conclusion follows from Theorem 15. ■

Remark 19 *Of the three hypothesis in the previous Theorem, hypothesis 3 is difficult to verify in general. Besides the integrability conditions in the definition of $\Theta_{\sup}(\lambda, W^\lambda)$, $\alpha(\cdot,\lambda)$ must be $\mathbb{F}^{W^\lambda} \vee \sigma(\lambda)$-adapted. This property will follow if $\mathbb{F}^W \vee \sigma(\lambda) = \mathbb{F}^{W^\lambda} \vee \sigma(\lambda)$. This problem seems to be difficult to solve in general.*

We deal with this problem in each of the examples to follow in the next sections. The general strategy is to show existence and uniqueness for s.d.e.'s of the form

$$X_t = \int_0^t \alpha\left(s, G, X_{[0,s]}\right) ds + V_t,$$

where V is a Brownian motion, α is a (degenerate) functional and G is a random variable independent of V. Therefore, X would be $\mathbb{F}^V \vee \sigma(G)$-adapted.

The following theorem gives a uniqueness result for the (L,μ)-weak equilibrium found in the previous theorem. Condition 6) in the following theorem deserves a comment. This assumption roughly says that two weak equilibriums have the same law whenever are obtained through a semimartingale decomposition of a Brownian motion with respect to a enlarged filtration. In other words, if in Condition 2) of Theorem 18 we use two different Brownian motions possibly defined in two diferent probability spaces, the two different weak equilibriums obtained have the same law. From the economic point of view, this assumption states that if the market maker knew the insider's aditional information then he would have exactly the same information flow as the insider.

Theorem 20 (Uniqueness in law) *Assume the same hypotheses of Theorem 18 and denote by $(Y^*, \theta^*, Z^*, H^*, \xi^*, \lambda^*)$ the (L,μ)-weak equilibrium. Suppose that there exists another probability space supporting processes (Y, θ, Z) such that*

1) $Y_t = \int_0^t \theta_s ds + Z_t$;

2) $\lambda \triangleq L(Y)$ *is independent of Z;*

3) *Z is a Brownian motion in its own filtration;*

4) $\theta \in \Theta_{\sup}(\lambda, Z)$;

5) $H^*(t, Y_t) = \mathbb{E}[H^*(1, Y_1) | \mathscr{F}_t^Y]$ *for* $t \in [0,1]$.

6) $\mathbb{F}^Z \vee \sigma(\lambda) = \mathbb{F}^Y \vee \sigma(\lambda)$.

Then, we have that $\mathscr{L}(Y^*, X^*, Z^*, \xi^*, \lambda^*) = \mathscr{L}(Y, X, Z, \xi, \lambda)$, where $\xi \triangleq H^*(1, Y_1)$, and therefore $\mathbb{E}[V(X,P,\xi)] = \mathbb{E}^*[V\left(X^*, P^*, \xi^*\right)]$.

Proof. Applying Itô's formula in the filtration $\mathbb{F}^I = \mathbb{F}^Z \vee \sigma(\lambda)$, we have that

$$\xi - H^*(t, Y_t) - \int_t^1 H_y^*(s, Y_s)\, \theta_s ds = \int_t^1 H_y^*(s, Y_s)\, dZ_s,$$

where in the last equality we have used that H^* satisfies equation (15). After taking conditional expectation, this yields

$$\mathbb{E}\left[\xi - H^*(t, Y_t) - \int_t^1 H_y^*(s, Y_s)\, \theta_s ds \mid \mathscr{F}_t^I\right] = 0.$$

Then, by Theorem 15 we have that $J(\eta) \leq J(\theta), \forall \eta \in \Theta_{\sup}(L(Y), Z)$. By hypothesis 5) and Proposition 16 one gets that Y is a Brownian motion in its own filtration. Therefore, $\mathscr{L}(Y, \lambda, \xi) = \mathscr{L}\left(Y^*, \lambda^*, \xi^*\right)$. As the process θ^* is adapted to $\mathbb{F}^{Y^*} \vee \sigma(\lambda^*)$, then it can be written as $\theta_t^* = \Lambda(t, Y_{[0,t]}^*, L(Y_{[0,1]}^*))$, $P \times \lambda$-a.s.. Then, defining $\hat{\theta}_t \triangleq \Lambda(t, Y_{[0,t]}, L(Y_{[0,1]}))$ and using that $\mathscr{L}(Y^*) = \mathscr{L}(Y)$, we have that

$$\mathbb{E}\left[Y_t - Y_s - \int_s^t \hat{\theta}_u du | \mathscr{F}_s^Y \vee \sigma(\lambda)\right] = 0.$$

Thus,

$$Y_t = \int_0^t \hat{\theta}_s ds + M_t = \int_0^t \theta_s ds + Z_t,$$

where M is a $\mathbb{F}^Y \vee \sigma(\lambda)$-martingale. Given the assumption 6), the uniqueness of the semimartingale decomposition of Y with respect to $\mathbb{F}^Y \vee \sigma(\lambda)$ proves that $\widehat{\theta} = \theta, P \times \lambda$ a.s. ■

The following result is helpful when proving that $\alpha \in \Theta_{\sup}(\lambda, Z)$.

Proposition 21 *Let Y be a Brownian motion and $\lambda = L(Y)$. Assume that Y has a semimartingale decomposition with respect to $\mathbb{F}^Y \vee \sigma(\lambda)$ given by $Y_t = \int_0^t \alpha_s ds + Z_t$, where Z is a $\mathbb{F}^Y \vee \sigma(\lambda)$-Brownian motion. Then,*

$$\exp\left(C \sup_{0\le t\le 1}\left|\int_0^t \alpha_s ds\right|\right) \in L^p(\Omega), \quad p \ge 1, \quad \forall C > 0,$$

and

$$\sup_{0\le t\le 1}\left|\int_0^t F(s, Y_s)\,\alpha_s ds\right| \in L^p(\Omega), \quad p \ge 1,$$

where F is any function satisfying an exponential growth condition.

Proof. To prove the first statement, notice that

$$\left|\exp\left(C \sup_{0\le t\le 1}\left|\int_0^t \alpha_s ds\right|\right)\right|^p \le \exp(pC \sup_{0\le t\le 1}|Y_t|)\exp(pC \sup_{0\le t\le 1}|Z_t|).$$

By the Cauchy-Schwarz inequality, taking into account that Y and Z are Brownian motions, we obtain that

$$\mathbb{E}\left[\left|\exp\left(C \sup_{0\le t\le 1}\left|\int_0^t \alpha_s ds\right|\right)\right|^p\right] \le \left(\mathbb{E}[\exp(2pC \sup_{0\le t\le 1}|Y_t|)]\right)^2 < \infty.$$

To prove the second statement, not that

$$\left|\int_0^t F(s, Y_s)\,\alpha_s ds\right|^p \le C(p)\left(\left|\int_0^t F(s, Y_s)\,dY_s\right|^p + \left|\int_0^t F(s, Y_s)\,dZ_s\right|^p\right)$$

Define

$$M_t^1 \triangleq \int_0^t F(s, Y_s)\,dY_s \quad \text{and} \quad M_t^2 \triangleq \int_0^t F(s, Y_s)\,dZ_s.$$

Here, M^1 is a $\mathbb{F}^Y$-local martingale. By the BDG inequality (see Theorem 73, pag. 222 in [19]), taking into account that F satisfies an exponential growth condition and that Y is a Brownian motion, we obtain that

$$\begin{aligned}
\mathbb{E}\left[\sup_{0\le t\le 1}\left|M_t^1\right|^p\right] &\le C_p \mathbb{E}\left[\left(\int_0^1 F(s, Y_s)^2 ds)\right)^{p/2}\right] \\
&\le C_p \mathbb{E}\left[\left(\int_0^1 A^2 \exp(2B|Y_s|)ds\right)^{p/2}\right] \\
&\le C_p A^p \mathbb{E}\left[\exp(pB \sup_{0\le t\le 1}|Y_t|)\right] < \infty.
\end{aligned}$$

Thus M^1 is a $\mathbb{F}^Y$-martingale and $\sup_{0\le t\le 1}\left|M_t^1\right| \in L^p(\Omega), p \ge 1$. We can repeat the same argument for M^2, taking into account that M^2 is a $\mathbb{F}^Y \vee \sigma(\lambda)$-local martingale. ■

6 Back's example and an example of binary information

In this section we comment on two known examples where the general result in Theorem 18 applies. Throughout this section we will consider a Brownian motion W defined on a complete probability space $(\Omega, \mathscr{F}, P)$. From now on, we denote by $\phi(x,t)$ the density of a centered Gaussian random variable with variance t, by $\Phi(x,t)$ its distribution function and $\overline{\Phi}(x,t) = 1 - \Phi(x,t)$.

In all the examples to follow in the next sections, we assume that μ is a probability measure on $\mathbb{R}$ with $\int_{\mathbb{R}} x^2 \mu(dx) < \infty$ and that there exists $H \in \mathscr{H}$ satisfying (15) and

$$\mu(A) = \int_{H(1,\cdot)^{-1}(A)} \phi(x,1)\, dx, \quad \forall A \in \mathscr{B}(\mathbb{R}).$$

Theorem 22 *Let W be a Brownian motion. Then W is a semimartingale respect to the filtration $\mathbb{F}^W \vee \sigma(W_1)$ with decomposition*

$$W_t = \int_0^t \alpha(u, W_1)\, du + W_t^{W_1}, \quad \forall t \in [0,1], \tag{17}$$

where W^{W_1} is a $\mathbb{F}^W \vee \sigma(W_1)$-Brownian motion,

$$\alpha(t, W_1) = \frac{W_1 - W_t}{1-t}, \tag{18}$$

for all $t \in [0,1)$.

The previous result is well known and its proof can be found, for instance, in [11], Théorème 1. In [11], Corollaire 1.1, it is also discussed the connection between the Brownian bridge $\{W_t - tW_1\}_{0 \le t < 1}$ and the Brownian motion $\{W_t^{W_1}\}_{0 \le t < 1}$, showing that these two processes have the same natural filtration and that W_1 is independent of $\{W_t^{W_1}\}_{0 \le t < 1}$. This idea is later used in order to consider equation (17) as a linear equation, where the unknown function is $W_\cdot(\omega)$ and $W_t^{W_1}(\omega)$ and $W_1(\omega)$ are given. The following result is slightly more general than Corollaire 1.1, in [11], in the sense that if we assume that we are given a Brownian motion B and a random variable G, independent from B and not necessarily Gaussian, we can construct a process X with terminal value G. In the particular case that $\mathscr{L}(G) = \mathscr{N}(0,1)$, the process X is a Brownian bridge with $X_1 = G$.

Theorem 23 *Let B be a Brownian motion and G a random variable independent of B, both defined in the same probability space $(\Omega, \mathscr{F}, P)$. Then there exists a unique strong solution X adapted to the filtration $\mathbb{F}^B \vee \sigma(G)$ of the following stochastic differential equation*

$$X_t = \int_0^t \frac{G - X_s}{1-s} ds + B_t, \quad t \in [0,1]. \tag{19}$$

Furthermore, if we assume that the law of G is $N(0,1)$, then X_t is a Brownian motion with respect its own filtration.

