Parametrix method for skew diffusions*

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Abstract

In this article, we apply the parametrix method in order to obtain the existence and the regularity properties of the density of a skew diffusion and provide a Gaussian upper bound. This expansion leads to a probabilistic representation.

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1 Introduction

A skew diffusion is the unique solution of the following one dimensional stochastic differential equation (SDE) with local time:

$$X_t(x) = x + \int_0^t b(X_s(x))ds + \int_0^t \sigma(X_s(x))dW_s + (2\alpha - 1)L_t^0(X),\ t \geq 0, \alpha \in (0, 1), \tag{1}$$

where $W = (W_t)_{t\geq 0}$ is a one-dimensional standard Brownian motion and $L^0(X) = (L_t^0(X))_{t\geq 0}$ is the symmetric local time of X at the origin. Here, we will assume that b is bounded and measurable and σ is uniformly elliptic, bounded and $a^2 = \sigma$ is a Hölder continuous function.

Suppose that $b = 0$ and $\sigma = 1$, then the solution of (1) is called the skew Brownian motion. Harrison and Shepp [8] prove that if $|2\alpha - 1| \leq 1$ then there is a unique strong solution and if $|2\alpha - 1| > 1$, there is no solution. The idea of the proof is a transformation technique to relate (1) with another stochastic differential equation without local time.

The equation (1) is linked with various applications as can be seen in Lejay [14] and the references therein. For example, Lejay and Martinez [15] introduce a numerical scheme for a skew diffusion, which is based on the simulation of skew Brownian motion. Martinez and Talay [16] prove that the expectation of a skew diffusion is a solution to a parabolic type partial differential equation with interface conditions at zero. They also introduce a transformed Euler scheme and provide the weak convergence rate for their numerical scheme. Another approximation scheme

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for a skew diffusion is introduced by Étoré [4] using a random walk approach. In [5], Étoré and Martinez introduce an exact simulation scheme for skew diffusions when the diffusion coefficient is constant.

Gairat and Shcherbakov in [7] give explicitly the joint density function of a skew diffusion with constant diffusion coefficient and some of its functionals. They apply their results to a mathematical finance model of stock prices with switching coefficients.

In this paper, our goal is to show that one can apply the parametrix method for stochastic differential equations of the type 1 and provide a probabilistic representation for the density. The other goal of this paper is to show as one possible application of this result that a Gaussian upper bound for the density exists and the differentiability of this density with respect to the initial variable x. The main mathematical difficulty we have to face is the fact that we have to deal with the local time term appearing in the equation (1).

The parametrix method (cf. Friedman [6]) is a classical method in order to construct fundamental solutions for parabolic type partial differential equations using a "Taylor-like" expansion argument. This method allows for coefficients to be less regular than in the Malliavin Calculus approach for the study of the density. On the other hand, this methodology is restricted to situations where the underlying process is Markov. As a sample of recent developments of this method, we refer the reader to Menozzi [17], Foschi et al [3] and the references therein.

Bally and Kohatsu-Higa [2] introduce the parametrix method using a semigroup approach and obtain the probabilistic representation for the density of the solution to a diffusion equation or for Lévy driven SDEs. They consider two kinds of parametrix methods already considered in [3]: the first one is called "forward parametrix method" and second one is called "backward parametrix method". In order to construct a forward parametrix expansion for diffusion process, we need to assume that the coefficients are C_b^2. On the other hand, a backward parametrix expansion converges if the drift coefficient is bounded and measurable and diffusion coefficient is bounded, uniformly elliptic and Hölder continuous.

To simplify the discussion, we will only consider the backward parametrix method. When one applies the parametrix method for the semigroup of diffusion equations, one uses the Euler scheme with coefficients evaluated at the arrival point of the density as the approximation process, in order to obtain the expansion for the semigroup around this approximation process.

For a skew diffusion case, we will take a generalized version of the skew Brownian motion (see, (10)) as "approximation process". The reason for this choice instead of the usual Euler scheme is because the latter is probably not suitable for this argument and that the density function of skew Brownian motion can still be written explicitly.

The parametrix expansion leads to a probabilistic representation for the density function of skew diffusions and therefore also provide a representation for the expectation of $f(X_T(x))$ for certain classes of functions f. Such a probabilistic representation can be used for many purposes, notably for Monte Carlo simulation or as an extension of the classical infinite dimensional analysis known as Malliavin Calculus. These and other applications such as lower bounds, differentiability with respect to time will be discussed elsewhere.

Finally, we note that our results for a skew diffusion process can also be extended to a diffusion process with dis-continuous coefficients by using the relation between the two processes (see Proposition 2.7).

This article is divided as follows: In Section 2, we give the notation and assumptions used throughout the paper. We will introduce the definition of the symmetric local time for continuous semi-martingale and the skew Brownian motion. We will also see the relation between skew diffusion and a SDE with discontinuous diffusion coefficient with an explicit construction of the skew diffusion flows. In Section 3, we obtain the generator associated with X and its domain of definition. In Section 4 we will give some key estimates in order to construct a parametrix expansion for a skew diffusion based on skew Brownian motion. In Section 5, we provide the parametrix method for the skew diffusion process using the semigroup approach to prove existence and Gaussian upper bound for its density function. Our main result is given in Theorem 5.1. In Section 6, we will show a regularity of the density function for a skew diffusion obtained in Section 5. This gives our second main result in the form of Theorem 6.1. In Section 7, we provide our probabilistic representation for the density of a skew diffusion process which based on a parametrix expansion. In a short Appendix, we provide an explicit calculation for beta type integrals.

2 Preliminaries

2.1 Notations and assumptions

We give some basic notations and definitions used throughout this paper. For a sequence of operators $(S_i)_{i=1,\dots,n}$, we define $\prod_{i=1}^{n} S_i = S_1 \dots S_n$ and $\prod_{i=n}^{1} S_i = S_n \dots S_1$. S^* will denote the adjoint operator of the operator S. The space of real valued infinitely differentiable functions with compact support contained in $\mathbb{R}$ is denoted by $C_c^\infty(\mathbb{R})$. Similarly, $L^\infty(\mathbb{R})$ denotes the space of all bounded measurable functions with the norm $||f||_\infty := \text{esssup}_{x\in\mathbb{R}}|f(x)|$. We define $\mathbb{R}_0 := \mathbb{R} \setminus \{0\}$. We denote by g_t^c the density function of the standard Brownian motion with variance c, i.e., $g_t^c(y) := \frac{e^{-\frac{y^2}{2tc}}}{\sqrt{2\pi tc}}$, $y \in \mathbb{R}$. The associated Hermite polynomials are defined respectively as $H_i(y, ct) := g_t^c(y)^{-1}\partial_y^i g_t^c(y)$, $i \in \mathbb{N}$. We define the Mittag-Leffler function $E_{\alpha,\beta}$ defined as $E_{\alpha,\beta}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+\beta)}$, $z \in \mathbb{R}, \alpha, \beta > 0$. Throughout this paper, we will use $t_0 := T$ as a fixed time where the densities will be evaluated.

We now state our main hypothesis on the coefficients of the SDE (1).

Assumption 2.1. The measurable functions b and $\sigma : \mathbb{R} \to \mathbb{R}$ satisfy the following conditions:

(i) σ is a positive, bounded and uniformly elliptic function. In particular, there exist positive constants $\overline{a}$ and $\underline{a}$, such that for any $x \in \mathbb{R}$, $\underline{a} \le a(x) := \sigma^2(x) \le \overline{a}$.

(ii) b is bounded and $a = \sigma^2$ is η-Hölder continuous for some $\eta \in (0, 1]$, i.e., there exist a positive constant K such that

$$\sup_{x\in\mathbb{R}} |b(x)| + \sup_{x,y\in\mathbb{R},x\neq y} \frac{|a(x) - a(y)|}{|x - y|^\eta} \le K.$$

Remark 2.2. If σ is continuous and uniformly elliptic, then σ is either positive or negative. If σ is negative, we can replace the Brownian motion W_t by $-W_t$. Therefore, the assumption that σ is positive is only for convenience.

Through the article, the constants C and c may change from line to line, where C may depend on $(K, \alpha, \underline{a}, \overline{a}, \eta)$ and c may depend on $(\alpha, \overline{a})$. As a particular constant with explicit dependence on time, we use the notation $C_T := C(1 + T^{\frac{1-\eta}{2}})$.

For solutions of SDE's, we write $X_t(x)$ or X_t indistinctly to denote the solution process.

2.2 Construction of the solution flow process X

In this section, we give a specific construction of the solution for (1) using a transformation method. The arguments of this section have an intersection with parts in Lejay [14] and Kulik [12]. We give them here for the sake of completeness so that the reader may follow easily the arguments.

Let us consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions. We first recall the definition of *symmetric local time* for a one-dimensional continuous semi-martingale $Y = (Y_t)_{t\geq 0}$. A stochastic process $(L_t^a(Y))_{t\geq 0}$ is called the symmetric local time of Y at $a \in \mathbb{R}$ if it satisfies

$$|Y_t - a| = |Y_0 - a| + \int_0^t (\mathbf{1}(Y_s > a) - \mathbf{1}(Y_s < a))dY_s + L_t^a(Y).$$

By Itô-Tanaka formula, the symmetric local time of Y exists and is unique (see, e.g. Karatzas and Shreve [9], Section 3.7).

Let us consider the one-dimensional stochastic differential equation

$$Z_t(z) = z + \int_0^t \rho(Z_s(z))dW_s, z \in \mathbb{R}, t \geq 0, \tag{2}$$

on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. In this section, we prove that the mapping $z \mapsto Z_t(z)$ is continuous for any $t \geq 0$. We will use this in the proof of Proposition 5.6.

Theorem 2.3. Assume that ρ is a measurable function and that there exist positive constants $c_0 > 1$ such that for any $z \in \mathbb{R}$, $c_0^{-1} \leq \rho^2(z) \leq c_0$. Then there exists a weak solution for the SDE (2) and the uniqueness in the sense of probability law holds. Moreover if ρ is continuous on $\mathbb{R}_0$ then the mapping $z \mapsto Z_t(z)$ is continuous for any $t \geq 0$ and $z \in \mathbb{R}$ almost surely.

In order to prove the above theorem, we first introduce the result of Engelbert and Schmidt.

Lemma 2.4 ([9], Theorems 5.5.4 and 5.5.7)**.** The stochastic differential equation (2) has a non-exploding weak solution if and only if

$$I(\rho) \subseteq \mathcal{Z}(\rho),$$

where

$$I(\rho) := \left\{ z \in \mathbb{R} \middle| \int_{-\varepsilon}^{\varepsilon} \frac{dy}{\rho^2(z+y)} = \infty, \forall \varepsilon > 0 \right\} \text{ and } \mathcal{Z}(\rho) := \{z \in \mathbb{R} | \rho(z) = 0\}.$$

Moreover, the SDE (2) has a weak solution and is unique in the sense of probability law if and only if

$$I(\rho) = \mathcal{Z}(\rho).$$

The explicit construction of this unique solution will be used in what follows. For explicit details, we refer the reader to [9], Chapter 5.

Let $B = (B_t, \mathcal{G}_t)_{t\geq 0}$ be a one dimensional Brownian motion on some probability space $(\Omega', \mathcal{F}', \mathbb{P}')$ and assume without loss of generality that the filtration $\mathcal{G} := (\mathcal{G}_t)_{t\geq 0}$ satisfies the usual conditions. For $(s, z) \in [0, \infty) \times \mathbb{R}$, we define

$$T_s(z) := \int_0^s \frac{du}{\rho^2(z + B_u)}.$$

Then $T_s(z)$ is strictly increasing and continuous with respect to s. Furthermore, from Problem 3.6.30 [9] (using the uniform ellipticity condition), we have that

$$\lim_{s\uparrow\infty} T_s(z) = \infty, \ a.s.$$

Define $A_\cdot(z)$ as the inverse of $T_\cdot(z)$, i.e.,

$$A_t(z) := \inf\{s \geq 0 | T_s(z) > t\}.$$

From Problem 3.4.5 (v) in [9], $A_t(z)$ is a $\mathcal{G}$-stopping time. We set

$$M_t(z) := B_{A_t(z)},\ Z_t(z) := z + M_t(z),\ \mathcal{F}_t := \mathcal{G}_{A_t},\ 0 \leq t. \tag{3}$$

Then there exists a Brownian motion $W = (W_t, \tilde{\mathcal{F}}_t)_{t\geq 0}$ on an extended probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ such that

$$Z_t(z) = z + M_t(z) = z + \int_0^t \rho(Z_s(z))dW_s,\ 0 \leq t,\ \tilde{\mathbb{P}}\text{-a.s.}$$

This means that $(Z(z), W)$ is a weak solution to the SDE (2). Using these notations, we prove Theorem 2.3.

Proof of Theorem 2.3. Since ρ is bounded and uniformly elliptic, we have from Lemma 2.4 that there exists a weak solution and uniqueness in the sense of probability law holds.

Now we prove that the mapping $z \mapsto Z_t(z)$ is continuous for any $t \geq 0$ almost surely. From (3), it suffices to prove that $z \mapsto A_t(z)$ is continuous for any $t \geq 0$. In order to obtain that result, we first need to prove that $z \mapsto T_t(z)$ is continuous.

