Models for insider trading with finite utility*

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Abstract

In these notes we review through simple examples recent results on models for insider trading based on the theory of enlargement of filtrations and on anticipating calculus. In particular, we concentrate on the case of strong type of insiders. That is, insiders that have additional information in the a.s. sense. We explain how to treat the utility maximization problem for insiders in order to obtain models where the utility is finite. In the anticipating framework, we introduce models where the signal of the insider can not be revealed to the small trader even though the insider has an effect on the price (large trader effect).

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1 Introduction

These notes are result of a short course delivered as part of a "cours de formation pour la recherche" at CREST during november-december of 2004. The goal of these lectures were to introduce young researchers and Ph D students to this area of research. I have decided to keep the spirit of the lectures in these notes although some of the concepts and results may seem to be repetitive through the text or may be well known to the advanced researcher.

The format throughout the text uses examples with concrete answers, in order to give the basic ideas behind the general results and the proofs, rather than the full generality and (sometimes non-trivial) technicalities that can be seen in the research papers mentioned in the references. We do this on the risk of becoming overly trivial but hopefully very clear in what the goals and the used techniques are.

These notes do not assume any previous knowledge of mathematical finance and they are largely self-contained. They do assume knowledge of basic stochastic calculus. In particular, I have decided to introduce discrete time models as an approximation of their continuous counterparts for two reasons: The first is to introduce various concepts through their discrete time counterparts. Most importantly, the second reason is to introduce the anticipating type models where the discrete counterpart becomes essential to understand the meaning and the difficulties in this setup.

These notes are designed to be read in two levels. The objective of the main line and first level of discussion is to give a brief overview of the theory. In a second level, we give more details and properties trough exercises which may assume more knowledge of stochastic calculus. Most of them are solved at the end of this survey. While I do not considered all of them to be trivial, I think they are useful for the person that wants to deepen his/her knowledge about this area of study. Still, part of the material appears here for the first time.

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Keywords: Asymmetric markets, utility maximization, enlargement of filtrations.

I have not tried my outmost to add historical notes or to put rightful due to each statement appearing in the text and probably many references may be missing. I apologize for all historical imprecisions and my ignorance in the exposition.

I would like to thank Nizar Touzi for inviting me to deliver these lectures and for his trust in my completing these notes. Also thanks to all the people who participated in the course with interesting comments, remarks and questions. In particular to Monique Jeanblanc for her careful review of a previous version of these notes.

This article deals with the modelling of financial markets where agents may have different information. That is, agents observe the same prices but their filtration may differ due to some extra information that has been obtained in some other fashion. In such a setting the natural mathematical question is what are the basic process properties when such change of filtration takes place.

This will take us naturally to the enlargement of filtration problem: That is, does a Wiener process remain a semimartingale in a larger filtration than the one generated by the process itself? In general, the answer to this question is negative but Jacod's theorem will tell us some cases when the answer is affirmative. The restriction of this theorem is that the only information allowed is the one given by a single random variable. So one of the goals of the present lectures is to try to show a possible route leading to the introduction of flows of information as possible differences between agents.

Another issue that we want to treat is that from the financial modelling point of view, there are insiders of two types. The first, called unlawful insiders are the agents that trade using the information they possess without any risk. In our theory these insiders will become infinitely rich. This situation corresponds to the typical application of Jacod's theorem.

The second, are insiders which try to hide the fact that they possess this perfect information or either their information does have some risks and therefore our search will be for models where utility is finite. We will try to concentrate on this type of insiders. Nevertheless not only for historical but rather for educational reasons we will have to review the theory of unlawful insiders.

We will first introduce the logarithmic utility optimization problem for the classical investor (the so called Merton problem), then we will introduce a first example of insider which knows exactly the price at the end of a time interval. This example is a first example of insider where the agent will make an infinite amount of money and there is arbitrage in the model.

Then we will give various alternatives in order to obtain finite utilities in models based on the Wiener process. Next we will show that without these modifications the optimal logarithmic utility of the insider is finite in the case of markets with jumps. This effect is due to the high risk imbedded in such models.

Next we will move into the study of insiders as large traders which will need in a natural manner the introduction of anticipating integrals. In particular, we will use the notion of forward integrals as defined by Russo-Vallois. With these models we will show that there is the possibility that the insider influences the prices but still the price information does not reveal its information to the traders.

2 The small investor problem

We will mainly consider the one dimensional setting to simplify the notation unless stated otherwise. The set-up will be the same for the discrete and continuous time cases. For this, we start considering a one dimensional Wiener process $W = \{W(t), 0 \le t \le T\}$ and a compound Poisson process $Z(t) = \sum_{i=0}^{N(t)} X_i$ where N is a simple Poisson process with intensity λ and X_i, $i = 1, ...$ are i.i.d. r.v.'s with density function f such that $E(e^X) < \infty$. These processes are defined on a complete probability space $(\Omega, \mathcal{F}, P)$. We denote by $(\mathcal{F}_t)_{t\in[0,T]}$ the natural filtration generated by the Wiener and the compound Poisson process and the sets of P-measure zero.

Suppose that $\{S(t); t \in [0, T]\}$ is a positive stochastic process defined on $(\Omega, \mathcal{F}, P)$. This process models the stock price. In the discrete time setting we suppose that trading only takes place at discrete times $0 = t_0 < ... < t_n = T$. Then the model for the price will be

$$\log(S(t_{i+1})/S(t_i)) = \mu(t_i)(t_{i+1} - t_i) + \sigma(t_i)(W(t_{i+1}) - W(t_i)) + (Z(t_{i+1}) - Z(t_i)) \tag{1}$$

where $\mu(t)$ and $\sigma(t) > c > 0$ are two $(\mathcal{F}_t)_{t\in[0,T]}$-adapted cádlág bounded processes.

The small agent who has as information at time t all the $\mathcal{F}_t$-measurable r.v.'s makes his investment decisions based on these r.v.'s. Another agent, called the insider, observes the same process S but he/she holds as information a bigger filtration $\mathcal{G}$ than the small agent (that is, $\mathcal{G} \supseteq \mathcal{F}$). Both investors are price takers. That is, none of them can change the value of S by using his/her trading strategies.

Nevertheless, insiders are usually large traders as they can change the price dynamics with his/her trades. This is another aspect of this area of research which is of interest. We neglect this aspect in the first part of our exposition (for more on this, see Section 11)

We start giving a brief exposition of the classical Merton problem for the small investor. That is, what is the optimal portfolio policy that a small investor should choose in order to maximize his/her utility.

Suppose that the small investor starts with a wealth of V_0 units of money and chooses a $\mathcal{F}$-adapted policy $p(t) = (p_0(t), p_1(t))$, where $p_0(t)$ denotes the money invested at his/her bank account at time t that gives an interest rate of r, and $p_1(t)$ denotes the number of shares held at time t. Suppose that we only allow trading at time points $\{0 = t_0 < t_1 < ... < t_n = T\}$. That is, p is a step process

$$p(t) = \sum_{i=0}^{n-1} p(t_i) 1_{(t_i, t_{i+1}]}(t)$$

An investment policy p is allowed for this investor, besides other technical conditions to be established later, if it is self-financing. That is, if the investor can change $p(t_i)$ to $p(t_{i+1})$ by using the proceedings of selling all his assets. Therefore, the self-financing condition is $p_0(t_0) + p_1(t_0)S(t_0) = V_0$ and for $i = 0, ..., n-2$

$$p_0(t_{i+1}) + p_1(t_{i+1})S(t_{i+1}) = p_0(t_i)e^{r(t_{i+1}-t_i)} + p_1(t_i)S(t_{i+1}). \tag{2}$$

That is, every sequence of possible vectors $p(t_i)$; $i = 0, ..., n-1$ have to satisfy the above $n-1$ equations. In fact with these restrictions the investor has only the freedom to choose $p_1(t_i)$, $i = 0, ..., n-1$. All the other variables $p_0(t_i)$, $i = 0, ..., n-1$ are determined by the self financing equations (2). In fact, we have the following result which is easy to prove.

Lemma 1 *Given an initial wealth V_0, then for every sequence of real values $p_1(t_i)$, $i = 0, ..., n-1$ there exists a unique sequence of values $p_0(t_i)$, $i = 0, ..., n-1$, such that the vectors $p(t_i)$ for $i = 0, ..., n-1$ form a self-financing policy.*

Now we can define the wealth of the investor as $\bar{V}(t) \equiv V^n(t) = p_0(t) + p_1(t)S(t)$ where $p_0(t) = p_0(t_i)e^{r(t-t_i)}$ for $t_i \leq t < t_{i+1}$. With this condition one can rewrite the wealth as the sum of earnings/losses for each interval $[t_i, t_{i+1}]$. That is, if the initial wealth of the small investor is V_0 then

$$\begin{aligned}
\bar{V}(t_i) &= V_0 + \sum_{j=0}^{i-1} \left(\bar{V}(t_{j+1}) - \bar{V}(t_j)\right) \qquad (3)\\
&= V_0 + \sum_{j=0}^{i-1} \left(p_0(t_j)\left(e^{r(t_{j+1}-t_j)} - 1\right) + p_1(t_j)\left(S(t_{j+1}) - S(t_j)\right)\right) \\
&= \bar{V}(t_{i-1}) + p_0(t_{i-1})\left(e^{r(t_i - t_{i-1})} - 1\right) + p_1(t_{i-1})\left(S(t_i) - S(t_{i-1})\right)
\end{aligned}$$

Note that the above formula is valid without any assumption on the model for S. The next restriction we will impose on the actions of the investor is that we will not allow uncovered losses (sometimes called the no-borrowing condition or tameness condition when written in a more general form). That is, p should satisfy that $\bar{V}(t) \geq 0$ for all $t \in [0, T]$.

Exercise 2 *Suppose that the support of the law of $S(t)$ is $(0,\infty)$ for all $t \in [0,T]$. Prove that $\bar{V}(t) \geq 0$ for all $t \in [0,T]$ if and only if $p_0(t) \geq 0$ and $p_1(t) \geq 0$ for all $t \in [0,T]$.*

This exercise proves that the no-borrowing restriction is quite strong for markets with transactions in discrete time. Nevertheless this will not be so in continuous time.

Now we have to discuss if there is the possibility for the small trader to make money without any risk. For this, we define discounted wealth process $\hat{V}(t_i) = e^{-rt_i}\bar{V}(t_i)$, $i = 0, ..., n$ and the discounted stock price process $\hat{S}(t) = e^{-rt}S(t)$.

Definition 3 *We say that there is the possibility of arbitrage for the small trader is there exists a policy $p = (p_0, p_1)$ such that $P(\hat{V}(T) \geq V_0) = 1$ and $P(\hat{V}(T) > V_0) > 0$.*

So far we have not made any use of the particular form of S. Now we will use equation (1) to prove that there is no possibility for arbitrage.

Proposition 4 *There is no possibility of arbitrage for the small trader in the space of admissible strategies*

$$\mathcal{A}(T) = \{p(t) = \sum_{i=0}^{n-1} p(t_i)1(t_i \leq t < t_{i+1}); \textit{adapted } E\left[|p_0(t_i)| + |p_1(t_i)|\right] < \infty,\ i = 0, ..., n-1\}.$$

Proof. First note that for $p \in \mathcal{A}(T)$, we have that $E\left|\widehat{V}(t_i)\right| < \infty$ for all $i = 1, ..., n$. Given that $(W(t_{i+1}) - W(t_i))_{i=0}^{n-1}$ is a Gaussian vector we can perform the following change of measure

$$\frac{dQ}{dP} = \exp\left(-\sum_{i=0}^{n-1} \frac{2\theta(t_i)(W(t_{i+1}) - W(t_i)) + \theta(t_i)^2(t_{i+1} - t_i)}{2}\right).$$

with

$$\theta(t_i) = \sigma(t_i)^{-1}\left(\mu(t_i) - r + \lambda(E(e^X) - 1)\right).$$

With this change we have that $\hat{V}$ is a discrete time martingale in $(\Omega, \mathcal{F}, Q)$. That is,

$$E_Q[\hat{V}(t_{i+1})\big/ \mathcal{F}_{t_i}] = \hat{V}(t_i).$$

In fact,

$$E_Q[\hat{V}(t_{i+1})\big/ \mathcal{F}_{t_i}] = \hat{V}(t_i)\left(1 + \pi(t_i)\left(E_Q\left[\frac{\hat{S}(t_{i+1})}{\hat{S}(t_i)}\Big/ \mathcal{F}_{t_i}\right] - 1\right)\right).$$

Furthermore using the change of variables theorem and the moment function for a Gaussian r.v. and a compound Poisson r.v. we have that

$$\begin{aligned}
&E_Q\left[\frac{\hat{S}(t_{i+1})}{\hat{S}(t_i)}\Big/ \mathcal{F}_{t_i}\right] \\
&= E\left[\exp\left(-\frac{\sigma(t_i)^2(t_{i+1} - t_i)}{2} + \sigma(t_i)(\tilde{W}(t_{i+1}) - \tilde{W}(t_i)) - \lambda(E(e^X) - 1)(t_{i+1} - t_1) + Z(t_{i+1}) - Z(t_i)\right)\right] \\
&= 1.
\end{aligned}$$

Here $\tilde{W}$ denotes a new Gaussian r.v. obtained after the change of variables.

Therefore $e^{-rT}E_Q[\bar{V}(T)] = E_Q[\hat{V}(t_n)] = e^{-rT}V_0$. Therefore arbitrage is not possible because otherwise we will have that $E_Q(\bar{V}(T)) > V_0 e^{rT}$. ■

The above proof is just a discrete version of a similar proposition in continuous time. In fact, we propose the following alternative proof.

Exercise 5 *Write the limit of the above measure Q as the partition size* $\max\{t_{i+1}-t_i; i=0,...,n-1\}$ *goes to zero. Use Girsanov's theorem to find the equation satisfied by* $\log(\hat{S}(t))$. *Prove that* $\hat{S}$ *is a martingale under this measure.*

Exercise 6 *Write an argument using stopping times to prove the above proposition without the assumption* $E[|p_0(t_i)|+|p_1(t_i)|]<\infty,\ i=0,...,n-1$.

Exercise 7 *Show that the change of measure used in the proof of Proposition 4 is not unique.*

Define $T_0=\inf\{t\in[0,T];\bar{V}(t)=0\}$. Then $\bar{V}(s)=0$ for all $s\geq T_0$. Suppose that $\bar{V}(s)>0$ for $s\leq t_{i-1}$ then we introduce the change of variables $\pi(t)=\frac{p_1(t)S(t)}{V(t)}$ for $t\leq t_{i-1}$. The variable π represents the fraction of wealth invested in stocks. $1-\pi$ represents the proportion invested in the bank account. Negative values of π are in general interpreted as loans of shares to the investor to be invested in the bank account.

This reduction of variables means that the investor can choose the values of $\pi(t_i)$ for $i=0,...,n-1$ so as to maximize his wealth. With these changes of variables one has that

$$\bar{V}(t_i)=\bar{V}(t_{i-1})\left(1+(1-\pi(t_{i-1}))\left(e^{r(t_i-t_{i-1})}-1\right)+\pi(t_{i-1})\left(\frac{S(t_i)}{S(t_{i-1})}-1\right)\right). \tag{4}$$

Therefore the no-borrowing condition takes the form

$$\pi(t_{i-1})\left(\frac{S(t_i)}{S(t_{i-1})}-e^{r(t_i-t_{i-1})}\right)\geq -e^{r(t_i-t_{i-1})}.$$

One has trivially that if $\pi(t)\in[0,1]$ for all $t\in[0,T]$ (no borrowing of stocks or from the bank is allowed. Try to link this with Exercise 2) then the no-borrowing condition is always satisfied.

