Estimates for the density of functionals of SDE's with irregular drift

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Abstract

We obtain upper and lower bounds for the density of a functional of a diffusion whose drift is bounded and measurable. The argument consists of using Girsanov's theorem together with an Itô-Taylor expansion of the change of measure. One then applies Malliavin calculus techniques in a non-trivial manner so as to avoid the irregularity of the drift. An integration by parts formula for this set-up is obtained.

Keywords: stochastic differential equations, density, Malliavin calculus, irregular drift

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1. Introduction

It is well known that the regularity of the fundamental solution of a linear second order parabolic PDE is related to the regularity of its coefficients. Malliavin calculus (or stochastic calculus of variations) has arised as a prob-

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abilistic tool for studying the existence and the smoothness of densities of diffusions, through the celebrated integration by parts formula. Although this formula requires regularity on the coefficients of the associated stochastic differential equation, it applies to a wide range of stochastic equations that do not necessarily have a PDE counterpart.

In this article, we consider a two-component multidimensional SDE, where the second component depends only on the first one:

$$X_t = x + \int_0^t b(X_u)du + \int_0^t \sigma_1(X_u)\mathrm{d}B_u + \int_0^t \sigma_2(X_u)\mathrm{d}B_u, \tag{1}$$

$$Y_t = y + \int_0^t \psi(X_u, Y_u)du + \int_0^t \varphi(X_u, Y_u)\mathrm{d}B'_u, \tag{2}$$

$t \leq T$, where $(B, B')^*$ denotes a $m = m_1 + m_2$-dimensional Brownian motion, $x \in \mathbb{R}^{d_1}$, $y \in \mathbb{R}^{d_2}$, $b : \mathbb{R}^{d_1} \to \mathbb{R}^{d_1}$, $\sigma_1 : \mathbb{R}^{d_1} \to \mathbb{R}^{d_1 \times m_1}$, $\sigma_2 : \mathbb{R}^{d_1} \to \mathbb{R}^{d_1 \times m_2}$, $\psi : \mathbb{R}^{d_1+d_2} \to \mathbb{R}^{d_2}$, $\varphi : \mathbb{R}^{d_1+d_2} \to \mathbb{R}^{d_2 \times m_2}$, and the stochastic integrals are Itô stochastic integrals. The drift b is supposed to be only measurable and bounded, while the other coefficients are smooth and bounded. The diffusions σ and φ are uniformly elliptic.

Our goal is to obtain upper and lower bounds for the density of Y_t, thus extending the use of Malliavin calculus to situations where the drift b is irregular. This type of results are useful in applications as they indicate that the behavior of the process Y is asymptotically similar to a Gaussian process. As a by-product of our analysis, we obtain a general integration by parts formula for this setting.

The motivation for this problem comes from finance (where Y is related to Asian options in the case that $\varphi = 0$) and filtering problems, where Y is the observed signal process and X is the state process to be inferred (see e.g.

Øksendal [11]).

For results on the weak existence and uniqueness of solutions of SDE's with non-Lipschitz coefficients, see e.g. [5] or [8] and the references therein.

On the other hand, for results on the weak existence and uniqueness of fundamental solutions of the associated PDE, the literature is vast, although most of it deals with weak existence, see e.g. Stroock [13] where a bounded fundamental solution in $W^{1,2}$ is obtained. One exception is Friedman [3], where the parametrix method is used for a diffusion with Hölder coefficients. This method was slightly improved by McKean and Singer [4].

We also note that bounds for the density of solution of SDE's have been obtained in several works under several regularity assumptions on the coefficients: for example Kusuoka and Stroock [9], Kohatsu-Higa [6] and Bally [2] use Malliavin Calculus, for the (locally) elliptic case. From the PDE theory point of view see Aronson [1] (uniformly elliptic case).

The methodology used in this article is related to Kohatsu-Higa and Tanaka [8]. In this reference, the authors have studied the system (1)-(2) with bounded irregular drift b, and have established the existence and smoothness of the density of Y_t, through the study of its characteristic function. In order to obtain this result, one first performs a change of probability measure, using Girsanov's theorem, in order to get rid of the irregular drift in (1), which leads to an SDE with smooth coefficients, driven by a new Brownian motion W. Then, since the change of measure is non-smooth, they expand it using Itô-Taylor expansion up to a sufficiently high order. Finally, in order to regularize the drift, they take the conditional expectation over some subinterval of the time interval in each multiple stochastic integral, and then apply conditional Malliavin calculus to derive the smoothness of the density.

This result relies on the truncation of the Itô-Taylor formula and the explicit form of the definition of characteristic function, therefore it does not lead to an integration by parts formula which is essential in order to obtain upper and lower bounds for the density. The goal of the present article is to show a variation of the argument which allows to obtain an integration by parts for any function and as result obtain upper and lower bounds for the density of Y_t.

Our proof follows the first two steps as in Kohatsu-Higa and Tanaka [8], but with a simpler and more general proof which leads to an integration by parts formula (that could not be derived in [8]). This will not only imply the existence and smoothness of the density of Y_t, but also implies upper bounds for the density of Y_t and its derivatives. Moreover, the general integration by parts formula that we state here enables us to derive at the same time a lower bound for the density, using the well-known lower bound of the law of smooth SDE's.

The present compact version of the integration by parts formula will clearly lead to various other applications to other stochastic equations.

2. Preliminaries, assumptions and notations

As in [8], a general problem which generalizes the system (1)-(2) will be considered. We describe that set-up in detail here although we will not be back into the setting of (1)-(2) until Section 4.

Let W be a Brownian motion defined on the probability space $(\Omega, \mathcal{F}, \mathbb{Q})$.