Fix $z \in \mathbb{R}$. We take a sequence $(z_k)_{k\in\mathbb{N}}$ which converges to z. Then for given $\delta > 0$, there exists $K \in \mathbb{N}$ such that for any $k \geq K$, $|z_k - z| \leq \delta$. Moreover, for any $\varepsilon > 0$ and $k \geq K$ we have that

$$\begin{aligned}
|T_t(z) - T_t(z_k)| \leq & \frac{1}{c_0}\int_0^t \mathbf{1}_{(-\varepsilon,\varepsilon)}(z + B_u)du + \frac{1}{c_0}\int_0^t \mathbf{1}_{(-\varepsilon,\varepsilon)}(z_k + B_u)du \\
& + \int_0^t \left| \frac{\mathbf{1}_{(-\varepsilon,\varepsilon)^c}(z + B_u)}{\rho^2(z + B_u)} - \frac{\mathbf{1}_{(-\varepsilon,\varepsilon)^c}(z_k + B_u)}{\rho^2(z_k + B_u)} \right| du \\
\leq & \frac{2}{c_0}\int_0^t \mathbf{1}_{A_{\varepsilon,\delta}}(B_u)du + \int_0^t \left| \frac{\mathbf{1}_{(-\varepsilon,\varepsilon)^c}(z + B_u)}{\rho^2(z + B_u)} - \frac{\mathbf{1}_{(-\varepsilon,\varepsilon)^c}(z_k + B_u)}{\rho^2(z_k + B_u)} \right| du,
\end{aligned} \tag{4}$$

where $A_{\varepsilon,\delta} := (-\varepsilon - \delta - z, \varepsilon + \delta - z)$. Since ρ is continuous on $\mathbb{R}_0$, the second term on the right hand side of (4) converges to 0 as $k \to \infty$. Therefore, we have

$$\limsup_{k\to\infty} |T_t(z) - T_t(z_k)| \le \frac{2}{c_0} \int_0^t \mathbf{1}_{A_{\varepsilon,\delta}}(B_u) du.$$

Since $\varepsilon, \delta > 0$ are arbitrary, by taking limits as ε, δ tend to 0, we have from the occupation formula (see [9], Problem 3.6.7),

$$\limsup_{k\to\infty} |T_t(z) - T_t(z_k)| \le \frac{2}{c_0} \int_0^t \mathbf{1}(B_u = -z) du = \frac{4}{c_0} \int_{\mathbb{R}} \mathbf{1}(y = -z) L_t^y(B) dy = 0, \ a.s.$$

Therefore we conclude that $\lim_{k\to\infty} |T_t(z) - T_t(z_k)| = 0$ a.s. Note that the right hand side of (4) is increasing with respect to t. Therefore, we also have that $z \mapsto T_t(z)$ is continuous for any $t \ge 0$ a.s.

Next we prove that for fixed $t_0 > 0$, $z \mapsto A_t(z)$ is continuous for any $t \in [0, t_0/c_0]$. We first note that since ρ is uniformly elliptic we have for any $z \in \mathbb{R}$,

$$\frac{t_0}{c_0} \le T_{t_0}(z).$$

Since $T_t(z)$ is strictly increasing with respect to $t \ge 0$, it holds that for any $(t, z) \in [0, t_0/c_0] \times \mathbb{R}$,

$$A_t(z) \le A_{t_0/c_0}(z) \le A_{T_{t_0}(z)}(z) = t_0.$$

Fix $(t, z) \in [0, t_0/c_0] \times \mathbb{R}$. We take a sequence $(z_k)_{k\in\mathbb{N}}$ which converges to z and define $s := A_t(z)$ and $s_k := A_t(z_k)$. Note that $s, s_k \in [0, t_0]$ for any $k \in \mathbb{N}$. Now, we assume by contradiction that s_k does not converge to s. Then there exists $\varepsilon_0 > 0$ such that for any $n \in \mathbb{N}$, there exists $k_n \ge n$ such that $|s_{k_n} - s| \ge \varepsilon_0$. On the other hand, $(s_{k_n})_{n\in\mathbb{N}}$ is a sequence on the compact set $[0, t_0]$. Therefore there exists a sub-sequence $(s_{k_n(m)})_{m\in\mathbb{N}}$ of $(s_{k_n})_{n\in\mathbb{N}}$ such that $\lim_{m\to\infty} s_{k_n(m)} = s' \ne s$. Since $\lim_{s\to t} \sup_{x\in\mathbb{R}} |T_t(z) - T_s(z)| = 0$ then $T_t(z)$ is continuous in time uniformly for $z \in \mathbb{R}$. This result together with the continuity of $T_t(z)$ with respect to z gives the joint continuity of $T_t(z)$ for $(t, z) \in [0, \infty) \times \mathbb{R}_0$. Therefore we conclude

$$T_s(z) = T_{A_t(z)}(z) = t = T_{A_t(z_{k_n(m)})}(z_{k_n(m)}) = T_{s_{k_n(m)}}(z_{k_n(m)}) = \lim_{m\to\infty} T_{s_{k_n(m)}}(z_{k_n(m)}) = T_{s'}(z).$$

Since $T_t(z)$ is strictly increasing with respect to t, we conclude $s = s'$. This is a contradiction, so $A_t(z)$ is continuous with respect to z for any $t \in [0, t_0/c_0]$ a.s. Furthermore, as t_0 is arbitrary, $A_t(z)$ is continuous with respect to z for any $t \ge 0$ a.s. Therefore the conclusion follows from (3). □

Now we consider the following one-dimensional stochastic differential equation with drift

$$Z_t(z) = z + \int_0^t \mu(Z_s(z)) ds + \int_0^t \rho(Z_s(z)) dW_s, \ z \in \mathbb{R}. \tag{5}$$

We introduce the method of removal of drift coefficient introduced in section 5.5 B in [9]. Assume that

(ND) : $\rho^2(z) > 0,\ z \in \mathbb{R}$,

(LI) : $\forall z \in \mathbb{R}, \exists \varepsilon > 0$ such that $\displaystyle\int_{z-\varepsilon}^{z+\varepsilon} \frac{|\mu(y)|}{\rho^2(y)} dy < \infty.$

For some constant $c \in \mathbb{R}$, we define the scale function

$$p(z) := \int_c^z \exp\left(-2\int_c^y \frac{\mu(r)}{\rho^2(r)} dr\right) dy,\ z \in \mathbb{R}. \tag{6}$$

The function p is continuous with strictly positive derivative and its second derivative, p'', exists and satisfies

$$p''(z) = -\frac{2\mu(z)}{\rho^2(z)} p'(z).$$

Moreover the function $p : \mathbb{R} \to (p(-\infty), p(\infty)))$ has a continuous and differentiable inverse function $q : (p(-\infty), p(\infty))) \to \mathbb{R}$. Then the following proposition holds:

Proposition 2.5 (Proposition 5.5.13 [9])**.** Suppose that (ND) and (LI) hold. A stochastic process $Z = (Z_t, \mathcal{F}_t)_{t\geq 0}$ is a weak (or strong) solution of equation (5) if and only if the stochastic process $Y = (Y_t := p(Z_t), \mathcal{F}_t)_{t\geq 0}$ is a weak (or strong) solution of the equation

$$Y_t = y_0 + \int_0^t \tilde{\rho}(Y_s) dW_s,$$

where

$$y_0 := p(z) \in (p(-\infty), p(\infty)))$$
$$\tilde{\rho}(y) := \begin{cases} p'(q(y))\rho(q(y)) & \text{if } y \in (p(-\infty), p(\infty))), \\ 0 & \text{otherwise .} \end{cases}$$

Theorem 2.6. Assume that μ is bounded, measurable and that ρ is a measurable function such that there exist positive constants $c_0 > 1$ for which $c_0^{-1} \leq \rho^2(z) \leq c_0$ for any $z \in \mathbb{R}$. Then there exists a unique weak solution of equation (5). Moreover if ρ is continuous on $\mathbb{R}_0$ then the mapping $z \mapsto Z_t(z)$ is continuous for any $t \geq 0$ and any $z \in \mathbb{R}$ almost surely.

Proof. Under the assumption, μ and ρ satisfy the conditions (ND) and (LI). Therefore from Proposition 2.5 and Theorem 2.3, it suffices to show that $\tilde{\rho}$ is bounded uniformly elliptic and continuous on $\mathbb{R}_0$. Since b is bounded and ρ is bounded and uniformly elliptic, we have $p(-\infty) = -\infty$ and $p(\infty) = \infty$ and p' is bounded and uniformly elliptic. Therefore $\tilde{\rho}$ is bounded and uniformly elliptic. If we choose $c = 0$ in (6), then $p(0) = q(0) = 0$. Hence $\tilde{\rho}$ is continuous on $\mathbb{R}_0$. □

Clearly, from the above statement one also obtains that the process Z is a Markov process.

We will prove that the skew diffusion (1) has a unique (weak or strong) solution by using an equivalent SDE without reflection. To this end, fix $\alpha \in (0, 1)$ and we define the following functions:

$$s_\alpha(x) := (1-\alpha)x\mathbf{1}(x \geq 0) + \alpha x \mathbf{1}(x < 0),$$

$$r_\alpha(x) := s_\alpha^{-1}(x) = \frac{x}{(1-\alpha)}\mathbf{1}(x \geq 0) + \frac{x}{\alpha}\mathbf{1}(x<0),$$
$$f_\alpha(x) := \frac{D_- s_\alpha(x) + D_+ s_\alpha(x)}{2} = (1-\alpha)\mathbf{1}(x>0) + \alpha\mathbf{1}(x<0) + \frac{1}{2}\mathbf{1}(x=0).$$

Here D_- and D_+ denote the left and right derivatives, respectively. Note that $f_\alpha \circ r_\alpha(x) = f_\alpha \circ s_\alpha(x) = f_\alpha(x)$. Using these notations, we have the following result.

Proposition 2.7. Suppose that $\alpha \in (0,1)$, and that Assumption 2.1 is satisfied. Define the coefficients

$$\mu(z) := f_\alpha(z) b(r_\alpha(z)) = (1-\alpha) b\left(\frac{z}{1-\alpha}\right)\mathbf{1}(z>0) + \alpha b\left(\frac{z}{\alpha}\right)\mathbf{1}(z<0) + \frac{b(0)}{2}\mathbf{1}(z=0).$$

Similarly, $\rho(z) := f_\alpha(z)\sigma(r_\alpha(z))$. Then there exists a unique solution for the equation

$$Z_t(z) = z + \int_0^t \mu(Z_s(z))ds + \int_0^t \rho(Z_s(z))dW_s,\ z \in \mathbb{R}. \tag{7}$$

Furthermore this solution defines an a.s. continuous flow $z \to Z_t(z)$ for all $t \geq 0$. Define $X_t(x) := r_\alpha(Z_t(z))$ with $z = s_\alpha(x)$. Then $X = (X_t(x))_{t\geq 0}$ is a solution of the SDE (1). Similarly, if X is a solution of (1) then $Z = (Z_t)_{t\geq 0} = (s_\alpha(X_t))_{t\geq 0}$ is the solution to (7) and the flow $x \to X_t(x)$ is a.s. continuous for all $t \geq 0$. In particular, for any bounded measurable function f,

$$\mathbb{E}[f(X_T(x))] = \mathbb{E}[f \circ r_\alpha(Z_T(z))],\ z = s_\alpha(x).$$

Proof. The proof of the first part is straightforward, we just show that if X is a solution of (1) then $Z = s_\alpha(X)$ is the solution of (7).

By the symmetric Itô-Tanaka formula (see, e.g. (32) of [14]), we have

$$\begin{aligned}
Z_t &:= s_\alpha(X_t) = s_\alpha(x) + \int_0^t f_\alpha(X_s)dX_s + \frac{1-2\alpha}{2}L_t^0(X) \\
&= s_\alpha(x) + \int_0^t f_\alpha(X_s)b(X_s)ds + \int_0^t f_\alpha(X_s)\sigma(X_s)dW_s + (2\alpha-1)\int_0^t f_\alpha(X_s)dL_s^0(X) + \frac{1-2\alpha}{2}L_t^0(X) \\
&= s_\alpha(x) + \int_0^t f_\alpha \circ r_\alpha \circ s_\alpha(X_s)b(r_\alpha(Z_s))ds + \int_0^t f_\alpha \circ r_\alpha \circ s_\alpha(X_s)\sigma(r_\alpha(Z_s))dW_s \\
&\quad + (2\alpha-1)f_\alpha(0)L_t^0(X) - \frac{2\alpha-1}{2}L_t^0(X) \\
&= s_\alpha(x) + \int_0^t f_\alpha(Z_s)b(r_\alpha(Z_s))ds + \int_0^t f_\alpha(Z_s)\sigma(r_\alpha(Z_s))dW_s \\
&= z + \int_0^t \mu(Z_s)ds + \int_0^t \rho(Z_s)dW_s.
\end{aligned}$$

Since $r_\alpha = s_\alpha^{-1}$, Z is a solution of (7) if and only if X is a solution of (1). The other statements follow from Theorem 2.6. □

Remark 2.8. Nakao [18] proved that if ρ is positive and of bounded variation function on any compact interval of $\mathbb{R}$, then the pathwise uniqueness holds for SDE (7) (see Le Gall [13] for a stronger result). Therefore similar statements can be made about existence and uniqueness of solutions for (1).

3 The generator of X

Definition 3.1. Let $\alpha \in (0,1)$. Let $D(\alpha)$ be the class of continuous bounded functions $f:\mathbb{R}\to\mathbb{R}$ with bounded continuous derivatives f' and f'' on $\mathbb{R}_0$ such that $f'(0+), f'(0-)$ exist and $\alpha f'(0+) = (1-\alpha)f'(0-)$.

For measurable functions b and σ, we define the differential operator L by

$$Lf(x) := b(x)f'(x) + \frac{\sigma^2(x)}{2}f''(x), f \in D(\alpha), x \in \mathbb{R}_0. \tag{8}$$

Define for any bounded measurable function f, the semigroup associated to the Markov process X as $P_t f(x) := \mathbb{E}[f(X_t(x))]$. The next proposition shows that the infinitesimal generator of P is L on $D(\alpha)$.