Equation (4) leads to a linear difference equation that can be solved by induction. One obtains that

$$\bar{V}(t_i)=V_0\prod_{j=0}^{i-1}\left(1+(1-\pi(t_j))\left(e^{r(t_{j+1}-t_j)}-1\right)+\pi(t_j)\left(\frac{S(t_{j+1})}{S(t_j)}-1\right)\right).$$

$\hat{V}$ satisfies the following linear difference equation which can also be solved by induction:

$$\hat{V}(t_i)=V_0+\sum_{j=0}^{i-1}\hat{V}(t_j)\pi(t_j)\left(\frac{\hat{S}(t_{j+1})}{\hat{S}(t_j)}-1\right) \tag{5}$$

$$\hat{V}(t_i)=V_0\prod_{j=0}^{i-1}\left(1+\pi(t_j)\left(\frac{\hat{S}(t_{j+1})}{\hat{S}(t_j)}-1\right)\right).$$

We will now take limits in the above arguments and in the same way as it is done in any introductory course on stochastic calculus, one obtains the Itô stochastic integral and the above arguments about the non-existence of arbitrage can be carried out similarly.

In fact, one has that S is a semimartingale and that if we consider π to be a predictable process then one can consider the limit of the equation (5). We obtain the wealth equation for continuous time trading (we will return to this issue in sections 9 and 12). This gives

$$\hat{V}(t)=V_0+\int_0^t\frac{\pi(s-)\hat{V}(s-)}{\hat{S}(s-)}d\hat{S}(s). \tag{6}$$

Note that there is a slight abuse of notation as we are using the same notation for the approximative wealth in equation (5) but it should be clear from the context if we are talking about the discrete time approximation or the continuous one above.

Exercise 8 *Prove that the limit of the sequence* $\hat{V}$ *is the solution of the above linear equations under enough conditions on* S *and* π.

This is a linear equation in V which can be explicitly solved. In the case that S has jumps the next calculation is a bit different (see Section 9). For this reason and to compute explicitly an optimal portfolio we take a particular continuous model for the stock price S. In what follows we further simplify our model and assume that the stock price is the geometric Brownian motion given by

$$S(t) = S(0)\exp\left(\left(\int_0^t \mu(s) - \frac{1}{2}\sigma^2(s)\right)ds + \int_0^t \sigma(s)dW_s\right)$$

where μ is the mean rate of return and $\sigma \geq c > 0$ is the volatility of the stock price which are uniformly bounded and adapted to the filtration generated by the Wiener process completed with respected to P. The process S also satisfies the linear equation

$$S(t) = S(0) + \int_0^t \mu(s)S(s)ds + \int_0^t \sigma(s)S(s)dW(s).$$

In such a case, equation (5) becomes

$$\hat{V}(t) = V_0 + \int_0^t \hat{V}(s)(\mu(s) - r)\pi(s)ds + \int_0^t \sigma(s)\pi(s)\hat{V}(s)dW(s). \tag{7}$$

This stochastic linear equation has an explicit solution given by

$$\hat{V}(t) = V_0 \exp\left(\int_0^t \left((\mu(s) - r)\pi(s) - \frac{1}{2}\sigma^2(s)\pi(s)^2\right)ds + \int_0^t \sigma(s)\pi(s)dW(s)\right). \tag{8}$$

Note that the restriction on the policy $\pi \in [0, 1]$ has disappeared. For any value of π, $V_t > 0$ for any $t \in [0, T]$. This will be further explained in section 9.

The small investor desires to optimize his/her portfolio policy by considering the problem

$$\max_{\pi \in \mathcal{F}} E\left[\log(\hat{V}(t))\right] \tag{9}$$

where the logarithmic function is used as a utility function. This particular utility function is used because the calculations to follow will become explicit. Note the inverse relationship between the logarithmic and the exponential function in (8) and (9). In what follows $\pi \in \mathcal{F}$ is a shorthand for π is $\mathcal{F}$-predictable process.

Other utility functions can also be used (see Exercise 12). Also note that if in the discrete time model we had $T_0 < t$ with positive probability then the $E(\log(\hat{V}(t))$ is defined as $-\infty$ therefore not giving the optimal value (Exercise: propose a portfolio with bigger value than $-\infty$ and compute its expected utility explicitly.). We now give an informal argument to obtain the optimal portfolio. A formal approach is given in the exercises.

To find the optimal portfolio solving the Merton problem (9) we note that if $E\int_0^t \pi(s)^2 ds < \infty$ then

$$\begin{aligned} E\left[\log(\hat{V}(t))\right] &= \log(V_0) + E\left[\int_0^t \left((\mu(s) - r)\pi(s) - \frac{1}{2}\sigma^2(s)\pi(s)^2\right)ds + \int_0^t \sigma(s)\pi(s)dW(s)\right] \\ &= \log(V_0) + E\left[\int_0^t \left((\mu(s) - r)\pi(s) - \frac{1}{2}\sigma^2(s)\pi(s)^2\right)ds\right], \end{aligned}$$

as $E\left[\int_0^t \sigma(s)\pi(s)dW(s)\right] = 0$. This expression says that the utility of the portfolio π is determined by the value of $f_s(\pi) = (\mu(s) - r)\pi - \frac{1}{2}\sigma^2(s)\pi^2$. This is a strictly concave function with domain $\pi \in \mathbb{R}$. Therefore its maximum value is obtained by differentiation. That is, $f_s'(\pi) = (\mu(s) - r) - \sigma^2(s)\pi = 0$ and therefore the maximum value is given by $\pi^*(s) = \frac{\mu(s)-r}{\sigma^2(s)}$. That is, the solution of Merton's problem says that the small investor has to keep the Sharpe ratio of his investments in stocks constant through the life of his investment.

Exercise 9 *Compute the maximum utility given by the optimal portfolio* $\pi^*(s) = \frac{\mu(s)-r}{\sigma^2(s)}$ *and write the explicit expression in the case that* μ *and* σ *are constants.*

Exercise 10 *Prove that within the class* $\mathcal{A}(t) = \{\pi : \pi$ *is* $\mathcal{F}$*-adapted,* $E\left[\int_0^t \pi(s)^2 ds\right] < \infty\}$ *the portfolio* $\pi(s) = \frac{\mu(s)-r}{\sigma^2(s)}$ *is the optimal portfolio for the problem* $\max_{\pi\in\mathcal{A}} E\log(\hat{V}(t))$.

Exercise 11 *(Cont.): Prove that the above portfolio is also optimal in the class* $\{\pi : \pi$ *is* $\mathcal{F}$*-adapted,* $\int_0^t \pi(s)^2 ds < \infty$ *a.s. and* $E\left[\log(\hat{V}(t))\right] < \infty\}$.

Exercise 12 *Use Girsanov's theorem to find the optimal portfolio for the problem* $\max_{\pi\in\mathcal{A}_\theta} E\left[\hat{V}(t)^\theta\right]$ *for* $\theta \in (0,1)$. *Show that the problem has no finite solution if* $\theta \notin (0,1)$. *Define the set* $\mathcal{A}_\theta$.

An issue that is important to discuss before continuing with the optimization problem for the insider is the issue of arbitrage: Can the investor make money without risking any loss?

Definition 13 *We say that there is arbitrage in the market if there exists a self-financing portfolio* π *with* $P(\hat{V}(T) \geq V_0) = 1$ *and* $P(\hat{V}(T) > V_0) > 0$.

A well known related theorem is

Theorem 14 *If there exists a measure* $Q \sim P$ *such that under* Q, $\hat{S}$ *is a martingale then there is no arbitrage for self-financing portfolios* π *satisfying* $\int_0^T (\pi(s)V(s))^2 ds < \infty$ *a.s..*

Exercise 15 *Prove that the assumption of the previous theorem is satisfied with*

$$\frac{dQ}{dP} = \exp\left(-\frac{1}{2}\int_0^t \left(\frac{\mu(s)-r}{\sigma(s)}\right)^2 - \int_0^t \frac{\mu(s)-r}{\sigma(s)} dW(s)\right).$$

Now we start to consider the same utility maximization problem but for the insider agent.

3 The portfolio problem for the insider. A toy example

To simplify our discussion suppose that the insider has as additional information the value of a random variable $I = S(T) \in \mathcal{F}_T$. This is equivalent to knowing W_T. This example is a "toy" example because it is hard to think of a real example where an insider knows exactly the value of $S(T)$. Nevertheless this example will give us basic information that will be important in what follows.

Exercise 16 *(Lévy's Theorem) Let* M *be a continuous local martingale in a filtration* $\mathcal{F}$ *such that* $\langle M\rangle_t = t$. *Then* M *is an* $\mathcal{F}$*-Wiener process. Hint: Use Itô's formula to prove that*

$$E\left[\exp\left(i\theta(M_t - M_s)\right)/\mathcal{F}_s\right] = \exp\left(-\frac{\theta^2(t-s)}{2}\right).$$

Therefore the increment $M_t - M_s$ *is conditionally independent of* $\mathcal{F}_s$ *and is a* $N(0, t-s)$ *random variable.*

The natural filtration for the insider is $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(I)$ (the smallest filtration satisfying the usual conditions which contains $\mathcal{F}$ and $\sigma(I)$).

Exercise 17 *Prove that* $\mathcal{G}_t = \cap_{\varepsilon\geq 0}\sigma\left(\mathcal{F}_{t+\varepsilon} \cup \sigma(I)\right)$.

In general, it is necessary to take the above intersection as the following simple example shows.

Exercise 18 *(Barlow-Perkins, also proposed in Williams [46], page 48) Let* Y_i *be iid Bernoulli r.v.'s with values* $\{-1, 1\}$*. Define* $X_n = \prod_{i=0}^{n} Y_i$ *define* $\mathcal{A} = \sigma(Y_i; i \geq 1)$*,* $\mathcal{B}_n = \sigma(X_r; r > n)$*. Prove that* $Y_0 \in \mathcal{B}_n \vee \sigma(\mathcal{A})$ *for all* n *but* Y_0 *is independent of* $(\cap \mathcal{B}_n) \vee \sigma(\mathcal{A})$*.*

Under the enlarged filtration $\mathcal{G}$ the process W is no longer a Wiener process but still a continuous semimartingale. Therefore we are interested in computing the semimartingale decomposition of $W = M + A$ where M is a $\mathcal{G}$-local martingale and A is a $\mathcal{G}$ adapted process of bounded variation. As the quadratic variation is still $\langle W \rangle_t^{\mathcal{G}} = t$ then, by Lévy's theorem M will be a $\mathcal{G}$ Wiener process.

We denote by P_t the regular conditional probability measure of $X = W(T)$ with respect to $\mathcal{F}_t$. That is, $P_t(dx) = P(W(T) \in dx / \mathcal{F}_t)$. Explicitly in our case

$$dP_t(x) = p_{T-t}(W_t, x)dx = \frac{1}{\sqrt{2\pi(T-t)}} \exp\left(-\frac{(x - W(t))^2}{2(T-t)}\right) dx$$

Theorem 19 *The semimartingale decomposition of* W *in* $\mathcal{G}$ *is given by*

$$W(t) = \widehat{W}(t) + \int_0^t \frac{W(T) - W(u)}{T - u} du \tag{10}$$

where $\widehat{W}$ *is a Wiener process in the initially enlarged filtration* $\mathcal{G}$ *in the interval* $[0, T]$*.*

Proof. We compute $E([W(t) - W(s)/\mathcal{G}_s]$ for $t \in [0, T)$ and $t > s$. The objective is to obtain an expression of bounded variation which will become the compensator of the process W in the enlarged filtration.

Alternatively we will compute for a measurable bounded function f and a $\mathcal{F}_s$ measurable r.v. h_s,

$$\begin{aligned} E\left[(W(t) - W(s))f(W(T))h_s\right] &= E\left[(W(t) - W(s)) \int f(x) dP_t(x) h_s\right] \\ &= E\left[(W(t) - W(s)) \int f(x) p_{T-t}(W_t, x) dx h_s\right]. \end{aligned}$$

Applying Itô's formula to $(W(u) - W(s))p_{T-u}(W_u, x)$, $u \in [s, t]$ and using that p_{T-t} solves the heat equation $\partial_t p_{T-t}(y, x) + \frac{1}{2}\partial_y^2 p_{T-t}(y, x) = 0$, we have

$$\begin{aligned} & E\left[(W(t) - W(s))f(W(T))h_s\right] \\ &= E\left[\int f(x) \left(\int_s^t p_{T-u}(W_u, x) + (W(u) - W(s))\partial_y p_{T-u}(W(u), x) dW(u)\right.\right. \\ &+ \left.\left.\int_s^t \partial_y p_{T-u}(W(u), x) du\right) dx h_s\right] \\ &= E\left[\int f(x) \int_s^t \partial_y p_{T-u}(W(u), x) du dx h_s\right] \\ &= E\left[\int_s^t \int f(x) \partial_y log(p_{T-u}(W(u), x)) p_{T-u}(W(u), x) dx du h_s\right] \\ &= E\left[f(W(T)) \int_s^t \partial_y log(p_{T-u}(W(u), W(T))) du h_s\right]. \end{aligned}$$

Therefore by a density argument, one has

$$E\left[W(t) - W(s) - \int_s^t \partial_y log(p_{T-u}(W(u), W(T))) du \Big/ \mathcal{G}_s\right] = 0.$$

As $\partial_y log(p_{T-u}(W(u), W(T))) = \frac{W(T) - W(u)}{T-u} \in \mathcal{G}_u$ then $\hat{W}(t) = W(t) - \int_0^t \frac{W(T) - W(u)}{T-u} du$ is a $\mathcal{G}$-continuous martingale with $\langle \hat{W} \rangle_t = \langle W \rangle_t = t$ and therefore by Lévy's theorem one has that $\hat{W}$ is a $\mathcal{G}$-Wiener process in $[0, T)$. We then define $\hat{W}(T) = \lim_{t \to T} \hat{W}(t)$ and all above properties follow for the closed interval $[0, T]$ as $E\left[\int_0^T \left|\frac{W(T) - W(u)}{T-u}\right| du\right] < \infty$. ■

Exercise 20 *Prove that* $E\left[\int_0^T \left|\frac{W(T)-W(u)}{T-u}\right|^r du\right] < \infty$ *if and only if* $r \in [0,2)$ *while* $E\left[\left|\int_0^T \frac{W(T)-W(u)}{T-u} du\right|^r\right] < \infty$ *for all* $r \geq 0$.

The methodology shown in Theorem 19 can be generalized much further in various directions. Some of them are shown in the following exercises.