Define the following change of measure

$$G_t := \exp\left(- \int_0^t \bar{b}(X_u) \mathrm{d}W_u - \frac{1}{2} \int_0^t \bar{b}^* \bar{b}(X_u) \mathrm{d}u \right).$$

Here $\bar{b} : \mathbb{R}^{d_1} \to \mathbb{R}^m$ is a bounded and measurable function.

The system we will discuss in the next sections is

$$X_t = x + \int_0^t \beta(X_u) du + \int_0^t \sigma(X_u) \mathrm{d}W_u, \tag{3}$$

$$Y_t = y + \int_0^t \psi(X_u, Y_u) du + \int_0^t \varphi(X_u, Y_u) \mathrm{d}W_u. \tag{4}$$

where all the coefficients $\beta, \sigma, \psi, \varphi$ are smooth, bounded functions with bounded derivatives. These assumptions are in force throughout the article.

In the next section, we work under the probability measure $\mathbb{Q}$. In particular, $\mathbb{E}$ denotes expectations under the probability measure $\mathbb{Q}$.

In order to consider the system (1)-(2), we will define a probability measure $\mathbb{P}$ by $\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{Q}} = G_T$. Then Girsanov's theorem will lead to the needed conclusions in Section 4 when $\beta = 0$ and $\bar{b} = -\begin{pmatrix} \sigma_1^*(\sigma_1\sigma_1^*)^{-1} b \\ 0 \end{pmatrix}$ and $B = W + \int_0^{\cdot} \bar{b}(X_s) ds$.

Therefore, we remark that we start our discussion from a general system before the Girsanov change of measure is to be performed, and G_t represents the change of measure that will lead the system (3)-(4) to become (1)-(2) with the measure $\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{Q}} = G_T$.

Throughout the paper, we use the following estimate for the moments of G_t.

Lemma 1. *For all $p \in \mathbb{R}$,*

$$\mathbb{E}[G_t^p] \leq \exp(\frac{1}{2}|p^2 - p| \left\|\bar{b}\right\|_\infty^2 t).$$

Proof.

$$
\begin{aligned}
\mathbb{E}[G_t^p] &= \mathbb{E}[\exp(-p\int_0^t \bar{b}(X_u)\mathrm{d}W_u - \frac{p}{2}\int_0^t \bar{b}^T\bar{b}(X_u)\mathrm{d}u)] \\
&= \mathbb{E}[\exp(\frac{p^2-p}{2}\int_0^t \bar{b}^T\bar{b}(X_u)\mathrm{d}u)\exp(-\int_0^t p\bar{b}(X_u)\mathrm{d}W_u - \frac{1}{2}\int_0^t p^2\bar{b}^T\bar{b}(X_u)\mathrm{d}u)] \\
&\le \exp(\frac{1}{2}|p^2-p|\left\|\bar{b}\right\|_\infty^2 t)\mathbb{E}[\exp(-\int_0^t p\bar{b}(X_u)\mathrm{d}W_u - \frac{1}{2}\int_0^t p^2\bar{b}^T\bar{b}(X_u)\mathrm{d}u)] \\
&= \exp(\frac{1}{2}|p^2-p|\left\|\bar{b}\right\|_\infty^2 t).
\end{aligned}
$$

□

We shall use the following assumptions:

Assumption $(\mathbf{A}_\sigma)$: σ is uniformly elliptic, i.e. there exists a nonnegative constant $\underline{\sigma}$ such that $\sigma\sigma^*(x) \ge \underline{\sigma}I$, for all $x \in \mathbb{R}^{d_1}$.

Assumption $(\mathbf{A}_\varphi)$: φ is uniformly elliptic, i.e. there exists a nonnegative constant $\underline{\varphi}$ such that $\varphi\varphi^*(x) \ge \underline{\varphi}I$, for all $x \in \mathbb{R}^{d_1+d_2}$.

Assumption $(\mathbf{A}_\mathrm{H})$: The system (3)-(4) satisfies the uniform Hörmander condition of order k. That is, define the vector fields

$$
\begin{aligned}
V_0 &:= \sum_{i=1}^{d_1}\beta^i\partial_{x^i} + \sum_{i=1}^{d_2}\psi^i\partial_{y^i} - \frac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{d_1}\sum_{l=1}^{m}\sigma_{jl}\partial_{x^j}\sigma_{il}\partial_{x^i} \\
V_i &:= \sum_{j=1}^{d_1}\sigma_{ji}\partial_{x^j} + \sum_{j=1}^{d_2}\varphi_{ji}\partial_{y^j}, 1 \le i \le m.
\end{aligned}
$$

Here we denote $\sigma_{ij} = \varphi_{i-d_1,j}$ if $i > d_1$ and ∂_{x^j} denotes the partial derivative operator wrt to the j-th coordinate, $j = 1, ..., d_1 + d_2$ and with some abuse of notation we let $x^{j+d_1} = y^j$.

We now define

$$A_0 := \{V_1, \ldots, V_m\};$$
$$A_n := \{[V_i, Z]; i = 0, \ldots, m, Z \in A_{n-1}\}, n \geq 1.$$

We assume that there exists $k \in \mathbb{N}$ and $c_0 > 0$ such that, for all $\xi \in \mathbb{R}^d$ and $x \in \mathbb{R}^{d^1}$,

$$\sum_{V \in \cup_{l=1}^{k} A_l} < V(x), \xi >^2 \geq c_0 \|\xi\|^2 .$$

Our objective is the study of the density of Y_t. This density under the initial probability $\mathbb{P}$ at the point y_0 is denoted by $p_{Y_t}(y_0) = \mathbb{E}^{\mathbb{P}}[\delta_{y_0}(Y_t)] = \mathbb{E}[\delta_{y_0}(Y_t)G_t]$, where $\delta_{y_0}(.)$ denotes the Dirac function at the point y_0. This is a slight abuse of notation, but one can take a sequence of functions approximating the Dirac function and pass to the limit, as we already know that the density of Y_t is smooth from [8]. For more details, see the proof of Proposition 2.1.6 in [10].