Proposition 3.2. Suppose that b and σ are bounded and continuous functions. Then the infinitesimal generator of $(P_t)_{t\geq 0}$ on $D(\alpha)$ is given by

$$\lim_{h\to 0}\frac{P_h f(x) - f(x)}{h} = Lf(x), f \in D(\alpha), x \in \mathbb{R}_0.$$

Proof. If $f \in D(\alpha)$, then f has a generalized second order derivative given by

$$\mu(dx) = f''(x)dx - \frac{2\alpha - 1}{\alpha}f'(0-)\delta_0(dx),$$

where δ_a is a point mass measure at $a \in \mathbb{R}$. From here, it also follows that f is the difference of two convex functions. Therefore, from the symmetric Itô-Tanaka formula, the fact that $f'(0+) = \frac{(1-\alpha)}{\alpha}f'(0-)$ and the occupation time formula, we have for any $x \in \mathbb{R}$,

$$\begin{aligned}
f(X_t) =& f(x) + \int_0^t f'(X_s)dX_s + \frac{1}{2}\int_{\mathbb{R}} L_t^x(X)\mu(dx) \\
=& f(x) + \int_0^t f'(X_s)\sigma(X_s)dW_s + \int_0^t f'(X_s)b(X_s)ds + (2\alpha-1)\int_0^t f'(X_s)dL_s^0(X) \\
&+ \frac{1}{2}\int_{\mathbb{R}} L_t^x(X)f''(x)dx - \frac{(2\alpha-1)f'(0-)}{2\alpha}L_t^0(X) \\
=& f(x) + \int_0^t f'(X_s)\sigma(X_s)dW_s + \int_0^t f'(X_s)b(X_s)ds \\
&+ (2\alpha-1)\frac{f'(0+)+f'(0-)}{2}L_t^0(X) + \frac{1}{2}\int_{\mathbb{R}} L_t^x(X)f''(x)dx - \frac{(2\alpha-1)f'(0-)}{2\alpha}L_t^0(X) \\
=& f(x) + \int_0^t f'(X_s)\sigma(X_s)dW_s + \int_0^t f'(X_s)b(X_s)ds + \frac{1}{2}\int_0^t f''(X_s)d\langle X\rangle_s \\
=& f(x) + \int_0^t f'(X_s)\sigma(X_s)dW_s + \int_0^t f'(X_s)b(X_s)ds + \frac{1}{2}\int_0^t f''(X_s)\sigma^2(X_s)ds.
\end{aligned}$$

Since f' and σ are bounded, we have that the above stochastic integral has expectation zero and therefore

$$P_t f(x) = f(x) + \int_0^t \mathbb{E}\left[b(X_s)f'(X_s) + \frac{f''(X_s)\sigma^2(X_s)}{2}\right] ds.$$

Hence we get

$$\begin{aligned}\frac{P_t f(x) - f(x)}{t} =& \frac{1}{t}\int_0^t \mathbb{E}\left[b(X_s)f'(X_s) + \frac{f''(X_s)\sigma^2(X_s)}{2}\right] ds\\ =& \frac{1}{t}\int_0^t P_s L f(x) ds.\end{aligned}$$

Therefore we have by continuity of X with respect to the time variable that for $x \neq 0$,

$$\left|\frac{P_t f(x) - f(x)}{t} - Lf(x)\right| \leq \frac{1}{t}\int_0^t |P_s Lf(x) - Lf(x)|\, ds \to 0, \text{ as } t \to 0,$$

Hence we conclude the proof. □

Remark 3.3. Note that the above proof also gives that

$$\frac{dP_t f}{dt}(x) = P_t Lf(x), f \in D(\alpha),\ x \in \mathbb{R}_0,\ t \geq 0. \tag{9}$$

4 Skew Brownian motion as the approximation process

We now define the approximation process that will be used in order to construct the parametrix in the next section. This is a slightly generalized version of the skew Brownian motion. We refer to [14] for general information about skew Brownian motion.

Proposition 4.1. Assume that σ is a measurable function, $z \in \mathbb{R}$ and $\alpha \in (0, 1)$. There exists an adapted stochastic process $X^{\alpha,z}$ which is the strong unique solution of the stochastic equation:

$$X_t^{\alpha,z} = x + \sigma(z)W_t + (2\alpha - 1)L_t^0(X^{\alpha,z}). \tag{10}$$

As $\sigma(z) > 0$, then the density of $X_t^{\alpha,z}$, denoted by $p_t^z(x, y)$, exists and can be explicitly written in separate cases as:
<u>Case A:</u> For $x \geq 0$,

$$p_t^z(x,y) = \left(g_t^{a(z)}(y-x) + (2\alpha-1)g_t^{a(z)}(y+x)\right)\mathbf{1}(y \geq 0) + 2(1-\alpha)g_t^{a(z)}(y-x)\,\mathbf{1}(y<0).$$

<u>Case B:</u> For $x < 0$

$$p_t^z(x,y) = \left(g_t^{a(z)}(y-x) + (1-2\alpha)g_t^{a(z)}(y+x)\right)\mathbf{1}(y < 0) + 2\alpha g_t^{a(z)}(y-x)\,\mathbf{1}(y \geq 0).$$

Furthermore $p_t^z(x, y)$ satisfies the following properties

- For any fixed $x, y, z \in \mathbb{R}$, $p^z_\cdot(x, y)$ is a continuous function on $(0, \infty)$.
- $p^z_t(\cdot, y)$ is a Lipschitz continuous function whose derivative is continuous everywhere except at $x = 0$ where we have $\alpha \partial_x p^z_t(0+, y) = (1-\alpha)\partial_x p^z_t(0-, y)$.
- If $\alpha \neq 1/2$, then for any $(t, x) \in (0, \infty) \times \mathbb{R}$, the mapping $y \mapsto p^z_t(x, y)$ is not continuous at $y = 0$. Indeed, we have $p^z_t(x, 0+) = 2\alpha g^{a(z)}_t(x)$ and $p^z_t(x, 0-) = 2(1-\alpha) g^{a(z)}_t(x)$.

Define the semigroup associated to $X^{\alpha,z}(x)$ as $P^z_t f(x) \equiv P^{\alpha,z}_t f(x) := \mathbb{E}[f(X^{\alpha,z}_t(x))]$. Then the infinitesimal generator of P^z_t on $D(\alpha)$ is given as

$$L^z f(x) \equiv L^{\alpha,z} f(x) = \frac{\sigma^2(z)}{2} f''(x), f \in D(\alpha), x \in \mathbb{R}_0.$$

Proof. We have that

$$\frac{X^{\alpha,z}_t}{\sigma(z)} = \frac{x}{\sigma(z)} + W_t + \frac{(2\alpha - 1)L^0_t(X^{\alpha,z})}{\sigma(z)} = \frac{x}{\sigma(z)} + W_t + (2\alpha - 1)L^0_t\left(\frac{X^{\alpha,z}}{\sigma(z)}\right).$$

This equation is a particular case of skew Brownian motion. That is,

$$X^\alpha_t = x + W_t + (2\alpha - 1)L^0_t(X^\alpha). \tag{11}$$

The solution of (11) is called the skew Brownian motion with parameter α. Harrison and Shepp [8] showed that there is no solution if $|2\alpha - 1| > 1$. By Revuz and Yor ([19], Chapter 3, Exercise 1.16), the density of X^α_t with $X^\alpha_0 = x$ is given explicitly. Then the density $p^z_t(x, y)$ is obtained by doing a change of variables. The properties of $p^z_t(x, y)$ are obtained from the explicit formula obtained and the statement about the generator is a particular case of Proposition 3.2. □

Remark 4.2. Assume that $\sigma(w) > \lambda_0 > 0$ for any $w \in \mathbb{R}$. Let $(X^{\alpha,w}_t)_{t\geq 0}$ be the unique solution of the stochastic equation (10). Define $Z^{\alpha,w}_t := s_\alpha(X^{\alpha,w}_t)$. Then from Proposition 2.7, $(Z^{\alpha,w}_t)_{t\geq 0}$ is the solution to the following SDE:

$$Z^{\alpha,w}_t = z + \int_0^t \rho_w(Z^{\alpha,w}_s) dW_s,$$

where $z = s_\alpha(x)$ and

$$\rho_w(x) := \sigma(w) f_\alpha(x) = \sigma(w)\left((1-\alpha)\mathbf{1}(x > 0) + \alpha \mathbf{1}(x < 0) + \frac{1}{2}\mathbf{1}(x = 0)\right).$$

Then since $X^{\alpha,w}_t$ has the density function $p^w_t(x, \cdot)$, using the change of variables theorem for densities of random variables, we can obtain the density of $Z^{\alpha,w}_t$ explicitly. This gives

$$p^w_{Z_t}(z, u) := \frac{p^w_t(r_\alpha(z), r_\alpha(u))}{1-\alpha}\mathbf{1}(u \geq 0) + \frac{p^w_t(r_\alpha(z), r_\alpha(u))}{\alpha}\mathbf{1}(u < 0).$$

Therefore, choosing $w = r_\alpha(u)$ on the "backward parametrix method", we can also get a parametrix expansion for Z.

4.1 Some auxiliary estimates

In this section, we introduce some key estimates (Lemmas 4.4 and 4.5) in order to construct a parametrix expansion for the skew diffusion (1).

We define $\Phi(t,x,y) := (L - L^y)\phi_t^y(x)$ and $\overline{p}(t,x,y) \equiv \phi_t^y(x) := p_t^y(x,y)$. Here, we need to explain why we need to use these three notations for the same mathematical object: The first $\overline{p}(t,x,y)$ is used to make clear how the time-space convolutions are taken. For more on this, see Section 5. The second is used in order to know to which variable the derivative operators are acting on. Finally, the third is used in order to note that the density is just a variant of the skew Brownian motion density.

Lemma 4.3. Let $\alpha \in (0,1)$. Then $\phi_t^y \in D(\alpha)$ for any $(t,y) \in (0,\infty) \times \mathbb{R}$.

The proof of the above statement follows directly from Proposition 4.1. Moreover the function ϕ_t^y satisfies the following Gaussian estimate.

Lemma 4.4. Under Assumption 2.1, there exist positive constants C and c such that for any $x, y \in \mathbb{R}$, $t > 0$,

$$\phi_t^y(x) \leq C g_t^c(y-x).$$

Proof. Since σ is bounded and uniformly elliptic, there exist positive constants C and c such that

$$\phi_t^y(x) \leq \begin{cases} C g_t^c(y) & \text{if } x = 0, \\ C g_t^c(y-x) & \text{if } y < 0 < x \text{ or } x < 0 \leq y, \\ C\left(g_t^c(y-x) + g_t^c(y+x)\right) & \text{if } y \geq 0, x > 0 \text{ or } y < 0, x < 0. \end{cases}$$

If the signs of y and x are the same, we have $g_t^c(y+x) \leq g_t^c(y-x)$, hence the proof is finished. □

We now give the essential estimate that will be used in order to prove the convergence of the parametrix method.

Lemma 4.5. Under Assumption 2.1, there exist positive constants $C_T = C(1 + T^{(1-\eta)/2})$ and c such that for any $x, y \in \mathbb{R}_0$, $t > 0$,

$$|\Phi(t,x,y)| \leq \frac{C_T}{t^{1-\eta/2}} g_t^c(y-x).$$

Proof. We first compute the action of the operators on the function ϕ_t^y explicitly. As before, we need to separate the study in various cases:

<u>Case 1:</u> If $x, y > 0$, then

$$\begin{aligned}(L - L^y)\phi_t^y(x) =& b(x)\Big(-H_1(y-x, ta(y))g_t^{a(y)}(y-x) + (2\alpha-1)H_1(y+x, ta(y))g_t^{a(y)}(y+x)\Big) \\ &+ \frac{a(x)-a(y)}{2}\Big(H_2(y-x, ta(y))g_t^{a(y)}(y-x) + H_2(y+x, ta(y))(2\alpha-1)g_t^{a(y)}(y+x)\Big).\end{aligned}$$

Case 2: If $x > 0 > y$, then

$$(L - L^y)\phi_t^y(x) = 2(1-\alpha)\left(-b(x)H_1(y-x, ta(y)) + \frac{a(x)-a(y)}{2}H_2(y-x, ta(y))\right) g_t^{a(y)}(y-x).$$

Case 3: If $x < 0 < y$, then

$$(L - L^y)\phi_t^y(x) = 2\alpha\left(-b(x)H_1(y-x, ta(y)) + \frac{a(x)-a(y)}{2}H_2(y-x, ta(y))\right) g_t^{a(y)}(y-x).$$

Case 4: If $x, y < 0$, then

$$\begin{aligned}(L - L^y)\phi_t^y(x) =& b(x)\left(-H_1(y-x, ta(y))g_t^{a(y)}(y-x) + (1-2\alpha)H_1(y+x, ta(y))g_t^{a(y)}(y+x)\right)\\ &+ \frac{a(x)-a(y)}{2}\left(H_2(y-x, ta(y))g_t^{a(y)}(y-x) + H_2(y+x, ta(y))(1-2\alpha)g_t^{a(y)}(y+x)\right).\end{aligned}$$

As all the cases follow similarly, we only consider the case $x, y > 0$.

From Assumption 2.1 and the inequality $|x|^p e^{-qx^2} \le (p/(2qe))^{p/2}$ for any $p, q > 0$ and $x \in \mathbb{R}$, we have

$$\begin{aligned}|(L - L^y)\phi_t^y(x)| \le& \frac{C}{t^{1/2}}\left(g_t^c(y-x) + g_t^c(y+x)\right)\\ &+ \frac{C|a(x)-a(y)|}{t}\left(g_t^c(y-x) + g_t^c(y+x)\right).\end{aligned}$$

Note that for $x, y > 0$, we have $g_t^c(y+x) \le g_t^c(y-x)$. Hence, using the Hölder continuity of a and $t > 0$, we have

$$\begin{aligned}|(L - L^y)\phi_t^y(x)| &\le \frac{C}{t^{1/2}}g_t^c(y-x) + \frac{C|x-y|^\eta}{t}g_t^c(y-x)\\ &\le \frac{C_T}{t^{1-\eta/2}}g_t^c(y-x).\end{aligned}$$

This concludes the proof. □

5 Parametrix for skew diffusion

In this section, we prove the existence of the density of the skew diffusion (1), using the parametrix method for the semigroup P.