Proof. As G is independent of B, one has that B is $\mathbb{F}^B \vee \sigma(G)$-Brownian motion. Using as an integrating factor $(1-t)^{-1}$, we obtain

$$d\left(\frac{X_t}{1-t}\right) = \frac{G}{(1-t)^{-2}} dt + \frac{dB_t}{1-t}.$$

Therefore, one has that $X_t = tG + (1-t)\int_0^t \frac{dB_s}{1-s}, 0 \le t < 1$. In lemma 6.9 of [13], pag. 358, it is proved that the process

$$\begin{aligned} \overline{B}_t &= (1-t)\int_0^t \frac{dB_s}{1-s}; \quad 0 \le t < 1, \\ \overline{B}_1 &= 0, \end{aligned}$$

is a continuous, centered Gaussian process with covariance function $s \wedge t - st$. Hence we have proved existence and uniqueness for the solutions of the equation (19). If we assume that $G \sim N(0,1)$, we have that tG is a continuous, centered Gaussian process with covariance function st. As the sum of two independent Gaussian processes is still a Gaussian process and B_t and G are independent, we obtain that X is a continuous, centered Gaussian process with covariance function $s \wedge t$, thus a standard Brownian motion. ■

The following property is important to determine the finiteness of optimal utilities. For $p > 0$, $\mathbb{E}[\int_0^1 |\alpha(t,G)|^p\, dt] < \infty$ if and only if $p < 2$, where

$$\alpha(t,x) = \frac{x - X_t}{1-t}, \quad \forall t \in [0,1].$$

Let's state the weak equilibrium result for this case.

Theorem 24 *Let $L(Y) = Y_1$. Then*

$$\left(Y^*, \theta^*, Z^*, H^*, \xi^*, \lambda^*\right) = (X, \alpha(\cdot, G), B, H, H(1,G), L(X))$$

is a (L,μ)-weak equilibrium.

In this particular case the above weak equilibrium is in fact a strong type equilibrium. For this, see Theorem 1 in [2] or Proposition 2 in [5].

Theorem 25 *Assume that we are given a Brownian motion Z and a strong information ξ. Assume that $H^* \in \mathscr{H}$ satisfies (15) and $\xi \sim H^*(1, N(0,1))$. Set $\theta_t^* \triangleq \alpha(t, (H^*)^{-1}(1,\xi))$. Then (H^*, θ^*) is an equilibrium. That is,*

- *$H^*(t,Y^*)$ is a rational price, that is $H^*(t,Y^*) = \mathbb{E}[\xi | \mathscr{F}_t^{Y^*}]$.*
- *For all $\theta \in \Theta_{\sup}(\xi, Z)$, one has*

$$\mathbb{E}[V(X,P,\xi)] \le \mathbb{E}[V(X^*,P^*,\xi)],$$

where $X_\cdot^{(*)} = \int_0^\cdot \theta_s^{(*)} ds$, $Y_\cdot^{(*)} = X^{(*)} + Z$ and $P_\cdot^{(*)} = H^*(\cdot, Y_\cdot^{(*)})$.

Now we consider the case in which the insider knows that the total demand at time 1 is greater or equal to a fixed constant a. The next two results are quoted from [12], example 4.6.

Theorem 26 *Let W be a Brownian motion. Then W is a semimartingale respect to the filtration $\mathbb{F}^W \vee \sigma\left(1_{[a,\infty)}(W_1)\right)$ with decomposition*

$$W_t = \int_0^t \alpha\left(u, 1_{[a,\infty)}(W_1)\right) du + W_t^a, \quad \forall t \in [0,1],$$

where W^a is a $\mathbb{F}^W \vee \sigma\left(1_{[a,\infty)}(W_1)\right)$-Brownian motion,

$$\alpha\left(t, 1_{[a,\infty)}(W_1)\right) = \frac{\phi(W_t - a, 1-t)}{\Phi(W_t - a, 1-t)} 1_{[a,\infty)}(W_1) + \frac{\phi(W_t - a, 1-t)}{\Phi(a - W_t, 1-t)} 1_{[a,\infty)^c}(W_1),$$

for all $t \in [0,1]$.

Lemma 27 *We have that $\mathbb{E}[\int_0^1 \left|\alpha\left(t, 1_{[a,\infty)}(W_1)\right)\right|^2 dt] < \infty$.*

Theorem 28 *Let B be a Brownian motion and G a Bernoulli random variable independent of B, both defined on the same probability space $(\Omega, \mathscr{F}, P)$. Then there exists a unique strong solution X adapted to the filtration $\mathbb{F}^B \vee \sigma(G)$ of the following stochastic differential equation*

$$X_t = \int_0^t \left(\frac{\phi(X_t - a, 1-t)}{\Phi(X_t - a, 1-t)} 1_{\{1\}}(G) + \frac{\phi(X_t - a, 1-t)}{\Phi(a - X_t, 1-t)} 1_{\{0\}}(G) \right) ds + B_t, \quad 0 \le t < 1. \tag{20}$$

Proof. First we will prove that $\Psi_a^1(x,t) \triangleq \phi(x-a,1-t)/\Phi(x-a,1-t)$ is Lipschitz in the x variable for $t \in [0,1)$, fixed. Note that we can take $a=0$, without loss of generality. Furthermore, $\Psi_0^1(x,t) = \Psi_0^1(x/\sqrt{1-t},0)/\sqrt{1-t}$.

We have that

$$\begin{aligned}
\partial_x \Psi_0^1(x,t) &= \frac{-\frac{x}{1-t}\phi(x,1-t)\Phi(x,1-t) - (\phi(x,1-t))^2}{(\Phi(x,1-t))^2} \\
&= -\frac{x}{1-t}\Psi_0^1(x,t) - \left(\Psi_0^1(x,t)\right)^2 \\
&= -\frac{1}{1-t}\left\{\frac{x}{\sqrt{1-t}}\Psi_0^1(x/\sqrt{1-t},0) + \left(\Psi_0^1(x/\sqrt{1-t},0)\right)^2\right\}.
\end{aligned}$$

Fix $t^* < 1$, then

$$\sup_{t\in[0,t^*],x\in\mathbb{R}} \left|\partial_x \Psi_0^1(x,t)\right| \le \frac{1}{1-t^*}\sup_{y\in\mathbb{R}}\left|y\Psi_0^1(y,0) + \left(\Psi_0^1(y,0)\right)^2\right|.$$

Applying l'Hospital's rule, it can be shown that

$$\begin{aligned}
\lim_{y\to-\infty} y\Psi_0^1(y,0) + \left(\Psi_0^1(y,0)\right)^2 &= 1, \\
\lim_{y\to\infty} y\Psi_0^1(y,0) + \left(\Psi_0^1(y,0)\right)^2 &= 0.
\end{aligned}$$

which entails that $\sup_{t\in[0,t^*],x\in\mathbb{R}} \left|\partial_x \Psi_0^1(x,t)\right| < \infty$. Therefore, $\Psi_a^1(x,t)$ is Lipschitz in the x variable uniformly in $t \in [0,t^*], t^* < 1$. To study the growth of $\Psi_a^1(x,t)$ we take $a=0$. Then,

$$\left|\Psi_0^1(x,t)\right| \le \frac{1}{\sqrt{1-t^*}}\sup_{y\in\mathbb{R}}\Psi_0^1(y,0),$$

for $t \in [0,t^*], t^* < 1$. It can be shown that $\lim_{y\to\infty}\Psi_0^1(y,0) = 0$ and $\lim_{y\to-\infty}\Psi_0^1(y,0)/y = -1$, which implies that $\sup_{y\in\mathbb{R}}\Psi_0^1(y,0) < \infty$. Hence, $\Psi_a^1(x,t)$ satisfies a linear growth condition, for $t \in [0,t^*], t^* < 1$. Using the classical results on s.d.e.'s, we have that there exists a unique strong solution to the following equation

$$Y_t^1 = \int_0^t \Psi_a^1\left(Y_s^1,s\right)ds + B_t, \quad 0 \le t < 1.$$

We can use a similar reasoning for $\Psi_a^2(x,t) \triangleq \phi(x-a,1-t)/\Phi(a-x,1-t)$ and get the same conclusions. Finally, the $\mathbb{F}^B \vee \sigma(G)$-adapted process $X_t \triangleq Y_t^1 1_{\{1\}}(G) + Y_t^2 1_{\{0\}}(G)$ solves our problem. ■

Theorem 29 *Let $L(Y) = 1_{[a,\infty)}(Y_1)$. Then*

$$\left(Y^*,\theta^*,Z^*,H^*,\xi^*,\lambda^*\right) = \left(W,\alpha\left(\cdot,1_{[a,\infty)}(W_1)\right),W^a,H,H(1,W_1),L(W)\right)$$

is a (L,μ)-weak equilibrium.

Proof. We apply Theorem 18. The first hypothesis of the theorem is assumed. The second hypothesis follows from Theorem 26. Finally that $\alpha \in \Theta_{\sup}(1_{[a,\infty)}(W_1),W^a)$ follows from Lemma 27, Proposition 21 and Theorem 28 (see Remark 19). ■

7 The maximum and its argument

In this section we deal with two examples that are more complicated, but by far more interesting. In particular, the second example is new in the literature of insider trading with initial strong information. Throughout this section we will consider a Brownian motion W defined on a complete probability space $(\Omega,\mathscr{F},P)$. We consider the maximum process in the interval $[s,t]$, $M_{s,t}, 0 \le s < t \le 1$ defined by $M_{s,t} \triangleq$

$\max_{s\le u\le t} W_u$. To simplify notation we use $M_t \triangleq M_{0,t}$, $\tau_t \triangleq \arg\max_{0\le s\le t} W_s$, $M \triangleq M_1$, $\tau \triangleq \tau_1$. and $\gamma_{s,t} \triangleq M_{s,t} - W_s$. The density and distribution function of $\gamma_{s,t}$ are given by $p_2(x,t-s) \triangleq 2\phi(x,t-s)\,1_{(0,\infty)}(x)$ and $\Pi_2(x,t-s) \triangleq \int_0^x p_2(z,t-s)\,dz$. Similarly, the density of the random vector $(\gamma_{s,t}, W_t - W_s)$ is given by

$$p_1(x,y,t-s) \triangleq \frac{2(2x-y)}{\sqrt{2\pi(t-s)^3}} \exp\left\{\frac{-(2x-y)^2}{2(t-s)}\right\} 1_{(0,\infty)\times(-\infty,x)}(x,y).$$

Let us recall a theorem by Lévy that links the maximum process M_t with the Brownian local time $L_t^x(W)$.

Theorem 30 *The pairs of processes $\{(M_t - W_t, M_t); 0 \le t < \infty\}$ and $\{(|W_t|, 2L_t^0(W)); 0 \le t < \infty\}$ have the same laws under P.*

For more details, see [13]; chapter 3, Theorem 6.17. Furthermore, it is easy to show that, for a fixed t, $M_t - W_t \sim \gamma_{0,t}$. Finally, we set

$$\varphi(x,t) \triangleq \frac{p_2(x,t)}{\Pi_2(x,t)} = \frac{e^{-\frac{x^2}{2t}}}{\int_0^x e^{-\frac{y^2}{2t}}\,dy} 1_{(0,\infty)}(x).$$

7.1 $L(Y) = \max_{t\in[0,1]} Y_t$

In this subsection we consider the case in which the insider knows the maximum of the total demand. A more general version of the following result is proved in Jeulin [10] (see Proposition 3.24, pag. 49). See also Mansuy and Yor [18] for an update reference on enlargement of filtrations theory.

Theorem 31 *Let W be a Brownian motion. Then W is a semimartingale respect to the filtration $\mathbb{F}^W \vee \sigma(M)$ with decomposition*

$$W_t = \int_0^t \alpha(u,M)\,du + W_t^M, \quad \forall t \in [0,1],$$

where W^M is a $\mathbb{F}^W \vee \sigma(M)$-Brownian motion,

$$\alpha(t,M) = \frac{M - W_t}{1-t} 1_{\{M_t < M\}} - \varphi(M - W_t, 1-t)\,1_{\{M_t = M\}}.$$

Note that $1_{\{M_t<M\}} = 1_{[0,\tau)}(t)$.