Exercise 21 *Prove that* $\{\hat{W}(t); t \in [0,T]\}$ *and* $W(T)$ *are independent random variables.*

Exercise 22 *(Föllmer-Imkeller, [15]) Prove that there exists a measure* $Q \sim P$ *such that, under* Q, $W(t)$ *and* $W(T)$ *are independent.*

Exercise 23 *(Harnesses) For* $s \leq a < b \leq T$, *prove that*

$$E\left[\frac{W_a - W_b}{a-b} \Big/ \mathcal{G}_s\right] = \frac{W_T - W_s}{T-s}.$$

Exercise 24 *Find the semimartingale decomposition of the Wiener process if* $I = \int_0^T h(r)W_r dr$ *for a deterministic function* $h \in L^2[0,T]$. *Impose conditions on* h *so that the semimartingale decomposition in the enlarged filtration is integrable.*

Exercise 25 *Suppose that* $I = X(T)$ *where* $X(\cdot)$ *is the solution of the stochastic differential equation*

$$X(t) = x_0 + \int_0^t b(X(s))ds + \int_0^t \sigma(X(s))dW(s).$$

Here b *and* $\sigma : \mathbb{R} \to \mathbb{R}$ *are smooth functions. Then* X *is a Markov process. Furthermore suppose that the transition density* $p_t(y,x)$ *exists, is smooth and satisfies the parabolic partial differential equation and the inequalities*

$$\partial_t p_t(y,x) = b(y)\partial_y p_t(y,x) + \frac{1}{2}\sigma(y)^2 \partial_x^2 p_t(y,x) \tag{11}$$

$$ct^{-a} \exp\left(-\frac{|x-y|^2}{ct}\right) \leq p_t(x,y) \leq Ct^{-a} \exp\left(-\frac{|x-y|^2}{Ct}\right)$$

$$|\partial_x p_t(x,y)| \leq Ct^{-a} \exp\left(-\frac{|x-y|^2}{Ct}\right)$$

for $t \in (0,T]$, $a, c, C > 0$ *and* x, $y \in \mathbb{R}$. *Prove that* $\hat{W}(t) = W(t) - \int_0^t \partial_x log(p_{T-u}(X(u), X(T)))du$ *is a* $(\mathcal{G}_t)_{t\in[0,T)}$*-Wiener process. Give conditions to obtain a Wiener process in* $[0,T]$. *If you know Malliavin Calculus:* $\partial_x log(p(u,x,y))$ *is called the logarithmic derivative of the density.*

Exercise 26 *(Ankirchner) Let* $I = |W(T)|$. *Prove that the compensator in this case is given by*

$$\int_0^t W(s)\left(-\frac{1}{T-s} + \frac{|W(T)|}{|W(s)|(T-s)} \tanh\left(\frac{|W(s)W(T)|}{T-s}\right)\right) ds.$$

Exercise 27 *Solve equation (10) in* $W(t)$ *to obtain that for* $t < T$

$$\frac{W_T - W_t}{T-t} = \frac{W_T}{T} - \int_0^t \frac{1}{T-r} d\widehat{W}_r.$$

Following the above theorem we have that the evolution of the stock price for the insider is

$$S(t) = S(0)\exp\left(\left(\int_0^t \mu(s) - \frac{1}{2}\sigma^2(s)\right) ds + \int_0^t \sigma(s)\widehat{W}(s) + \int_0^t \sigma(u)\frac{W_T - W_u}{T-u} du\right).$$

Note that the values of S observed by both the small investor and the insider are the same. Given that the insider has some extra information about the driving process, this information should give him/her an advantage that can be expressed through the amount of money that he can make using this extra information. To quantify this mathematically, let $\pi = \{\pi(s); 0 \le s \le T\}$ be an $\mathcal{G}$-adapted process that denotes, as before, the proportion of the total wealth that the insider invests in stocks. Then the discounted wealth process $\hat{V}$ satisfies the equation (7). The solution of this linear equation is

$$\hat{V}(t) = V_0 \exp\left(\int_0^t \left((\mu(s) - r + \sigma(s)\frac{W_T - W_s}{T-s})\pi(s) - \frac{1}{2}\sigma^2(s)\pi(s)^2\right) ds + \int_0^t \sigma(s)\pi(s) d\widehat{W}(s)\right).$$

Now we compute the average utility to obtain that (using that $E\left[\int_0^t \sigma(s)\pi(s)d\widehat{W}(s)\right] = 0$)

$$E\left[\log(\hat{V}(t))\right] = \log(V_0) + \int_0^t E\left[\left(\mu(s) - r + \sigma(s)\frac{W_T - W_s}{T-s}\right)\pi(s) - \frac{1}{2}\sigma^2(s)\pi(s)^2\right] ds.$$

where π is $\mathcal{G}$-adapted. As before, it is enough to find the maximum of the function within the expectation. In the case of the insider agent we have to maximize

$$f_s(\pi) = \left(\mu(s) - r + \sigma(s)\frac{W_T - W_s}{T-s}\right)\pi - \frac{1}{2}\sigma^2(s)\pi^2.$$

Note that this function is concave and the optimum is given for

$$\hat{\pi}(s) = \frac{1}{\sigma^2(s)}\left(\mu(s) - r + \sigma(s)\frac{W_T - W_s}{T-s}\right).$$

The maximum utility is

$$E\left[\log(\hat{V}^*(t))\right] = \log(V_0) + \int_0^t E\left[\frac{1}{2\sigma^2(s)}\left(\mu(s) - r + \sigma(s)\frac{W_T - W_s}{T-s}\right)^2\right] ds \tag{12}$$

$$= \log(V_0) + \int_0^t E\left[\frac{(\mu(s) - r)^2}{2\sigma^2(s)}\right] ds + \frac{1}{2}\log\left(\frac{T}{T-t}\right). \tag{13}$$

Exercise 28 *Formalize the above argument defining explicitly the class $\mathcal{I}$ of admissible strategies and prove that the optimal portfolio is the one found in the previous informal calculation.*

The difference in utility between the insider and the small agent is $\frac{1}{2}\log\left(\frac{T-t}{T}\right)$ which does not depend on the parameters μ or σ. This is easily explained by the fact that the information that the insider possesses does not depend on these parameters. In more complex models this is not expected although a clear example is not available yet (see Example 45). This result also shows that the additional information of the insider provides him/her with an infinite wealth as $t \to T$. This is due to the fact that the insider can make use of his information in all the oscillations around the value of $S(T)$. Therefore this market with a small agent and an insider allows for arbitrage in the interval $[0, T]$. This is natural, given the good quality of the extra information.

For any time interval $[0, t]$, $t < T$, it is also clear that the underlying model for the insider becomes a geometric Wiener process with random drift. Therefore all the classical mathematical financial theory applies. For example, option prices (which are the same as Black Scholes prices), hedging strategies, equilibrium theory, portfolio choice theory, etc.

Exercise 29 *Find an explicit arbitrage strategy for the portfolio of the insider in* $[0, T]$.

This model in the time interval $[0, T]$ corresponds to insider trading in the sense of an agent holding information which is prohibited by law. This point of view may be somewhat interesting if one is interested in detecting this type of insiders.

Instead, we are interested in obtaining models where insiders have finite utility. In fact most insiders are lawful actors in financial markets. So in these notes we will rather discuss how to modify these models so as to obtain finite utilities for insiders. This line will be developed in future sections. First, we will study the mathematical theory around these developments which are based on enlargement of filtrations. Before that, let us propose some exercises.

Exercise 30 *Find the optimal strategy for the insider for the utility function* $U(x) = x^\theta$ *for* $\theta \in (0,1)$.

Note that if one knows the value of W_T then W becomes a Brownian bridge.

Exercise 31 *Use an alternative expression for the Brownian bridge (e.g. see page 300 in Protter's book) to obtain the result (12). That is, the process* X *defined as*

$$X(t) = a\left(1 - \frac{t}{T}\right) + b\frac{t}{T} + (T-t)\int_0^t \frac{1}{T-s}dB(s)$$

is a Brownian bridge starting from $X_0 = a$ *and ending at* $X(T) = b$ *for* $t = T$. *Compare this Exercise with Exercise 27 and point out what is the difference with the present approach?*

Exercise 32 *Compute the optimal logarithmic utility for the insider in the interval* $[0,t]$ *conditioned on the value of* $S(T) = x$ *for* $t < T$.

Exercise 33 *Define* $u^{\mathcal{F}}(t, V_0)$ *and* $u^{\mathcal{G}}(t, V_0)$ *the optimal logarithmic utility (in* $[0,t]$*) of the small investor and the insider respectively. Obviously as* $\mathcal{G}$ *is a filtration bigger than* $\mathcal{F}$ *we have that* $u^{t,\mathcal{F}}(x) \leq u^{t,\mathcal{G}}(x)$. *Define the fair price of the information* $I = W(T)$ *as the real number* $\rho_t(V_0, I)$ *such that* $u^{\mathcal{F}}(t, V_0) = u^{\mathcal{G}}(t, V_0 - \rho_t(V_0, I))$. *Prove that* $\rho_t(V_0, I) = V_0\left(1 - \sqrt{1 - \frac{t}{T}}\right)$. *The fair price of information is a fraction that corresponds to how much less money we need in order to make the same amount of wealth as the small trader. Note that* $\rho_T(V_0, I) = V_0$ *which reflects the fact that there is arbitrage at time* T.

Generalizing the previous example leads us to the theory of initial enlargement of filtration and in particular to Jacod's theorem.

4 Jacod's Theorem

As before, let I be a $\mathcal{F}_T = \sigma(W(s); s \leq T)$ adapted random variable and let $P_t(dx)$ denote the regular conditional probability of I with respect to $\mathcal{F}_t$. Define $\mathcal{G}_t$ as the smallest filtration that includes $\mathcal{F}_t$ and $\sigma(I)$ i.e. $\mathcal{G} = \mathcal{F} \vee \sigma(I)$.

Theorem 34 *If there exists a deterministic measure* $\eta(dx)$ *such that* $P_t(dx) << \eta(dx)$, *then* W *is a* $\mathcal{G}$*-semimartingale.*

Exercise 35 *Use the ideas given in Theorem 19 to give an sketch of the proof of the above theorem if we further assume that* $E\left[\int_0^t |\alpha(u)|\, du\right] < \infty$ *to show that*

$$W(t) = \widehat{W}(t) + \int_0^t \alpha(u)du$$

is a $\mathcal{G}$*-Wiener process, where*

$$\alpha(u) = \frac{\frac{d}{du}\left\langle W, \frac{dP_\cdot(I)}{d\eta}\right\rangle_u}{\frac{dP_u(I)}{d\eta}}.$$

In fact, it is better to try to generalize the above result without using the reference measure η. This will become clear later. Before doing the generalization we want to introduce a further example:

Exercise 36 *Given a filtration $\mathcal{G} \supseteq \mathcal{F}$. Assume that W is a semimartingale in $\mathcal{G}$ and that its decomposition is given by*

$$W_t = \widehat{W}_t + \int_0^t \alpha_u du$$

with $E\left[\int_0^t |\alpha_u|^2 du\right] < +\infty$ a.s. for all $t < T$. Prove that the optimal portfolio is $\hat{\pi}(s) = \frac{\mu(s)-r}{\sigma^2(s)} + \frac{\alpha_s}{\sigma(s)}$ and the additional utility of the insider in the interval $[0,t]$ is given by $\frac{1}{2}\int_0^t E\left[\alpha_u^2\right] du$.

We also remark that Jacod's Theorem as stated in [26] is in the semimartingale framework and with slightly less stringent conditions.

5 An approach using the integration by parts formula

Jacod's Theorem (see Jacod, [26]) is the basic tool to characterize when a semimartingale keeps this property in a enlarged filtration and what its new decomposition is. Another way to approach this problem is using the integration by parts formula of Malliavin calculus. We show this in the next theorem as a way of illustration. For this reason we do not give full details in this section.

We denote by D the stochastic derivative and by δ_a^b the Skorohod integral from a to b, the dual operator of D in $L^2([a,b] \times \Omega)$. For a general introduction to Malliavin calculus and the notation used here, see Nualart [37]. Now, we define a general integration by parts formula.

Definition 37 *Let (I, Y) be a random vector, measurable with respect to $\mathcal{F}_T$, such that there exists a random variable $H_{u,T}(I,Y) \in L^2(\Omega)$ with the property that for any $f \in C_b^1$ and $A \in \mathcal{F}_u$ it satisfies that*

$$E\left[f'(I)Y1_A\right] = E\left[f(I)H_{u,T}(I,Y)1_A\right].$$

Then we say that there is an integration by parts formula (ibp) for (I,Y) in $[u,T]$ and $H_{u,T}$ is the weight associated with the ibp for (I,Y) in $[u,T]$.

Theorem 38 *Let I be a random variable such that there is an ibp for $(I, D_u I)$ in $[u,T]$. Then W is a semimartingale in the enlarged filtration $\{\mathcal{F}_s \vee \sigma(I); s \leq T\}$ and*

$$W_t = \widehat{W}_t + \int_0^t E\left[H_{u,T}(I, D_u I)/\,\mathcal{F}_u \vee \sigma(I)\right] du$$

where $\widehat{W}$ is a Wiener process in the enlarged filtration.

Proof. First, we compute $E\left[W_t/\mathcal{F}_s \vee \sigma(I)\right]$. For this consider for a regular bounded function f and $A \in \mathcal{F}_s$

$$\begin{aligned} E\left[(W_t - W_s)f(I)1_A\right] &= E\left[\int_s^t D_u f(I) du 1_A\right] \\ &= \int_s^t E\left[f'(I)D_u I 1_A\right] du \\ &= \int_s^t E\left[f(I)H_{u,T}(I, D_u I)1_A\right] du \\ &= E\left[f(I)\int_s^t E\left[H_{u,T}(I, D_u I)/\,\mathcal{F}_u \vee \sigma(I)\right] du 1_A\right]. \end{aligned}$$

In conclusion we have that

$$E([W_t/\mathcal{F}_s \vee \sigma(I)] = W_s + \int_s^t E\left[H_{u,T}(I, D_u I)/\,\mathcal{F}_u \vee \sigma(I)\right] du.$$

This means that W is a semimartingale in the filtration $\{\mathcal{F}_s \vee \sigma(I); s \leq T\}$ and one concludes that its martingale part has to be a Wiener process due to the Lévy characterization theorem. ■

Under certain conditions we have that the standard ibp formula is satisfied if there exists a process h such that $\frac{D_u I h(\cdot)}{\int_u^T D_v I h(v) dv} \in Dom(\delta_u^T)$ and then

$$H_{u,T}(I, D_u I) = \int_u^T \frac{D_u I h(s)}{\int_u^T D_v I h(v) dv} dW_s.$$

This is the formula of integration by parts using h as a localization function. The usual integration by parts formula is obtained using $h = DI$ which leads to the Malliavin covariance matrix, $\int_u^T (D_u I)^2 du$, in the denominator of the above Skorohod integral. In such a case, if $I \in \mathbb{D}^{2,8}$ and $E\left[\left(\int_u^T (D_u I)^2 du\right)^{-8}\right] < \infty$ then the ibp formula is satisfied for $(X, D_u X)$.

Exercise 39 *Obtain Theorem 10 as a consequence of the previous Theorem.*

6 Weak information approach

It is clear that we have the following probability decomposition that represents the Brownian bridge

$$P(W \in A) = \int P(W \in A/\,W(T) = x) P^{W(T)}(dx).$$

That is, under the law $P(W \in \cdot/\,W(T) = x)$, W is described by a Brownian bridge. Fabrice Baudoin (see [5]) used this property to construct a "Brownian bridge" where the law of the final random variable is any measure $\nu \sim P^{W(T)}$. That is we define

$$P^\nu(W \in A) = \int P(W \in A/\,W(T) = x)\nu(dx).$$

We denote by E^ν the expectation with respect to P^ν. To obtain the analogue of Theorem 19 in this setting, we compute the Radon Nikodym derivative $\frac{dP^\nu}{dP}\big|_{\mathcal{F}_t}$. Let h_t be an $\mathcal{F}_t$-measurable random variable, then

$$E^\nu[h_t] = E\left[\frac{d\nu}{dP^{W(T)}}(W(T)) h_t\right] = E\left[E\left[\frac{d\nu}{dP^{W(T)}}(W(T))\Big/\,\mathcal{F}_t\right] h_t\right].$$

Therefore

$$\left.\frac{dP^\nu}{dP}\right|_{\mathcal{F}_t} = E\left[\frac{d\nu}{dP^{W(T)}}(W(T))\Big/\,\mathcal{F}_t\right].$$

Furthermore using the same technique as in Theorem 19, we have

Theorem 40 *$W(t) = \hat{W}(t) + \int_0^t \alpha(u) du$ where $\hat{W}$ is a Wiener process in $(\Omega, (\mathcal{F}_t)_{t\in[0,T]}, P^\nu)$ and*

$$\alpha(u) = E\left[\frac{W(T) - W_u}{T - u}\frac{d\nu}{dP^{W(T)}}(W(T))\Big/\,\mathcal{F}_u\right] E\left[\frac{d\nu}{dP^{W(T)}}(W(T))\Big/\,\mathcal{F}_u\right]^{-1}$$