We first define the following time-space convolutions $\circledast$ for functions $f, g : [0,\infty) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and time dependent operators A and B

$$\begin{aligned}f \circledast g(t, x, y) &:= \int_0^t ds \int_{\mathbb{R}} dz f(s, x, z)g(t-s, z, y),\\ (A \circ B)_t f &:= \int_0^t A_s B_{t-s} f ds.\end{aligned}$$

We denote $f^{\circledast 1} = f$, $f^{\circledast k} = f \circledast f^{\circledast (k-1)}$ and $f \circledast g^{\circledast 0} = f$ and similarly for the time convolution of operators. That is, $A^{\circ k} = A \circ A^{\circ (k-1)}$ for $k \in \mathbb{N}$.

Now we introduce the following operators for $f \in L^\infty(\mathbb{R})$ and $y \in \mathbb{R}_0$,

$$\hat{Q}_t f(y) := (P_t^y)^* f(y) = \int_{\mathbb{R}} f(x) p_t^y(x,y) dx = \int_{\mathbb{R}} f(x) \overline{p}(t,x,y) dx,$$
$$\hat{S}_t f(y) := \int_{\mathbb{R}} f(x)(L - L^y)\phi_t^y(x) dx = \int_{\mathbb{R}} f(x) \Phi(t,x,y) dx,$$

$$\begin{aligned}\hat{I}_{t_0}^n(f)(y) &:= \begin{cases} (\hat{S}^{\circ n} \circ \hat{Q})_{t_0} f(y), & \text{if } n \geq 1, \\ \hat{Q}_{t_0} f(y), & \text{if } n = 0, \end{cases} \\ &= \int_{\mathbb{R}} f(x) \overline{p} \circledast \Phi^{\circledast n}(t,x,y) dx.\end{aligned}$$

Moreover, we define $\hat{I}_{t_0}^0(y_0, y_1) := p_{t_0}^{y_0}(y_1, y_0)$ and for $n \geq 1$,

$$\begin{aligned}\hat{I}_{t_0}^n(y_0, y_{n+1}) :=& \overline{p} \circledast \Phi^{\circledast n}(t_0, y_{n+1}, y_0) \\ =& \int_0^{t_0} dt_1 \cdots \int_0^{t_{n-1}} dt_n \int_{\mathbb{R}^n} dy_1 \cdots dy_n \prod_{i=0}^{n-1} \Phi(t_i - t_{i+1}, y_{i+1}, y_i) p_{t_n}^{y_n}(y_{n+1}, y_n).\end{aligned}$$

Furthermore, we define the adjoint operators for $x \in \mathbb{R}_0$ and $f \in L^\infty(\mathbb{R})$:

$$\begin{aligned}\hat{Q}_t^* f(x) &:= \int_{\mathbb{R}} f(y) p_t^y(x,y) dy = \int_{\mathbb{R}} f(y) \overline{p}(t,x,y) dy, \\ \hat{S}_t^* f(x) &:= \int_{\mathbb{R}} f(y)(L - L^y)\phi_t^y(x) dy = \int_{\mathbb{R}} f(y) \Phi(t,x,y) dy, \\ \hat{I}_{t_0}^{n,*}(f)(x) &:= \begin{cases} (\hat{Q}^* \circ (\hat{S}^*)^{\circ n})_{t_0} f(x), & \text{if } n \geq 1, \\ \hat{Q}_{t_0}^* g(x), & \text{if } n = 0, \end{cases} \\ &= \int_{\mathbb{R}} f(y) \overline{p} \circledast \Phi^{\circledast n}(t_0, x, y) dy.\end{aligned}$$

We will extend the definition of $\hat{Q}_t^* f(x)$ at $x = 0$ by continuity. Under Assumption 2.1 and using Lemmas 4.5 and 4.4, there exists a positive constant C such that for any $f \in L^\infty(\mathbb{R})$,

$$\sup_{t \geq 0} \max\{t^{1-\eta/2} ||\hat{S}_t f||_\infty, t^{1-\eta/2} ||\hat{S}_t^* f||_\infty, ||\hat{Q}_t f||_\infty, ||\hat{Q}_t^* f||_\infty\} \leq C_T ||f||_\infty. \tag{12}$$

Therefore $\hat{S}_t f$, $\hat{S}_t^* f$, $\hat{Q}_t f$ and $\hat{Q}_t^* f$ are well defined. Now, we will prove that $\hat{I}_{t_0}^n(f)$, $\hat{I}_{t_0}^{n,*}(f)$ and $\hat{I}_{t_0}(y_0, y_{n+1})$ are well-defined. Indeed, from (12) and Lemma 4.5,

$$\left|\left| \left(\prod_{i=0}^{n-1} \hat{S}_{t_i - t_{i-1}} \right) \hat{Q}_{t_n} f \right|\right|_\infty \leq ||f||_\infty \prod_{i=0}^{n-1} \frac{C_T}{(t_i - t_{i+1})^{1-\eta/2}} = C_T^n ||f||_\infty \prod_{i=0}^{n-1} \frac{1}{(t_i - t_{i+1})^{1-\eta/2}},$$

hence $\hat{I}^n_{t_0}(f)(y)$ is well defined due to Lemma 8.1. In fact,

$$\begin{aligned}\sum_{n=0}^{\infty}\left|\left|\hat{I}^n_{t_0}(f)\right|\right|_{\infty} &\leq \sum_{n=0}^{\infty} C_T^n ||f||_{\infty}\int_0^{t_0} dt_1\cdots\int_0^{t_{n-1}} dt_n \prod_{i=0}^{n-1}\frac{1}{(t_i-t_{i+1})^{1-\eta/2}}\\ &= ||f||_{\infty}\sum_{n=0}^{\infty} t_0^{n(1-\eta/2)}C_T^n\prod_{i=0}^{n-1}B(1+i\eta/2,\eta/2)\\ &= ||f||_{\infty}\sum_{n=0}^{\infty}\left(t_0^{(1-\eta/2)}C_T\Gamma(\eta/2)\right)^n\frac{1}{\Gamma(1+n\eta/2)}\\ &= ||f||_{\infty}E_{\eta/2,1}(t_0^{(1-\eta/2)}C_T\Gamma(\eta/2))<\infty.\end{aligned}$$

So $\sum_{n=0}^{\infty}\hat{I}^n_{t_0}(g)(y)$ converges absolutely and uniformly for $(t,y)\in(0,T]\times\mathbb{R}$. Due to a similar argument, we have

$$\left|\left|\hat{Q}^*_{t_n}\left(\prod_{i=n-1}^{0}\hat{S}^*_{t_i-t_{i-1}}\right)f\right|\right|_{\infty}\leq C_T^n||f||_{\infty}\prod_{i=0}^{n-1}\frac{1}{(t_i-t_{i+1})^{1-\eta/2}}. \tag{13}$$

So $\hat{I}^{n,*}_{t_0}(f)(x)$ is well defined and

$$\sum_{n=0}^{\infty}\left|\left|\hat{I}^{n,*}_{t_0}(f)\right|\right|_{\infty}<\infty. \tag{14}$$

From here, we conclude that $\sum_{n=0}^{\infty}\hat{I}^{n,*}_t(f)(x)$ converges absolutely and uniformly for $(t,x)\in(0,T]\times\mathbb{R}$.

Now we state our main results for the parametrix expansion of skew diffusions.

Theorem 5.1. Suppose that Assumption 2.1 holds and that the drift coefficient b is continuous. Define

$$p_T(x,y):=\sum_{n=0}^{\infty}\hat{I}^n_T(y,x),\ x\in\mathbb{R},y\in\mathbb{R}_0.$$

Then $p_T(x,y)$ converges absolutely and uniformly for $x\in\mathbb{R},y\in\mathbb{R}_0$ and it is continuous at any $x\in\mathbb{R}$, and it has Gaussian upper bounds. That is, there exists positive constants C and c such that for any $x\in\mathbb{R}$, $y\in\mathbb{R}_0$

$$p_T(x,y)\leq E_{\eta/2,1}(C(1\vee T)^{(3-2\eta)/2})g^c_T(y-x).$$

Moreover, for any bounded measurable function f and $x\in\mathbb{R}$,

$$\mathbb{E}[f(X_T(x))]=\sum_{n=0}^{\infty}\hat{I}^{n,*}_T(f)(x)=\int_{\mathbb{R}}f(y)p_T(x,y)dy. \tag{15}$$

Therefore, $p_T(x,\cdot)$ is the probability density function of $X_T(x)$ for any $(T,x)\in(0,\infty)\times\mathbb{R}$.

Proof. The idea of the proof requires two steps: In the first step, done in Proposition 5.2, we use the parametrix method for the semigroup associated to X. We first prove that (15) holds for any $f \in C_c^\infty(\mathbb{R})$ and almost every $x \in \mathbb{R}_0$. So $p_T(x, \cdot)$ is the density function of $X_T(x)$ for almost every $x \in \mathbb{R}_0$.

This weakness in the argument is due to the duality that is used in order to apply the backward parametrix method. For more details, see the proof of Proposition 5.2.

In the second step, done in Proposition 5.3, using the continuity of flows of $X_t(\cdot)$ and the continuity of $p_T(\cdot, y)$, we will obtain the density of $X_t(x)$ for all $x \in \mathbb{R}$.

As a consequence, we also get similar results for $Z_t(z)$ in Corollary 5.6. □

Proposition 5.2. Assume that Assumption 2.1 holds and that the drift coefficient b is a continuous function. Then for any $f \in C_c^\infty(\mathbb{R})$, $\sum_{n=0}^\infty \hat{I}_T^{n,*}(f)(x)$ converges absolutely and uniformly for $x \in \mathbb{R}_0$ and the following expansion holds:

$$\mathbb{E}[f(X_T(x))] = \sum_{n=0}^\infty \hat{I}_T^{n,*}(f)(x), \quad \text{a.e. } x \in \mathbb{R}_0.$$

Proposition 5.3. Assume that Assumption 2.1 holds and that the drift coefficient b is a continuous function. Then $p_T(x, y)$ converges absolutely and uniformly for $x, y \in \mathbb{R}_0$ and is continuous at any $x \in \mathbb{R}$, and it has a Gaussian upper bound. That is, there exists positive constants C and c such that

$$p_T(x, y) \leq E_{\eta/2,1}(C(1 \vee T)^{(3-2\eta)/2}) g_T^c(y - x).$$

Furthermore the density of $X_T(x)$ is given by $p_T(x, \cdot)$ for all $x \in \mathbb{R}$.

We first prove the following continuity lemma.

Lemma 5.4. Let f be a bounded and continuous function on $\mathbb{R}$. Then for any $x \in \mathbb{R}$, we have

$$\lim_{t \to 0+} \hat{Q}_t^* f(x) = \lim_{t \to 0+} \int_{\mathbb{R}} f(y) \phi_t^y(x) dy = f(x).$$

Proof. We first consider $(t, x) \in [0, T] \times \mathbb{R}_0$. Then we have

$$\begin{aligned}
\hat{Q}_t^* f(x) &= \int_{\mathbb{R}} f(y) \phi_t^y(x) dy \\
&= \int_{-\infty}^0 f(y) \phi_t^y(x) dy \mathbf{1}_{(-\infty,0)}(x) + \int_0^\infty f(y) \phi_t^y(x) dy \mathbf{1}_{(-\infty,0)}(x) \\
&\quad + \int_{-\infty}^0 f(y) \phi_t^y(x) dy \mathbf{1}_{(0,\infty)}(x) + \int_0^\infty f(y) \phi_t^y(x) dy \mathbf{1}_{(0,\infty)}(x) \\
&=: J_1(t, x) + J_2(t, x) + J_3(t, x) + J_4(t, x).
\end{aligned}$$

First, we consider the limit of $J_1(t, x)$. From the definition of $\phi_t^y(x) = p_t^y(x, y)$, we have

$$J_1(t, x) =$$

$$\begin{aligned}
&=\int_{-\infty}^{0} f(y)g_t^{a(y)}\,(y-x)\,dy\mathbf{1}_{(-\infty,0)}(x)+(1-2\alpha)\int_{-\infty}^{0} f(y)g_t^{a(y)}\,(y+x)\,dy\mathbf{1}_{(-\infty,0)}(x)\\
&=:J_{1,1}(t,x)+J_{1,2}(t,x).
\end{aligned}$$

By a similar proof of Theorem 1 of [6], $\lim_{t\to 0} J_{1,1}(t,x)=f(x)\mathbf{1}_{(-\infty,0)}(x)$. Since σ is bounded and uniformly elliptic, using the change of variables $z=(y+x)/\sqrt{t}$, we have

$$J_{1,2}(t,x)\le C||f||_\infty \int_{-\infty}^{0} g_t^c(x+y)dy\mathbf{1}_{(-\infty,0)}(x)=C||f||_\infty \int_{-\infty}^{\frac{x}{\sqrt{t}}} g_1^c(y)dy\mathbf{1}_{(-\infty,0)}(x)\to 0,\ \text{as } t\to 0+.$$

So we conclude $J_1(t,x)+J_2(t,x)\to f(x)\mathbf{1}_{(-\infty,0)}(x)$ as $t\to 0+$. In the same way, $J_3(t,x)+J_4(t,x)\to f(x)\mathbf{1}_{(0,\infty)}(x)$. Therefore, we have $\lim_{t\to 0+}\hat{Q}_t^* f(x)=f(x)$ for any $x\in\mathbb{R}_0$.

In a similar fashion, one deals with the particular case $x=0$. □

Proof of Proposition 5.2. Let $t_0=T$ and $x\neq 0$. We first prove that

$$\partial_t(P_tP_{T-t}^{y_2})\phi_\varepsilon^{y_0}(x)=P_t(L-L^{y_1})\phi_{T-t+\varepsilon}^{y_1}(x). \tag{16}$$

Since X_t^z is a time homogeneous Markov process, by the Chapman-Kolmogorov equation, we have

$$P_t^z\phi_\varepsilon^z(x)=\int_{\mathbb{R}}\phi_\varepsilon^z(y)p_t^z(x,y)dy=\int_{\mathbb{R}}p_\varepsilon^z(y,z)p_t^z(x,y)dy=p_{\varepsilon+t}^z(x,z)=\phi_{\varepsilon+t}^z(x). \tag{17}$$

From equation (17), using Lemma 4.3 and (9), we have for $x\neq 0$,

$$\begin{aligned}
\partial_t(P_tP_{T-t}^{y_1})\phi_\varepsilon^{y_1}(x)&=(\partial_tP_t)P_{T-t}^{y_1}\phi_\varepsilon^{y_1}(x)-P_t\partial_tP_{T-t}^{y_1}\phi_\varepsilon^{y_1}(x)\\
&=(\partial_tP_t)\phi_{T-t+\varepsilon}^{y_1}(x)-P_tL^{y_1}P_{T-t}^{y_1}\phi_\varepsilon^{y_1}(x)\\
&=P_t(L-L^{y_1})\phi_{T-t+\varepsilon}^{y_1}(x),
\end{aligned}$$

which gives (16).