Lemma 32 *We have that $\mathbb{E}\left[\int_0^1 |\alpha(t,M)|\,dt\right] < \infty$ and $\mathbb{E}\left[\int_0^1 |\alpha(t,M)|^2\,dt\right] = \infty$.*

Proof. To deduce the convergence of the first expectation, notice that

$$\mathbb{E}\left[\int_0^1 \alpha(t,M)\,dt\right] = \mathbb{E}[W_1] - \mathbb{E}[W_1^M] = 0,$$

which implies

$$\mathbb{E}\left[\int_0^1 1_{\{M_t<M\}} \frac{M-W_t}{1-t}\,dt\right] = \mathbb{E}\left[\int_0^1 1_{\{M_t=M\}} \varphi(M-W_t,1-t)\,dt\right].$$

As the integrands in the above expectations are positive, the problem is reduced to show

$$\mathbb{E}\left[\int_0^1 1_{\{M_t<M\}} \frac{M-W_t}{1-t}\,dt\right] < \infty.$$

Let's compute this expectation

$$\mathbb{E}\left[\int_0^1 1_{\{M>M_t\}}\frac{M-W_t}{1-t}dt\right]=\mathbb{E}\left[\int_0^1 1_{\left\{M_{t,1}>M_t\right\}}\frac{M_{t,1}-W_t}{1-t}dt\right]$$
$$=\mathbb{E}\left[\int_0^1 1_{\left\{\gamma_{t,1}>M_t-W_t\right\}}\frac{\gamma_{t,1}}{1-t}dt\right].$$

Conditioning with respect $\mathscr{F}_t^W$ and using Lemma 50, this expectation is equal to

$$\int_0^1\int_0^\infty\int_y^\infty \frac{x}{1-t}p_2(x,1-t)\,p_2(y,t)\,dxdydt=\sqrt{\frac{2}{\pi}}<\infty.$$

To show the divergence of the second moment, notice that

$$\mathbb{E}\left[\int_0^1\alpha(t,M)^2dt\right]$$
$$=\mathbb{E}\left[\int_0^1 1_{\{M>M_t\}}\left(\frac{M-W_t}{1-t}\right)^2dt\right]+\mathbb{E}\left[\int_0^1 1_{\{M=M_t\}}\left(\varphi(M-W_t,1-t)\right)^2dt\right].$$

Therefore, it suffices to show the divergence of one of the above expectations. The second expectation above is equal to

$$\mathbb{E}\left[\int_0^1 1_{\left\{M_t>M_{t,1}\right\}}\left(\varphi(M_t-W_t,1-t)\right)^2dt\right]$$
$$=\mathbb{E}\left[\int_0^1\int_0^{M_t-W_t}\left(\varphi(M_t-W_t,1-t)\right)^2p_2(x,1-t)\,dxdt\right]$$
$$=\int_0^1\int_0^\infty\int_0^y\left(\varphi(y,1-t)\right)^2p_2(x,1-t)\,p_2(y,t)\,dxdydt$$
$$=\int_0^1\int_0^\infty\frac{\left(p_2(y,1-t)\right)^2}{\Pi_2(y,1-t)}p_2(y,t)\,dydt.$$

But this integral is infinite, because

$$\lim_{y\to0^+}y\frac{\left(p_2(y,1-t)\right)^2}{\Pi_2(y,1-t)}p_2(y,t)=p_2(0,1-t)\,p_2(0,t)=\frac{2}{\pi\sqrt{t(1-t)}}\neq0,$$

and this implies that $\int_0^\infty\frac{(p_2(y,1-t))^2}{\Pi_2(y,1-t)}p_2(y,t)\,dy=\infty,\forall t\in[\varepsilon,1-\varepsilon]$, which is a set of positive Lebesgue measure provided $\varepsilon<1/2$. ■

In order to verify that $\alpha(\cdot,M)$ is $\mathbb{F}^{W^M}\vee\sigma(M)$-adapted we prove that W is $\mathbb{F}^{W^M}\vee\sigma(M)$-adapted, which follows from the following result.

Theorem 33 *Let B be a Brownian motion and G a positive random variable independent of B, both defined in the same probability space $(\Omega,\mathscr{F},P)$. Then there exist a unique strong solution X adapted to the filtration $\mathbb{F}^B\vee\sigma(G)$ of the following stochastic differential equation*

$$X_t=\int_0^t\left(\frac{G-X_s}{1-s}1_{\left\{M_s^X<G\right\}}-\varphi(G-X_s,1-s)\,1_{\left\{M_s^X=G\right\}}\right)ds+B_t,\tag{21}$$

where $M_t^X\triangleq\max_{0\le s\le t}X_s$.

Proof. Our approach to the solution of (21) is to write $X_t=X_t^1 1_{[0,\rho)}(t)+X_t^2 1_{[\rho,1)}(t)$, where $\rho\triangleq\inf\left\{t:X_t^1=G\right\}$, X_t^1 and X_t^2 are the solutions to the following s.d.e.'s

$$\mathscr{E}_1:\qquad X_t^1=\int_0^t\frac{G-X_s^1}{1-s}ds+B_t,\quad 0\le t<\rho;$$

and

$$\mathscr{E}_2: \qquad X_t^2 = G - \int_\rho^t \varphi\left(G - X_s^2, 1-s\right) ds + B_t - B_\rho, \quad \rho \le t < 1;$$

which we denote by $\mathscr{E}_1$ and $\mathscr{E}_2$, respectively. The next step is to show the existence and uniqueness of the solutions to $\mathscr{E}_1$ and $\mathscr{E}_2$. Note, that $\rho \le 1$ is a $\mathbb{F}^B \vee \sigma(G)$-stopping time.

- *Existence and uniqueness for the solution of* $\mathscr{E}_1$: Follows as in the case of the Brownian bridge (see lemma 6.9 of [13], pag. 358).
- *Existence and uniqueness for the solution of* $\mathscr{E}_2$: Note that the drift has a singularity at $t = \rho$. That is, $\lim_{x\to 0^+} \varphi(x,t) = \infty, t > 0$. Instead of proving existence and uniqueness for $\mathscr{E}_2$, we will prove it for the following equivalent s.d.e.

  $$\mathscr{E}_2': \qquad R_t = \int_0^t \varphi(R_s, 1-\rho-s)\, ds + N_t, \quad 0 \le t < 1-\rho.$$

  The s.d.e. $\mathscr{E}_2'$ is obtained from $\mathscr{E}_2$ through the change of variables $R_t = G - X_{t+\rho}^2$ and $N_t = -(B_{t+\rho} - B_\rho)$. The existence is proved in Proposition 34. To prove the uniqueness, we may consider $\Delta_t \triangleq R_t^1 - R_t^2$ the difference of two positive solutions R^1 and R^2 of $\mathscr{E}_2'$. Then, applying Itô's formula to $\Delta^2_{t\wedge(1-\rho)}$, we obtain that P-a.s.

  $$\begin{aligned} &\Delta^2_{t\wedge(1-\rho)} \\ = \;& 2\int_0^{t\wedge(1-\rho)} \left(R_s^1 - R_s^2\right)\left(\varphi\left(R_s^1, 1-\rho-s\right) - \varphi\left(R_s^2, 1-\rho-s\right)\right) ds \le 0, \end{aligned}$$

  as $\varphi(x,t) \ge \varphi(y,t)$ if $x \le y$ for all $t \in (0,1)$.

■

Proposition 34 *There exists a positive, continuous, strong solution with respect to* $\mathbb{F}^N \vee \sigma(\rho)$ *to*

$$R_t = \int_0^t \varphi(R_s, 1-\rho-s)\, ds + N_t, \quad 0 \le t < 1-\rho, \tag{22}$$

where N is a Wiener process and $\rho \in (0,1)$ is a random variable independent of $\mathbb{F}^N$.

Proof. First of all, note that

$$x\varphi(x,t) \le 1, \forall t > 0, x \in \mathbb{R}. \tag{23}$$

We define $\varphi^n(x,t) \triangleq \exp\{-\frac{1}{n}(\frac{1}{x}+\frac{1}{t})\}\varphi(x,t)$, which satisfies (23) with φ^n instead of φ. This sequence of functions is monotone increasing in n, bounded and converges to $\varphi(x,t)$ for each $x \in \mathbb{R}$, $t > 0$ such that $x^{-1} + t^{-1} > 0$. Furthermore,

$$\begin{aligned} \partial_x \varphi^n(x,t) &= \exp\{-\frac{1}{n}(\frac{1}{x}+\frac{1}{t})\}\left(\frac{1}{n}x^{-2}\varphi(x,t) + \partial_x\varphi(x,t)\right) \\ &= \left(\frac{1}{nx^2} - \frac{x}{t} - \varphi(x,t)\right)\varphi^n(x,t). \end{aligned}$$

Using inequality (23), one obtains that $\sup_{x\in[0,\infty),t\in[0,1]} |\partial_x \varphi^n(x,t)| < \infty$, which implies that $\varphi^n(x,t)$ is a Lipschitz function. Therefore, for a fixed $n \in \mathbb{N}$, we have the existence and uniqueness of solutions for the following s.d.e.

$$R_t^n = \int_0^t \varphi^n(R_s^n, 1-\rho-s)\, ds + N_t, \quad 0 \le t < 1-\rho.$$

By a comparison theorem, we have that $P\left(R_t^{n+1} \ge R_t^n; 0 \le t < 1-\rho\right) = 1$, which shows that $R_t \triangleq \lim_{n\to\infty} R_t^n$, $0 \le t < 1-\rho$ exists almost surely in $(-\infty, \infty]$ and it is a measurable process as it is a limit of measurable processes. Now, we show that for $t \in [0,1-\rho), R_t < \infty, P$-a.s. and R satisfies equation $\mathscr{E}_2'$. In order to prove the first property, we show the uniform integrability in $n \in \mathbb{N}$ of $R_t^n, 0 \le t < 1-\rho$. Applying Itô's formula, we obtain

$$(R_t^n)^2 = t + 2\int_0^t R_s^n \varphi^n (R_s^n, 1-\rho-s)\, ds + 2\int_0^t R_s^n dN_s, \quad 0 \le t < 1-\rho.$$

Next, we bound the expectation of the second term above. We obtain

$$\begin{aligned}
&\mathbb{E}\left[\int_0^{t\wedge(1-\rho)} R_s^n \varphi^n (R_s^n, 1-\rho-s)\, ds\right] \le \mathbb{E}\left[\int_0^{t\wedge(1-\rho)} |R_s^n|\, \varphi (R_s^n, 1-\rho-s)\, ds\right] \\
&= \mathbb{E}\left[\int_0^{t\wedge(1-\rho)} 1_{\{R_s^n>0\}} R_s^n \varphi (R_s^n, 1-\rho-s)\, ds\right] \le \mathbb{E}\left[\int_0^{t\wedge(1-\rho)} 1_{\{R_s^n>0\}} ds\right] \le t.
\end{aligned}$$