Proof. As before we need to compute $E^{\nu}\left[W(t)-W(s)/\mathcal{F}_s\right]$. Instead we compute for h_s a $\mathcal{F}_s$ measurable random variable

$$
\begin{aligned}
&E^{\nu}\left[\left(W(t)-W(s)\right)h_s\right] \\
&= E\left[\frac{d\nu}{dP^{W(T)}}(W(T))\left(W(t)-W(s)\right)h_s\right] \\
&= E\left[\frac{d\nu}{dP^{W(T)}}(W(T))\int_s^t \frac{W(T)-W(u)}{T-u}du h_s\right] \\
&= E^{\nu}\left[\int_s^t \frac{W(T)-W(u)}{T-u}du h_s\right] \\
&= E^{\nu}\left[\int_s^t E^{\nu}\left[\frac{W(T)-W(u)}{T-u}\Big/\mathcal{F}_u\right]du h_s\right]. \\
&= E^{\nu}\left[\int_s^t E\left[\frac{W(T)-W(u)}{T-u}\frac{d\nu}{dP^{W(T)}}(W(T))\Big/\mathcal{F}_u\right]E\left[\frac{d\nu}{dP^{W(T)}}(W(T))\Big/\mathcal{F}_u\right]^{-1}du h_s\right].
\end{aligned}
$$

where the last equality follows directly if we consider

$$
\begin{aligned}
&E^{\nu}\left[E\left[\frac{W(T)-W(u)}{T-u}\frac{d\nu}{dP^{W(T)}}(W(T))\Big/\mathcal{F}_u\right]E\left[\frac{d\nu}{dP^{W(T)}}(W(T))\Big/\mathcal{F}_u\right]^{-1}h_u\right] \\
&= E\left[\frac{d\nu}{dP^{W(T)}}(W(T))E\left[\frac{W(T)-W(u)}{T-u}\frac{d\nu}{dP^{W(T)}}(W(T))\Big/\mathcal{F}_u\right]E\left[\frac{d\nu}{dP^{W(T)}}(W(T))\Big/\mathcal{F}_u\right]^{-1}h_u\right] \\
&= E\left[E\left[\frac{W(T)-W(u)}{T-u}\frac{d\nu}{dP^{W(T)}}(W(T))\Big/\mathcal{F}_u\right]h_u\right] \\
&= E\left[\frac{W(T)-W(u)}{T-u}\frac{d\nu}{dP^{W(T)}}(W(T))h_u\right] \\
&= E^{\nu}\left[\frac{W(T)-W(u)}{T-u}h_u\right].
\end{aligned}
$$

The rest of the argument follows as in Theorem 19. ■

Other cases where ν is not equivalent to the Lebesgue measure can also be treated on a case by case basis. The idea remains the same. The application of this result is also clear. The insider has the information that the law of the final random variable $W(T)$ is ν. Then this restriction means that the insider "knows" the final law of the process $W(T)$ while the small investor thinks it is a Wiener process.

Exercise 41 *Find the optimal portfolio and the maximal logarithmic utility of the insider if the law ν is given by a $N(\mu,\sigma^2)$. Find out if the utility is finite in such a case.*

Exercise 42 *Suppose that*

$$
E^{\nu}\left[\left(\frac{\partial}{\partial x}\log\left(\frac{d\nu}{dP^{W(T)}}\right)\right)^2(W(T))\right]<\infty.
$$

Prove that the optimal logarithmic utility for the insider possessing the information that the law of $W(T)$ is ν, is finite. Prove that the measure ν given by a $N(\mu,\sigma^2)$ satisfies the above condition.

7 The entropy characterization of additional information

Here we introduce the characterization of additional utility of the insider in the interval $[0,t]$ as the expectation of the entropy of the conditional measure of $W(T)$ with respect to $\mathcal{F}_t$ obtained in

Amendiger et. al. [2]. To make things simpler suppose that $P_t \sim P^I$ and that $p_t(x) = \frac{dP_t}{dP^I}(x) > 0$ a.s. (for more on this, see Jacod [26], Lemma 1.8 and Corollary 1.11).

By the Itô representation theorem we further have that there exists an adapted process $\{\alpha_t(x); t \in [0,T)\}$ such that

$$p_t(x) = 1 + \int_0^t \alpha_s(x) dW_s.$$

Theorem 43 *Assume that* $E\int_0^t \left(\frac{\alpha_s}{p_s}(I)\right)^2 ds < \infty$ *for all* $t \in [0,T)$. *Then the additional logarithmic utility of the insider in the interval* $[0,t]$ *is* $E[\log p_t(I)]$.

Proof. We compute the compensator of W in $\mathcal{G}$. As before, we have for a measurable bounded function f and an $\mathcal{F}_s$ measurable random variable h_s that

$$\begin{aligned}
E\left[(W(t) - W(s))f(I)h_s\right] &= E\left[(W(t) - W(s)) \int f(x)p_t(x)dP^I(x)h_s\right] \\
&= E\left[\int_s^t \int \alpha_\theta(x)f(x)dP^I(x)d\theta h_s\right] \\
&= E\left[\int_s^t \int \frac{\alpha_\theta}{p_\theta}(x)f(x)dP_\theta(x)d\theta h_s\right] \\
&= E\left[\int_s^t \frac{\alpha_\theta}{p_\theta}(I)d\theta f(I)h_s\right]
\end{aligned}$$

Therefore $W(t) = \hat{W}(t) + \int_0^t \frac{\alpha_s}{p_s}(I)ds$ where $\hat{W}$ is a $\mathcal{G}$ Wiener process. Then by exercise 36 we have that the additional logarithmic utility of the insider in the interval $[0,t]$ is $\frac{1}{2}E\left[\int_0^t \left(\frac{\alpha_s}{p_s}(I)\right)^2 ds\right]$. Furthermore, applying Itô's formula to $\log(p_t(x))$, we have

$$E\left[\log p_t(I)\right] = \frac{1}{2}E\left[\int_0^t \left(\frac{\alpha_s}{p_s}(I)\right)^2 ds\right].$$

The quantity $E\left[\log p_t(I)\right] = E\left[\int p_t(x)\log p_t(x)dP^I(x)\right]$. That is, the gain of the insider is the expectation of the conditional entropy of the random variable I. ■

Exercise 44 *In the case* $\mu = r$, *apply the same reasoning as above to prove that the optimal logarithmic utility of a weak insider in the sense of Baudoin is given by*

$$E\left[\left(\frac{d\nu}{dP^{W(T)}}\log\frac{d\nu}{dP^{W(T)}}\right)(W(T))\right].$$

Prove that this is not the case if $\mu \neq r$.

8 Finite utilities for insiders.

8.1 Finite number of trades.

In stock markets there are heterogenous agents with different types of information which coexist in equilibrium. In this sense the example in section 3 does not correspond too well to the reality of the coexistence of insiders and small investors. Nevertheless, if one is interested in detecting unlawful insiders then the toy example in section 3 describes this situation.

Our goal in the sections to follow is to describe ways of modifying the toy example in section 3 in order that the insider attains a finite optimal utility. The most important reason why the insider does not achieve infinite utility in reality is that the information about $S(T)$ is better for the insider than for the small agent but not an almost sure type of information.

That is, we will consider a model where there is a deformation of information of the insider. We will treat this in the next section (also see the weak information approach in [6]).

In this section, we discuss what happens when the insider is allowed to trade only a fixed number of times up to the date T. Suppose without loss of generality that the times where transactions are allowed are $0 = t_0 < t_1 < ... < t_{n-1}$. Then, using the formulae for wealth from section 2, we have that

$$E\left[\log(V_T)\right] = \log(V_0) + \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} E\left[\left(\mu - r + \sigma \frac{W_T - W_s}{T-s}\right)\pi(t_j) - \frac{1}{2}\sigma^2\pi(t_j)^2\right] ds.$$

Now note that the decisions of the insider can only be based on his/her information at that time. That is, $\pi(t_j) \in \mathcal{G}_{t_j}$. Therefore, we have to maximize

$$\sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} E\left[\left(\mu - r + \sigma E\left(\frac{W_T - W_s}{T-s} \Big/ \mathcal{G}_{t_j}\right)\right)\pi(t_j) - \frac{1}{2}\sigma^2\pi(t_j)^2\right] ds.$$

As (see exercise 23)

$$E\left[\frac{W_T - W_s}{T-s} \Big/ \mathcal{G}_{t_j}\right] = \frac{W_T - W_{t_j}}{T - t_j}.$$

Therefore the function to maximize is

$$f_s(\pi) = \left(\mu(s) - r + \sigma(s)\frac{W_T - W_{t_j}}{T - t_j}\right)\pi - \frac{1}{2}\sigma^2(s)\pi^2.$$

As before the optimal portfolio value is $\hat{\pi}(s) = \frac{\mu(s)-r}{\sigma^2(s)} + \frac{W_T - W_{t_j}}{\sigma(s)(T-t_j)}$ and the optimal logarithmic utility is

$$\log(V_0) + E\left[\int_0^t \frac{\mu(s) - r}{2\sigma^2(s)} ds\right] + \sum_{j=0}^{n-1} \frac{(t_{j+1} - t_j)}{2\,(T - t_j)}.$$

This quantity is finite as long as $t_{n-1} < T$. In fact this quantity tends to infinity as $t_{n-1} \to T$, therefore this proposed solution is only partial but it also reflects the fact that one possible way to model insiders with finite utility is to allow them to trade only at a finite number of times.

If we want to insist on continuous trades then a possibility is to model the information of the insider in a different fashion. This will be done in the next section. Before this, we give the following interesting exercise.

Exercise 45 *Consider the bidimensional model*

$$S^i(t) = S^i(0)\exp\left(\left(\mu_i - \frac{\sigma_i^2}{2}\right)t + \sigma_i W^i(t)\right)$$

where (W^1, W^2) is a bidimensional Wiener process with correlation $EW^1(t)W^2(t) = \rho t$. Suppose that the insider has as information $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(W^1(T))$, but that there is a restriction on trading only in the second asset. Find the optimal portfolio with logarithmic utility for the insider. Do the same if the information in the market is only given by $\mathcal{F}_t^2 = \sigma(W^2(s); s \leq t)$.

There are various other simple options to try to limit the behavior of the insider in order to make its utility finite.

Exercise 46 *Find the optimal portfolio and the maximal logarithmic utility of the insider if we introduce the restriction $\pi \in [0,1]$ (no borrowing from the bank or stocks allowed) and prove that the utility is finite in this case.*

Exercise 47 *Use Exercise 24 to prove that in the case $I = \int_0^T h(s)W(s)ds$, then the optimal logarithmic utility of the insider in the interval $[0,T]$ is ∞. Although this example does not correspond exactly to the average of the stock price, it does show that the utility will also be infinite if the insider has an information on the form of an average.*

8.2 Towards a dynamic model for insider information

Another possibility to obtain finite utilities for the insider is to model the additional information of the insider as $I = f(S(T))$ where f is not a bijection. This alternative modelling has also its weak points. The most important being that the information of the insider does not improve as t approaches T.

Exercise 48 *Use Theorem 43 to prove that the optimal logarithmic utility for the insider in the following cases is finite.*
1. $I = 1(W(T) \geq a)$ *for* $a > 0$.
2. $I = W(T) + \varepsilon$ *where* ε *is a* $N(0,1)$ *r.v. independent of* W.

This exercise shows a model which is closer to reality. The information held by the insider is blurred by an additional noise. Nevertheless, this noise does not disappear even when $t = T$. This problem also appears in the weak information approach of F. Baudoin. This is a problem related with the fact that we are doing an initial enlargement of filtration. That is, the filtration is enlarged only once at time $t = 0$.

In relation with this problem we have recently proposed a model of additional information of the type $I(t) = W_T + W'((T-t)^\theta)$ where W' is a Wiener process independent of W (see [10]). This model contains a deformation of information through time. This is equivalent to say that the information of the insider is $S(T)\exp\left(\sigma W'((T-t)^\theta)\right)$. That is, the blurring is done at logarithmic scale. The filtration, denoted by $\mathcal{G}$, is the smallest filtration that satisfies the usual conditions and that includes $\mathcal{F}_t \vee \sigma(I(s); s \leq t)$.

Theorem 49 *Let* $I(t) = W_T + W'((T-t)^\theta)$ *and* $\mathcal{G}_t = \mathcal{F}_t \vee \sigma\left(I(s); s \leq t\right)$. *Then* $\{W(t); t \in [0,T]\}$ *is a semimartingale in the enlarged filtration* $\mathcal{G}$ *and the decomposition is given by*

$$W_t = \hat{W}_t + \int_0^t \frac{I(s) - W(s)}{T - s + (T-s)^\theta} ds$$

where $\{\hat{W}(t); t \in [0,T]\}$ *is a* $\mathcal{G}$ *- Wiener process.*

Proof. We try a slight variant of the proofs given before. Consider for $s \leq u \leq t$

$$\begin{aligned}
&E\left[W_t - W_u \middle| \mathcal{G}_s\right] \\
&= E\left[W_t - W_u \middle| \mathcal{F}_s \vee \sigma(I(s) - W_s, W'((T-s)^\theta) - W'((T-v)^\theta); v \leq s\right] \\
&= E\left[W_t - W_u \middle| I(s) - W_s\right] \\
&= \frac{t-u}{T - s + (T-s)^\theta}\left(I(s) - W_s\right)
\end{aligned}$$

where we have used the independence of $W_t - W_u$, $\mathcal{F}_s$ and $\sigma(W'((T-s)^\theta) - W'((T-v)^\theta); v \leq s)$ and that conditional expectation of $E(X/Y) = \frac{cov(X,Y)}{Var(Y)} Y$ for a mean zero Gaussian random vector (X,Y). Similarly,

$$E\left[W'((T-u)^\theta) \middle| I(s) - W_s\right] = \frac{(T-u)^\theta}{T - s + (T-s)^\theta}\left(I(s) - W_s\right).$$

Then

$$E\left[W_t - W_u - \int_u^t \frac{I(r) - W(r)}{T - r + (T-r)^\theta} dr \middle| \mathcal{G}_s\right] = 0.$$

■

Exercise 50 *Prove that if* $I(s) = W_T + W'((T-s)^\theta)$ *, then*

$$E\left[W_t - W_u \middle| I(s) - W_s\right] = \frac{t-u}{T-s+(T-s)^\theta}\left(I(s) - W_s\right)$$

using the time homogeneity property of the Wiener process. Prove that this result is also valid for Lévy processes with finite mean.

As before we can also compute the insider's optimal utility which gives

$$\begin{aligned} E\left[\log(\hat{V}^*(t))\right] &= \log(x) + \int_0^t E\left[\frac{1}{2\sigma^2(s)}\left(\mu(s) - r + \sigma(s)\frac{I(s) - W_s}{T-s+(T-s)^\theta}\right)^2\right] ds \\ &= \log(x) + \int_0^t E\left[\frac{(\mu(s)-r)^2}{2\sigma^2(s)}\right] ds + \frac{1}{2}\int_0^t \frac{1}{T-s+(T-s)^\theta} ds. \end{aligned}$$

This shows that for $\theta < 1$, the utility for this model is finite. In fact, one can even prove that there is absence of arbitrage therefore answering our previous question regarding the coexistence between the insider and the small investor in the same model for the time interval $[0, T]$.