$\mathbb{E}^{\mathcal{F}_s}$ denotes the conditional expectation, under $\mathbb{Q}$, w.r.t. $\mathcal{F}_s$ $((\mathcal{F}_s)_{s\geq 0}$ being the filtration generated by the Brownian motion W). $m\mathcal{F}$ denotes the space of $\mathcal{F}$-measurable functions. $\sigma\{X_a, a \in A\}$ denotes the smallest σ-field generated by the family of random variables $\{X_a, a \in A\}$, $A \subseteq [0, T]$.

The constants that appear troughout the text, only depend on the data (x, y, β, σ, ψ, φ and T) and may change from one line to another (and whenever we want to emphasize the dependence of a constant C on a parameter p, we write it either as $C(p)$ or as C_p).

The high order partial mixed derivative of a function $f(y) = f(y^1, \ldots, y^{d_2})$ w.r.t $y^{\alpha_1}, \ldots, y^{\alpha_k}$ (with $\alpha_1, \ldots, \alpha_k \in \{1, \ldots, d_2\}$) is denoted by $f^{\alpha}(y)$ where $\alpha = (\alpha_1, ..., \alpha_k)$. The length k of α is denoted by $|\alpha|$ and the set of all multi-indices of length less or equal than k is denoted by $\mathcal{A}_k$. From now on, all these index manipulations will refer to the variables associated with the process Y and this index set is partially ordered using the inclusion operator. That is, we say that $(\beta_1, ..., \beta_j) \subseteq (\alpha_1, ..., \alpha_k)$ if $\{\beta_1, ..., \beta_j\} \subseteq \{\alpha_1, ..., \alpha_k\}$ and the $\sum_{l=1}^{j} 1(\beta_l = a) \leq \sum_{l=1}^{k} 1(\alpha_l = a)$ for any $a \in \{1, ..., d_2\}$. In this sense we associate the identity operator to the zero-length empty index. That is, $f^{\emptyset} = f$, $|\emptyset| = 0$ and $\emptyset \subseteq \alpha$ for any index α.

$\mathcal{C}_b^{\alpha}$ denotes the space of bounded functions which are $|\alpha|$-times continuously differentiable (w.r.t. $y^{\alpha_1}, \ldots, y^{\alpha_k}$) with bounded derivatives. We denote the indicator function of the set $[y_0^1, +\infty) \times \cdots \times [y_0^{d_2}, +\infty)$ (where $(y_0^1, \ldots, y_0^{d_2})^* = y_0$) by $\mathbb{1}_{[y_0, +\infty)}$. Function spaces and their notation may change dimension parameter from d_1 to $d_1 + d_2$ without any further specific mention.

$\mathbb{N}^* := \mathbb{N} \backslash \{0\}$. Vector components are denoted with superscripts and matrix components by subscripts.

We use strongly the integration by parts formula of Malliavin Calculus. We refer the reader to [10] for results and notation used in this article.

3. Integration by parts formula

The following formula gives the basic decomposition that will be used in order to obtain our integration by parts formula.

Lemma 2. *Let $\{t_n, n \geq 0\}$ be a strictly decreasing sequence of strictly positive real numbers with $t_0 = t$. Then,*

$$G_t \stackrel{\mathbf{L}^2}{=} \sum_{n=0}^{\infty} G_{t_{n+1}} Z_t^n.$$

Here

$$\begin{aligned} Z_t^n &= \int_{t_1}^{t}\int_{t_2}^{s_2}\dots\int_{t_n}^{s_n}\bar{b}(X_{s_{n-1}})\mathrm{d}W_{s_{n-1}}\dots\bar{b}(X_{s_2})\mathrm{d}W_{s_2}, \\ Z_t^0 &\equiv 1. \end{aligned}$$

PROOF. First note that $\mathrm{d}G_s = \bar{b}(X_s)G_s\mathrm{d}W_s$. Using the SDE satisfied by G iteratively, we obtain :

$$\begin{aligned} G_t &= G_{t_1} + \int_{t_1}^{t}\bar{b}(X_{s_2})G_{s_2}\mathrm{d}W_{s_2} \\ &= G_{t_1} + \int_{t_1}^{t}\bar{b}(X_{s_2})\left(G_{t_2} + \int_{t_2}^{s_2}\bar{b}(X_{s_3})G_{s_3}\mathrm{d}W_{s_3}\right)\mathrm{d}W_{s_2} \\ &= G_{t_1} + G_{t_2}\int_{t_1}^{t}\bar{b}(X_{s_2})\mathrm{d}W_{s_2} + \int_{t_1}^{t}\int_{t_2}^{s_2}\bar{b}(X_{s_2})\bar{b}(X_{s_3})G_{s_3}\mathrm{d}W_{s_3}\mathrm{d}W_{s_2} \\ &\overset{\vdots}{=} \sum_{n=0}^{N} G_{t_{n+1}}\int_{t_1}^{t}\int_{t_2}^{s_2}\dots\int_{t_n}^{s_n}\bar{b}(X_{s_{n+1}})\mathrm{d}W_{s_{n+1}}\dots\bar{b}(X_{s_2})\mathrm{d}W_{s_2} + R_N, \end{aligned}$$

where

$$R_N = \int_{t_1}^{t}\int_{t_2}^{s_2}\dots\int_{t_{N+1}}^{s_{N+1}}\bar{b}(X_{s_{N+2}})G_{s_{N+2}}\mathrm{d}W_{s_{N+2}}\bar{b}(X_{s_{N+1}})\mathrm{d}W_{s_{N+1}}\dots\bar{b}(X_{s_2})\mathrm{d}W_{s_2}.$$

Clearly (using Lemma 1),

$$\mathbb{E}\left|R_N\right|^2 \le \exp(t\left\|\bar{b}\right\|_\infty^2)\frac{(t\left\|\bar{b}\right\|_\infty^2)^{N+1}}{(N+1)!}\underset{N\to+\infty}{\longrightarrow} 0.$$

□

Our aim now is to derive an integration by parts formula that will enable us to estimate p_{Y_t} (the density of Y_t under the probability $\mathbb{P}$) and its derivatives.