We will now consider an argument by duality. Therefore we need to introduce the following inner product notation $\langle f,g\rangle:=\int_{\mathbb{R}}f(x)g(x)dx$, for two measurable functions $f,g:\mathbb{R}\to\mathbb{R}$ such that $fg\in L^1(\mathbb{R})$.

Fix for the moment, $f,g\in C_c^\infty(\mathbb{R})$. Using (16), we have

$$\begin{aligned}
\int_{\mathbb{R}}dy_1f(y_1)\left(\langle g,P_T\phi_\varepsilon^{y_1}\rangle-\langle g,P_T^{y_1}\phi_\varepsilon^{y_1}\rangle\right)&=\int_{\mathbb{R}}dy_1f(y_1)\int_{\mathbb{R}}dxg(x)\left(P_T-P_T^{y_1}\right)\phi_\varepsilon^{y_1}(x)\\
&=\int_{\mathbb{R}}dy_1f(y_1)\int_{\mathbb{R}}dxg(x)\int_0^T dt\partial_t(P_tP_{T-t}^{y_1})\phi_\varepsilon^{y_1}(x)\\
&=\int_{\mathbb{R}}dy_1f(y_1)\int_{\mathbb{R}}dxg(x)\int_0^T dtP_t\left((L-L^{y_1})\phi_{T-t+\varepsilon}^{y_1}(\cdot)\right)(x).
\end{aligned} \tag{18}$$

Now we consider the limit of both sides of (18). From Lemma 4.5

$$\begin{aligned}&\int_{\mathbb{R}} dx|g(x)|P_t\left(\int_{\mathbb{R}} dy_1|f(y_1)|\left|(L-L^{y_1})\phi^{y_1}_{T-t+\varepsilon}(\cdot)\right|\right)(x)\\
\leq&\frac{C_T||f||_\infty}{(T-t+\varepsilon)^{1-\eta/2}}\int_{\mathbb{R}} dx|g(x)|P_t\left(\int_{\mathbb{R}} dy_1\left(g^c_{T-t+\varepsilon}(y_1-\cdot)\right)\right)(x)\\
=&\frac{C_T||f||_\infty}{(T-t+\varepsilon)^{1-\eta/2}}\int_{\mathbb{R}} dx|g(x)|P_t\mathbf{1}_{\mathbb{R}}(x)\leq\frac{C_T||f||_\infty||g||_{L^1}}{(T-t)^{1-\eta/2}}.\end{aligned}$$

Since $1-\eta/2\in[1/2,1)$, the above expression is integrable on $[0,T]$ and by the dominated convergence theorem, the right hand side of (18) converges to

$$\begin{aligned}&\lim_{\varepsilon\to 0}\int_0^T dt\int_{\mathbb{R}} dxg(x)P_t\left(\int_{\mathbb{R}} dy_1f(y_1)(L-L^{y_1})\phi^{y_1}_{T-t+\varepsilon}(\cdot)\right)(x)\\
=&\int_0^T dt\int_{\mathbb{R}} dxg(x)P_t\left(\int_{\mathbb{R}} dy_1f(y_1)(L-L^{y_1})\phi^{y_1}_{T-t}(\cdot)\right)(x).\end{aligned}\tag{19}$$

Next, we consider the left hand side of (18). We define $P_t(x,A):=\mathbb{P}(X_t(x)\in A)$ for any $A\in\mathcal{B}(\mathbb{R})$ and $t\geq 0$. Then from Fubini's theorem, Lemma 5.4, (12) and dominated convergence theorem, we have

$$\begin{aligned}&\lim_{\varepsilon\to 0}\int_{\mathbb{R}} f(y_1)\langle g,P_T\phi^{y_1}_\varepsilon\rangle dy_1=\lim_{\varepsilon\to 0}\int_{\mathbb{R}} dy_1f(y_1)\int_{\mathbb{R}} dxg(x)P_T\phi^{y_1}_\varepsilon(x)\\
=&\lim_{\varepsilon\to 0}\int_{\mathbb{R}} dy_1f(y_1)\int_{\mathbb{R}} dxg(x)\int_{\mathbb{R}} P_T(x,dw)\phi^{y_1}_\varepsilon(w)=\lim_{\varepsilon\to 0}\int_{\mathbb{R}} dxg(x)\int_{\mathbb{R}} P_T(x,dw)\hat{Q}^*_\varepsilon f(w)\\
=&\lim_{\varepsilon\to 0}\int_{\mathbb{R}} g(x)P_T\hat{Q}^*_\varepsilon f(x)dx=\int_{\mathbb{R}} g(x)P_Tf(x)dx=\langle P_Tf,g\rangle.\end{aligned}\tag{20}$$

Finally, we consider the second term on the left hand side of (18). Before doing that note that

$$\lim_{\varepsilon\to 0}(P^y_{T+\varepsilon})^*g(x)=\lim_{\varepsilon\to 0}\int_{\mathbb{R}} g(z)p^{(y)}_{T+\varepsilon}(z,x)dz.$$

Therefore using Lemma 4.4 and the dominated convergence theorem we have that for any $x\in\mathbb{R}_0$, $\lim_{\varepsilon\to 0}(P^y_{T+\varepsilon})^*g(x)=(P^y_T)^*g(x)$. Applying this result, we obtain

$$\begin{aligned}&\lim_{\varepsilon\to 0}\int_{\mathbb{R}} f(y_1)\langle g,P^{y_1}_T\phi^{y_1}_\varepsilon\rangle dy_1=\lim_{\varepsilon\to 0}\int_{\mathbb{R}} f(y_1)\langle (P^{y_1}_T)^*g,\phi^{y_1}_\varepsilon\rangle dy_1\\
=&\lim_{\varepsilon\to 0}\int_{\mathbb{R}} dy_1f(y_1)\int_{\mathbb{R}} dx(P^{y_1}_T)^*g(x)\phi^{y_1}_\varepsilon(x)=\lim_{\varepsilon\to 0}\int_{\mathbb{R}} f(y_1)(P^{y_1}_\varepsilon)^*(P^{y_1}_T)^*g(y_1)dy_1\\
=&\lim_{\varepsilon\to 0}\int_{\mathbb{R}} f(y_1)(P^{y_1}_{T+\varepsilon})^*g(y_1)dy_1=\langle\hat{Q}^*_Tf,g\rangle.\end{aligned}\tag{21}$$

Therefore from (18), (19), (20) and (21), we conclude

$$\langle P_Tf,g\rangle=\langle\hat{Q}^*_Tf,g\rangle+\int_0^T dt\int_{\mathbb{R}} dxg(x)P_t\left(\int_{\mathbb{R}} dy_1f(y_1)(L-L^{y_1})\phi^{y_1}_{T-t}(\cdot)\right)(x)$$

$$
\begin{aligned}
=&\langle \hat{Q}_T^* f, g\rangle + \int_0^T dt \int_{\mathbb{R}} dx g(x) \int_{\mathbb{R}} P_t(x, dy_2) \int_{\mathbb{R}} dy_1 f(y_1)(L - L^{y_1})\phi_{T-t}^{y_1}(y_2) \\
=&\langle \hat{Q}_T^* f, g\rangle + \int_0^T dt \int_{\mathbb{R}} dx g(x) \int_{\mathbb{R}} P_t(x, dy_2) \hat{S}_{T-t}^* f(y_2) \\
=&\langle \hat{Q}_T^* f, g\rangle + \int_0^T dt \int_{\mathbb{R}} dx g(x) P_t \hat{S}_{T-t}^* f(x) \\
=&\langle \hat{Q}_T^* f, g\rangle + \int_0^T \langle g, P_t \hat{S}_{T-t}^* f\rangle dt. \qquad (22)
\end{aligned}
$$

Note that, by taking an appropriate sequence of smooth bounded functions, one can claim that the equation (22) holds for any $f \in L^\infty(\mathbb{R})$ and $g \in C_c^\infty(\mathbb{R})$.

Hence by replacing f by $\hat{S}_{T-t}^* f$ in the equation (22) and iterating, we have

$$
\begin{aligned}
\langle P_T f, g\rangle =&\langle \hat{Q}_T^* f, g\rangle + \sum_{n=1}^{N-1} \int_0^{t_0} dt_1 \cdots \int_0^{t_{n-1}} dt_n \langle \hat{Q}_{t_n}^* \hat{S}_{t_{n-1}-t_n}^* \cdots \hat{S}_{t_0-t_1}^* f, g\rangle \\
&+ \int_0^{t_0} dt_1 \cdots \int_0^{t_{N-1}} dt_N \langle P_{t_N} \hat{S}_{t_{N-1}-t_N}^* \cdots \hat{S}_{t_0-t_1}^* f, g\rangle.
\end{aligned}
$$

Now, we apply Fubini's theorem in order to exchange time and space integrals which is assured by the hypothesis that $f \in C_c^\infty(\mathbb{R})$ and the estimates in Lemma 4.5, so that

$$
\langle P_T f, g\rangle =\langle \hat{Q}_T^* f, g\rangle + \sum_{n=1}^{N-1} \langle (\hat{Q}^* \circ (\hat{S}^*)^{\circ n})_{t_0} f, g\rangle + \langle \hat{R}_{t_0}^N(f), g\rangle,
$$

where

$$
\hat{R}_{t_0}^N(f)(z) := (P \circ (\hat{S}^*)^{\circ N})_{t_0} f(z) = \int_0^{t_0} dt_1 \cdots \int_0^{t_{N-1}} dt_N P_{t_N} \hat{S}_{t_{N-1}-t_N}^* \cdots \hat{S}_{t_0-t_1}^* f(z).
$$

From (13) and Lemma 8.1 with $a = 1 - \eta/2$ and $b = 0$, we have

$$
||\hat{R}_{t_0}^N(f)||_\infty \le C_T^N \int_0^{t_0} dt_1 \cdots \int_0^{t_{N-1}} dt_N \prod_{i=0}^{N} \frac{||f||_\infty}{(t_i - t_{i+1})^{1-\eta/2}} \to 0, \text{ as } N \to \infty.
$$

Hence we have

$$
|\langle \hat{R}_{t_0}^N(f), g\rangle| \le ||\hat{R}_{t_0}^N(f)||_\infty ||g||_{L^1} \to 0, \text{ as } N \to \infty.
$$

Therefore, using (14), we get the infinite sum expansion

$$
\begin{aligned}
\langle P_T f, g\rangle, =&\langle \hat{Q}_T^* f, g\rangle + \sum_{n=1}^{\infty} \langle (\hat{Q}^* \circ (\hat{S}^*)^{\circ n})_{t_0} f, g\rangle \\
=&\langle \hat{Q}_T^* f, g\rangle + \left\langle \sum_{n=1}^{\infty} \int_0^{t_0} dt_1 \cdots \int_0^{t_{n-1}} dt_n \hat{Q}_{t_n}^* \hat{S}_{t_{n-1}-t_n}^* \cdots \hat{S}_{t_0-t_1}^* f, g \right\rangle
\end{aligned}
$$

$$= \left\langle \sum_{n=0}^{\infty} \hat{I}_T^{n,*}(f), g \right\rangle.$$

This implies that

$$P_T f = \sum_{n=0}^{\infty} \hat{I}_T^{n,*}(f), \text{ a.e. } x \in \mathbb{R}_0.$$

This concludes the statement. $\square$

Proof of Proposition 5.3. We first prove that $p_T(x,y)$ is well defined. Let $y_0 = y$, $t_0 = T$. Define for $x \in \mathbb{R}$ and $y \in \mathbb{R}_0$,

$$\hat{K}_{t_0}^n(y,x) := \bar{p} \circledast |\Phi|^{\circledast n}(t_0, x, y).$$

Then from Lemma 4.5 and 4.4, we have

$$\hat{K}_{t_0}^n(y,x) \le \int_0^{t_0} dt_1 \cdots \int_0^{t_{n-1}} dt_n \int_{\mathbb{R}^n} dy_1 \cdots dy_n \prod_{i=0}^{n-1} \frac{C_T}{(t_i - t_{i+1})^{1-\eta/2}} g^c_{t_i - t_{i+1}} (y_i - y_{i+1})\, g^c_{t_n} (y_n - x).$$

By the Chapman-Kolmogorov property, we get

$$\hat{K}_{t_0}^n(y,x) \le C_T^n \int_0^{t_0} dt_1 \cdots \int_0^{t_{n-1}} dt_n \prod_{i=0}^{n-1} \frac{1}{(t_i - t_{i+1})^{1-\eta/2}} g^c_{t_0} (y - x). \tag{23}$$

Since $1 - \eta/2 \in [1/2, 1)$, we have

$$\begin{aligned}\sum_{n=1}^{\infty} C_T^n \int_0^{t_0} dt_1 \cdots \int_0^{t_{n-1}} dt_n \prod_{i=0}^{n-1} \frac{1}{(t_i - t_{i+1})^{1-\eta/2}} &= \sum_{n=1}^{\infty} t_0^{n(1-\eta/2)} C_T^n \prod_{i=0}^{n-1} B(1 + i\eta/2, \eta/2) \\ &= \sum_{n=1}^{\infty} \frac{\left(t_0^{(1-\eta/2)} C_T \Gamma(\eta/2)\right)^n}{\Gamma(1 + n\eta/2)} \\ &< E_{\eta/2,1}(t_0^{(1-\eta/2)} C_T \Gamma(\eta/2)) < \infty.\end{aligned}$$

Therefore, $p_T(x,y)$ is well defined and (23) gives

$$p_T(x,y) \le E_{\eta/2,1}(C(1 \vee T)^{(3-2\eta)/2}))g^c_T(y - x).$$

Since for any $n \ge 0$ and $f \in C_c^\infty(\mathbb{R})$, $\hat{I}_T^{n,*}(f)(x) = \int_{\mathbb{R}} f(y) \hat{I}_T^n(y,x) dy$ is satisfied we obtain that

$$P_T f(x) = \sum_{n=0}^{\infty} \hat{I}_T^{n,*}(f)(x) = \sum_{n=0}^{\infty} \int_{\mathbb{R}} f(y) \hat{I}_t^n(y,x) dy = \int_{\mathbb{R}} f(y) p_T(x,y) dy, \quad \text{a.e. } x \in \mathbb{R}_0,$$

which implies that $p_T(x,y)$ is a density of $X_T(x)$ for almost every $x \in \mathbb{R}_0$. As $p^{y_n}_{t_n}(x, y_n)$ is continuous at $x \in \mathbb{R}$ (see Proposition 4.1), then $p_T(x,y)$ is also continuous for $x \in \mathbb{R}$.