Exercise 51 *Prove that the fair price of the flow of information characterized by* $\{I(t); t \in [0, T]\}$ *(see Exercise 33) is given by*

$$p(t, T, I) = V_0\left(1 - \exp\left(-\frac{1}{2}\int_0^t \frac{1}{T-s+(T-s)^\theta} ds\right)\right).$$

Note that this quantity is not V_0 *for* $t = T$ *and* $\theta \in (0, 1)$*. This result can be interpreted as the fact that the fair price for the insider's information is less than the wealth of all the other market agents which is the case in example 33.*

The semimartingale decomposition for W in the enlarged filtration obtained here is a result of a projection formula. In fact, we have the following result from Corcuera et. al [10]:

Theorem 52 *Let* I *be an* $\mathcal{F}_T$*-measurable random variable and assume that there exists an* $\mathcal{F} \vee \sigma(I)$*-progressively measurable process* $\alpha = \{\alpha_t,\ t \in [0, T)\}$ *locally in* L^1*, such that* $W_t - \int_0^t \alpha_s ds,\ t \in [0, T)$ *is an* $\mathcal{F} \vee \sigma(I)$*-Wiener process with* $\int_0^T |\alpha_u|\, du < +\infty$ *a.s., then* $W_t - \int_0^t E\left[\alpha_s | \mathcal{G}_s\right] ds,\ t \in [0, T)$ *is an* $\mathcal{G}$*-Wiener process and the additional utility of the insider in the interval* $[0, t]$ *is given by* $\frac{1}{2}\int_0^t E\left[E\left[\alpha_s | \mathcal{G}_s\right]^2\right] du$*.*

Proof. Since W' is independent of $\mathcal{F}_T$, then $\hat{W}_t = W_t - \int_0^t \alpha_s ds$ is a $\mathcal{J}$-Wiener process, with $\mathcal{J} = (\mathcal{F}_t \vee \sigma(I) \vee \sigma(W'_s, s \le t))_{t \in [0,T)}$. We have that $E\left[\hat{W}_t | \mathcal{G}_t\right] = W_t - \int_0^t E\left[\alpha_s | \mathcal{G}_s\right] ds$, where we can consider an $\mathcal{G}$-progressively measurable version of $E\left(\alpha_s | \mathcal{G}_s\right),\ s \in [0, T)$ (see Dellacherie and Meyer (1980), page 113), and this will be an $\mathcal{G}$-martingale. In fact, for $0 \le s < t < T$

$$E\left[E\left[\hat{W}_t | \mathcal{G}_t\right] | \mathcal{G}_s\right] = E\left[\hat{W}_t | \mathcal{G}_s\right] = E\left[E\left[\hat{W}_t | \mathcal{J}_s\right] | \mathcal{G}_s\right] = E\left[\hat{W}_s | \mathcal{G}_s\right].$$

Finally, one concludes using Lévy's characterization theorem. ■

This idea of adding a vanishing independent Wiener process is useful not only in the example treated in Section 2, but in general in any situation where the semimartingale decomposition of W in the enlarged filtration has an information drift which degenerates at some point.

Nevertheless one awkward point still remains: The optimal portfolios of the insider are highly oscillating. That is, $\pi(s) = \mu - r + \sigma\frac{I(s) - W_s}{T-s+(T-s)^\theta}$ is the optimal portfolio of the insider and $\limsup_{s \to T} \pi(s) = +\infty$ and $\liminf_{s \to T} \pi(s) = -\infty$. We will see in the next section that one way to solve this problem is to consider riskier markets. That is, markets with jumps.

9 Insiders in markets with jumps

So far all the examples of insider trading have considered $Z \equiv 0$. This on one side is because the Wiener process has has an explicit density.

In fact, even if this is not the case some explicit compensator can be calculated. For example, if Z is a Lévy process with finite expectation we have the following theorem. A Lévy process is a càdlàg stochastically continuous stochastic process with independent stationary increments. Basic examples of Lévy process are the Wiener process and the compounded Poisson process introduced in Section 2. Furthermore for any Lévy process we have through the Lévy-Khintchine representation that

$$E\left[e^{i\theta Z_T}\right] = e^{T\psi(\theta)}$$
$$\psi(\theta) = i\theta b - \frac{\sigma^2\theta^2}{2} + \int_{-\infty}^{+\infty}\left(e^{i\theta x} - 1 - 1\left(|x| \le 1\right) i\theta x\right)\nu(dx)$$

where $\int_{-\infty}^{+\infty}\left(|x|^2 \wedge 1\right)\nu(dx) < \infty$.

Theorem 53 *Let Z be a Lévy process with $E|Z_T| < \infty$. If $\mathcal{F}$ denotes the filtration generated by Z and $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(Z_T)$ then*

$$Z_t = \hat{Z}_t + \int_0^t \frac{Z_T - Z_s}{T - s} ds$$

for $t \le T$ where $\hat{Z}$ is a $\mathcal{G}$-integrable martingale.

Proof. First note that $E|Z_T| < \infty$ is equivalent to $\int_{|x|\ge 1}|x|\,\nu(dx) < \infty$ (see Proposition 25.4 page 159 in [44]). Consider for a $\mathcal{F}_s$-measurable bounded random variable h_s the quantity

$$\begin{aligned} E\left[(Z_t - Z_s)e^{i\theta Z_T}h_s\right] &= \frac{\partial}{\partial\mu_2}E\left[e^{i\mu_1(Z_T - Z_t) + i\mu_2(Z_t - Z_s)}e^{i\theta Z_T}h_s\right]\Bigg|_{\mu_1=\mu_2=0} \\ &= \frac{\partial}{\partial\mu_2}e^{(T-t)\psi(\mu_1+\theta)+(t-s)\psi(\mu_2+\theta)}\Bigg|_{\mu_1=\mu_2=0} E\left[e^{i\theta Z_s}h_s\right] \\ &= (t-s)\psi'(\theta)E\left[e^{i\theta Z_s}h_s\right]. \end{aligned}$$

From here it follows that

$$E\left[(Z_t - Z_s)f(Z_T)h_s\right] = \frac{t-s}{T-s}E\left((Z_T - Z_s)f(Z_T)h_s\right).$$

From this equality one obtains using Fubini's theorem that

$$E\left[(Z_t - Z_s - \int_s^t \frac{Z_T - Z_u}{T-u}du)f(Z_T)h_s\right] = 0$$

The integrability property follows directly from the definition of $\hat{Z}$. ■

Exercise 54 *(Mansuy-Yor) Define $\mathcal{F}_{s,t} = \mathcal{F}_s \vee \sigma(Z_u; u \ge t)$. Under the same conditions of Theorem 53, prove that for $v \le s \le t \le u$, one has*

$$E\left[\frac{Z_u - Z_v}{u - v}\Big/ \mathcal{F}_{s,t}\right] = \frac{Z_t - Z_s}{t - s}.$$

Exercise 55 *(P. Tankov) Prove that in the case that Z is a simple Poisson process, then $\langle Z\rangle_t^{\mathcal{G}} = \int_0^t \frac{Z_T - Z_u}{T-u}du$. Guess the extension of this result for square integrable Lévy process*

Consider a pure jump case in order to simplify calculations. Let us suppose that we are given two independent simple Poisson processes N^+ and N^- which count two types of jumps. One of size $a^+ = a \in (0, \ln 2)$ and the other of size $a^- = \ln(2 - e^a) < 0$. The size of the jumps is not very important except that one has to be positive and the other negative. This particular choice simplifies the calculations. Furthermore suppose that the rates of jumps for each type is λ^+ and λ^- respectively. Then, we take the model $S(t) = S_0 \exp(\mu t + N_t)$ where $N_t = a^+ N_t^+ + a^- N_t^-$. As explained in Section 2, the approximative wealth process for transaction at time t_j, $j = 0, ..., i-1$ is

$$V(t_i) = V_0 \prod_{j=0}^{i-1} \left(1 + (1 - \pi(t_j)) \left(e^{r(t_{j+1}-t_j)} - 1\right) + \frac{\pi(t_j)}{S(t_j)} \left(S(t_{j+1}) - S(t_j)\right)\right).$$

Then

$$\begin{aligned}&\log(V(t_i))\\ &= \log(V_0) + \sum_{j=0}^{i-1} \log\left(1 + (1 - \pi(t_j)) \left(e^{r(t_{j+1}-t_j)} - 1\right) + \pi(t_j) \left(e^{\mu(t_{j+1}-t_j)+\left(N_{t_{j+1}}-N_{t_j}\right)} - 1\right)\right).\end{aligned}$$

Given that $\left(e^{r(t_{j+1}-t_j)} - 1\right) \approx r(t_{j+1} - t_j)$, $e^{\mu(t_{j+1}-t_j)} - 1 \approx \mu(t_{j+1} - t_j)$ and

$$\begin{aligned}&\log\left(1 + \frac{\pi(t_j)\left(e^{\left(N_{t_{j+1}}-N_{t_j}\right)} - 1\right)}{1 + (1 - \pi(t_j))\left(e^{r(t_{j+1}-t_j)} - 1\right) + \pi(t_j)\left(e^{\mu(t_{j+1}-t_j)} - 1\right)e^{\left(N_{t_{j+1}}-N_{t_j}\right)}}\right)\\ &\approx \log\left(1 + \pi(t_j)\left(e^{\left(N_{t_{j+1}}-N_{t_j}\right)} - 1\right)\right),\end{aligned}$$

we have that

$$\begin{aligned}\log(V(t_i)) \approx \log(V_0) &+ \sum_{j=0}^{i-1} \left\{(1 - \pi(t_j))r(t_{j+1} - t_j) + e^{\left(N_{t_{j+1}}-N_{t_j}\right)}\pi(t_j)\mu(t_{j+1} - t_j)\right\}\\ &+ \sum_{j=0}^{i-1} \log\left(1 + \pi(t_j)\left(e^{\left(N_{t_{j+1}}-N_{t_j}\right)} - 1\right)\right).\end{aligned}$$

For a càdlàg process π such that $\pi(s) \in (-(e^a - 1), e^a - 1)$, this converges to

$$\log(V_t) = \log(V_0) + \int_0^t ((1 - \pi(s))r + \pi(s)\mu)\, ds + \sum_{s \le t} \log(1 + \pi(s-)(e^{\Delta N(s)} - 1)).$$

This is a simple form of Itô's formula for jump processes (for a general formulation, see [21] Theorem 5.1, page 66). As $\Delta N(s) \in \{a^+, a^-\}$ we have that the logarithmic utility becomes

$$\begin{aligned}&E(\log(\hat{V}_t))\\ &= \log(V_0) + E\int_0^t (\mu - r)\pi(s)ds + E\sum_{s \le t} \log(1 + \pi(s-)(e^{\Delta N(s)} - 1))\\ &= \log(V_0) + E\int_0^t (\mu - r)\pi(s)ds + \sum_{i=-}^{+} \lambda_i E\int_0^t \log(1 + \pi(s)(e^{a_i} - 1))ds.\end{aligned}$$

Exercise 56 *Use an approximation argument to prove that for any càdlàg process π such that the expectations are finite we have that*

$$\begin{aligned}&E\left[\sum_{s \le t} \log(1 + \pi(s-)(e^{\Delta N(s)} - 1))\right] =\\ &\lambda^+ E\left[\int_0^t \log(1 + \pi(s)(e^a - 1))ds\right] + \lambda^- E\left[\int_0^t \log(1 + \pi(s)(1 - e^a))ds\right].\end{aligned}$$

Use Itô's formula for jump processes to derive this result.

As before the solution to the portfolio optimization for the small agent (i.e. non-insider) is obtained by analyzing the function

$$f(\pi) = (\mu - r)\pi + \int_{\mathbb{R}} \log\left(1 + (e^x - 1)\,\pi\right) F(dx)$$

where $F(dx) = \delta_{a^+}(dx)\lambda^+ + \delta_{a^-}(dx)\lambda^-$ and $\pi \in (-\frac{1}{e^a-1}, \frac{1}{e^a-1})$. The function f is a strictly concave function with respect to π and the first order condition $f'(\pi) = 0$ gives

$$\mu - r + \lambda^+ \frac{(e^a - 1)}{1 + (e^a - 1)\pi} + \lambda^- \frac{(1 - e^a)}{1 + (1 - e^a)\pi} = 0.$$

This equation reduces to a quadratic equation with two solutions. Let

$$\pi_+ = -\frac{\lambda^+ + \lambda^-}{2(\mu - r)} + \sqrt{\left(\frac{\lambda^+ + \lambda^-}{2(\mu - r)}\right)^2 + \frac{\lambda^+ - \lambda^-}{(\mu - r)(e^a - 1)} + \frac{1}{(e^a - 1)^2}}.$$

and π_- be the other solution. Then the restriction, $\frac{1}{e^a-1} > \pi > -\frac{1}{e^a-1}$, determines the optimal portfolio, denoted by π^*, as

$$\pi^* = \begin{cases} \pi_+ & \text{if } \mu > r \\ \frac{\lambda^+ - \lambda^-}{\lambda^+ + \lambda^-}(e^a - 1)^{-1} & \text{if } \mu = r \\ \pi_- & \text{if } \mu < r \end{cases}$$

The optimal logarithmic utility is finite (as the portfolio values are bounded) and given by

$$\log(V_0) + \left((\mu - r)\pi^* + \lambda^+ \log(1 + (e^a - 1)\pi^*) + \lambda^- \log(1 + (1 - e^a)\pi^*)\right) T.$$

Note that this result is valid as long as $\lambda^+ > 0$ and $\lambda^- > 0$.

Exercise 57 *If $\lambda^- = 0$ prove that the optimal logarithmic utility is infinite if $\mu \geq r$. What happens if $\mu < r$?*

The comparison with the Merton problem (see Section 2 and Figure 1) is as follows: While in the continuous model the ratio of investment on the stock grows linearly with the difference between the appreciation rate and the interest rate, in the jump model such growth is limited by the possible jump size in the opposite direction which may make our portfolio not admissible (that is, we may lose all our investment) with positive probability.

An interesting consequence of this analysis is that in models with bounded jump sizes the borrowing/loaning of shares/money is limited according to how big are jumps. Therefore models where the Lévy measure has unbounded support restrict $\pi \in [0, 1]$.

This characteristic implies that models with jumps are high risk models. In fact, this will lead to the insider to behave cautiously even if he/she has as information the final price of the asset.

That is, define the insider problem as one where the additional information is of the form $I = N(T)$ and $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(I)$. Using Theorem 53, we have that

$$N_t - \int_0^t \frac{N_T - N_s}{T - s} ds = \hat{N}_t$$

is a $\mathcal{G}$-martingale. Nevertheless this will not be enough to compute the logarithmic utility. In fact, we will further enlarge the filtration to $\mathcal{H}_t = \mathcal{F}_t \vee \sigma(N_T^+, N_T^-)$.

Exercise 58 *Prove that $\mathcal{H}_t = \mathcal{G}_t$ if it does not exist $a \in (0, \ln 2)$ such that $k_1 a + k_2 \ln(2 - e^a) = 0$ for a pair of integers k_1 and k_2.*

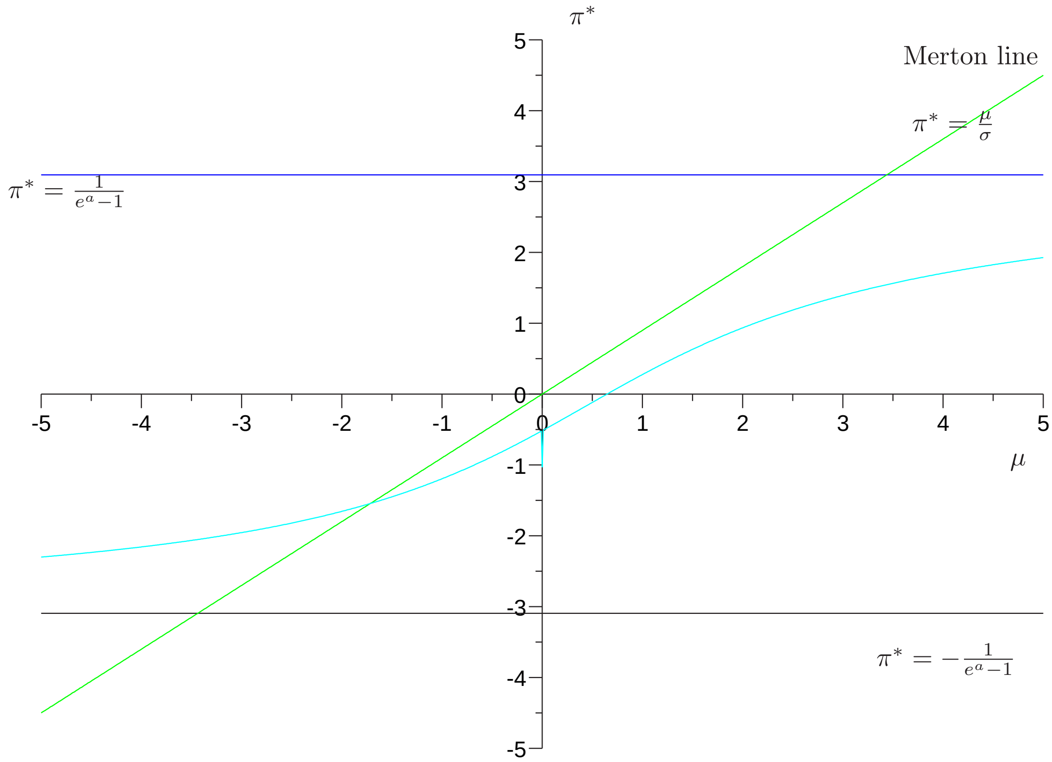

Figure 1: The Merton's line and the optimal portfolio for markets with jumps in the case $\lambda_- > \lambda_+$

Working in the filtration $\mathcal{H}$ we have that

$$N_t^i - \int_0^t \frac{N_T^i - N_s^i}{T - s} ds = \hat{N}_t^i$$

where $\hat{N}^i$ is a $\mathcal{H}$-martingale for $i = +, -$. Then following our previous calculations, we have that the logarithmic utility for the insider can be computed similarly. In fact we have the following result.