Let us first recall the following classic result on the integration by parts for SDE's with smooth coefficients (see [9]).

Theorem 1. *Let* $U_s := (X_s, Y_s)$ *defined by the system* (3)-(4)*, and assume* (A_{H}) *and the smoothness of the coefficients* $\beta, \sigma, \psi, \varphi$*. Let* $0 \le s_1 < s_2 \le t$*. Then,* $U_{s_2} \in \mathbb{D}^\infty$*, and, for every* $V \in \mathbb{D}^\infty$ *and* α *any multi-index of length* k*, there exists a random variable* $H^\alpha_{[s_1,s_2]}(U_{s_2}, V)$ *such that, for any* $\phi \in \mathcal{C}^\alpha_b$*,*

$$\mathbb{E}^{\mathcal{F}_{s_1}}\left[\phi^\alpha(U_{s_2})V\right] = \mathbb{E}^{\mathcal{F}_{s_1}}\left[\phi(U_{s_2})H^\alpha_{[s_1,s_2]}(U_{s_2}, V)\right].$$

Moreover, there exist positive constants C_k *(depending on the norms* $\|V\|_{k,p}$*) and* μ_k *such that (for* $p > 1$*)*

$$\left\|H^\alpha_{[s_1,s_2]}(U_{s_2}, V)\right\|_{\mathbf{L}^p} \le C_k(1 + \|(x,y)\|)^{\mu_k}(s_2 - s_1)^{-\mu_k}. \tag{5}$$

As a natural extension we define $H^\emptyset_{[s_1,s_2]}(U_{s_2}, V) \equiv 1$.

The following lemma is the important element in the proofs of our main results.

Lemma 3. *Let* $0 \le s_1 < s_2 \le s_3$ *and* $k \in \mathbb{N}^*$*, and assume* (A_{H})*. We have the following integration by parts formula that holds for any measurable function* $f \in \mathcal{C}^\alpha_b$*, and any random variable* $Z^{s_2}_{s_3} \in L^2(\Omega)$ *that is measurable w.r.t.* $\mathcal{F}^{s_2}_{s_3} := \sigma\{X_{s_2}, W_r - W_{s_2}, r \in [s_2, s_3]\}$*,*

$$\mathbb{E}^{\mathcal{F}_{s_1}}\left[\partial^\alpha_y f(Y_{s_3})Z^{s_2}_{s_3}\right] = \mathbb{E}^{\mathcal{F}_{s_1}}\left[f(Y_{s_3})Z^{s_2}_{s_3}\sum_{\beta\subseteq\alpha} R^\beta_{s_2,s_3}H^\beta_{[s_1,s_2]}((X_{s_2}, Y_{s_2}), 1)\right],$$

where $R^\beta_{s_2,s_3} \in \bigcap_{p>1} L^p(\Omega)$ *for any* $\beta \subseteq \alpha$*.*

Proof. For the sake of clarity, we first deal with the one-dimensional case and the first derivative: we want an integration by parts for $\mathbb{E}^{\mathcal{F}_{s_1}}[f'(Y_t)Z^{s_2}_{s_3}]$. $Y_{s_3}(s_2, x, y)$ denotes the second component of the stochastic flow defined by (X, Y) (starting from (x, y) at time s_2). We define

$$g(s_2, x, y) := \mathbb{E}^{\mathcal{F}_{s_2}}[f(Y_{s_3}(s_2, x, y))(\partial_y Y_{s_3}(s_2, x, y))^{-1}Z^{s_2}_{s_3}|X_{s_2} = x].$$

Then

$$\partial_y g(s_2,x,y) = \mathbb{E}^{\mathcal{F}_{s_2}}[f'(Y_{s_3}(s_2,x,y))Z^{s_2}_{s_3}|X_{s_2}=x]$$
$$- \mathbb{E}^{\mathcal{F}_{s_2}}[f(Y_{s_3}(s_2,x,y))(\partial_y Y_{s_3}(s_2,x,y))^{-2}\partial_y^2 Y_{s_3}(s_2,x,y)Z^{s_2}_{s_3}|X_{s_2}=x].$$

This leads to the following IBP:

$$\begin{aligned}
\mathbb{E}^{\mathcal{F}_{s_1}}[f'(Y_{s_3})Z^{s_2}_{s_3}] &= \mathbb{E}^{\mathcal{F}_{s_1}}[\mathbb{E}^{\mathcal{F}_{s_2}}[f'(Y_{s_3})Z^{s_2}_{s_3}]] \\
&= \mathbb{E}^{\mathcal{F}_{s_1}}[\partial_y g(s_2,X_{s_2},Y_{s_2})] \\
&+ \mathbb{E}^{\mathcal{F}_{s_1}}[f(Y_{s_3})(\partial_y Y_{s_3}(s_2,x,y))^{-2}\partial_y^2 Y_{s_3}(s_2,x,y)\Big|_{x=X_{s_2},y=Y_{s_2}} Z^{s_2}_{s_3}] \\
&= \mathbb{E}^{\mathcal{F}_{s_1}}[f(Y_{s_3})(\partial_y Y_{s_3}(s_2,x,y))^{-1}\Big|_{x=X_{s_2},y=Y_{s_2}} Z^{s_2}_{s_3} H_{[s_1,s_2]}(X_{s_2},Y_{s_2},1)] \\
&+ \mathbb{E}^{\mathcal{F}_{s_1}}[f(Y_{s_3})(\partial_y Y_{s_3}(s_2,x,y))^{-2}\partial_y^2 Y_{s_3}(s_2,x,y)\Big|_{x=X_{s_2},y=Y_{s_2}} Z^{s_2}_{s_3}].
\end{aligned}$$

Therefore defining $R^{(1)}_{s_2,s_3} = (\partial_y Y_{s_3}(s_2,x,y))^{-1}\Big|_{x=X_{s_2},y=Y_{s_2}}$ and

$$R^{\emptyset}_{s_2,s_3} = (\partial_y Y_{s_3}(s_2,x,y))^{-2}\partial_y^2 Y_{s_3}(s_2,x,y)\Big|_{x=X_{s_2},y=Y_{s_2}}$$

finishes the proof in this particular case.