For the third term, one has $\mathbb{E}\left[\int_0^{t\wedge(1-\rho)} R_u^n dN_u\right] = 0$. Thus, $\sup_{n\in\mathbb{N}} \mathbb{E}\left[(R^n_{t\wedge(1-\rho)})^2\right] \le 3t$. This implies the uniform integrability of $R^n_{t\wedge(1-\rho)}$ and therefore $R_{t\wedge(1-\rho)} \in L^1(\Omega)$. Next, we show that R_t satisfies $\mathscr{E}_2'$. First note that

$$\begin{aligned}
R_{t\wedge(1-\rho)} &= \lim_{n\to\infty} R^n_{t\wedge(1-\rho)} \\
&= \lim_{n\to\infty} \int_0^{t\wedge(1-\rho)} \varphi^n (R_s^n, 1-\rho-s)\, ds + N_{t\wedge(1-\rho)}.
\end{aligned}$$

To conclude the proof we show that

$$\lim_{n\to\infty} \int_0^{t\wedge(1-\rho)} \varphi^n (R_s^n, 1-\rho-s)\, ds = \int_0^{t\wedge(1-\rho)} \varphi (R_s, 1-\rho-s)\, ds,$$

$0 \le t < 1$, with probability 1. This will also give the continuity for the paths of R. Fix $\varepsilon > 0$ and define

$$\begin{aligned}
\rho_0^\varepsilon &\triangleq \inf\{t \in (0, 1-\rho) : N_t = \varepsilon\}, \\
\rho_l^\varepsilon &\triangleq \inf\{t \in \left(\rho_{l-1}^\varepsilon, 1-\rho\right) : N_t - N_{\rho_{l-1}^\varepsilon} = -R^1_{\rho_{l-1}^\varepsilon}/2\}, \quad l \ge 1.
\end{aligned}$$

By construction, the sequence $\{\rho_l^\varepsilon\}_{l\in\mathbb{N}}$ is nondecreasing and therefore we can define $\sigma^\varepsilon \triangleq \lim_{l\to\infty} \rho_l^\varepsilon$. For fixed $\omega \in \Omega$, we apply the dominated convergence theorem in each interval $[\rho_{l-1}^\varepsilon, \rho_l^\varepsilon]$, $l \ge 1$. One has that,

$$R_t^1 = R^1_{\rho_{l-1}^\varepsilon} + \int_{\rho_{l-1}^\varepsilon}^t \varphi^1 \left(R_s^1, 1-\rho-s\right) ds + N_t - N_{\rho_{l-1}^\varepsilon} > \frac{R^1_{\rho_{l-1}^\varepsilon}}{2} \ge \frac{\varepsilon}{2^l},$$

for $t \in [\rho_{l-1}^\varepsilon, \rho_l^\varepsilon]$ and $l \ge 1$, due to the positivity of the integral. Then, using inequality (23), we have for $s \in [\rho_{l-1}^\varepsilon, \rho_l^\varepsilon]$ that

$$\varphi^n (R_s^n, 1-\rho-s) \le \varphi (R_s^n, 1-\rho-s) \le \varphi\left(R_s^1, 1-\rho-s\right) \le \frac{1}{R_s^1} \le \frac{2^l}{\varepsilon}.$$

Hence, by the dominated convergence theorem

$$\lim_{n\to\infty} \int_{\rho_{l-1}^\varepsilon}^{\rho_l^\varepsilon} \varphi^n (R_s^n, 1-\rho-s)\, ds = \int_{\rho_{l-1}^\varepsilon}^{\rho_l^\varepsilon} \varphi (R_s, 1-\rho-s)\, ds.$$

This implies that

$$R_t = R_{\rho_0^\varepsilon} + \int_{\rho_0^\varepsilon}^t \varphi (R_s, 1-\rho-s)\, ds + N_t - N_{\rho_0^\varepsilon}, \quad \rho_0^\varepsilon \le t < \sigma^\varepsilon.$$

We prove now that $\sigma^{\varepsilon}=1-\rho$. If $\omega\in\Omega$ is such that there exists l for which $\rho_l^{\varepsilon}=1-\rho$, we have finished. By contradiction, assume that the sequence $\left\{\rho_l^{\varepsilon}\right\}_{l\in\mathbb{N}}$ is strictly increasing. First of all, by the definition of $\left\{\rho_l^{\varepsilon}\right\}_{l\in\mathbb{N}}$ and the fact that the sequence is strictly increasing, one has that $N_{\rho_l^{\varepsilon}}-N_{\rho_{l-1}^{\varepsilon}}=-R^1_{\rho_{l-1}^{\varepsilon}}/2$. Taking limits we obtain that $R^1_{\sigma^{\varepsilon}}=0$, due to the continuity of Brownian paths. Then $R_t^1+\int_t^{\sigma^{\varepsilon}}\varphi^1\left(R_s^1,1-\rho-s\right)ds=N_t-N_{\sigma^{\varepsilon}}$, but this contradicts the law of iterated logarithm when t tends to σ^{ε}, because the left hand side is positive almost surely for $t\in[\rho_0^{\varepsilon},\sigma^{\varepsilon})$. Hence we can conclude that the set of $\omega\in\Omega$ for which does not exist a finite l such that $\rho_l^{\varepsilon}=1-\rho$ is a null set. Now, notice that $\rho_0^{\varepsilon}\downarrow 0$ when $\varepsilon\downarrow 0$. Hence, $N_{\rho_0^{\varepsilon}}\underset{\varepsilon\downarrow 0}{\longrightarrow}0$ and by monotone convergence

$$\lim_{\varepsilon\downarrow 0}\int_{\rho_0^{\varepsilon}}^t\varphi\left(R_s,1-\rho-s\right)ds=\int_0^t\varphi\left(R_s,1-\rho-s\right)ds.$$

Therefore,

$$R_t=\lim_{\varepsilon\downarrow 0}R_{\rho_0^{\varepsilon}}+\int_0^t\varphi\left(R_s,1-\rho-s\right)ds+N_t,\quad 0\le t<1-\rho.$$

As $R_0=\lim_{n\to\infty}R_0^n=0$, making $t=0$ in the above equation we obtain $\lim_{\varepsilon\downarrow 0}R_{\rho_0^{\varepsilon}}=0$. Furthermore, as $|R_t|<\infty$, P-a.s. we obtain that $\int_0^t\varphi\left(R_s,1-\rho-s\right)ds<\infty, P\text{-}a.s.$, for $t<\sigma$. Hence we have showed that R satisfies equation (22). Note that in particular, we have also proved that $R_t>0$. ■

Theorem 35 *Let $L(Y)=\max_{0\le t\le 1}Y_t$. Then*

$$\left(Y^*,\theta^*,Z^*,H^*,\xi^*,\lambda^*\right)=\left(W,\alpha\left(\cdot,M\right),W^M,H,H\left(1,W_1\right),L\left(W\right)\right)$$

satisfies all the requirements to be a (L,μ)-weak equilibrium except the càglàd property in the condition v).

Proof. Properties i) through iv) in the definition of weak equilibrium follow directly. Property v) with the exception of the càglàd property follows from Lemma 32, Proposition 21 and Theorem 33 (see Remark 19). From the assumptions on H and μ and equation (14), we have that $H\left(\cdot,W_{\cdot}\right)$ is a $\mathbb{F}^W$-martingale. As $H\left(t,W_t\right)=\mathbb{E}[H\left(1,W_1\right)|\mathscr{F}_t^W]$, property vi) follows. Let's check property vii). To simplify the notation we set $\alpha_t\triangleq\alpha\left(t,M\right),0\le t\le 1$. Note that $\alpha_t\ge 0$ if $t\le\tau$ and $\alpha_t\le 0$ if $t>\tau$. From this property, it easily follows the following inequality

$$\int_0^1|\alpha_t|\,dt\le\int_0^{\tau}\alpha_t dt-\int_{\tau}^1\alpha_t dt\le 3\sup_{0\le t\le 1}\left|\int_0^t\alpha_s ds\right|,\tag{24}$$

which combined with Proposition 21 gives that

$$\int_0^1|\alpha_t|\,dt\in L^p\left(\Omega\right),p\ge 1.\tag{25}$$

For $\varepsilon\in(0,1)$, define $\tau^{\varepsilon,+}=(\tau+\varepsilon)\wedge 1$. Then the process $\alpha^{\varepsilon}=\{\alpha_t^{\varepsilon}\triangleq\alpha_t 1_{(\tau,\tau^{\varepsilon,+}]^c}\left(t\right),t\in[0,1]\}$ converges $P\times\lambda$, a.e. to α as $\varepsilon\downarrow 0$ and it satisfies $|\alpha^{\varepsilon}|\le|\alpha|$. Now we will prove that $\alpha^{\varepsilon}\in\Theta_{\sup}(M,W^M),\forall\varepsilon\in(0,1)$. First, the càglàd property of α^{ε} follows from the fact that this approximation avoids the essential discontinuity of α_t in $t=\tau$. The integrability property (1) is trivial. Property (4) follows from equation (24). The proof of properties (2) and (3) are similar. We will prove property (2). We have that

$$\sup_{0\le t\le 1}\left|\int_0^t H(s,Y_s^{\alpha^{\varepsilon}})\alpha_s^{\varepsilon}ds\right|\le\sup_{0\le t\le 1}\left|H(t,Y_t^{\alpha^{\varepsilon}})\right|\int_0^1|\alpha_t|\,dt,$$

which belongs to $L^1\left(\Omega\right)$ by the Cauchy-Schwarz inequality, property (25) and Lemma 44. According to Proposition 48, $\lim_{n\to\infty}J\left(\alpha^{\varepsilon,n}\right)=J\left(\alpha^{\varepsilon}\right)$ for all $\varepsilon\in(0,1)$ where $\alpha^{\varepsilon,n}$ is defined according to Definition 47 with $\theta=\alpha^{\varepsilon}$. As the functional J is concave in $\Theta_b(M,W^M)$, we obtain that $J\left(\eta\right)\le J\left(\alpha^{\varepsilon,n}\right)+D_{\eta-\alpha^{\varepsilon,n}}J\left(\alpha^{\varepsilon,n}\right)$ for $\eta\in\Theta_b(M,W^M)$.

- $\underline{\lim_{\varepsilon\downarrow 0} J(\alpha^{\varepsilon}) = J(\alpha)}$: This is analogous to the proof of Proposition 48. Note that using property (24), we have that $\left(\int_0^1 \alpha_t - \alpha_t^{\varepsilon} dt\right)^2 \leq C \sup_{0\leq t\leq 1}\left(\int_0^1 \alpha_t\right)^2$ and

$$\begin{aligned}
&\left|\int_0^1 H(t, Y_t^{\alpha})\,\alpha_t - H(t, Y_t^{\alpha^{\varepsilon}})\alpha_t^{\varepsilon} dt\right| \\
&\leq \left|\int_0^1 H(t, Y_t^{\alpha})(\alpha_t - \alpha_t^{\varepsilon})\, dt\right| + \left|\int_0^1 \left(H(t, Y_t^{\alpha}) - H(t, Y_t^{\alpha^{\varepsilon}})\right)\alpha_t^{\varepsilon} dt\right| \\
&\leq C \sup_{0\leq t\leq 1} |H(t, Y_t^{\alpha})| \int_0^1 |\alpha_t|\, dt \\
&+ C \sup_{0\leq t\leq 1} \int_0^1 \left|H_y(t, Y_t^{\alpha^{\varepsilon}+r(\alpha-\alpha^{\varepsilon})})\right| dr \left(\int_0^1 |\alpha_t|\, dt\right)^2.
\end{aligned}$$

This gives sufficient integrability properties to apply the dominated convergence theorem. Note that as in the proof of Lemma 44,

$$\sup_{0\leq t\leq 1}\left|H_y(t, Y_t^{\alpha^{\varepsilon}+r(\alpha-\alpha^{\varepsilon})})\right| \leq C \exp\left\{9B \sup_{0\leq t\leq 1}\left|\int_0^t \alpha_s ds\right|\right\} \exp\{B \sup_{0\leq t\leq 1} |Z_t|\}. \tag{26}$$