Exercise 59 *Prove that the utility for the insider is given by*

$$E\left[\log(\hat{V}_t)\right] \tag{14}$$

$$= \log(V_0) + E\left[\int_0^t (\mu - r)\pi(s)ds\right] + \sum_{i=-}^{+} E\left[\int_0^t \log(1 + \pi(s)(e^{a_i} - 1))\frac{N_T^i - N_s^i}{T - s}ds\right]$$

$$= \log(V_0) + E\left[\int_0^t (\mu - r)\pi(s)ds\right] + \sum_{i=-}^{+} E\left[\int_0^t \log(1 + \pi(s)(e^{a_i} - 1))B_i(s)ds\right], \tag{15}$$

where $B_i(s) = E\left[\frac{N_T^i - N_s^i}{T-s} \Big/ \mathcal{G}_s\right]$. *Note that the rates* λ_i *are replaced by the random rates* $B_i(s)$.

In general, $B_i(s)$ is positive unless we are in the case described in Exercise 58. In that case, the insider can "count"the jumps and $B_i(s)$ becomes zero after the last jump before T.

As before our objective function is

$$f_s(\pi) = (\mu - r)\pi + \sum_{i=-}^{+} \log(1 + \pi(e^{a_i} - 1))B_i(s)$$

for

$$\pi \in \begin{cases} (-(e^a-1)^{-1}, (e^a-1)^{-1}) & \text{if } B_i(s) > 0 \text{ for } i = -, + \\ (-(e^a-1)^{-1}, +\infty) & \text{if } B_+(s) > 0 \text{ and } B_-(s) = 0 \\ (-\infty, (e^a-1)^{-1}) & \text{if } B_-(s) > 0 \text{ and } B_+(s) = 0 \\ (-\infty, +\infty) & \text{if } B_+(s) = 0 \text{ and } B_-(s) = 0. \end{cases}$$

The function f_s is strictly concave in the first three cases and linear in the last.

CASE I: $B_i(s) > 0$ for $i = -, +$ for all $s \in [0, T]$. In this case the optimal solution is obtained as the solution of $f_s'(\pi) = 0$ which gives for $\mu \neq r$

$$\hat{\pi}_\pm = -\frac{B_+ + B_-}{2(\mu - r)} \pm \sqrt{\left(\frac{B_+ + B_-}{2(\mu - r)}\right)^2 + \frac{B_+ - B_-}{(\mu - r)(e^a - 1)} + \frac{1}{(e^a-1)^2}}. \tag{16}$$

As before, the right solutions for the optimal problem are determined with the restriction $\frac{1}{e^a-1} > \pi^*(s) > -\frac{1}{e^a-1}$

$$\hat{\pi}^*(s) = \begin{cases} \hat{\pi}_+(s) & \text{if } \mu > r \\ \frac{B_+ - B_-}{B_+ + B_-}(s)(e^a-1)^{-1} & \text{if } \mu = r \\ \hat{\pi}_-(s) & \text{if } \mu < r \end{cases}$$

In particular, we have the following result:

Theorem 60 *If π is a bounded $\mathcal{G}$-adapted portfolio with $\frac{1}{e^a-1} > \pi(s) > -\frac{1}{e^a-1}$ and $P\{\omega; B_+(s) > 0$ and $B_-(s) > 0$ for all $s \in [0, T]\} = 1$. Then the utility associated with π is finite.*

Proof. In fact, from (14) using that $\log(1 + \pi(s)(e^{a_i} - 1)) \leq \pi(s)(e^{a_i} - 1)$ we obtain that

$$\begin{aligned} E\left[\int_0^t \log(1 + \pi(s)(e^{a_i} - 1))B_i(s)ds\right] &\leq E\left[\int_0^t \pi(s)(e^{a_i} - 1)B_i(s)ds\right] \\ &\leq C(a_i)E\left[\int_0^t \left|\frac{N_T^i - N_s^i}{T - s}\right| ds\right] \\ &\leq C(a_i)\lambda_i t < +\infty. \end{aligned}$$

■

CASE II(a): In the case that $\lambda \times P\{(s, \omega); B_+(s) > 0$ and $B_-(s) = 0\} > 0$ and $\mu \geq r$ we have that as $\lim_{\pi \downarrow +\infty} f_s(\pi) = +\infty$ then the maximal utility will be infinite.

CASE II(b): If $\mu < r$ and $\lambda \times P\{(s, \omega); B_+(s) = 0\} = 0$ then $\pi^*(s) = -(e^a - 1)^{-1} - (\mu - r)^{-1}B_-(s)$ and utility is finite.

Exercise 61 *Prove something similar for the case $\lambda \times P\{(s, \omega); B_+(s) = 0$ and $B_-(s) > 0\} > 0$ with $\mu < r$. Also for the case $\lambda \times P\{(s, \omega); B_+(s) = 0$ and $B_-(s) = 0\} > 0$ prove that the optimal value for $f(\pi)$ is infinite. Do an analysis as in Case II, for the case that $\lambda \times P\{(s, \omega); B_i(s) = 0\} > 0$ for $i = +, -$.*

The situation most typical in financial markets should be that portfolios are bounded a.s. and the logarithmic utility of the $\mathcal{G}$-investor will be finite.

In summary we have proven that the fact that utility is finite or not depends on whether B_+or B_- are zero or not. To study this problem as we have already seen in exercise 58 is a matter of obtaining an algebraic characterization of the jump structure.

We divide the study in two cases:
Case 1: Assume that there exists $k_1,\ k_2 \in \mathbb{N}$ such that $k_1 a + k_2 \ln(2 - e^a) = 0$ then for any $x \in \mathbb{N}a + \mathbb{N}\ln(2 - e^a)$, $B_+(s) > 0$ and $B_-(s) > 0$ conditioned on $N_T = x$. Therefore portfolios are bounded and by a similar reasoning as in Theorem 60 maximal logarithmic utility is finite. The existence of a such that there exists $k_1,\ k_2 \in \mathbb{N}$ with $k_1 a + k_2 \ln(2 - e^a) = 0$ is assured by the continuity of the function $h(a) = -a^{-1}\ln(2 - e^a)$ for $a \in (0, \ln(2))$.
Case 2: On the contrary, if there are no $k_1,\ k_2 \in \mathbb{N}$ such that $k_1 a + k_2 \ln(2 - e^a) = 0$ then $P(B_+(s) = 0 \text{ for some subinterval of } [0,T]) > 0$ and $P(B_-(s) = 0 \text{ for some subinterval of } [0,T]) > 0$. In fact, there is always a time when given the value of $N(T)$ there can be no more both positive and negative jumps but only jumps of one type. Both probabilities being positive assures that the logarithmic utility of the $\mathcal{G}$-investor will be infinite.

Exercise 62 *Assume that there exists $k_1,\ k_2 \in \mathbb{N}$ such that $k_1 a + k_2 \ln(2 - e^a) = 0$, prove that the optimal conditional logarithmic utility of the insider in $[0,T]$ is finite.*

Exercise 63 *Prove that if the insider knows $N_+(T)$ and $N_-(T)$ then the insider achieves infinite utility.*

Exercise 64 *Prove that if we add a Wiener process to a Poisson process with positive jumps then there is a case where the utility is infinite.*

10 Enlargement of filtrations for random times

10.1 Jacod's Theorem for random times

Now we start to describe a general set-up that will become useful later when we treat another case of insider information: That is, the case of positive random variables which we will also called random times. In particular note that, for a simple Poisson process with parameter $\lambda_1 > 0$, and letting T_n denote the time of the n-th jump, we have that

$$P(T_n \geq x / \mathcal{F}_t) = 1\{x \leq T_n \leq t\} + 1\{T_n > t\} \int_{(x-t)\vee 0}^{\infty} \frac{\lambda_1 (\lambda_1 u)^{n-1-N_t} e^{-\lambda_1 u}}{(n-1-N_t)!} du \tag{17}$$
$$P(T_n \geq x) = \int_x^{\infty} \frac{\lambda_1 (\lambda_1 u)^{n-1} e^{-\lambda_1 u}}{(n-1)!} du.$$

Here the filtration $\mathcal{F}_t$ is the one generated by the Poisson process N. Therefore the conditional law of the random variable T_n ($P(T_n \geq x / \mathcal{F}_t)$) is not absolutely continuous with respect to a fixed measure ($P(T_n \geq x)$) but to a random one. This case can not be handled by Jacod's Theorem as stated. Note that, in this case, N_t does not have a density. In the financial application, this corresponds to the insider that knows the time of the n-th jump of the stock price of size bigger than a certain number (see example 10.2), which is a rough example of market timers. To achieve some generality we work in a semimartingale environment.

Let $Z = \{Z_t, 0 \leq t \leq T\}$ be a d dimensional semimartingale defined on a complete probability space $(\Omega, \mathcal{F}, P)$. Here, $(\mathcal{F}_t)_{t\in[0,T]} \equiv \left(\mathcal{F}_t^Z\right)_{t\in[0,T]}$ is the filtration generated by the process Z. We will assume through this Section unless stated otherwise that Z satisfies

$$\sup_{t\in[0,T]} E\left[|Z_t|\right] < \infty. \tag{18}$$

From now on, $\mathcal{G}$ denotes the smallest filtration including $\mathcal{F}$ and $\sigma(I)$.

For each $t \in [0,T]$, we denote by $P_t(\omega, dx)$ a regular version of the conditional law of a random variable τ given the σ-field $\mathcal{F}_t$, abbreviating it by $P_t(dx)$ if its nature as a measure is emphasized. We can choose this version in such a way that the following conditions are satisfied:

1. For every Borel set B on $\mathbb{R}^d$, $\{P_t(B), t \in [0,T]\}$ is an $(\mathcal{F}_t)_{t\in[0,T]}$-progressively measurable process.

2. For every $(t,\omega) \in [0,T] \times \Omega$, $P_t(\omega, dx)$ is a probability measure on $\mathbb{R}^d$.

3. For any bounded and $(\mathcal{F}_t)_{t\in[0,T]}$-adapted process $h : \Omega \times [0,T] \to \mathbb{R}$ and for any bounded and measurable function $f : \mathbb{R}^d \to \mathbb{R}$, we have

$$E\left[f(\tau)\int_0^T h_t dt\right] = E\left[\int_0^T \int_{\mathbb{R}^d} f(x)P_t(dx)h_t dt\right].$$

First, we consider a setup for initial enlargement of filtrations. That is, $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(\tau)$. In most situations when a random time is considered, one does not have that the measure $P_t^{(1)} << P_t$. Nevertheless, this is mostly due to the possible point measure at time τ. Therefore we consider a version of Jacod's theorem which excludes this point. Let $P_t(dx)$ be the regular conditional probability of τ given $\mathcal{F}_t$.

Definition 65 *We say that a random time τ belongs to the class $\mathcal{L}^*$, denoted by $\tau \in \mathcal{L}^*$, if there exists random kernels $P_t^{(i)}(\omega, dx)$, $i = 1,2$ such that*

1. *For every Borel set B in the positive real line, $\{P_t^{(i)}(B), t \in [0,T)\}$ is an $(\mathcal{F}_t)_{t\in[0,T)}$-progressively measurable process.*

2. *For every $(t,\omega) \in [0,T) \times \Omega$, $P_t^{(i)}(\omega, dx)$ is a signed measure on the real line.*

3. *For every $t \in [0,T)$, $E\left[\int_0^t \left|P_u^{(i)}\right| du\right] < \infty$.*

4. *For any bounded and $(\mathcal{F}_t)_{t\in[0,T]}$-adapted process $h : \Omega \times [0,T] \to \mathbb{R}$, for any bounded and measurable function $f : [0,\infty] \to \mathbb{R}$, and for every $t \in [0,T)$, we have*

$$E\left[f(\tau)1(\tau < s)\left(Z_t - Z_s\right)h_s\right] = E\left[\int_s^t \int_0^s f(x)P_u^{(1)}(dx)du h_s\right],$$
$$E\left[f(\tau)1(t < \tau)\left(Z_t - Z_s\right)h_s\right] = E\left[\int_s^t \int_t^T f(x)P_u^{(2)}(dx)du h_s\right].$$

Theorem 66 *Suppose that τ is a random time in the class $\mathcal{L}^*$ and Z is a semimartingale satisfying (18) such that $E\left|\Delta Z(\tau)\right| < \infty$. Assume that for almost all (t,ω), the signed measures $P_t^{(i)}(dx)$, $i = 1,2$ are absolutely continuous with respect to $P_t(dx)$, and set*

$$\alpha_t^{(i)}(x) = \frac{dP_t^{(i)}}{dP_t}(x).$$

We can choose a version of $\alpha_t^{(i)}(x)$ which is $\mathcal{P}\otimes\mathcal{B}(\mathbb{R})$-measurable, where $\mathcal{P}$ denotes the $\mathcal{F}_t$-progressive σ-field. Define

$$\beta(u) = \alpha_u^{(1)}(\tau)1(u \geq \tau) + \alpha_u^{(2)}(\tau)1(u < \tau).$$

Then $Z_t - \int_0^t \beta(u)du - \Delta Z(\tau)1(t \geq \tau)$ is a martingale with respect to the filtration $(\mathcal{G}_t)_{t\in[0,T)}$.

Proof. We choose versions of $\alpha_t^{(i)}(x)$ which is $\mathcal{P} \otimes \mathcal{B}(\mathbb{R})$-measurable for $i = 1,2$. Let h be a measurable adapted bounded process and f a bounded measurable function on $\mathbb{R}$. Set $F = f(\tau)$.