In the multidimensional case, we first state the following identity for multiple chain-rule ($f^{(\alpha)}$ denotes the multiple derivative of f w.r.t. the y variable, with multi-index $\alpha = (\alpha_1,\ldots,\alpha_{|\alpha|})$):

Lemma 4. *Let α be any multi-index. Let $f:\mathbb{R}^d\to\mathbb{R}$, $\varphi:\mathbb{R}^d\to\mathbb{R}^d$ be functions which are $|\alpha| = k\in\mathbb{N}$ times continuously differentiable, such that $\nabla\varphi$ is invertible. Then,*

$$f^{(\alpha)}\circ\varphi = \sum_{\beta\in\mathcal{A}_k}\left\{(f\circ\varphi)(\det\nabla\varphi)^{-\gamma_\beta}P_\beta(\varphi)\right\}^{(\beta)}, \tag{6}$$

where (γ_β) are positive integer exponents and $P_\beta(\varphi)$ are polynomial functions of $\{(D^j\varphi);\ j\le k\}$.

Proof (of Lemma 4). First, one shows by induction on the length of α. Before this, one proves two basic properties. These are (we use the notation $(i) := (0, \dots 0, 1, 0, \dots, 0)$, 1 being in the i-th coordinate)

$$\begin{aligned} f^{(i)}\psi &= (f\psi)^{(i)} - f\psi^{(i)} \\ f^{(i)} \circ \varphi &= \sum_{k=1}^{d} (\nabla\varphi)^{-1}_{ik} (f \circ \varphi)^{(k)} \end{aligned} \tag{7}$$

In fact for $\alpha = (j)$ one easily proves that

$$(f \circ \varphi)^{(j)} = \sum_{k=1}^{d} \left\{ (f \circ \varphi (\nabla\varphi)^{-1}_{jk})^{(k)} - f \circ \varphi ((\nabla\varphi)^{-1}_{jk})^{(k)} \right\}. \tag{8}$$

Therefore (6) follows from

$$(\nabla\varphi)^{-1} = (\det \nabla\varphi)^{-1} \mathrm{Adj}(\nabla\varphi),$$

where $\mathrm{Adj}(\nabla\varphi)$ denotes the adjoint of $\nabla\varphi$. Now we assume that (6) is valid for any α with $|\alpha| \leq k$. Then in order to prove (6) for a multi-index of length $k+1$ we apply the formula (6) for $f^{(\alpha_1)}$ instead of f. Then applying successively (8) and (7), one obtains the result. □

Remark 1. Much more detailed information may be obtained on the polynomials appearing in (6) but as this will make the proof longer we do not mention it here.

We want to apply Lemma 4 with $d = d_2$ and $\varphi := Y_{s_3}(s_2, x, .)$ (therefore all derivatives are wrt the variables in y). It is known that $\det(\nabla_y Y_{s_3}(s_2, x, y))^{-1} \in L^p(\Omega)$ for any $p > 1$. In fact, $\det \nabla_y Y_{s_3}(s_2, x, y)$ satisfies a linear SDE with intial condition equal to 1: see [12] page 150.

For $\beta \in \mathcal{A}_k$, define

$$g_\beta(s_2, x, y) := \mathbb{E}^{\mathcal{F}_{s_2}}[f(Y_{s_3}(s_2, x, y)) R_{\beta,\alpha}(Y_{s_3}(s_2, x, y)) Z^{s_2}_{s_3} | X_{s_2} = x].$$

where (using the same notations as in Lemma 4)

$$R_{\beta,\alpha}(Y_{s_3}(s_2,x,y)) := (\det \nabla_y Y_{s_3}(s_2,x,y))^{-\gamma_\beta} P_\beta(Y_{s_3}(s_2,x,y)).$$

Then the function g_β is jointly measurable, regular in y and integrable wrt to the joint law of (X_{s_2}, Y_{s_2}). Therefore using a standard argument, one can approximate the function $g_\beta(s_2,\cdot,\cdot)$ (s_2 being fixed) in the $\mathbf{L}^2(\mu)$ sense w.r.t. μ, the joint law of (X_{s_2}, Y_{s_2}) (whose density under $\mathbb{Q}$ is smooth) using functions in $\mathcal{C}_b^\infty$. Therefore the argument that follows is applied first to the approximation and then the same result follows by taking limits. We have

$$\begin{aligned}
\mathbb{E}^{\mathcal{F}_{s_1}}[f^{(\alpha)}(Y_{s_3})Z_{s_3}^{s_2}] &= \mathbb{E}^{\mathcal{F}_{s_1}}[\mathbb{E}^{\mathcal{F}_{s_2}}[f^{(\alpha)}(Y_{s_3})Z_{s_3}^{s_2}]] \\
&= \sum_{\beta\in\mathcal{A}_k} \mathbb{E}^{\mathcal{F}_{s_1}}[g_\beta^{(\beta)}(s_2, X_{s_2}, Y_{s_2})] \qquad (9) \\
&= \sum_{\beta\in\mathcal{A}_k} \mathbb{E}^{\mathcal{F}_{s_1}}[g_\beta(s_2, X_{s_2}, Y_{s_2}) H^\beta_{[s_1,s_2]}(X_{s_2}, Y_{s_2}, 1)] \qquad (10) \\
&= \sum_{\beta\in\mathcal{A}_k} \mathbb{E}^{\mathcal{F}_{s_1}}[f(Y_{s_3}) R_{\beta,\alpha}(Y_{s_3}) Z_{s_3}^{s_2} H^\beta_{[s_1,s_2]}(X_{s_2}, Y_{s_2}, 1)].
\end{aligned}$$

As mentioned before, notice that in order to apply Theorem 1, we need to approximate the function g_β as noted above. Using the approximation argument and by passing to the limit for the equality (9)=(10) the result follows in the general case.