- $\underline{\lim_{\varepsilon\downarrow 0}\lim_{n\to\infty} D_{\eta-\alpha^{\varepsilon,n}} J(\alpha^{\varepsilon,n}) = 0}$: Repeating the proof of Proposition 49, we obtain $\left|D_{\eta-\alpha^{\varepsilon,n}} J(\alpha^{\varepsilon,n})\right| \leq B_1^{\varepsilon,n} + B_2^{\varepsilon,n}$, where

$$B_1^{\varepsilon,n} \triangleq \left|\mathbb{E}\left[\int_0^1 \left(\eta_t - \alpha_t^{\varepsilon,n}\right)\left(H(t, Y_t^{\alpha}) - H(t, Y_t^{\alpha^{\varepsilon,n}})\right) dt\right]\right|$$

and

$$B_2^{\varepsilon,n} \triangleq \left|\mathbb{E}\left[\int_0^1 \left(\int_0^t (\eta_s - \alpha_s^{\varepsilon,n})\, ds\right)\left(H_y(t, Y_t^{\alpha})\,\alpha_t - H_y(t, Y_t^{\alpha^{\varepsilon,n}})\alpha_t^{\varepsilon,n}\right) dt\right]\right|.$$

Let's show that $\lim_{\varepsilon\downarrow 0}\lim_{n\to\infty} B_1^{\varepsilon,n} = 0$. This follows by dominated convergence, once we have shown that $\sup_{\varepsilon,n}\int_0^1 \left(\eta_t - \alpha_t^{\varepsilon,n}\right)\left(H(t, Y_t^{\alpha}) - H(t, Y_t^{\alpha^{\varepsilon,n}})\right) dt \in L^1(\Omega)$, because $\lim_{n\to\infty}\alpha^{\varepsilon,n} = \alpha^{\varepsilon}$, $P\times\lambda$-a.s. and $\lim_{\varepsilon\downarrow 0}\alpha^{\varepsilon} = \alpha$ $P\times\lambda$-a.s. Using inequalities (24) and (26) we obtain

$$\begin{aligned}
&\left|\int_0^1 \left(\eta_t - \alpha_t^{\varepsilon,n}\right)\left(H(t, Y_t^{\alpha}) - H(t, Y_t^{\alpha^{\varepsilon,n}})\right) dt\right| \\
&\leq \int_0^1 \left|\eta_t - \alpha_t^{\varepsilon,n}\right| \int_0^1 H_y(t, Y_t^{\alpha^{\varepsilon,n}+r(\alpha-\alpha^{\varepsilon,n})}) dr \left|Y_t^{\alpha} - Y_t^{\alpha^{\varepsilon,n}}\right| dt \\
&\leq \left\{C + \int_0^1 |\alpha_t|\, dt\right\} \sup_{0\leq t\leq 1}\int_0^1 H_y(t, Y_t^{\alpha^{\varepsilon,n}+r(\alpha-\alpha^{\varepsilon,n})}) dr \int_0^1 \left|\alpha_t - \alpha_t^{\varepsilon,n}\right| dt,
\end{aligned}$$

which is in $L^1(\Omega)$, because as in Lemma 44, $\sup_{0\leq t\leq 1}|Z_t|$ and $\sup_{0\leq t\leq 1}\left|\int_0^t \alpha_s ds\right|$ have exponential moments. The proof of $\lim_{\varepsilon\downarrow 0}\lim_{n\to\infty} B_2^{\varepsilon,n} = 0$ can be obtained similarly. Therefore, we have proved that $J(\eta) \leq J(\alpha), \forall \eta \in \Theta_b(M, W^M)$. The final result follows from the application of Proposition 48, using an argument as in the end of the proof of Theorem 15.

■

7.2 $L(Y) = \arg\max_{t\in[0,1]} Y_t$

In this section we consider the case in which the insider knows the time at which the total demand achieves its maximum. The first part of this subsection is devoted to obtaining the compensator of W with respect to the filtration $\mathbb{F}^W \vee \sigma(\tau)$, which we will denote by $\mathbb{F}^{\tau} = \{\mathscr{F}_t^{\tau}, 0 \leq t \leq 1\}$. This will be done dividing the problem into two parts: before the random time τ and after it. But first, we give the conditional law of τ given $\mathscr{F}_t^W$.

Proposition 36 *The conditional law of τ given $\mathscr{F}_t^W$, is*

$$P\left(\tau > u|\mathscr{F}_t^W\right) = \begin{cases} 1 - 1_{\left\{M_u \geq \gamma_{u,t} + W_u\right\}} \overline{p}_2\left(M_u - W_t, 1-t\right) & \text{if} \quad u < t \\ \int_u^1 r\left(M_t - W_t, v - t, 1 - v\right) dv & \text{if} \quad u \geq t \end{cases},$$

where $r(x,s,t)$ is given by $r(x,s,t) \triangleq \frac{2}{\sqrt{2\pi t}} \phi(x,s) 1_{(0,\infty)}(x) = \frac{1}{\sqrt{2\pi t}} p_2(x,s)$. Moreover, $P\left(\tau > u|\mathscr{F}_t^W\right)$ is continuous in u, P-a.s.

Proof. If $u < t$, then

$$\begin{aligned}
& P\left(\tau > u|\mathscr{F}_t^W\right) \\
= \;& P\left(M_u < M_{u,1}|\mathscr{F}_t^W\right) = P\left(M_u < M_{u,t} \vee M_{t,1}|\mathscr{F}_t^W\right) \\
= \;& P\left(M_u < M_{u,t}, M_{u,t} > M_{t,1}|\mathscr{F}_t^W\right) + P\left(M_u < M_{t,1}, M_{u,t} \leq M_{t,1}|\mathscr{F}_t^W\right) \\
= \;& 1_{\left\{M_u < M_{u,t}\right\}} P\left(M_{u,t} > \gamma_{t,1} + W_t|\mathscr{F}_t^W\right) + P\left(\gamma_{t,1} > (M_u \vee M_{u,t}) - W_t|\mathscr{F}_t^W\right) \\
= \;& 1_{\left\{M_u < M_{u,t}\right\}} \int_0^{M_{u,t} - W_t} p_2(z, 1-t)\, dz + \int_{\left(M_u \vee M_{u,t}\right) - W_t}^{\infty} p_2(z, 1-t)\, dz \\
= \;& 1_{\left\{M_u < M_{u,t}\right\}} \int_0^{M_{u,t} - W_t} p_2(z, 1-t)\, dz + 1 - \int_0^{\left(M_u \vee M_{u,t}\right) - W_t} p_2(z, 1-t)\, dz \\
= \;& 1 - 1_{\left\{M_u \geq \gamma_{u,t} + W_u\right\}} \int_0^{M_u - W_t} p_2(z, 1-t)\, dz.
\end{aligned}$$

If $u > t$, the calculations are more involved, the idea is to break the maximum processes into pieces that are independent of $\mathscr{F}_t^W$ and pieces that are $\mathscr{F}_t^W$-measurable.

$$\begin{aligned}
& P\left(\tau > u|\mathscr{F}_t^W\right) \\
= \;& P\left(M_u < M_{u,1}|\mathscr{F}_t^W\right) = P\left(M_t \vee M_{t,u} < M_{u,1}|\mathscr{F}_t^W\right) \\
= \;& P\left(M_t < M_{u,1}, M_t \geq M_{t,u}|\mathscr{F}_t^W\right) + P\left(M_{t,u} < M_{u,1}, M_t < M_{t,u}|\mathscr{F}_t^W\right) \\
= \;& P\left(M_t - W_t < \gamma_{u,1} + W_u - W_t, M_t - W_t \geq \gamma_{t,u}|\mathscr{F}_t^W\right) \\
& + P\left(\alpha_{t,u} < \gamma_{u,1} + W_u - W_t, M_t - W_t < \gamma_{t,u}|\mathscr{F}_t^W\right).
\end{aligned}$$

Hence,

$$\begin{aligned}
& P\left(M_t - W_t < \gamma_{u,1} + W_u - W_t, M_t - W_t \geq \gamma_{t,u}|\mathscr{F}_t^W\right) \\
= \;& \int_0^{M_t - W_t} \int_{-\infty}^{x} \int_{M_t - W_t - y}^{\infty} p_1(x, y, u-t)\, p_2(z, 1-u)\, dz dy dx \\
= \;& \int_0^{M_t - W_t} \int_{-\infty}^{x} 2 p_1(x, y, u-t)\, \overline{\Phi}(M_t - W_t - y, 1-u)\, dy dx \\
= \;& \int_{-\infty}^{M_t - W_t} \int_{0 \vee y}^{M_t - W_t} 2 p_1(x, y, u-t)\, \overline{\Phi}(M_t - W_t - y, 1-u)\, dx dy \\
= \;& \int_{-\infty}^{M_t - W_t} 2(\phi(|y|, u-t) - \phi(2(M_t - W_t) - y, u-t)) \overline{\Phi}(M_t - W_t - y, 1-u)\, dy \\
= \;& \int_0^{\infty} 2(\phi(|M_t - W_t - z|, u-t) - \phi(M_t - W_t + z, u-t)) \overline{\Phi}(z, 1-u)\, dz
\end{aligned}$$

On the other hand,

$$
\begin{aligned}
& P\left(\gamma_{t,u} < \gamma_{u,1} + W_u - W_t, M_t - W_t < \gamma_{t,u} | \mathscr{F}_t^W\right) \\
= & \int_{M_t - W_t}^{\infty} \int_{-\infty}^{x} \int_{x-y}^{\infty} p_1(x,y,u-t)\, p_2(z,1-u)\, dz dy dx \\
= & \int_{M_t - W_t}^{\infty} \int_{-\infty}^{x} 2 p_1(x,y,u-t)\, \overline{\Phi}(x-y,1-u)\, dy dx \\
= & \int_{M_t - W_t}^{\infty} \int_{0}^{\infty} 2 p_1(x,x-z,u-t)\, \overline{\Phi}(z,1-u)\, dz dx \\
= & \int_{0}^{\infty} 4\phi(M_t - W_t + z, u-t)\, \overline{\Phi}(z,1-u)\, dz.
\end{aligned}
$$

Summing up, and taking into account that $\phi(|z|, u-t) = \phi(z, u-t)$, we obtain

$$
\begin{aligned}
& P\left(\tau > u | \mathscr{F}_t^W\right) \\
= & \int_0^{\infty} 2\left\{\phi(M_t - W_t - z, u-t) + \phi(M_t - W_t + z, u-t)\right\} \overline{\Phi}(z,1-u)\, dz.
\end{aligned}
$$

Differentiating under the integral sign, we obtain that there exists a density function r such that

$$
P\left(\tau > u | \mathscr{F}_t^W\right) = \int_u^1 r(M_t - W_t, v-t, 1-v)\, dv.
$$

Furthermore, this density is smooth in all its variables due to the regularity of ϕ and $\overline{\Phi}$. For the explicit computation of this density we refer to [14]. To conclude the proof we only need to show that $P\left(\tau > u | \mathscr{F}_t^W\right)$, as a function of u, is continuous in $u = t$. We have that

$$
\lim_{u \to t} P\left(\tau > u | \mathscr{F}_t^W\right) = \lim_{u \to t} P\left(M_u < M_{u,1} | \mathscr{F}_t^W\right) = P\left(M_t \le M_{t,1} | \mathscr{F}_t^W\right),
$$

where we have used the dominated convergence theorem for conditional expectations and the P-$a.s.$ continuity in t of the paths of M_t and $M_{t,1}$. ■