Moreover, the law of $X_T(x)$ is absolutely continuous with respect to the Lebesgue measure and for almost every $x \in \mathbb{R}_0$, $p_T(x,\cdot)$ is its corresponding probability density function. Therefore, we conclude that for any bounded measurable function f,

$$\mathbb{E}[f(X_T(x))] = \sum_{n=0}^{\infty} \hat{I}_T^{n,*}(f)(x) = \int_{\mathbb{R}} f(y)p_T(x,y)dy, \text{ a.e. } x \in \mathbb{R}_0.$$

Next, we use Proposition 2.7 which gives the continuity of $\mathbb{E}[f(X_T(x))]$ with respect to $x \in \mathbb{R}$ and $f \in C_c^\infty(\mathbb{R})$ and then finally leads to the conclusion. □

By using appropriate approximation arguments, we can extend the statement of Theorem 5.1 for bounded measurable drift coefficients.

Corollary 5.5. Under Assumption 2.1, all the statements of Theorem 5.1 hold.

Proof. By Theorem 174 of Kestelman [10], page 111, there exists a sequence of continuous functions $(b_N)_{N\in\mathbb{N}}$ such that

$$\lim_{N\to\infty} b_N = b, \text{ a.e.}, \tag{24}$$

$$\sup_{N\in\mathbb{N}} ||b_N||_\infty \leq ||b||_\infty. \tag{25}$$

Let $X^{(N)} = (X_t^{(N)})_{t\geq 0}$ be the unique weak solution to the following SDE

$$X_t^{(N)} = x + \int_0^t b_N(X_s^{(N)})ds + \int_0^t \sigma(X_s^{(N)})dW_s + (2\alpha-1)L_t^0(X^{(N)}), \ t \geq 0, \alpha \in (0,1).$$

Let $T > 0$. We first prove that there exists a subsequence $(X_T^{(N_k)})_{k\in\mathbb{N}}$ of the sequence $(X_T^{(N)})_{N\in\mathbb{N}}$ such that for any $f \in C_c^\infty(\mathbb{R})$,

$$\lim_{k\to\infty} \mathbb{E}[f(X_T^{(N_k)})] = \mathbb{E}[f(X_T)]. \tag{26}$$

Let $Z_t := s_\alpha(X_t)$ and $Z_t^{(N)} := s_\alpha(X_t^{(N)})$, $t \in [0,T]$. Note that from Lemma 2.7, the drift coefficients of Z and $Z^{(N)}$ are bounded measurable functions and the diffusion coefficient $\rho(z) = f_\alpha(z)\sigma(r_\alpha(z))$ of Z and $Z^{(N)}$ is a bounded, uniformly elliptic function. The proof of Theorem 2 of Krylov [11] shows that there exists a subsequence $(Z_T^{(N_k)})_{k\in\mathbb{N}}$ of the sequence $(Z_T^{(N)})_{N\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} \mathbb{E}[|Z_T^{(N_k)} - Z_T|^2] = 0.$$

Since r_α is a Lipschitz continuous function, by using Jensen's inequality, it holds that

$$|\mathbb{E}[f(X_T^{(N_k)})] - \mathbb{E}[f(X_T)]| \leq ||f'||_\infty \mathbb{E}[|X_T^{(N_k)} - X_T|^2]^{1/2} \leq C\mathbb{E}[|Z_T^{(N_k)} - Z_T|^2]^{1/2} \to 0,$$

as $k \to \infty$ for some constant C. This implies (26).
Let $p_T^{(N)}(x,\cdot)$ be the density function of $X_T^{(N)}$. Then from Theorem 5.1, it holds that

$$p_T^{(N)}(x,y) = \sum_{n=0}^{\infty} \hat{I}_T^{n,N}(y,x),\ x \in \mathbb{R}, y \in \mathbb{R}_0, \tag{27}$$

$$p_T^{(N)}(x,y) \le E_{\eta/2,1}(C(1\vee T)^{(3-2\eta)/2}) g_T^c(y-x). \tag{28}$$

Here $\hat{I}_{t_0}^{0,N}(y_0,y_1) := p_{t_0}^{y_0}(y_1,y_0)$ and for $n \ge 1$,

$$\begin{aligned}\hat{I}_{t_0}^{n,N}(y_0,y_{n+1}) :=& \overline{p} \circledast \Phi_N^{\circledast n}(t_0,y_{n+1},y_0)\\ =& \int_0^{t_0} dt_1 \cdots \int_0^{t_{n-1}} dt_n \int_{\mathbb{R}^n} dy_1 \cdots dy_n \prod_{i=0}^{n-1} \Phi_N(t_i - t_{i+1}, y_{i+1}, y_i) p_{t_n}^{y_n}(y_{n+1},y_n),\end{aligned}$$

$\Phi_N(t,x,y) := (L_N - L^y)\phi_t^y(x)$ and L_N is the differentiable operator defined on (8) with drift coefficient b_N. Note that from (25), the constants C and c in (28) do not depend on N. From (24) and (28), by using dominated convergence theorem, we have

$$p_T(x,y) := \lim_{N\to\infty} p_T^{(N)}(x,y) = \sum_{n=0}^{\infty} \lim_{N\to\infty} \hat{I}_T^{n,N}(y,x) = \sum_{n=0}^{\infty} \hat{I}_T^{n}(y,x),\ x \in \mathbb{R}, y \in \mathbb{R}_0.$$

This fact and the equation (26) imply that $p_T(x,\cdot)$ is the density function of X_T. □

Now, due to the functional relation between X and Z, we can also present similar results for the density of Z.

Corollary 5.6. Suppose that Assumption 2.1 holds. Let $Z_T(z) := s_\alpha(X_T(x))$ where $z = s_\alpha(x)$. Then $Z_T = Z_T(z)$ has the probability density function, $p_{Z_T}(z,\cdot)$ for any $z \in \mathbb{R}_0$, and its given by

$$p_{Z_T}(z,u) = \frac{p_T(r_\alpha(z), r_\alpha(u))}{1-\alpha} \mathbf{1}(u \ge 0) + \frac{p_T(r_\alpha(z), r_\alpha(u))}{\alpha} \mathbf{1}(u < 0).$$

Moreover $p_{Z_T}(z,\cdot)$ satisfies the following Gaussian upper bound:

$$p_{Z_T}(z,u) \le \max\left\{\frac{1}{1-\alpha}, \frac{1}{\alpha}\right\} E_{\eta/2,1}(C(1\vee T)^{(3-2\eta)/2})) g_T^c(u-z).$$

for some $C, c > 0$.

Proof. We prove that $p_{Z_T}(z,\cdot)$ is the density function of $Z_T(z)$ for every $z \in \mathbb{R}$. For any $f \in C_c^\infty(\mathbb{R})$, by the change of variables $u = s_\alpha(y)$ we have for every $z \in \mathbb{R}$,

$$\begin{aligned}\mathbb{E}[f(Z_T(z))] =& \mathbb{E}[f(s_\alpha(X_T(x)))]\\ =& \int_0^\infty \frac{f(u)}{1-\alpha} p_T(r_\alpha(x), r_\alpha(u)) du + \int_{-\infty}^0 \frac{f(u)}{\alpha} p_T(r_\alpha(x), r_\alpha(u)) du\\ =& \int_{\mathbb{R}} f(u) p_{Z_T}(z,u) du.\end{aligned}$$

Therefore, $p_{Z_T}(z,\cdot)$ is the density function of $Z_T(z)$ for all $z \in \mathbb{R}$. Now we prove the Gaussian upper bound for $p_{Z_T}(z,\cdot)$. From Proposition 5.3, we have for some $C, c > 0$,

$$\begin{aligned} p_{Z_T}(z,u) &\le \max\left\{\frac{1}{1-\alpha},\frac{1}{\alpha}\right\} E_{\eta/2,1}(C(1\vee T)^{(3-2\eta)/2}) g^c_T\left(r_\alpha(u)-r_\alpha(z)\right) \\ &\le \max\left\{\frac{1}{1-\alpha},\frac{1}{\alpha}\right\} E_{\eta/2,1}(C(1\vee T)^{(3-2\eta)/2}) \begin{cases} g^c_T\left(u-z\right), & \text{if } z,u>0, \\ g^c_T\left((1-\alpha)u-\alpha z\right), & \text{if } z>0>u, \\ g^c_T\left(\alpha u-(1-\alpha)z\right), & \text{if } z<0<u, \\ g^c_T\left(u-z\right), & \text{if } z,u<0. \end{cases} \end{aligned}$$

Let $z > 0 > u$. If $\alpha/(1-\alpha) \ge 1$, then $-\frac{(1-\alpha)u-\alpha z}{1-\alpha} \ge z-u > 0$ and if $\alpha/(1-\alpha) < 1$, then $-\frac{(1-\alpha)u-\alpha z}{\alpha} > z-u > 0$. Let $z < 0 < u$. If $\alpha/(1-\alpha) \ge 1$, then $\frac{\alpha u-(1-\alpha)z}{1-\alpha} \ge u-z > 0$ and if $\alpha/(1-\alpha) < 1$, then $\frac{\alpha u-(1-\alpha)z}{\alpha} > u-z > 0$. Therefore, we obtain that

$$p_{Z_T}(z,u) \le \max\left\{\frac{1}{1-\alpha},\frac{1}{\alpha}\right\} E_{\eta/2,1}(C(1\vee T)^{(3-2\eta)/2}) g^c_T\left(u-z\right),$$

which concludes the statement. □

6 Regularity of the density for a skew diffusion

In this section, we prove that $x \mapsto p_T(x,y)$ is differentiable function on $\mathbb{R}_0$.

Theorem 6.1. Suppose Assumption 2.1 holds. For any $y \in \mathbb{R}$, $p_T(x,y)$ is differentiable with respect to $x \in \mathbb{R}_0$ and we have

$$\partial_x p_T(x,y) = \sum_{n=0}^{\infty} \partial_x \hat{I}^n_{t_0}(y,x) = \sum_{n=0}^{\infty} (\partial_x \bar{p}) \circledast \Phi^{\circledast n}(t_0,x,y)$$

where $t_0 = T$. Moreover, we have

$$|\partial_x p_T(x,y)| \le \frac{E_{\eta/2,1/2}(C(1\vee T)^{1+\eta/2})}{T^{1/2}} g^c_T\left(y-x\right) \tag{29}$$

and for any $x, y \in \mathbb{R}_0$, we have

$$\alpha \partial_x p_T(0+,y) = (1-\alpha)\partial_x p_T(0-,y) \tag{30}$$

Proof of Theorem 6.1. We first prove that for $y_0 := y$,

$$\partial_x \hat{I}^n_{t_0}(y,x) = \int_0^{t_0} dt_1 \cdots \int_0^{t_{n-1}} dt_n \int_{\mathbb{R}^n} dy_1 \cdots dy_n \prod_{i=0}^{n-1} \Phi(t_i-t_{i+1}, y_{i+1}, y_i) \partial_x p^{y_n}_{t_n}(x,y_n). \tag{31}$$

For any $x, y \in \mathbb{R}_0$,

$$\partial_x p^y_t(x,y) = \begin{cases} -H_1(y-x,ta(y)) g^{a(y)}_t\left(y-x\right) + (2\alpha-1)H_1(y+x,ta(y)) g^{a(y)}_t\left(y+x\right), & \text{if } x,y>0, \\ -2(1-\alpha)H_1(y-x,ta(y)) g^{a(y)}_t\left(y-x\right), & \text{if } x>0>y, \\ -2\alpha H_1(y-x,ta(y)) g^{a(y)}_t\left(y-x\right), & \text{if } x<0<y, \\ -H_1(y-x,ta(y)) g^{a(y)}_t\left(y-x\right) + (1-2\alpha)H_1(y+x,ta(y)) g^{a(y)}_t\left(y+x\right), & \text{if } x,y<0. \end{cases}$$

Therefore, from $|x|^p e^{-qx^2} \leq (p/(2qe))^{p/2}$ for any $p, q > 0$ and $x \in \mathbb{R}$, we have for any $x, y \in \mathbb{R}_0$,

$$|\partial_x p_t^y(x,y)| \leq \frac{C}{t^{1/2}} g_t^c(y-x).$$

As in the proof of Proposition 5.3, from Lemma 4.4 and Chapman-Kolmogorov's equation, we have

$$\begin{aligned}
&\int_0^{t_0} dt_1 \cdots \int_0^{t_{n-1}} dt_n \int_{\mathbb{R}^n} dy_1 \cdots dy_n \prod_{i=0}^{n-1} |\Phi(t_i - t_{i+1}, y_{i+1}, y_i)||\partial_x p_{t_n}^{y_n}(x, y_n)| \\
&\leq \int_0^{t_0} dt_1 \cdots \int_0^{t_{n-1}} dt_n \int_{\mathbb{R}^n} dy_1 \cdots dy_n \prod_{i=0}^{n-1} \frac{C_T}{(t_i - t_{i+1})^{1-\eta/2}} g_{t_i - t_{i+1}}^c(y_i - y_{i+1}) \frac{C}{t_n^{1/2}} g_{t_n}^c(y_n - x) \\
&\leq C_T^n \int_0^{t_0} dt_1 \cdots \int_0^{t_{n-1}} dt_n \prod_{i=0}^{n-1} \frac{1}{(t_i - t_{i+1})^{1-\eta/2}} \frac{1}{t_n^{1/2}} g_{t_0}^c(y_0 - x) \\
&= \frac{C_T^n t_0^{n\eta/2} \Gamma^n(\eta/2) \Gamma(1/2)}{t_0^{1/2} \Gamma(1/2 + n\eta/2)} g_{t_0}^c(y_0 - x) < \infty,
\end{aligned}$$

where we used Lemma 8.1 with $b = -1/2$ and $a = 1 - \eta/2$ in the last equation. Hence the, right hand side of (31) is well defined. Then from the dominated convergence theorem, we obtain (31) and

$$|\partial_x \hat{I}_{t_0}^n(y,x)| \leq \frac{C_T^n t_0^{n\eta/2} \Gamma^n(\eta/2)}{t_0^{1/2}} \frac{\Gamma(1/2)}{\Gamma(1/2 + n\eta/2)} g_{t_0}^c(y_0 - x) < \infty.$$

Therefore,

$$\sum_{n=0}^{\infty} \sup_{x,y \in \mathbb{R}_0} |\partial_x \hat{I}_{t_0}^n(y,x)| < \infty.$$

From here, we conclude the first two statements of the Theorem. Finally, since for any $y \in \mathbb{R}_0$ and $t > 0$, $p_t^y(\cdot, y) \in D(\alpha)$, then one obtains (30). $\square$

From Theorem 6.1, we have the following gradient estimate for the semigroup.