From the above equality one defines $R^\beta_{s_2,s_3} = R_{\beta,\alpha}(Y_{s_3})$. Its moment properties follow from the regularity of the process (X, Y). □

The following result is the integration by parts formula.

Theorem 2. *Assume* (A_{H})*, and let* $k \in \mathbb{N}$*.*

(i) For any multi-index α *of length* $|\alpha| = k$*, there exists a random variable*

H_t^α such that, for any $f \in \mathcal{C}_b^\alpha(\mathbb{R}^{d_2};\mathbb{R})$,

$$\mathbb{E}\left[f^\alpha(Y_t)G_t\right] = \mathbb{E}\left[f(Y_t)H_t^\alpha\right],$$

(ii) There exists a positive constant $C_k \equiv C_k(T)$ and μ_k such that,

$$\left(\mathbb{E}\left|H_t^\alpha\right|^2\right)^{\frac{1}{2}} \leq C_k t^{-\mu_k}.$$

It follows that, for any $f \in \mathcal{C}_b^\alpha(\mathbb{R}^{d_2};\mathbb{R})$ and multi-index α such that $|\alpha| = k$,

$$\left|\mathbb{E}\left[f^\alpha(Y_t)G_t\right]\right| \leq C_k t^{-\mu_k}\left(\mathbb{E}\left|f(Y_t)\right|^2\right)^{\frac{1}{2}}. \tag{11}$$

Proof of Theorem 2. Let $t_0 = t > t_1 > \cdots > t_n > \cdots > 0$ (to be chosen later), and recall that $Z_t^n = \int_{t_1}^t \int_{t_2}^{s_2} \ldots \int_{t_n}^{s_n} \bar{b}(X_{s_{n+1}})\mathrm{d}W_{s_{n+1}} \ldots \bar{b}(X_{s_2})\mathrm{d}W_{s_2}$, for $n \geq 1$, and $Z_t^0 = 1$. It is clear that Z_t^n is $\mathcal{F}_t^{t_n}$-measurable.

From Lemma 2,

$$\mathbb{E}\left[f^\alpha(Y_t)G_t\right] = \mathbb{E}\left[\sum_{n=0}^{\infty} G_{t_{n+1}} f^\alpha(Y_t) Z_t^n\right].$$

In order to exchange $\mathbb{E}$ and $\sum_{n=0}^\infty$, let us study $\sum_{n=0}^\infty |e_n|$, where $e_n := \mathbb{E}\left[G_{t_{n+1}} f^\alpha(Y_t) Z_t^n\right] = \mathbb{E}\left[G_{t_{n+1}} \mathbb{E}^{\mathcal{F}_{t_{n+1}}}\left[f^\alpha(Y_t)Z_t^n\right]\right]$. From Lemma 3 (recall that Z_t^n are $\mathcal{F}_t^{t_n}$-measurable), one has using Lemma 3, with $s_3 = t$, $s_2 = t_n$ and $s_1 = t_{n+1}$

$$\mathbb{E}^{\mathcal{F}_{t_{n+1}}}\left[f^\alpha(Y_t)Z_t^n\right] = \mathbb{E}^{\mathcal{F}_{t_{n+1}}}\left[f(Y_t)Z_t^n \sum_{\beta\in\mathcal{A}_k} R_{t_n,t}^\beta H_{[t_{n+1},t_n]}^\beta((X_{t_n},Y_{t_n}),1)\right],$$

and then

$$e_n = \mathbb{E}\left[f(Y_t)Z_t^n \sum_{\beta\in\mathcal{A}_k} R_{t_n,t}^\beta H_{[t_{n+1},t_n]}^\beta((X_{t_n},Y_{t_n}),1)G_{t_{n+1}}\right].$$

Furthermore,

$$|e_n| \leq \|f(Y_t)\|_{\mathbf{L}^2} \sum_{\beta\in\mathcal{A}_k} \left\|H^{\beta}_{[t_{n+1},t_n]}((X_{t_n},Y_{t_n}),1)\right\|_{\mathbf{L}^6} \left\|Z^n_t R^{\beta}_{t_n,t}\right\|_{\mathbf{L}^6} \left\|G_{t_{n+1}}\right\|_{\mathbf{L}^6} \tag{12}$$

$$\leq \|f(Y_t)\|_{\mathbf{L}^2} \frac{C_k}{(t_n - t_{n+1})^{\mu_k}} \frac{(Ct\left\|\bar{b}\right\|^2_{\infty})^{n/2}}{(n-1)!^{1/6}} \exp\left(\frac{5}{2} t_{n+1} \left\|\bar{b}\right\|^2_{\infty}\right),$$

where we have used, respectively, estimate (5) from Theorem 1, classical estimate (BDG inequality) for Z^n_t and Lemma 1 for $G_{t_{n+1}}$.