Proposition 37 *If* $0 \le s \le t \le 1$, *we have that*

$$
\mathbb{E}\left[1_{\{\tau > t\}}\left(W_t - \int_0^t \frac{M_u - W_u}{\tau - u} du\right) | \mathscr{F}_s^{\tau}\right] = 1_{\{\tau > t\}}\left(W_s - \int_0^s \frac{M_u - W_u}{\tau - u} du\right).
$$

Proof. Let $A \in \mathscr{F}_s^W$ and f a bounded Borel measurable function, then taking into account that τ has a conditional density given $\mathscr{F}_t^W$, in the set $\{\tau > t\}$, we have that

$$
\begin{aligned}
& \mathbb{E}\left[1_A f(\tau) 1_{\{\tau > t\}}(W_t - W_s)\right] = \mathbb{E}[1_A \mathbb{E}\left[f(\tau) 1_{\{\tau > t\}} | \mathscr{F}_t^W\right](W_t - W_s)] \\
& = \mathbb{E}\left[1_A \int_t^1 f(u)\, r(M_t - W_t, u-t, 1-u)\, du\, (W_t - W_s)\right].
\end{aligned}
$$

Applying Theorem 30 and Tanaka's formula, we obtain that the last expectation is equal to

$$
\begin{aligned}
& \mathbb{E}\left[1_A \int_t^1 f(u)\, r(|W_t|, u-t, 1-u)\, du \left(2\int_s^t dL_v^0(W) - \int_s^t d|W_v|\right)\right] \\
& = \mathbb{E}\left[1_A \int_t^1 f(u)\, r(|W_t|, u-t, 1-u)\, du \left(-\int_s^t \operatorname{sgn}(W_v)\, dW_v\right)\right].
\end{aligned}
$$

Notice that $r(|W_t|, u-t, 1-u) = \frac{2}{\sqrt{2\pi(1-u)}}\phi(W_t, u-t)$. Using Itô's formula, we can write

$$
r(|W_t|, u-t, 1-u) = \frac{2}{\sqrt{2\pi(1-u)}}\phi(0,u) + \int_0^t \frac{2}{\sqrt{2\pi(1-u)}}\partial_x \phi(W_v, u-v)\, dW_v,
$$

Then, the former expectation is equal to

$$\mathbb{E}\left[1_A\int_t^1 f(u)\int_s^t -\operatorname{sgn}(W_v)\frac{2}{\sqrt{2\pi(1-u)}}\partial_x\phi(W_v,u-v)\,dvdu\right]$$
$$=\mathbb{E}\left[1_A\int_t^1 f(u)\int_s^t \frac{|W_v|}{u-v}r(|W_v|,u-v,1-u)\,dvdu\right]$$
$$=\mathbb{E}\left[1_A\int_t^1 f(u)\int_s^t \frac{M_v-W_v}{u-v}r(M_v-W_v,u-v,1-u)\,dvdu\right]$$
$$=\mathbb{E}\left[1_A\int_s^t \mathbb{E}\left[1_{\{\tau>t\}}f(\tau)\frac{M_v-W_v}{\tau-v}|\mathscr{F}_v\right]dv\right]=\mathbb{E}\left[1_A f(\tau)1_{\{\tau>t\}}\int_s^t\frac{M_v-W_v}{\tau-v}dv\right].$$

As the σ-algebra $\mathscr{F}_s^\tau$ is generated by elements of the form $1_A f(\tau)$, where $A\in\mathscr{F}_s^W$ and f is a bounded Borel function, we obtain the result using elementary properties of the conditional expectation. Note also, that (M_t-W_t,M_t) and $(|W_t|,2L_t(0))$ are not the same processes. We can interchange them because we are dealing with expectations, and therefore they only depend on the law of the processes, which are equal by Theorem 30. ■

Now, we are going to prove an analogous result for the case after the time τ. In the proof we will use the decomposition of W with respect to $\mathbb{F}^W\vee\sigma(M)$ (see Theorem 31).

Proposition 38 *If $0\le s\le t\le 1$, we have that*

$$\mathbb{E}\left[1_{\{\tau\le s\}}\left(W_t+\int_0^t\varphi(M-W_u,1-u)\,du\right)|\mathscr{F}_s^\tau\right]=1_{\{\tau\le s\}}\left(W_s+\int_0^s\varphi(M-W_u,1-u)\,du\right)$$

Proof. Let $A\in\mathscr{F}_s^W$ and $f(\tau)=1_{\{\tau\le r\}}$, where $0\le r\le 1$. We have that

$$\mathbb{E}\left[1_A f(\tau)1_{\{\tau\le s\}}(W_t-W_s)\right]=\mathbb{E}\left[1_A 1_{\{\tau\le r\}}1_{\{\tau\le s\}}(W_t-W_s)\right]$$
$$=\mathbb{E}\left[1_A 1_{\{\tau\le r\wedge s\}}1_{\{\tau\le s\}}(W_t-W_s)\right]=\mathbb{E}\left[1_A 1_{\{M_{s\wedge r}=M\}}1_{\{\tau\le s\}}(W_t-W_s)\right]$$
$$=-\mathbb{E}\left[1_A f(\tau)1_{\{\tau\le s\}}\int_s^t\varphi(M-W_u,1-u)\,du\right].$$

Notice that $1_{\{\tau\le r\wedge s\}}=1_{\{M_{s\wedge r}=M\}}$ is $\mathscr{F}_s^W\vee\sigma(M)$-measurable, and that $\varphi(M-W_u,1-u)$ is $\mathscr{F}_u^\tau$-measurable because $M=M_\tau$. The elements of the form $1_A f(\tau)$,where $A\in\mathscr{F}_s^W$ and $f(\tau)=1_{\{\tau\le r\}}$, $0\le r\le 1$, generate the σ-algebra $\mathscr{F}_s^\tau$. Therefore as in the proof of the previous proposition we obtain the result using elementary properties of conditional expectations. ■

The next lemma gives us an integrability result for the drift term in the $\mathbb{F}^\tau$-decomposition of W.

Lemma 39 *We have that $\mathbb{E}\left[\int_0^1|\alpha(t,\tau)|\,dt\right]<\infty$ and $\mathbb{E}\left[\int_0^1|\alpha(t,\tau)|^2\,dt\right]=\infty$.*

Proof. As in Lemma 32, we have that

$$\mathbb{E}\left[\int_0^1 1_{[0,\tau)}\frac{M_t-W_t}{\tau-t}dt\right]=\mathbb{E}\left[\int_0^1 1_{[\tau,1]}(t)\,\varphi(M-W_t,1-t)\,dt\right],$$

where the integrands are positive. The second part of the statement follows as in Lemma 32. ■

The main result of this section is the following theorem which gives the semimartingale decomposition of W in the filtration $\mathbb{F}^\tau$.

Theorem 40 *W is a $\mathbb{F}^\tau$-semimartingale with the following decomposition*

$$W_t=\int_0^t\alpha(u,\tau)\,du+W_t^\tau, \tag{27}$$

where

$$\alpha(u,\tau)=\frac{M_u-W_u}{\tau-u}1_{[0,\tau)}(u)-\varphi(M-W_u,1-u)1_{[\tau,1]}(u)$$

and W^τ is a $\mathbb{F}^\tau$-Brownian motion.

Proof. If we define $W_t^\tau \triangleq W_t - \int_0^t \alpha(u,\tau)\,du$, we have a process in $L^1(\Omega)$, because $\mathbb{E}[\,|\int_0^t \alpha(u,\tau)\,du|] \leq \mathbb{E}[\int_0^t |\alpha(u,\tau)|\,du] < \infty$, by Lemma 39. Furthermore, the quadratic variation of W^τ is t, because W is a $\mathscr{F}_t^W$-Brownian motion and $\int_0^\cdot \alpha(u,\tau)\,du$ is a process of finite variation. Hence, by the Levy's characterization of the Brownian motion, we only need to prove that W^τ is a $\mathbb{F}^\tau$- local martingale. To show this, using Proposition 37 and Proposition 38, we obtain the conclusion as in the proof of Theorem 2, in [15]. ■

Theorem 41 *The Brownian motion W in the decomposition* (27) *is* $\mathbb{F}^{W^\tau} \vee \sigma(\tau)$*-adapted.*

Proof. First we will show that the following s.d.e. has a unique strong solution.

$$\mathscr{E}_1: \qquad X_t = \int_0^t \frac{M_s^X - X_s}{\rho - s}ds + B_t, \quad 0 \leq t < \rho,$$

where B is a Brownian motion with respect its own filtration and ρ is a random variable independent of B and taking values in $[0,1]$. We will prove first that $\Psi\left(t, X_{[0,t]}\right) = \frac{M_t^X - X_t}{\tau - t}$ is functional Lipschitz. We have that

$$\left|\Psi\left(t, X_{[0,t]}\right) - \Psi\left(t, Y_{[0,t]}\right)\right| \leq \frac{1}{\tau - t}\left\{|X_t - Y_t| + \left|M_t^X - M_t^Y\right|\right\}.$$

Obviously, $|X_t - Y_t| \leq M_t^{|X-Y|}$. On the other hand $M_t^X \leq M_t^{X-Y} + M_t^Y$, which gives that $M_t^X - M_t^Y \leq M_t^{|X-Y|}$. We also have that $M_t^Y \leq M_t^{Y-X} + M_t^X$, which yields $M_t^Y - M_t^X \leq -M_t^{X-Y} \leq -M_t^{|X-Y|}$. Hence, $\left|\Psi\left(t, X_{[0,t]}\right) - \Psi\left(t, Y_{[0,t]}\right)\right| \leq \frac{2}{\tau - t}M_t^{|X-Y|}$. By Theorem 7 in chapter V of Protter [19], we obtain the existence and uniqueness of solutions for $\mathscr{E}_1$.

Now, if we take $B_t = W_t^\tau$ and $\rho = \tau$, we obtain that X must coincide with W_t for $t < \tau$. This means that W_t is $\mathscr{F}_t^{W^\tau} \vee \sigma(\tau)$-adapted. Furthermore, as $\lim_{t\to\tau} W_t = W_\tau$, one also has that W_τ is $\mathscr{F}_t^{W^\tau} \vee \sigma(\tau)$-adapted.