Then we have

$$\begin{aligned}
&E\left[(Z_t - Z_s)F1(t<\tau)h_s\right] = E\left[(Z_t - Z_s)f(\tau)1(t<\tau)h_s\right] \\
&= E\left[\int_s^t \int_t^T f(x)P_u^{(2)}(dx)du h_s\right] \\
&= E\left[\int_s^t \int_t^T f(x)\alpha_u^{(2)}(x)P_u(dx)du h_s\right] \\
&= E\left[\int_s^t f(\tau)\alpha_u^{(2)}(\tau)du 1(t<\tau)h_s\right] \\
&= E\left[\int_s^t \alpha_u^{(2)}(\tau)du F1(t<\tau)h_s\right].
\end{aligned}$$

Similarly, one obtains that

$$E\left[(Z_t - Z_s)F1(\tau<s)h_s\right] = E\left[\int_s^t \alpha_u^{(1)}(\tau)du F1(\tau<s)h_s\right].$$

To finish the proof we consider the general case. Let $\pi = \{t_0 < s = t_1 < ... < t_{n-1} = t < t_n\}$ be a partition with $|\pi| = \max\{t_k - t_{k-1}; 1 \le k \le n\}$.

$$\begin{aligned}
&E\left[(Z_t - Z_s)Fh_s\right] \\
&= E\left[\left(1(\tau \le t_0)\int_s^t \alpha_u^{(1)}(\tau)du + 1(t_n < \tau)\int_s^t \alpha_u^{(2)}(\tau)du\right)Fh_s\right] \\
&+ \sum_{j=1}^{n-2}\sum_{k=0}^{n-1} E\left[(Z_{t_{j+1}} - Z_{t_j})F1(t_k < \tau \le t_{k+1})h_s\right].
\end{aligned}$$

Let's consider the last term

$$\begin{aligned}
&\sum_{j=1}^{n-2}\sum_{k=0}^{n-1} E\left[(Z_{t_{j+1}} - Z_{t_j})F1(t_k < \tau \le t_{k+1})h_s\right] \\
&= E\sum_{j<k}\left[(Z_{t_{j+1}} - Z_{t_j})F1(t_k < \tau \le t_{k+1})h_s\right] \\
&+ E\sum_{k=1}^{n-2}\left[(Z_{t_{k+1}} - Z_{t_k})F1(t_k < \tau \le t_{k+1})h_s\right] \\
&+ \sum_{j>k} E\left[(Z_{t_{j+1}} - Z_{t_j})F1(t_k < \tau \le t_{k+1})h_s\right].
\end{aligned}$$

Now each term can be rewritten as follows:

$$\begin{aligned}
&E\left[\sum_{j<k}(Z_{t_{j+1}} - Z_{t_j})F1(t_k < \tau \le t_{k+1})h_s\right] \\
&= E\left(\sum_{j<k}\int_{t_j}^{t_{j+1}} \alpha_u^{(2)}(\tau)du F1(t_k < \tau \le t_{k+1})h_s\right) \\
&= E\left(\sum_{j=1}^{n-2}\int_{t_j}^{t_{j+1}} \alpha_u^{(2)}(\tau)du F1(t_{j+1} < \tau \le t_n)h_s\right) \\
&\to E\left(\int_s^t 1(u<\tau\le t)\alpha_u^{(2)}(\tau)du F h_s\right),
\end{aligned}$$

$$\sum_{k=1}^{n-2} E\left[(Z_{t_{k+1}} - Z_{t_k})F1(t_k < \tau \le t_{k+1})h_s\right] \to E\left[\Delta Z(\tau)F1(s < \tau \le t)h_s\right]$$

and

$$\begin{aligned}
&E\left[\sum_{j>k}(Z_{t_{j+1}} - Z_{t_j})F1(t_k < \tau \le t_{k+1})h_s\right] \\
&= E\left[\sum_{j>k} F1(t_k < \tau \le t_{k+1}) \int_{t_j}^{t_{j+1}} \alpha_u^{(1)}(\tau)du h_s\right] \\
&= E\left[\sum_{j=1}^{n-1} F1(t_0 < \tau \le t_j) \int_{t_j}^{t_{j+1}} \alpha_u^{(1)}(\tau)du h_s\right] \\
&\to E\left[\int_s^t F1(s < \tau \le u)\alpha_u^{(1)}(\tau)du h_s\right]
\end{aligned}$$

as $|\pi| \downarrow 0$. Therefore $Z_t - B(t)$ is a martingale in the filtration $(\mathcal{G}_t)_{t\in[0,T)}$ where $B(t) = \int_0^t \beta(u)du + \Delta Z(\tau)1(t \ge \tau)$. ■

10.2 Example of market timers: n-th price jump

Now we consider some simple examples of the above situation. This example treats the situation where the filtration is enlarged by the time of the n-th jump of positive size. To simplify consider the case of the model introduced in Section 9 let N be a compound Poisson process with two types of jumps: $N_t = a^+N_t^+ + a^-N_t^-$.

Let T_n be the random time associated with the n-th jump associated with the process N_t^+. That is, $T_n = \inf\{s; N_s^+ = n\}$. The random variable T_n is a $\mathcal{F}$ stopping time and in this case, we define $\mathcal{F}_t = \sigma(N_u^i; u \le t, i = -, +)$ and $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(T_n)$. That is, the insider knows in advance the time at which the stock will jump positively for the n-th time.

Theorem 67 *We have the following decomposition*

$$N_t = \widehat{N}_t + a^-\lambda^-t + a^+\int_0^{t\wedge T_n} \frac{N_{T_n-}^+ - N_u^+}{T_n - u}du + a^+\left(\lambda^+(t - T_n) + 1\right)1(t \ge T_n)$$

where $\widehat{N}$ is a $\mathcal{G}$-martingale (Note that $N_{T_n-}^+ = n-1$ so that $\int_0^{t\wedge T_n} \frac{N_{T_n-}^+ - N_u^+}{T_n-u}du = \int_0^{t\wedge T_{n-1}} \frac{N_{T_n-}^+ - N_u^+}{T_n-u}du$).

Proof. We apply Theorem 66 and we start calculating $P_u^{(1)}$:

$$\begin{aligned}
E\left[f(T_n)1(T_n < s)\left(N_t - N_s\right)h_s\right] &= E\left[f(T_n)1(T_n < s)E\left[N_t - N_s/\,\mathcal{F}_s\right]h_s\right] \\
&= \lambda(t-s)E\left[f(T_n)1(T_n < s)h_s\right].
\end{aligned}$$

Therefore $dP_u^{(1)} = \lambda dP_u$. Next we compute

$$\begin{aligned}
&E\left[f(T_n)1(T_n > t)\left(N_t - N_s\right)h_s\right] \\
&= E\left[E\left[f(T_n)1(T_n > t)/\,\mathcal{F}_t\right]\left(N_t - N_s\right)h_s\right] \\
&= E\left[\int_0^{+\infty} f(u+t)\frac{\lambda^+\left(\lambda^+u\right)^{n-1-N_t^+}e^{-\lambda^+u}}{\left(n-1-N_t^+\right)!}du\left(N_t - N_s\right)h_s\right] \\
&= a^-\lambda^-(t-s)E\left[f(T_n)1(T_n > t)h_s\right] \\
&+ a^+E\left[\int_0^{+\infty} f(u+t)\frac{\lambda^+\left(\lambda^+u\right)^{n-1-N_t^+}e^{-\lambda^+u}}{\left(n-1-N_t^+\right)!}du\left(N_t^+ - N_s^+\right)h_s\right].
\end{aligned}$$

Here we have used the conditional distribution of τ given $\mathcal{F}_t$ (see equation (17)) and the independence of N^+ and N^-. Now we compute the second expectation on the right hand side above. This gives using the probability distribution of $N_t^+ - N_s^-$

$$\begin{aligned}
&E\left[\int_0^{+\infty} f(u+t)\frac{\lambda^+ (\lambda^+ u)^{n-1-N_t^+} e^{-\lambda^+ u}}{(n-1-N_t^+)!}du\left(N_t^+ - N_s^+\right)h_s\right] \\
&= \sum_{j=1}^{\infty} \frac{e^{-\lambda^+(t-s)}(\lambda^+(t-s))^j}{(j-1)!}E\left[\int_0^{+\infty} f(u+t)\frac{\lambda^+ (\lambda^+ u)^{n-1-j-N_s^+} e^{-\lambda^+ u}}{(n-1-j-N_s^+)!}duh_s\right] \\
&= \left(\lambda^+\right)^2(t-s)E\left[\int_0^{+\infty} f(u+t)\frac{(\lambda^+ u)^{n-2-N_t^+} e^{-\lambda^+ u}}{(n-2-N_t^+)!}duh_s\right] \\
&= (t-s)E\left[\int_0^{+\infty} f(u+t)\frac{N_{T_n-}^+ - N_t^+}{u}\frac{\lambda^+ (\lambda^+ u)^{n-1-N_t^+} e^{-\lambda^+ u}}{(n-1-N_t^+)!}duh_s\right] \\
&= (t-s)E\left[\frac{N_{T_n-}^+ - N_t^+}{T_n - t}f(T_n)1(T_n > t)h_s\right].
\end{aligned}$$

Then, one can verify that

$$E\left[f(T_n)1(T_n > t)\left(N_t - N_s - a^-\lambda^-(t-s) - a^+\int_s^t \frac{N_{T_n-}^+ - N_u^+}{T_n - u}du\right)h_s\right] = 0. \tag{19}$$

Therefore

$$dP_u^{(2)} = a^+\frac{N_{T_n-}^+ - N_u^+}{T_n - u}dP_u,$$

and the conclusion follows from Theorem 66. ■

Exercise 68 *Verify the previous equality (19).*

Note that the above Theorem is also valid for $\lambda^- = 0$ or $\lambda^+ = 0$To adapt to the fact that T_n takes values in $[0, +\infty)$ we maximize the following utility

$$\max_{\pi \in \mathcal{G}} \int_0^{+\infty} e^{-rs}E\left[\log(\hat{V}_s)\right]ds.$$

Following a similar discussion as in the previous example one finds that the optimal portfolio for $s \leq T_{n-1}$ is given as in the previous example by

$$\hat{\pi}^*(s) = \begin{cases} \hat{\pi}_+(s) & \text{if } \mu > r \\ \frac{n-1-N_s^+ - \lambda^-(T_n - s)}{n-1-N_s^+ + \lambda^-(T_n - s)}(e^a - 1)^{-1} & \text{if } \mu = r \\ \hat{\pi}_-(s) & \text{if } \mu < r \end{cases}$$

where $\hat{\pi}_\pm(s)$ is also the solution of the corresponding quadratic equation (16) with $B_+(u) = \frac{N_{T_n-}^+ - N_u^+}{T_n - u}$ and $B_-(u) = \lambda^-$.

Exercise 69 *Prove that the optimal utility of the insider in the interval* $[0, T_{n-1}]$ *is finite for* $n \geq 2$.

Next for $t \in (T_{n-1}, T_n)$ we have that if $\mu > r$ then $\hat{\pi}^*(s) = (e^a - 1)^{-1} - \lambda^-(\mu - r)^{-1}$. In contrast, if $\mu \leq r$ then there is no optimal value and the maximal logarithmic utility is infinite. The optimal portfolio after T_n of the $\mathcal{G}$-investor and the $\mathcal{F}$-investor coincide. In conclusion if $\mu > r$ then the optimal logarithmic utility of the insider in the interval $[0, T_n)$ is finite if and only if $\mu > r$. Clearly the analysis of the optimal portfolio and the utility after T_n is like in the beginning of Section 9. In the interval $[0, T_n]$ there is no optimal portfolio in the case $\mu > r$.

Exercise 70 *Provide a careful analysis to prove that the logarithmic utility of the insider is finite in the interval* $[0, T_n)$ *if and only if* $\mu > r$.

11 The insider as a large trader

In this chapter we initiate a first glance of the interactions between three distinguished elements which where somewhat independent in the previous chapters: the stock price, the insider strategy and the small trader.

First we will start with a study of the effect of the insider's strategy on the stock price dynamics. That is, suppose that the insider information is modelled using a filtration $\mathcal{G}$ and that the insider's strategy, π, is adapted to $\mathcal{G}$. Then we first consider the following discrete model

$$S_n(t_{i+1}) = S_n(t_i)\left(1 + (\mu + b\pi(t_i))\,(t_{i+1} - t_i) + \sigma\left(W(t_{i+1}) - W(t_i)\right)\right)$$

then we have that

$$\begin{aligned}
S_n(t_n) &= S_0 \prod_{i=0}^{n-1} \left(1 + (\mu + b\pi(t_i))\,(t_{i+1} - t_i) + \sigma\left(W(t_{i+1}) - W(t_i)\right)\right) \\
&= S_0 \exp\left(\sum_{i=0}^{n-1} \log\left(1 + (\mu + b\pi(t_i))\,(t_{i+1} - t_i) + \sigma\left(W(t_{i+1}) - W(t_i)\right)\right)\right) \\
&\approx S_0 \exp\left(\sum_{i=0}^{n-1} \left(\left(\mu + b\pi(t_i) - \frac{\sigma^2}{2}\right)(t_{i+1} - t_i) + \sigma\left(W(t_{i+1}) - W(t_i)\right)\right)\right).
\end{aligned}$$

Therefore taking limits we have that S_n converges a.s. to

$$S(t) = S_0 \exp\left(\int_0^t \left(\mu + b\pi(s) - \frac{\sigma^2}{2}\right) ds + \sigma W(t)\right).$$

In order to formalize the idea that S is the solution of a linear stochastic differential equation we need the notion of the forward integral.

Definition 71 *Let $\phi : [0,T] \times \Omega \to$ be a measurable (non necessarily adapted) continuous process. The* forward integral *of ϕ with respect $W(.)$ is defined by*

$$\int_0^T \phi(t) d^- W(t) = \lim_{n\to+\infty} \sum_{i=0}^{n-1} \phi(t_i)(W(t_{i+1}) - W(t_i)), \tag{20}$$

if the limit exists in $L^1(\Omega)$ and is independent of the partition sequence taken.

This definition does not coincide exactly with the definition of Russo-Vallois. Under some more assumptions one can prove that this definition coincides with the original definition of Russo-Vallois.

Now, we can say that the class of admissible portfolios for the large trader are the ones such that S is the unique solution to the following model for the stock price

$$S(t) = S(0) + \int_0^t (\mu + b\pi(s))\, S(s) ds + \int_0^t \sigma S(s) d^- W(s). \tag{21}$$

Note that since π is not adapted to the filtration generated by W, $\mathcal{F}$, the usual rules of stochastic calculus do not apply. In particular, one has the following result

Exercise 72 *For $f \in C_b^1([0,T] \times \mathbb{R})$ prove that*

$$E\Big[\int_0^T f(t, W(T)) d^- W(t)\Big] = E\Big[\int_0^T \frac{\partial}{\partial x} f(t, W(T)) dt\Big].$$

That is, forward integrals do not have expectation zero.

The situation described above models the fact that the insiders policies have an effect on the price dynamics. From now on we assume that $0 < b < \sigma^2/2$. The condition $b > 0$ expresses that as the insider increases his/her portfolio holdings then the price of the stock increases. The condition $b < \sigma^2/2$ expresses that the volatility has to be big enough to "hide" the insider's behavior.

Now in order to express the wealth process we need to further assume that W is a semimartingale in $\mathcal{G}$ with the decomposition $W(t) = \hat{W}(t) + \int_0^t \alpha(s)ds$ where α is a $\mathcal{G}$-adapted process and $\hat{W}$ is a $\mathcal{G}$ Wiener process. Then we have in the above expression that

$$\int_0^t \sigma S(s)dW(s) = \int_0^t \sigma S(s)d\widehat{W}(s) + \int_0^t \sigma S(s)\alpha(s)ds.$$

As in section 2, one defines the optimal logarithmic utility problem for the insider. In such a case, we obtain that the logarithmic utility is

$$E(\left[\log(\hat{V}(t))\right] = \log(V_0) + \int_0^t E\left[\pi(s)(\mu - r + \sigma\alpha(s)) + \left(b - \frac{1}{2}\sigma^2\right)\pi(s)^2\right] ds.$$

As before, one considers the strictly concave function $f_s(\pi) = \pi(\mu - r + \sigma\alpha(s)) + \left(b - \frac{1}{2}\sigma^2\right)\pi^2$. Note that here we use that $b < \sigma^2/2$. Then the optimal portfolio for the insider is $\hat{\pi}(s) = \frac{\mu - r + \sigma\alpha(s)}{\sigma^2 - 2b}$ and the optimal logarithmic utility is $\log(V_0) + \frac{(\mu-r)^2 t}{2(\sigma^2-2b)} + \int_0^t E\left[\frac{\sigma^2\alpha(s)^2}{2(\sigma^2-2b)}\right] ds$.