By choosing $t_n := \frac{t}{2^n}, \forall n \geq 0$, it is clear that $\sum_{n=0}^{\infty} |e_n| < \infty$, and

$$|\mathbb{E}\left[f^{\alpha}(Y_t)G_t\right]| = \left|\sum_{n=0}^{\infty} e_n\right| \leq \sum_{n=0}^{\infty} |e_n| \leq C_k t^{-\mu_k} \|f(Y_t)\|_{\mathbf{L}^2}.$$

Thus, we have established that

$$\mathbb{E}\left[f^{\alpha}(Y_t)G_t\right] = \mathbb{E}\left[f(Y_t)H^{\alpha}_t\right],$$

with

$$H^{\alpha}_t := \sum_{n=0}^{\infty} \sum_{\beta\in\mathcal{A}_k} G_{t_{n+1}} Z^n_t R^{\beta}_{t_n,t} H^{\beta}_{[t_{n+1},t_n]}((X_{t_n},Y_{t_n}),1),$$

$$\left(\mathbb{E}\left|H^{\alpha}_t\right|^2\right)^{\frac{1}{2}} \leq C_k t^{-\mu_k}.$$

□

4. Bounds for the density of Y_t in (2)

In this section, we go back to the consideration of the system (1)-(2). To obtain the upper and lower bounds for the density of Y we will link this system to the system (3)-(4) by using the Girsanov's theorem. In fact, let

$\beta = 0$ and $\bar{b} = -\begin{pmatrix} \sigma_1^*(\sigma_1\sigma_1^*)^{-1}b \\ 0 \end{pmatrix}$ and $\sigma = \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix}$. Then by Girsanov's theorem and

$$\begin{pmatrix} B \\ B' \end{pmatrix} = W + \int_0^{\cdot} \begin{pmatrix} \bar{b}(X_s) \\ 0 \end{pmatrix} ds$$

is a Brownian motion under $\mathbb{P}$. Then the solution of the system (3)-(4) under $\mathbb{P}$ is equal in law to the weak solution of the system (1)-(2).

4.1. Upper bounds

Theorem 3. *Let Y be defined by (2), and assume (A_σ) and (A_{H}) with $\beta \equiv 0$. Then*

1. *The density p_{Y_t} (under the initial probability $\mathbb{P}$) of Y_t exists and is smooth.*
2. *Assume moreover (A_φ) and that φ and ψ are bounded. Then, for any multi-index α of length $|\alpha| = k$,*

$$\left|p_{Y_t}^{\alpha}(y_0)\right| \leq \bar{C}_k t^{-\mu_k} \exp\left(-\frac{|y_0 - y|^2}{\bar{C}t}\right).$$

Proof of Theorem 3. The first statement follows from Theorem 2, using $f(y) := e^{i<\theta,y>}$, and the characterization of the existence and smoothness of the density of a random variable in terms of the integrability of its characteristic function (this statement is the main result in [8]).

The second statement immediately follows from (11) with $f(y) := \mathbb{1}_{[y_0,+\infty)}(y)$, using the classical exponential martingale inequality (see (A.5) in [10]) after a Girsanov's transform to get rid of the drift in (4) (for which we need the extra ellipticity assumption on φ). □

4.2. Lower bound

Theorem 4. *Assume (A_σ) and (A_φ), and let $T > 0$. Then, there exists a positive constant $\underline{C} \equiv \underline{C}(T)$ such that, for all $t \in (0, T]$,*

$$p_{Y_t}(y_0) \geq \underline{C} t^{-\frac{d_2}{2}} \exp\left(-\frac{|y_0 - y|^2}{\underline{C} t}\right).$$

Proof of Theorem 4. Let $t_0 = t$, $t_1 \in [\frac{t}{2}, t)$ to be chosen later, and $t_n := \frac{t}{2^n}, \forall n \geq 2$. Besides, we use the same notations as in the proof of Theorem 2, with $f := \mathbb{1}_{[y_0,+\infty)}$ and $\alpha := (1, \ldots, 1)$. One has

$$p_{Y_t}(y_0) = \mathbb{E}\left[\delta_{y_0}(Y_t)G_t\right] = \mathbb{E}\left[\delta_{y_0}(Y_t)G_{t_1}\right] + \sum_{n=1}^{\infty} e_n.$$

By the triangular inequality,

$$\mathbb{E}\left[\delta_{y_0}(Y_t)G_t\right] \geq \mathbb{E}\left[\delta_{y_0}(Y_t)G_{t_1}\right] - \sum_{n=1}^{\infty} |e_n|. \tag{13}$$

- First term:

Let $p > 1$ to be chosen later too, and q such that $\frac{1}{p} + \frac{1}{q} = 1$. Using Hölder's inequality, one has

$$\begin{aligned}\mathbb{E}\left[\left(\mathbb{E}^{\mathcal{F}_{t_1}}\left[\delta_{y_0}(Y_t)\right]\right)^{\frac{1}{p}}\right] &= \mathbb{E}\left[\left(\mathbb{E}^{\mathcal{F}_{t_1}}\left[\delta_{y_0}(Y_t)\right]\right)^{\frac{1}{p}} G_{t_1}^{\frac{1}{p}} G_{t_1}^{-\frac{1}{p}}\right] \\ &\leq \left(\mathbb{E}\left[\delta_{y_0}(Y_t)G_{t_1}\right]\right)^{\frac{1}{p}} \left(\mathbb{E}\left[G_{t_1}^{-\frac{q}{p}}\right]\right)^{\frac{1}{q}}.\end{aligned}$$