The next step is to show existence and uniqueness for solutions of

$$\mathscr{E}_2: \qquad X_t = G - \int_\rho^t \varphi(G - X_s, 1-s)\,ds + B_t - B_\rho, \quad \rho \leq t < 1,$$

where B is a Brownian motion with respect its own filtration, ρ is a random variable independent of B and taking values in $[0,1]$ and G is a $\mathscr{F}_\rho^B$-adapted random variable. The existence is proved in Proposition 34 and the uniqueness in Theorem 33. If we take $B_t = W_t^\tau, \rho = \tau$ and $G = M$ we obtain that X must coincide with W_t for $t \geq \tau$. This means that W_t is $\mathscr{F}_t^{W^\tau} \vee \sigma(\tau)$-adapted. ■

Theorem 42 *Let* $L(Y) = \arg\max_{0\leq t\leq 1} Y_t$*.Then*

$$\left(Y^*, \theta^*, Z^*, H^*, \xi^*, \lambda^*\right) = \left(W, \alpha(\cdot,\tau), W^\tau, H, H(1, W_1), L(W)\right)$$

satisfies all the requirement to be a (L,μ)*-weak equilibrium except the càglàd property in the condition v).*

Proof. The proof of this result is exactly the same as the one for Theorem 35, except for the càglàd approximations used. In this case, for $\varepsilon \in (0,1)$, define $\tau^{\varepsilon,+} = (\tau + \varepsilon) \wedge 1$ and $\tau^{\varepsilon,-} = (\tau - \varepsilon) \wedge 0$. Then the process $\alpha^\varepsilon = \{\alpha_t^\varepsilon \triangleq \alpha_t 1_{(\tau^{\varepsilon,-},\tau^{\varepsilon,+}]^c}(t), t \in [0,1]\}$ converges $P \times \lambda$,a.e. to α as $\varepsilon \downarrow 0$ and it satisfies $|\alpha^\varepsilon| \leq |\alpha|$. ■

7.3 Comparing the expected wealth of M and τ

In this subsection we show using numerical calculations that the information about the time at which the total demand achieves its maximum gives less expected profit that the information about the maximum. That is,

$$J(\alpha(\cdot.\tau)) \leq J(\alpha(\cdot.M)).$$

We can write

$$J\left(\alpha\left(\cdot,M\right)\right)=\mathbb{E}\left[\int_0^1\left(\mathbb{E}\left[\xi|\mathscr{F}_1^W\right]-H\left(t,W_t\right)\right)\alpha\left(t,M\right)dt\right]$$
$$=\mathbb{E}\left[\int_0^1\left(H\left(1,W_1\right)-H\left(t,W_t\right)\right)\alpha\left(t,M\right)dt\right].$$

Note that

$$\mathbb{E}\left[\int_0^1 H\left(t,W_t\right)\alpha\left(t,M\right)dt\right]=\mathbb{E}\left[\int_0^1 H\left(t,W_t\right)dW_t\right]-\mathbb{E}\left[\int_0^1 H\left(t,W_t\right)dW_t^M\right]=0,$$

because the integrability properties of H yield that $\int_0^t H\left(s,W_s\right)dW_s$ and $\int_0^t H\left(s,W_s\right)dW_s^M$ are a $\mathbb{F}^W$-martingale and a $\mathbb{F}^W\vee\sigma\left(M\right)$-martingale, respectively. As the same arguments work for $\alpha\left(\cdot,\tau\right)$, we obtain that for $\lambda\in\{M,\tau\}$

$$J\left(\alpha\left(\cdot,\lambda\right)\right)=\mathbb{E}\left[H\left(1,W_1\right)\int_0^1\alpha\left(t,\lambda\right)dt\right].$$

Note also that, after τ, the compensators of M and τ coincide. Hence, the problem is reduced to verify if

$$A(M)\triangleq\mathbb{E}\left[H\left(1,W_1\right)\int_0^\tau\frac{M-W_t}{1-t}dt\right]\geq\mathbb{E}\left[H\left(1,W_1\right)\int_0^\tau\frac{M_t-W_t}{\tau-t}dt\right]\triangleq A(\tau).$$

7.3.1 Computation of $A(M)$ and $A(\tau)$

An exact computation of $A(M)$ and $A(\tau)$ is difficult. This is due to the fact that we need to compute integrals with respect to the joint density of (W_1,τ) conditioned to $\mathscr{F}_t^W$, which is unknown. Although we have computed explicitly this density, it turns out that it is useless because of its complicated expression. Therefore, we perform a Monte Carlo simulation.

First of all, we have considered a uniform partition $\pi_m=\left\{t_i=\frac{i}{m}\right\}_{i=0,\ldots,m}$ of the interval $[0,1]$. We sample the paths of a Brownian motion in this partition and approximate the integral inside the expectation by its upper Riemann sum in π_m. We have used $H\left(1,W_1\right)$ as a control variate to reduce the variance of our estimators. Recall that the variance of $H\left(1,W_1\right)$ can be computed analytically and the covariances $\mathrm{Cov}(H\left(1,W_1\right),H\left(1,W_1\right)\int_0^\tau\frac{M-W_t}{1-t}dt)$ and $\mathrm{Cov}(H\left(1,W_1\right),H\left(1,W_1\right)\int_0^\tau\frac{M_t-W_t}{\tau-t}dt)$ have been estimated doing a pilot simulation with number of simulations $n=1000$. The main simulations, including the control variate, have length $n=10^5$. We have repeated the simulations for different partitions. We quote here the results with $m=10000$ and we also compute a 99% confidence interval $[L,U]$ for each simulation. The results are showed in the following tables. Here β denotes the value of the control variate. The pricing rules that we use in our experiments are $H(t,y)=y$ and $H(t,Y)=e^{y+(1-t)/2}$, which are solutions to the heat equation (15). We denote them by the letters L and E respectively. These examples of pricing rule are the examples considered in Back [3] and yields that the prices process follows a Brownian motion and a geometric Brownian motion, respectively. In the first case, note that price or demand information are the same.

Monte Carlo estimation of optimal utilities

	$\widehat{A}$	L	U	$\widehat{\sigma}_n$	β	H
$\widehat{A}(M)$	0.684	0.675	0.693	1.074	1.062	L
$\widehat{A}(\tau)$	0.189	0.186	0.192	0.376	0.616	L
$\widehat{A}(M)$	2.656	2.625	2.687	3.775	1.955	E
$\widehat{A}(\tau)$	1.386	1.379	1.393	0.829	0.718	E

From these simulations one is inclined to postulate that $J\left(\alpha\left(\cdot.\tau\right)\right)\leq J\left(\alpha\left(\cdot.M\right)\right)$.

Remark 43 *It is worth pointing out that these examples can also be considered in the Karatzas-Pikovsky setting, see [12]. In this setting, one studies the portfolio optimization problem of an agent with additional information, with respect to the small investor. This model assumes that the price dynamics is given exogenously and that the insider can not influence the price process (for more information, on this type of formulation, see e.g.[12], [1], [9], [8], [7], [3] and [6], among others). In fact, in this framework the finiteness of logarithmic utilities for the insider is determined by the quantity* $\mathbb{E}\left[\int_0^1 |\alpha(s)|^2 ds\right]$. *By Lemmas 39 and 32, we have that in both cases the logarithmic utility for the insider is infinite. However, as the compensator is the same after* τ, *we decided to compute the previous expectation until* τ. *The result is*

$$\mathbb{E}\left[\int_0^\tau |\alpha(s,M)|^2 ds\right] < \mathbb{E}\left[\int_0^\tau |\alpha(s,\tau)|^2 ds\right] = \infty,$$

which is the reverse conclusion as the one shown in the above table. Nevertheless, there are two major differences between our approach and the Karatzas-Pikovsky's one. First, in Karatzas-Pikovsky's approach the insider is risk averse, while in our approach is risk neutral. Moreover, in Karatzas-Pikovsky's approach the insider has no influence in the price dynamics, while in our model the price process is driven by the insider's demand. Therefore, it would be interesting to extend our model to risk averse insiders to try to examine this issue further.

8 Conclusions

In this paper we construct a model which allows the existence of a rational expectations equilibrium, in a weaker sense than that of Kyle-Back's setting, with an insider possessing information different from the value of the asset at the end of the trading interval. We provide sufficient conditions for the existence and uniqueness in law of a weak equilibrium. Our model allows to compare the expected wealth obtained by insiders with different kinds of information. We study in some detail the examples of the maximum and the time at which the maximum of the demand is achieved, finding that the first provides more expected final wealth than the second. In order to deal with these examples we prove a new initial enlargement formula for the argument of the maximum of a Brownian motion. Moreover, we prove the existence and uniqueness of a strong solution for a stochastic differential equation with a drift degenerating at a random time.

9 Appendix

Lemma 44 *Let* F *be a function satisfying an exponential growth condition and* θ *a process satisfying* (4), *then* $\sup_{0\leq t\leq 1}\left|F(t,Y_t^\theta)\right|$ *belongs to* $L^p(\Omega)$, *for any* $p\geq 0$, *where* $Y_t^\theta = \int_0^t \theta_s ds + Z_t$ *and* Z *is a Brownian motion.*

Proof. Thanks to the exponential growth condition on F, one has that

$$\begin{aligned}\sup_{0\leq t\leq 1}\left|F(t,Y_t^\theta)\right|^p &\leq A \sup_{0\leq t\leq 1} \exp\left\{pB\left|\int_0^t \theta_s ds + Z_t\right|\right\} \\ &\leq A\exp\{pB\sup_{0\leq t\leq 1}\left|\int_0^t \theta_s ds\right|\}\exp\{pB\sup_{0\leq t\leq 1}|Z_t|\}.\end{aligned}$$

The result follows from (4) and the fact that the law of $\sup_{0\leq t\leq 1}|Z_t|$ has finite exponential moments. ■

Proposition 45 *Let* Θ *be a convex real linear space and* J *a functional defined on* Θ. *Assume that for any* $\theta\in\Theta$ *there exists the Gâteaux derivative of* J. *That is, for all* $v\in\Theta$ *the following limit exists*

$$D_v J(\theta) \triangleq \lim_{\varepsilon\to 0}\frac{J(\theta+\varepsilon v)-J(\theta)}{\varepsilon},$$

and the application $v\mapsto D_v J(\theta)$ *is linear for every* $\theta\in\Theta$. *Then, the following statements are equivalent:*

1) J is concave;

2) $J\left(\theta^2\right) \le J\left(\theta^1\right) + D_{\theta^2-\theta^1}J\left(\theta^1\right), \quad \forall \theta^1, \theta^2 \in \Theta.$

Proof. If J is concave then $J\left(\alpha\theta^2 + (1-\alpha)\theta^1\right) \ge \alpha J\left(\theta^2\right) + (1-\alpha)J(\theta^1), \forall \theta^1, \theta^2 \in \Theta, \alpha \in [0,1]$. This implies

$$\frac{J\left(\theta^1 + \alpha\left(\theta^2 - \theta^1\right)\right) - J\left(\theta^1\right)}{\alpha} \ge J\left(\theta^2\right) - J\left(\theta^1\right), \quad \forall \theta^1, \theta^2 \in \Theta, \alpha \in [0,1].$$

Taking the limit when α tends to zero, by the assumption on the existence of the Gâteaux derivative, we obtain $D_{\theta^2-\theta^1}J\left(\theta^1\right) \ge J\left(\theta^2\right) - J\left(\theta^1\right), \forall \theta^1, \theta^2 \in \Theta$. Conversely, assume that the statement 2) is satisfied. Set $\theta = \alpha\theta^2 + (1-\alpha)\theta^1$, then we have that

$$\begin{aligned} J\left(\theta^1\right) &\le J(\theta) + D_{\theta^1-\theta}J(\theta), \\ J\left(\theta^2\right) &\le J(\theta) + D_{\theta^2-\theta}J(\theta). \end{aligned}$$

Multiplying the first inequality by α and the second one by $(1-\alpha)$ and adding them one obtains

$$\alpha J\left(\theta^1\right) + (1-\alpha)J\left(\theta^2\right) \le J(\theta) + D_{\alpha(\theta^1-\theta)+(1-\alpha)\left(\theta^2-\theta\right)}J(\theta).$$

As $\alpha(\theta^1 - \theta) + (1-\alpha)\left(\theta^1 - \theta\right) = 0$, the result follows. ■

Lemma 46 *Assume that $H \in \mathscr{H}$ and $\theta, v \in \Theta_b(M,Z)$. Then, for all $\varepsilon \in (-1,1)$, we have that for $i = 1,2$*

$$\frac{d^i}{d\varepsilon^i}\left(\mathbb{E}\left[\int_0^1 (\xi - H(t, Y_t^{\theta+\varepsilon v}))(\theta_t + \varepsilon v_t)\,dt\right]\right) = \mathbb{E}\left[\int_0^1 \frac{d^i}{d\varepsilon^i}\left((\xi - H(t, Y_t^{\theta+\varepsilon v}))(\theta_t + \varepsilon v_t)\right)dt\right].$$