Corollary 6.2. Suppose Assumption 2.1 holds. We assume that f is a measurable function such that $\int_{\mathbb{R}} |f(y)| g_T^c(y-x) dy < \infty$ for any $c > 0$. Then $\mathbb{E}[f(X_T(x))]$ is differentiable with respect to $x \in \mathbb{R}_0$ and it holds that

$$\partial_x P_T f(x) = \int_{\mathbb{R}} f(y) \partial_x p_T(x,y) dy.$$

Proof. For $x \in \mathbb{R}_0$ and $h \in (-1,1)$ with $x \pm h \neq 0$, it follows from the mean value theorem and the upper bound for $\partial_x p_T(x,y)$ in (29) that

$$|p_T(x+h,y) - p_T(x,y)| = |h| \left| \int_0^1 \partial_x p_T(x + \theta h, y) d\theta \right|$$

$$\leq |h| \frac{E_{\eta/2,1/2}(C(1\vee T)^{1+\eta/2})}{T^{1/2}} \int_0^1 \frac{e^{-\frac{(y-x-h\theta)^2}{2cT}}}{\sqrt{2\pi T}} d\theta.$$

Using the inequality $(a-b)^2 \geq a^2/2 - b^2$, we have

$$\left| \frac{p_T(x+h,y) - p_T(x,y)}{h} \right| \leq \frac{E_{\eta/2,1/2}(C(1\vee T)^{1+\eta/2})}{T^{1/2}} \frac{e^{-\frac{(y-x)^2}{4cT}}}{\sqrt{2\pi T}}.$$

Therefore, from the dominated convergence theorem, we have

$$\lim_{h\to 0} \frac{\mathbb{E}[f(X_T(x+h))] - \mathbb{E}[f(X_T(x))]}{h} = \int_{\mathbb{R}} f(y) \lim_{h\to 0} \frac{p_T(x+h,y) - p_T(x,y)}{h} dy = \int_{\mathbb{R}} f(y) \partial_x p_T(x,y) dy,$$

which concludes the statement. □

Theorem 6.3. Suppose that Assumption 2.1 holds. Let $Z(z)$ be the solution to SDE (7). For any $(z,u) \in \mathbb{R}_0 \times \mathbb{R}$, $p_{Z_T}(z,u)$ is differentiable as a function of $z \in \mathbb{R}_0$ and we have

$$\partial_z p_{Z_T}(z,u) = \begin{cases} \frac{1}{1-\alpha} \partial_x p_T(z/(1-\alpha), r_\alpha(u)) \left(\frac{\mathbf{1}(u\geq 0)}{1-\alpha} + \frac{\mathbf{1}(u<0)}{\alpha} \right), & \text{if } z>0, \\ \frac{1}{\alpha} \partial_x p_T(z/\alpha, r_\alpha(u)) \left(\frac{\mathbf{1}(u\geq 0)}{1-\alpha} + \frac{\mathbf{1}(u<0)}{\alpha} \right), & \text{if } z<0. \end{cases}$$

As a consequence of the above result and 29 one can obtain a similar result as Corollary 5.6.

7 Probabilistic representation

In this section, we introduce a probabilistic representation of the density function of the skew diffusion $X_T(x)$ and $\mathbb{E}[f(X_T(x))]$.

We first define the following counting process.

Definition 7.1. Let $R_t := \sum_{n=1}^{\infty} \mathbf{1}(\tau_n \leq t)$ where $(\tau_n - \tau_{n-1})_{n\in\mathbb{N}}$ with $\tau_0 = 0$ are independent and identically distributed random variables with density function ζ. Then $R = (R_t)_{t\geq 0}$ is called the counting process with jump times $(\tau_n)_{n\in\mathbb{N}}$.

Remark 7.2. Usual choices for the density function ζ are: $\zeta(t) = \lambda e^{-\lambda t} \mathbf{1}_{[0,\infty)}(t)$ then, $R = (R_t)_{t\geq 0}$ is a Poisson process with parameter $\lambda > 0$. Another choice is $\zeta(t) := \frac{A}{t^\beta} \mathbf{1}_{[0,2T]}(t)$ where $A := (1-\beta)/(2T)^{1-\beta}$ and $\beta \in (0,1)$. For more on this, see Andersson and Kohatsu [1].

Lemma 7.3. Let $R = (R_t)_{t\geq 0}$ be the counting process with jumps times $(\tau_n)_{n\in\mathbb{N}}$. Then for any $t > 0$, $n \in \mathbb{N}$ and any measurable bounded function $V_n : \mathbb{R}^n \to \mathbb{R}$,

$$\begin{aligned} &\mathbb{E}[\mathbf{1}(R_t = n) V_n(\tau_1, \ldots, \tau_n)] \\ &= \int_0^t ds_n \int_0^{s_n} ds_{n-1} \cdots \int_0^{s_2} ds_1 V_n(s_1, \ldots, s_n)(1 - F_\zeta(t-s_n)) \prod_{i=0}^{n-1} \zeta(s_{i+1} - s_i), \end{aligned}$$

where $F_\zeta(x) := \int_{-\infty}^x \zeta(y)dy$ and $s_0 = 0$. In particular, if R is a Poisson process with parameter $\lambda > 0$, then we have

$$\mathbb{E}[\mathbf{1}(R_t = n)V_n(\tau_1, \dots, \tau_n)] = \lambda^n e^{-\lambda t} \int_0^t ds_n \int_0^{s_n} ds_{n-1} \cdots \int_0^{s_2} ds_1 V_n(s_1, \dots, s_n).$$

Proof. We first prove by induction that the joint probability density function of $(\tau_1, \dots, \tau_n)$ is given by

$$\prod_{i=0}^{n-1} \zeta(s_{i+1} - s_i)\mathbf{1}(0 < s_1 < \cdots < s_n). \tag{32}$$

If $n = 1$, the statement holds by the definition of ζ. Assume that (32) holds for $n > 1$. Then since $\tau_{n+1} - \tau_n$ is independent from τ_i, for any $i = 1, ..., n$, we have for any $x_1, ..., x_{n+1} > 0$,

$$\begin{aligned}
&\mathbb{P}(\tau_1 < x_1, \dots, \tau_{n+1} < x_{n+1}) = \mathbb{P}(\tau_1 < x_1, \dots, \tau_n + (\tau_{n+1} - \tau_n) < x_{n+1}) \\
&= \int_0^{x_1} ds_1 \cdots \int_0^{x_n} ds_n \int_0^\infty dt_{n+1} \prod_{i=0}^{n-1} \zeta(s_{i+1} - s_i)\mathbf{1}(0 < s_1 < \cdots < s_n)\zeta(t_{n+1})\mathbf{1}(s_n + t_{n+1} < x_{n+1}) \\
&= \int_0^{x_1} ds_1 \cdots \int_0^{x_{n+1}} ds_{n+1} \prod_{i=0}^{n} \zeta(s_{i+1} - s_i),
\end{aligned}$$

where in the last equation we use the change of variable $s_{n+1} = s_n + t_{n+1}$. Hence (32) holds for any $n \in \mathbb{N}$.

From (32) and the Fubini theorem, we have

$$\begin{aligned}
&\mathbb{E}[\mathbf{1}(R_t = n)V_n(\tau_1, \dots, \tau_n)] = \mathbb{E}[\mathbf{1}(\tau_n \le t < \tau_{n+1})\Phi(\tau_1, \dots, \tau_n)] \\
&= \int_t^\infty ds_{n+1} \int_0^t ds_n \int_0^{s_n} ds_{n-1} \cdots \int_0^{s_2} ds_1 V_n(s_1, \dots, s_n) \prod_{i=0}^{n} \zeta(s_{i+1} - s_i) \\
&= \int_0^t ds_n \int_0^{s_n} ds_{n-1} \cdots \int_0^{s_2} ds_1 \int_t^\infty ds_{n+1}\zeta(s_{n+1} - s_n)V_n(s_1, \dots, s_n) \prod_{i=0}^{n-1} \zeta(s_{i+1} - s_i) \\
&= \int_0^t ds_n \int_0^{s_n} ds_{n-1} \cdots \int_0^{s_2} ds_1 V_n(s_1, \dots, s_n)(1 - F_\zeta(t - s_n)) \prod_{i=0}^{n-1} \zeta(s_{i+1} - s_i),
\end{aligned}$$

which concludes the proof of Lemma 7.3. □

Let $\varphi_t^z(x)$ be a strictly positive density function of a time-homogeneous Markov process Y_t^z with $Y_0^z = z$. We define a function $\hat{\theta}_t(x, z)$ as

$$\hat{\theta}_t(x, z) := \begin{cases} \dfrac{(L - L^z)\phi_t^z(x)}{\varphi_t^z(x)} & \text{if } x \text{ and } z \neq 0, \\ 0 & \text{if } x \text{ or } z = 0. \end{cases}$$

Note that when $\alpha = 1/2$ and $\varphi_t^z(x) := \phi_t^z(x)$, i.e., the diffusion case, it holds that

$$\hat{\theta}_t(x,z)\phi_t^z(x) = (L - L^z)\phi_t^z(x).$$

However, in the general skew diffusion case, this kind of property does not hold.

Let $\pi_0 = (s_i \wedge T)_{i\in\mathbb{N}}$ with $0 =: s_0 \leq s_1 < \cdots < s_n < \cdots$. Suppose that for any partition π_0 there exists a time continuous Markov chain $Y^{*,\pi_0}(y_0)$ such that

$$\begin{aligned} Y_0^{*,s_0}(y_0) &= y_0, \\ \mathbb{P}(Y_{s_i}^{*,\pi_0}(y_0) \in dy_{i+1} | Y_{s_{i-1}}^{*,\pi_0}(y_0) = y_i) &= \varphi_{s_i - s_{i-1}}^{y_i}(y_{i+1}) dy_{i+1}. \end{aligned} \tag{33}$$

Let $y_0 := y$ and $t_0 := T$. From the definition of $p_T(x,y)$ and using the change of variables $s_n = t_0 - t_n$, we have

$$p_{t_0}(x,y) = \sum_{n=0}^{\infty} \int_0^{t_0} ds_n \int_0^{s_n} ds_{n-1} \cdots \int_0^{s_2} ds_1 h_n(s_1,\ldots,s_n,y,x). \tag{34}$$

Here,

$$h_n(s_1,\ldots,s_n,y,x) := \int_{\mathbb{R}^n} dy_1 \cdots dy_n \prod_{i=0}^{n-1} \hat{\theta}_{s_{i+1}-s_i}(y_{i+1},y_i)\varphi_{s_{i+1}-s_i}^{y_i}(y_{i+1}) p_{t_0 - s_n}^{y_n}(x,y_n).$$

Then from Lemmas 4.4, 4.5 and the Chapman-Kolmogorov's equation, we have

$$|h_n(s_1,\ldots,s_n,y,x)| \leq \prod_{i=0}^{n-1} \frac{C_T}{(s_{i+1}-s_i)^{1-\eta/2}} g_{t_0}^c(y-x).$$

This gives the needed integrability properties for the convergence of the sum (34). In the probabilistic representation to follow this condition will imply the $L^1(\Omega)$-integrability of the probabilistic representation.

Theorem 7.4. Assume that Assumption 2.1 holds. Let $R = (R_t)_{t\geq 0}$ be the counting process with jump times $\pi := (\tau_n)_{n\in\mathbb{N}}$ independent of $(Y^{*,\pi_0})_{\pi_0}$. Let $f : \mathbb{R} \to \mathbb{R}$ and define $D := \operatorname{supp} f \subseteq \mathbb{R}$. Also, let Z be a D-valued random variable independent of R and $(Y^{*,\pi_0})_{\pi_0}$. Assume that g is the density function for Z such that $g > 0$ on D. Suppose that $\int_{\mathbb{R}} |f(y)| g_T^c(y) dy < \infty$ for any $c > 0$.