Hence, using twice the classical lower bound for the transition density of SDE with uniformly elliptic diffusion coefficient (see e.g. [9], [6], [2] and [7]), and using Lemma 1,

$$
\begin{aligned}
&\mathbb{E}\left[\delta_{y_0}(Y_t)G_{t_1}\right] \geq \left(\mathbb{E}\left[G_{t_1}^{-\frac{q}{p}}\right]\right)^{-\frac{p}{q}} \left(\mathbb{E}\left[\left(\mathbb{E}^{\mathcal{F}_{t_1}}\left[\delta_{y_0}(Y_t)\right]\right)^{\frac{1}{p}}\right]\right)^p \\
&\geq \exp\left(-(\frac{q}{2p}+\frac{1}{2})t_1 \left\|b\right\|_\infty^2\right) \underline{C} \left(\mathbb{E}\left[(2\pi(t-t_1))^{-\frac{d_2}{2p}} \exp\left(-\frac{|y_0-Y_{t_1}|^2}{p\underline{C}(t-t_1)}\right)\right]\right)^p \\
&= \exp\left(-(\frac{q}{2p}+\frac{1}{2})t_1 \left\|b\right\|_\infty^2\right) \underline{C}(t-t_1)^{\frac{(p-1)d_2}{2}} \left(\mathbb{E}\left[(2\pi(t-t_1))^{-\frac{d_2}{2}} \exp\left(-\frac{|y_0-Y_{t_1}|^2}{p\underline{C}(t-t_1)}\right)\right]\right)^p \\
&\geq \exp\left(-(\frac{q}{2p}+\frac{1}{2})t \left\|b\right\|_\infty^2\right) (t-t_1)^{\frac{(p-1)d_2}{2}} Ct^{-\frac{pd_2}{2}} \exp\left(-\frac{|y_0-y|^2}{Ct}\right)
\end{aligned}
$$

(here, we have used the formula for the convolution of two Gaussian densities).

By choosing $p := 1+\frac{1}{2d_2}$, one obtains (we bound $\exp\left(-(\frac{q}{2p}+\frac{1}{2})t\left\|b\right\|_\infty^2\right)$ from below by $\exp\left(-(\frac{q}{2p}+\frac{1}{2})T\left\|b\right\|_\infty^2\right)$)

$$
\mathbb{E}\left[\delta_{y_0}(Y_t)G_{t_1}\right] \geq (t-t_1)^{\frac{1}{4}} \underline{C} t^{-\frac{1}{4}-\frac{d_2}{2}} \exp\left(-\frac{|y_0-y|^2}{\underline{C}t}\right). \tag{14}
$$

- Second term:

From (12) with $f(y) := \mathbb{1}_{[y_0,+\infty)}(y)$ and $\alpha := (1,\dots,1)$, and for $n \geq 1$, one has (note that $\mu_{d_2} = \frac{d_2}{2}$ in the elliptic case)

$$
\begin{aligned}
|e_n| &\leq \left\|\mathbb{1}_{[y_0,+\infty)}(Y_t)\right\|_{\mathbf{L}^2} \left\|H^{\alpha}_{[t_{n+1},t_n],y}\right\|_{\mathbf{L}^6} \left\|Z^n_t R^n_t\right\|_{\mathbf{L}^6} \left\|G_{t_{n+1}}\right\|_{\mathbf{L}^6} \\
&\leq \bar{C}\exp\left(-\frac{|y_0-y|^2}{\bar{C}t}\right) \frac{C_1}{(\frac{t}{2^{n+1}})^{\frac{d_2}{2}}} \frac{(t-t_1)^{\frac{1}{2}}(Ct\left\|\bar{b}\right\|_\infty^2)^{\frac{n-1}{2}}}{(n-1)!^{1/6}3^{\frac{n-1}{6}}} \exp\left(\frac{5}{2}t_{n+1}\left\|\bar{b}\right\|_\infty^2\right).
\end{aligned}
$$

Hence,

$$
\sum_{n=1}^{\infty} |e_n| \leq (t-t_1)^{\frac{1}{2}} \bar{C} t^{-\frac{d_2}{2}} \exp\left(-\frac{|y_0-y|^2}{\bar{C}t}\right). \tag{15}
$$

- Conclusion:

From (14) and (15), one can choose $t_1 \in [\frac{t}{2}, t)$ such that

$$
\sum_{n=1}^{\infty} |e_n| \leq \frac{1}{2}\mathbb{E}\left[\delta_{y_0}(Y_t)G_{t_1}\right].
$$

Indeed, this is achieved by setting

$$t_1 := t - \min\{\frac{t}{2}, h(t)\},$$

where

$$h(t) := \frac{1}{16}\left(\frac{\underline{C}}{\bar{C}}\right)^4 t^{-1}\exp\left(-4(\underline{C}^{-1} - \bar{C}^{-1})\frac{|y_0 - y|^2}{t}\right).$$

Hence, from (13), one has

$$\mathbb{E}\left[\delta_{y_0}(Y_t)G_t\right] \geq \frac{1}{2}\mathbb{E}\left[\delta_{y_0}(Y_t)G_{t_1}\right].$$

Using the lower bound in (14),

$$\begin{aligned}
&\mathbb{E}\left[\delta_{y_0}(Y_t)G_t\right] \\
&\geq \begin{cases} \frac{1}{2}\underline{C}(\frac{t}{2})^{1/4}t^{-1/4-d_2/2}\exp\left(-\frac{|y_0-y|^2}{\underline{C}t}\right), \text{ if } t_1 = \frac{t}{2}; \\ \frac{1}{2}\underline{C}(\frac{1}{16})^{\frac{1}{4}}\frac{\underline{C}}{\bar{C}}t^{-1/4}\exp\left(-(\underline{C}^{-1} - \bar{C}^{-1})\frac{|y_0-y|^2}{t}\right)t^{-1/4-d_2/2}\exp\left(-\frac{|y_0-y|^2}{\underline{C}t}\right), \\ \quad \text{if } t_1 = t - h(t). \end{cases}
\end{aligned}$$

Therefore,

$$\mathbb{E}\left[\delta_{y_0}(Y_t)G_t\right] \geq Ct^{-\frac{d_2}{2}}\exp\left(-C\frac{|y_0 - y|^2}{t}\right).$$

□

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