Proof. We do the proof for $i = 2$. First, we estimate

$$\begin{aligned} &\left|\frac{d^2}{d\varepsilon^2}\left\{(\xi - H(t, Y_t^{\theta+\varepsilon v}))(\theta_t + \varepsilon v_t)\right\}\right| \\ &\le \left|H_{yy}(t, Y_t^{\theta+\varepsilon v})\left(\int_0^t v_s ds\right)^2 (\theta_t + \varepsilon v_t)\right| + \left|2H_y(t, Y_t^{\theta+\varepsilon v})\left(\int_0^t v_s ds\right) v_t\right| \\ &\le C\left\{\left|H_{yy}(t, Y_t^{\theta+\varepsilon v})\right| + \left|H_y(t, Y_t^{\theta+\varepsilon v})\right|\right\}, \end{aligned}$$

where C is a constant which is independent of ε. These quantities are bounded in $L^p(\Omega)$ as Lemma 44 shows. Hence, the result follows by the dominated convergence theorem. ■

Definition 47 *Let $\theta \in \Theta_{\sup}(M,Z)$. For every $n \in \mathbb{N}$, define $\theta_t^n = \theta_t 1_{\{\sup_{s\le t}|\theta_s|\le n\}}$. Clearly, the sequence $\{\theta^n\}_{n\in\mathbb{N}} \subseteq \Theta_b(M,Z)$. We also have that $\{\theta^n\}_{n\in\mathbb{N}}$ converges $P \times \lambda$ a.s. to θ. Furthermore, $\{\theta^n\}_{n\in\mathbb{N}}$ converges to θ in $L^1(P \times \lambda)$ by dominated convergence, because $|\theta^n| \le |\theta|$.*

Proposition 48 *Assume that $\theta \in \Theta_{\sup}(M,Z)$. Then, $\lim_{n\to\infty} J(\theta^n) = J(\theta)$, where J is the functional defined in (7).*

Proof. We can define the following sequence of $\mathbb{F}^I$-stopping times $\tau^n = \inf\{t \le 1 : \sup_{s\le t}|\theta_s| > n\}$. In the set $\{\tau^n > t\}$, one has that for all $s \le t, |\theta_s| \le n$ and $\theta_s^n = \theta_s$. On the other hand, in the set $\{\tau^n \le t\}$, one has that $\sup_{s\le t}|\theta_s| > n$ and $\theta_t^n = 0$. Moreover, $\tau^n \uparrow 1$, P-a.s., when n tends to infinity. We have that

$$\begin{aligned} |J(\theta) - J(\theta^n)| &\le \left|\mathbb{E}\left[\xi\int_0^1 (\theta_t - \theta_t^n)\,dt\right]\right| + \left|\mathbb{E}\left[\int_0^1 \left(H(t, Y_t^\theta)\theta_t - H(t, Y_t^{\theta^n})\theta_t^n\right)dt\right]\right| \\ &\triangleq A_1^n + A_2^n. \end{aligned}$$

Applying Cauchy-Schwarz, we obtain

$$A_1^n \leq \mathbb{E}\left[|\xi|\left|\int_0^1 (\theta_t - \theta_t^n)\,dt\right|\right] \leq \mathbb{E}[|\xi|^2]^{1/2}\mathbb{E}\left[\left(\int_0^1 (\theta_t - \theta_t^n)\,dt\right)^2\right]^{1/2}.$$

The first expectation is finite, because ξ has moments of second order. For the second expectation, notice that if we fix $\omega \in \Omega$, by dominated convergence, we have that $\lim_{n\to\infty}\left(\int_0^1 (\theta_t - \theta_t^n)\,dt\right)^2 = 0, P$-a.s., because $\theta \in L^1(\Omega \times [0,1])$. Furthermore,

$$\left(\int_0^1 (\theta_t - \theta_t^n)\,dt\right)^2 = \left(\int_{\tau^n}^1 \theta_t dt\right)^2 \leq C \sup_{0\leq t\leq 1}\left(\int_0^t \theta_s ds\right)^2,$$

which is in $L^1(\Omega)$, by hypothesis (4). Therefore, also by dominated convergence one has that

$$\lim_{n\to\infty}\mathbb{E}\left[\left(\int_0^1 (\theta_t - \theta_t^n)\,dt\right)^2\right] = 0.$$

For the term A_2^n, we have that

$$\int_0^1 \left(H(t,Y_t^\theta)\theta_t - H((t,Y_t^{\theta^n})\theta_t^n\right)dt = \int_{\tau^n}^1 H(t,Y_t^\theta)\theta_t dt, \quad P\text{-a.s.}$$

Notice that,

$$\int_0^1 \left|H(t,Y_t^\theta)\theta_t\right| dt \leq \sup_{0\leq t\leq 1}\left|H(t,Y_t^\theta)\right|\int_0^1 |\theta_t|\,dt < \infty, \quad P-a.s.,$$

because $\theta \in L^1(\Omega \times [0,1])$. Hence, $\lim_{n\to\infty}\int_{\tau^n}^1 H(t,Y_t^\theta)\theta_t dt = 0, P$-a.s., by dominated convergence. Furthermore, $\left|\int_{\tau^n}^1 H(t,Y_t^\theta)\theta_t dt\right| \leq C\sup_{0\leq t\leq 1}\left|\int_0^t H(s,Y_s^\theta)\theta_s ds\right|$ which is in $L^1(\Omega)$, by hypothesis (2). Thus, by dominated convergence theorem we obtain that $\lim_{n\to\infty} A_2^n = 0$. ■

Proposition 49 *Assume that $\theta \in \Theta_{\sup}(M,Z)$ satisfies the optimality equation* (9)*. Then,* $\lim_{n\to\infty} D_{\eta-\theta^n}J(\theta^n) = 0$*, for all $\eta \in \Theta_b(M,Z)$, where $D_{\eta-\theta^n}J(\theta^n)$ is given by* (8).

Proof. As $\theta \in \Theta_{\sup}(M,Z)$ satisfies equation (9), Remark 13 yields

$$\mathbb{E}\left[\int_0^1 (\eta_t - \theta_t^n)\{(\xi - H(t,Y_t^\theta)) - \int_t^1 H_y(s,Y_s^\theta)\theta_s ds\}dt\right] = 0.$$

Therefore,

$$\begin{aligned}\left|D_{\eta-\theta^n}J(\theta^n)\right| &\leq \left|\mathbb{E}\left[\int_0^1 (\eta_t - \theta_t^n)(H(t,Y_t^\theta) - H(t,Y_t^{\theta^n}))dt\right]\right| \\ &+ \left|\mathbb{E}\left[\int_0^1\left(\int_0^t (\eta_s - \theta_s^n)\,ds\right)(H_y(t,Y_t^\theta)\theta_t - H_y(t,Y_t^{\theta^n})\theta_t^n)dt\right]\right| \\ &\triangleq B_1^n + B_2^n.\end{aligned}$$

For the term B_1^n, one has that

$$\begin{aligned}&\mathbb{E}\left[\int_0^1 (\eta_t - \theta_t^n)(H(t,Y_t^\theta) - H(t,Y_t^{\theta^n}))dt\right] \\ &= \mathbb{E}\left[\int_0^1 1_{\{t>\tau^n\}}\eta_t(H(t,Y_t^\theta) - H(t,Y_{\tau^n}^\theta + Z_t - Z_{\tau^n}))dt\right].\end{aligned}$$

When n tends to infinity the integrand in the last equation tends to 0, $P\otimes\lambda$-a.s. So we only need to justify the application of the dominated convergence theorem. We have, $P\otimes\lambda$-a.s., that

$$\left|\eta_t(H(t,Y_t^\theta)-H(t,Y_{\tau^n}^\theta+Z_t-Z_{\tau^n}))\right|\le C\{\sup_{0\le t\le 1}\left|H(t,Y_t^\theta)\right|+\left|H(t,Y_{\tau^n}^\theta+Z_t-Z_{\tau^n})\right|\},$$

By Lemma 44, $\sup_{0\le t\le 1}\left|H(t,Y_t^\theta)\right|$ is an integrable random variable. These quantities are bounded in $L^p(\Omega)$ as the proof of Lemma 44 shows. Hence by dominated convergence $\lim_{n\to\infty}B_1^n=0$. For the term B_2^n, one has

$$\begin{aligned}
&\mathbb{E}\left[\int_0^1\left(\int_0^t(\eta_s-\theta_s^n)\,ds\right)(H_y(t,Y_t^\theta)\theta_t-H_y(t,Y_t^{\theta^n})\theta_t^n)dt\right]\\
&=\mathbb{E}\left[\int_{\tau^n}^1\left(\int_0^t(\eta_s-\theta_s^n)\,ds\right)H_y(t,Y_t^\theta)\theta_t dt\right]\\
&=\mathbb{E}\left[\int_{\tau^n}^1\left(\int_{\tau^n}^t\eta_s ds\right)H_y(t,Y_t^\theta)\theta_t dt\right]+\mathbb{E}\left[\int_{\tau^n}^1\left(\int_0^{\tau^n}(\eta_s-\theta_s)\,ds\right)H_y(t,Y_t^\theta)\theta_t dt\right]\\
&\triangleq B_{2,1}^n+B_{2,2}^n.
\end{aligned}$$

The term $B_{2,1}^n$ converges to zero due to the dominated convergence theorem as

$$\begin{aligned}
&\left|\int_{\tau^n}^1\left(\int_{\tau^n}^t\eta_s ds\right)H_y(t,Y_t^\theta)\theta_t dt\right|=\left|\int_{\tau^n}^1\eta_s\left(\int_s^1 H_y(t,Y_t^\theta)\theta_t dt\right)ds\right|\\
&\le C\sup_{0\le t\le 1}\left|\int_0^t H_y(s,Y_s^\theta)\theta_s ds\right|\int_{\tau^n}^1|\eta_s|\,ds\le C\sup_{0\le t\le 1}\left|\int_0^t H_y(s,Y_s^\theta)\theta_s ds\right|\in L^1(\Omega),
\end{aligned}$$

thanks to condition (2) and that $\int_{\tau^n}^1|\eta_s|\,ds$ converges to zero as n tends to infinity. The term $B_{2,2}^n$ converges to zero due to the dominated convergence theorem as

$$\begin{aligned}
\left|\int_{\tau^n}^1\left(\int_0^{\tau^n}(\eta_s-\theta_s)\,ds\right)H_y(t,Y_t^\theta)\theta_t dt\right|&\le C\sup_{0\le t\le 1}\left|\int_0^t(\eta_s-\theta_s)\,ds\right|\\
&\times\sup_{0\le t\le 1}\left|\int_0^t H_y(s,Y_s^\theta)\theta_s ds\right|\in L^1(\Omega),
\end{aligned}$$

thanks to conditions (3) and (4), that $\eta\in\Theta_b(M,Z)$ and that $\left|\int_{\tau^n}^t H_y(s,Y_s^\theta)\theta_s ds\right|$ converges to zero as n tends to infinity. ■

Lemma 50 *Let $a,b\in\mathbb{R}$, and $s,t>0$, then*

$$\int_0^\infty\phi(x+a,s)\,\phi(x+b,t)\,dx=\phi(b-a,s+t)\left(1-\Phi\left(\frac{at+bs}{s+t},\frac{st}{s+t}\right)\right).$$

Proof. To prove this statement, rewrite the product of the two density functions $\phi(x+a,s)\,\phi(x+b,t)$ as a single density function by completing squares in the exponent. ■

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