Then for every $x \in \mathbb{R}$, we have

$$\mathbb{E}[f(X_T(x))] = \mathbb{E}\left[\frac{f(Z)}{g(Z)} \frac{p_{T-\tau_T}^{Y_{\tau_T}^{*,\pi}(Z)}(x, Y_{\tau_T}^{*,\pi}(Z))}{1 - F_\zeta(T-\tau_T)} \prod_{i=0}^{R_T - 1} \frac{\hat{\theta}_{\tau_{i+1}-\tau_i}(Y_{\tau_{i+1}}^{*,\pi}(Z), Y_{\tau_i}^{*,\pi}(Z))}{\zeta(\tau_{i+1}-\tau_i)} \right].$$

Here, $\tau_T := \tau_{R_T}$. Furthermore the density of X_T can be represented as

$$p_T(x,y) = \mathbb{E}\left[\frac{p_{T-\tau_T}^{Y_{\tau_T}^{*,\pi}(y)}(x, Y_{\tau_T}^{*,\pi}(y))}{1 - F_\zeta(T-\tau_T)} \prod_{i=0}^{R_T - 1} \frac{\hat{\theta}_{\tau_{i+1}-\tau_i}(Y_{\tau_{i+1}}^{*,\pi}(y), Y_{\tau_i}^{*,\pi}(y))}{\zeta(\tau_{i+1}-\tau_i)} \right].$$

Proof. First, note that since the density $p_T(x,\cdot)$ of $X_T(x)$ satisfies a Gaussian upper bound then using Young's inequality, we get that $\mathbb{E}[|f(X_T(x))|] < \infty$.

We will obtain a representation formula for h_n. For $n=1$, since $\varphi^{y_0}_{s_1}$ is a density function of $Y^{*,\pi_0}_{s_1}(y_0)$, we have

$$
\begin{aligned}
h_1(s_1,y,x) =& \int_{\mathbb{R}} \hat{\theta}_{s_1}(y_1,y_0)\varphi^{y_0}_{s_1}(y_1)p^{y_1}_{t_0-s_1}(x,y_1)dy_1 \\
=&\mathbb{E}\left[p^{Y^{*,\pi_0}_{s_1}(y)}_{t_0-s_1}(x,Y^{*,\pi_0}_{s_1}(y))\hat{\theta}_{s_1}(Y^{*,\pi_0}_{s_1}(y),y)\right].
\end{aligned}
$$

For $n=2$, we have

$$
\begin{aligned}
h_2(s_1,s_2,y,x) =& \int_{\mathbb{R}} \hat{\theta}_{s_1}(y_1,y_0)\varphi^{y_0}_{s_1}(y_1)\int_{\mathbb{R}} \hat{\theta}_{s_2-s_1}(y_2,y_1)\varphi^{y_1}_{s_2-s_1}(y_2)p^{y_2}_{t_1-s_2}(x,y_2)dy_2 \\
=& \int_{\mathbb{R}} \hat{\theta}_{s_1}(y_1,y_0)\varphi^{y_0}_{s_1}(y_1)\mathbb{E}\left[\hat{\theta}_{s_2-s_1}(Y^{*,\pi_0}_{s_2}(y),y_1)p^{Y^{*,\pi_0}_{s_2}(y)}_{t_1-s_2}(x,Y^{*,\pi_0}_{s_2}(y))\middle|Y^{*,\pi_0}_{s_1}(y)=y_1\right] \\
=&\mathbb{E}\left[p^{Y^{*,\pi_0}_{s_2}(y)}_{t_1-s_2}(x,Y^{*,\pi_0}_{s_2}(y))\hat{\theta}_{s_1}(Y^{*,\pi_0}_{s_2}(y),y_0)\hat{\theta}_{s_2-s_1}(Y^{*,\pi_0}_{s_2}(y),Y^{*,\pi_0}_{s_1}(y))\right].
\end{aligned}
$$

In the same way as in the case $n=2$, we obtain that

$$
h_n(s_1,\dots,s_n,y,x) =\mathbb{E}\left[p^{Y^{*,\pi_0}_{s_n}(y)}_{t_0-s_n}(x,Y^{*,\pi_0}_{s_n}(y))\prod_{i=0}^{n-1}\hat{\theta}_{s_{i+1}-s_i}(Y^{*,\pi_0}_{s_{i+1}}(y),Y^{*,\pi_0}_{s_i}(y))\right].
$$

Let $R=(R_t)_{t\geq 0}$ be the counting process with jump times $(\tau_n)_{n\in\mathbb{N}}$ independent from Y^{*,π_0}. From Lemma 7.3, we get for every $x\in\mathbb{R}$

$$
p_{t_0}(x,y) = \sum_{n=0}^{\infty}\int_0^{t_0} ds_n\int_0^{s_n} ds_{n-1}\cdots\int_0^{s_2} ds_1 V_n(s_1,\dots,s_n,y,x)(1-F_\zeta(T-s_n))\prod_{i=0}^{n-1}\zeta(s_{i+1}-s_i),
$$

where

$$
V_n(s_1,\dots,s_n,y,x) := \frac{h_n(s_1,\dots,s_n,y,x)}{(1-F_\zeta(T-s_n))\prod_{i=0}^{n-1}\zeta(s_{i+1}-s_i)}.
$$

Therefore, we have

$$
\begin{aligned}
p_T(x,y) =& \sum_{n=0}^{\infty}\mathbb{E}[\mathbf{1}(R_T=n)V_n(\tau_1,\dots,\tau_n,y,x)] = \sum_{n=0}^{\infty}\mathbb{E}[\mathbf{1}(R_T=n)V_n(\tau_1,\dots,\tau_n,y,x)] \\
=&\mathbb{E}[V_{R_T}(\tau_1,\dots,\tau_T,y,x)] \\
=&\mathbb{E}\left[\frac{p^{Y^{*,\pi}_{\tau_T}(y)}_{T-\tau_T}(x,Y^{*,\pi}_{\tau_T}(y))}{1-F_\zeta(T-\tau_T)}\prod_{i=0}^{R_T-1}\frac{\hat{\theta}_{\tau_{i+1}-\tau_i}(Y^{*,\pi}_{\tau_{i+1}}(y),Y^{*,\pi}_{\tau_i}(y))}{\zeta(\tau_{i+1}-\tau_i)}\right].
\end{aligned}
$$

Since $X_T(x)$ has a density $p_T(x,\cdot)$ and Z is independent from R and $Y^{*,\pi}$, we have

$$
\mathbb{E}[f(X_T(x))] = \int_D \frac{f(y)}{g(y)}g(y)p_T(x,y)dy = \mathbb{E}\left[\frac{f(Z)}{g(Z)}p_T(x,Z)\right]
$$

$$=\mathbb{E}\left[\frac{f(Z)}{g(Z)}\mathbb{E}\left[\frac{p_{T-\tau_T}^{Y^{*,\pi}_{\tau_T}(y)}(x,Y^{*,\pi}_{\tau_T}(y))}{1-F_\zeta(T-\tau_T)}\prod_{i=0}^{R_T-1}\frac{\hat{\theta}_{\tau_{i+1}-\tau_i}(Y^{*,\pi}_{\tau_{i+1}}(y),Y^{*,\pi}_{\tau_i}(y))}{\zeta(\tau_{i+1}-\tau_i)}\right]\Bigg|_{y=Z}\right]$$
$$=\mathbb{E}\left[\frac{f(Z)}{g(Z)}\mathbb{E}\left[\frac{p_{T-\tau_T}^{Y^{*,\pi}_{s_n}(Z)}(x,Y^{*,\pi}_{\tau_T}(Z))}{1-F_\zeta(T-\tau_T)}\prod_{i=0}^{R_T-1}\frac{\hat{\theta}_{\tau_{i+1}-\tau_i}(Y^{*,\pi}_{\tau_{i+1}}(Z),Y^{*,\pi}_{\tau_i}(Z))}{\zeta(\tau_{i+1}-\tau_i)}\Bigg|\sigma(Z)\right]\right].$$

This implies the statement. □

For example, if we choose $\zeta(t)=\lambda e^{-\lambda t}\mathbf{1}_{[0,\infty)(t)}$, then we have

$$p_T(x,y)=e^{\lambda T}\mathbb{E}\left[\lambda^{-R_T}p_{T-\tau_T}^{Y^{*,\pi}_{\tau_T}(y)}(x,Y^{*,\pi}_{\tau_T}(y))\prod_{i=0}^{R_T-1}\hat{\theta}_{\tau_{i+1}-\tau_i}(Y^{*,\pi}_{\tau_{i+1}}(y),Y^{*,\pi}_{\tau_i}(y))\right],$$

We also have the following probabilistic representation for the derivative of the density of X_T.

Theorem 7.5. Assume that Assumption 2.1 holds. Let $R=(R_t)_{t\geq 0}$ be the counting process with $\pi:=(\tau_n)_{n\in\mathbb{N}}$ independent of $(Y^{*,\pi_0})_{\pi_0}$. Then for any $x\in\mathbb{R}_0$,

$$\partial_x p_T(x,y)=\mathbb{E}\left[\frac{\partial_x p_{T-\tau_T}^{Y^{*,\pi}_{\tau_T}(y)}(x,Y^{*,\pi}_{\tau_T}(y))}{1-F_\zeta(T-\tau_T)}\prod_{i=0}^{R_T-1}\frac{\hat{\theta}_{\tau_{i+1}-\tau_i}(Y^{*,\pi}_{\tau_{i+1}}(y),Y^{*,\pi}_{\tau_i}(y))}{\zeta(\tau_{i+1}-\tau_i)}\right].$$

Moreover, if we assume the same hypothesis as in Theorem 7.4 then we have the following probabilistic representation:

$$\partial_x\mathbb{E}[f(X_T(x))]=\mathbb{E}\left[\frac{f(Z)}{g(Z)}\frac{\partial_x p_{T-\tau_T}^{Y^{*,\pi}_{\tau_T}(y)}(x,Y^{*,\pi}_{\tau_T}(y))}{1-F_\zeta(T-\tau_T)}\prod_{i=0}^{R_T-1}\frac{\hat{\theta}_{\tau_{i+1}-\tau_i}(Y^{*,\pi}_{\tau_{i+1}}(y),Y^{*,\pi}_{\tau_i}(y))}{\zeta(\tau_{i+1}-\tau_i)}\right].$$

In the same way the following probabilistic representation for the density of Z_t holds.

Theorem 7.6. Assume that Assumption 2.1 holds and also the same hypothesis as in Theorem 7.4. Then for any $x\in\mathbb{R}_0$,

$$p_{Z_T}(z,u)=\mathbb{E}\left[\frac{p_{T-\tau_T}^{Y^{*,\pi}_{\tau_T}(r_\alpha(u))}(r_\alpha(z),Y^{*,\pi}_{\tau_T}(r_\alpha(u)))}{1-F_\zeta(T-\tau_T)}\prod_{i=0}^{R_T-1}\hat{\theta}_{\tau_{i+1}-\tau_i}(Y^{*,\pi}_{\tau_{i+1}}(r_\alpha(u)),Y^{*,\pi}_{\tau_i}(r_\alpha(u)))\right]$$
$$\times\left(\frac{\mathbf{1}(u\geq 0)}{1-\alpha}+\frac{\mathbf{1}(u<0)}{\alpha}\right).$$

Example 7.1. One example of a Markov process Y^z satisfying the assumptions in this section is

$$Y_t^z=z+\sigma(z)W_t$$

and its density is given by

$$\varphi_t^z(x)=g_t^{a(z)}(x-z)=\frac{e^{-\frac{(x-z)^2}{2t\sigma(z)^2}}}{\sqrt{2\pi\sigma^2(z)t}}.$$

In this case, the Markov chain $Y^{*,\pi_0}(y_0)$ is given by

$$Y_0^{*,\pi_0}(y_0) = y_0 \text{ and } Y_{s_i}^{*,\pi_0}(y_0) = Y_{s_{i-1}}^{*,\pi_0}(y_0) + \sigma(Y_{s_{i-1}}^{*,\pi_0}(y_0))(W_{s_i} - W_{s_{i-1}})$$

and we have the following cases:
Case 1: If $x, y > 0$,

$$\begin{aligned}\hat{\theta}_t(x,y) =& b(x)\left(-H_1(y-x, ta(y)) + (2\alpha-1)H_1(y+x, ta(y))\exp\left(-\frac{2xy}{a(y)t}\right)\right) \\ &+ \frac{a(x)-a(y)}{2}\left(H_2(y-x, ta(y)) + (2\alpha-1)H_2(y+x, ta(y))\exp\left(-\frac{2xy}{a(y)t}\right)\right).\end{aligned}$$

Case 2: If $x > 0 > y$,

$$\hat{\theta}_t(x,y) = 2(1-\alpha)\left(-b(x)H_1(y-x, ta(y)) + \frac{a(x)-a(y)}{2}H_2(y-x, ta(y))\right).$$

Case 3: If $x < 0 < y$,

$$\hat{\theta}_t(x,y) = 2\alpha\left(-b(x)H_1(y-x, ta(y)) + \frac{a(x)-a(y)}{2}H_2(y-x, ta(y))\right).$$

Case 4: If $x, y < 0$,

$$\begin{aligned}\hat{\theta}_t(x,y) =& b(x)\left(-H_1(y-x, ta(y)) + (1-2\alpha)H_1(y+x, ta(y))\exp\left(-\frac{2xy}{a(y)t}\right)\right) \\ &+ \frac{a(x)-a(y)}{2}\left(H_2(y-x, ta(y)) + (1-2\alpha)H_2(y+x, ta(y))\right).\end{aligned}$$

8 Appendix

8.1 On some Beta type integral

Lemma 8.1. Let $b > -1$ and $a \in [0,1)$. Then for any $t_0 > 0$,

$$\int_0^{t_0} dt_1 \cdots \int_0^{t_{n-1}} dt_n t_n^b \prod_{j=0}^{n-1}(t_j - t_{j+1})^{-a} = \frac{t_0^{b+n(1-a)}\Gamma^n(1-a)\Gamma(1+b)}{\Gamma(1+b+n(1-a))}.$$

Proof. Let $b > -1$ and $a \in [0,1)$. Using the change of variables $s = ut$, we have

$$\int_0^t s^b(t-s)^{-a}ds = t^{b+1-a}\int_0^1 u^b(1-u)^{-a}du = t^{b+1-a}B(1+b, 1-a),$$

where $B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}$ is the standard Beta function. Using this repeatedly, we obtain the statement. □

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