Exercise 73 *Consider the case that the large investor is not an insider. That is, the admissible portfolios are* $(\mathcal{F}_t)_{t\in[0,T]}$ *−adapted portfolios. Prove that the optimal portfolio in this case is* $\hat{\pi} = \frac{\mu-r}{\sigma^2-2b}$ *and the optimal logarithmic utility is* $\log(V_0) + \frac{(\mu-r)^2 t}{2(\sigma^2-2b)}$*. In this case the model for the underlying increases its instantaneous return if and only if* $\mu \geq r$*.*

Note that in the case that $b \geq \sigma^2/2$ then the system explodes as the function f_s becomes convex. One first example of application may be the classical case $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(W(T))$. Nevertheless this example loses some of its interest because it is clear that $\sigma(W(T)) \neq \sigma(S(T))$. Therefore the information held by the insider is not clearly interpretable from a financial point of view.

Exercise 74 *Prove that* $\sigma(S(T)) = \sigma(\int_0^T \frac{b\alpha(r)}{\sigma^2-2b}dr + W(T))$*.*

Clearly the problem we are proposing here is a fixed point problem and related with some type of equilibrium concept. If the information of the insider is $\mathcal{F}_t \vee \sigma(S(T))$, but the price is also influenced by the information itself through the portfolio π. In order to solve this situation we will use Example 24 to obtain the following result:

Theorem 75 *Suppose that* $0 < b < \sigma^2/2$ *and define the following portfolio for the insider*

$$\begin{aligned}
\hat{\pi}(t) &= \frac{\mu - r}{\sigma^2 - 2b} + \frac{\sigma}{\sigma^2 - 2b} a(t)A(t)^{-1} \int_t^T a(r)dW(r) \\
a(t) &= (T-t)^\theta \\
A(t) &= \int_t^T a(r)^2 dr \\
\theta &= -b\sigma^{-2} \in (-0.5, 0).
\end{aligned}$$

Then the portfolio $\hat{\pi}$ *is the optimal portfolio for the large investor under model (21) with information* $\mathcal{F}_t \vee \sigma(S(T))$*.*

Proof. First suppose that we are given the portfolio $\hat{\pi}$. Then, we have that

$$\log(S(T)/S(0)) = \int_0^T \left(\mu + b\hat{\pi}(s) - \frac{\sigma^2}{2}\right) ds + \sigma W(T).$$

Therefore the sigma field generated by $S(T)$ is the same as the sigma field generated by the random variable

$$\begin{aligned} Y &= \frac{b\sigma}{\sigma^2 - 2b} \int_0^T a(t)A(t)^{-1} \int_t^T a(r)dW(r)dt + \sigma W(T) \\ &= \frac{b\sigma}{\sigma^2 - 2b} \int_0^T a(r) \int_0^r a(t)A(t)^{-1} dtdW(r) + \sigma W(T). \end{aligned}$$

After some calculations, we obtain that

$$\begin{aligned} A(t) &= \int_t^T (T-u)^{2\theta} du = (2\theta+1)^{-1}(T-t)^{2\theta+1} \\ \int_0^r a(t)A(t)^{-1}dt &= (2\theta+1)\theta^{-1}\left((T-r)^{-\theta} - T^{-\theta}\right) \\ Y &= \int_0^T \left(\frac{b\sigma}{\sigma^2-2b}(2\theta+1)\theta^{-1}\left(1 - T^{-\theta}(T-r)^{\theta}\right) + \sigma \right) dW(r) \\ &= \sigma T^{-\theta} \int_0^T (T-r)^{\theta} dW(r) \end{aligned}$$

Therefore the filtration $\mathcal{F}_t \vee \sigma(Y)$ is of the type of Exercise 24 which gives that the compensator is

$$\alpha(t) = a(t)A(t)^{-1} \int_t^T a(r)dW(r).$$

Note that $\widehat{\pi}$ is admissible as it is generated through an enlargement of filtrations procedure. ■

Exercise 76 *Prove that* $\sigma(S(s); s \le t) \vee \sigma(S(T)) = \mathcal{F}_t \vee \sigma(Y)$.

One natural question after this calculation is what the small investor can do in this situation? One should then note that the small investor does not have access to $\hat{\pi}$ or to the filtration $\mathcal{G}$. That is, the small investor may try to do his "best" possible model with the data he/she possesses which is $\mathcal{H}_t = \sigma(S(s); s \le t) = \sigma\left(\int_0^s \frac{b\alpha(r)}{\sigma^2-2b}dr + W(s); s \le t\right) \subseteq \mathcal{G}$. Then the model that the small insider will use is

$$S(t) = S_0 + \int_0^t E\left[\mu + b\widehat{\pi}(s)/\mathcal{H}_s\right] S(s)ds + \int_0^t \sigma S(s)d\widetilde{W}(s) \tag{22}$$

where $\widetilde{W}$ is a Wiener process on $\mathcal{H}$ (here we are assuming that $\mathcal{H}$ can support a Wiener process). In this situation the small investor will use as optimal portfolio

$$\begin{aligned} \widetilde{\pi}^*(s) &= \frac{\mu - r}{\sigma^2} + \frac{b}{\sigma^2} E\left[\widehat{\pi}(s)/\mathcal{H}_s\right] \\ &= \frac{(\mu-r)(\sigma^2-b)}{\sigma^2(\sigma^2-2b)} + \frac{b}{\sigma(\sigma^2-2b)} E\left[\alpha(s)/\mathcal{H}_s\right], \end{aligned}$$

and the optimal logarithmic utility in $[0,t]$ the small trader expects to gain with model (22) is

$$A = \log(V_0) + \frac{(\mu-r)^2(\sigma^2-b)^2}{2\sigma^2\left(\sigma^2-2b\right)^2} t + \frac{b^2}{2\left(\sigma^2-2b\right)^2} \int_0^t E\left[E\left[\alpha(s)/\mathcal{H}_s\right]^2\right] ds.$$

Nevertheless as the actual model for S is the one in (21) we have that the actual utility of the small investor is

$$A + \frac{b}{\sigma^2 - 2b} \int_0^t E\left[E\left[\alpha(s)/\mathcal{H}_s\right]^2\right] ds.$$

The interpretation of this result is the following: If you make more money than what you expect with your adapted model, it may be because there is a large investor exerting an influence on the price of the market. This may look at bit odd but in fact the risk of the small trader is also bigger than he/she thinks it is.

Obviously, if the small trader is able to guess the model of the large trader his expected logarithmic utility will further increase (but always finite) and the optimal portfolio will be

$$\pi^*(t) = \widetilde{\pi}^*(s) + \sigma^{-1} E\left[\alpha(s)/\mathcal{H}_s\right]$$

One of the remaining problems with this model is that it still explodes when the logarithmic utility for the insider/large trader is considered in the interval $[0, T]$. Nevertheless, the problem can also be solved using the techniques explained in Section 8.

For this reason, we study some simpler models to try to understand better the structure of the enlargement of filtrations approach within this model. Somewhat this is the goal of the next section. Before that we make a remark.

Remark 77 *We have assumed that the filtration $\mathcal{H}$ is rich enough to support a Wiener process $\widetilde{W}$. In fact, one can prove that if there exists an optimal portfolio $\widetilde{\pi}^*$ for the small trader leading to a finite utility, then this is the case and furthermore the drift of the model is*

$$E\left[\mu + b\widehat{\pi}(s)/\mathcal{H}_s\right] = \sigma^2 \widetilde{\pi}^*(s) + r.$$

That is, the projection of the anticipating large trader insider model into the filtration given by the price gives as result that the optimization problem for the small trader is the Merton problem.

12 Continuous stream of information

In the particular case where $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(W(T))$, the optimal portfolio for the insider is a function of $\frac{W(T)-W(t)}{\sigma(T-t)}$ (similarly for Theorem 75, the optimal portfolio is anticipating). Roughly speaking, the above model allows to introduce the anticipation through the drift of the sde defining the stock price model. Therefore one way to introduce a continuous information in the market is by taking a drift that depends on the variables representing his/her additional information.

To look at a concrete "toy" example (think of various reasons why this is a toy example) consider for $\delta > T$ fixed

$$S(t) = S(0) + \int_0^t (\mu + bW(s+\delta))\, S(s)ds + \int_0^t \sigma S(s) d^- W(s). \tag{23}$$

In this model, the insider has an effect on the drift of the diffusion through information that is δ units of time in the future. This continuous deformation of information may be used to model streams of information rather than one single piece of information. In this case, it is difficult to see what is the information held by the insider but his/her effect on the market is known.

Exercise 78 *Prove that W is not a semimartingale on the filtration $(\mathcal{F}_{t+\delta})_{t\in[0,T]}$.*

In such a situation we are interested in looking at the optimal policy of the small investor. That is, the small investor filtration is $\mathcal{H}_t = \sigma(S_s; s \le t)$. The above stochastic integral can be treated as in the previous section and this gives as solution

$$S(t) = S(0) \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right) t + b\int_\delta^{t+\delta} W(s)ds + \sigma W(t)\right).$$

Therefore $\mathcal{H}_t = \sigma\left(b\int_\delta^{s+\delta} W(r)dr + \sigma W(s); s \le t\right)$.

We now describe the wealth process for a strategy π such that $E\int_0^T \pi^2(s)ds < \infty$ through the discrete time argument, as in (4) (recall also a similar argument in Section 9), to obtain that

$$\log(V(t_i)) \approx \log(V_0) + \sum_{j=0}^{i-1}\left(\left((1-\pi(t_j))r + \pi(t_j)\left(\mu - \frac{1}{2}\sigma^2\right)\right)(t_{j+1}-t_j) + \pi(t_j)b\int_{t_j+\delta}^{t_{j+1}+\delta} W(s)ds\right)$$
$$+ \sum_{j=0}^{i-1}\log\left(1+\pi(t_j)\left(\exp\left(\sigma\left(W(t_{j+1})-W(t_j)\right)\right)-1\right)\right)$$

As before the first integral will converge to the Lebesgue integral. For the second, we will have to make assumptions on π to obtain the convergence as this sum will tend to an anticipating stochastic integral (that is, there may be correlations between $\pi(t_j)$ and the increments $(W(t_{j+1}) - W(t_j))$). To see this consider the Taylor expansion approximation of the last term above

$$\sum_{j=0}^{i-1}\left(\pi(t_j)\left(\sigma\left(W(t_{j+1})-W(t_j)\right)\right) + \frac{1}{2}\pi(t_j)\left(1-\pi(t_j)\right)\sigma^2\left(W(t_{j+1})-W(t_j)\right)^2\right).$$

Now we suppose that $\sum_{j=0}^{i-1}\left(\pi(t_j)\left(W(t_{j+1})-W(t_j)\right)\right)$ converges in L^1 to an integrable random variable denoted by $\int_0^t \pi(s)dW(s)$. Note in particular that the expectation of this random variable is not necessarily zero as there may be covariances between $\pi(t_j)$ and $(W(t_{j+1}) - W(t_j))$. Also suppose that $\sum_{j=0}^{i-1}\left(\frac{1}{2}\pi(t_j)\left(1-\pi(t_j)\right)\sigma^2\left(\left(W(t_{j+1})-W(t_j)\right)^2 - (t_{j+1}-t_j)\right)\right)$ converges in L^1 to zero. We will later show that the optimal portfolios proposed satisfy this condition. With these assumptions, we have that the limit of the logarithmic wealth process can be written as

$$J(t,\pi) := E\left[\log(\hat{V}(t))\right] \tag{24}$$
$$= \log(V_0) + E\left[\int_0^t\left(\pi(s)(\mu - r + bW(s+\delta)) - \frac{1}{2}\sigma^2\pi(s)^2\right)ds\right] + \sigma E\left[\int_0^t \pi(s)d^-W(s)\right]. \tag{25}$$

Here $d^-W(s)$ denotes the forward stochastic integral of Russo-Vallois.

The previous discussion can be carried out in full generality without going through the above approximative argument (see Kohatsu-Sulem) but we have preferred the above approach as to convince the reader that the concept of forward integral is natural for the above financial problem and that it is not an artificial mathematical construct.

For the rest of the discussion suppose that the optimization problem $\max_\pi J(t,\pi)$ has a solution. Then we apply a variational argument to J to obtain for any $\mathcal{H}$ adapted process v so that the forward integral of this process exists. Then the first order condition is

$$\left.\frac{\partial J(t,\pi^*+\epsilon v)}{\partial \epsilon}\right|_{\epsilon=0} = E\left[\int_0^t\left(v(s)(\mu - r + bW(s+\delta)) - \sigma^2\pi^*(s)v(s)\right)ds\right] + \sigma E\left[\int_0^t v(s)d^-W(s)\right] = 0 \tag{26}$$

Note that the second order condition is satisfied (J is a concave functional). Now we consider $v(s) = X1(s \geq \theta)$ for $\theta \leq t$ fixed and X an $\mathcal{H}_\theta$ measurable random variable which is forward integrable. Then we have that

$$E\left[X\left(\int_\theta^t\left((\mu - r + bW(s+\delta)) - \sigma^2\pi^*(s)\right)ds + \sigma\left(W(t)-W(\theta)\right)\right)\right] = 0.$$

This gives that

$$E\left[\int_\theta^t\left((\mu - r + bW(s+\delta)) - \sigma^2\pi^*(s)\right)ds + \sigma\left(W(t)-W(\theta)\right)\Big/\mathcal{H}_\theta\right] = 0. \tag{27}$$

Therefore this also proves that $E\left(W(t) - W(\theta)/\mathcal{H}_\theta\right)$ is differentiable in t and furthermore that

$$\pi^*(\theta) = \frac{\mu - r}{\sigma^2} + \frac{b}{\sigma^2} E\left(W(\theta+\delta)/\mathcal{H}_\theta\right) + \frac{1}{\sigma} \lim_{t \to \theta} E\left[\frac{W(t) - W(\theta)}{t - \theta} \Big/ \mathcal{H}_\theta\right].$$

We are then reduced to the computation of $E\left[W(s)/\mathcal{H}_t\right]$ for $s \geq t$.

Lemma 79 *Define* $Y(t) = b\int_\delta^{t+\delta} W(r)dr + \sigma W(t)$*. Then for* $\delta \geq T$

$$\lim_{s \downarrow t} E\left[\frac{W(s) - W(t)}{s - t} \Big/ \mathcal{H}_t\right] = bM \int_0^t g(t,u)dY(u).$$

$$E\left[W(t+\delta)/\mathcal{H}_t\right] = (b(t+\delta) + \delta)M \int_0^t g(t,u)dY(u)$$

where $M \equiv M_t = \sigma^{-1} b\left((b\delta + 2\sigma)\left(e^{\frac{2bt}{\sigma}} - 1\right) + \sigma\left(e^{\frac{2bt}{\sigma}} + 1\right)\right)^{-1}$ *and* $g(t,u) = e^{\frac{b}{\sigma}(2t-u)} + e^{\frac{b}{\sigma}u}$.

Proof. First note that Y is a Gaussian process. Therefore $E\left[W(s)/\mathcal{H}_t\right] = \int_0^t h(s,t,u)dY(u)$ for a deterministic function h. To compute h we compute the covariances between $W(s)$ and the stochastic integral and $Y(v)$ for some $v \leq t \leq s$. First

$$E\left[W(s)Y(v)\right] = bsv + \sigma(s \wedge v). \tag{28}$$

Also

$$E\left[\int_0^t h(s,t,u)dY(u)Y(v)\right] = b^2 \int_0^t \int_0^u h(s,t,\theta_1)(\theta_1 \wedge \theta_2 + \delta)d\theta_2 d\theta_1 \tag{29}$$

$$+ 2b\sigma v \int_0^t h(s,t,\theta)d\theta + \sigma^2 \int_0^u h(s,t,\theta)d\theta. \tag{30}$$

Therefore the above two expressions have to be equal. After differentiation of the equality wrt $v \leq t$ three times, we obtain

$$-b^2 h(s,t,u) + \sigma^2 \frac{\partial^2 h}{\partial u^2}(s,t,u) = 0.$$

Solving this differential equation gives

$$h(s,t,u) = C_1(s,t)e^{-\frac{b}{\sigma}u} + C_2(s,t)e^{\frac{b}{\sigma}u}. \tag{31}$$

Next one verifies that for the following constants, equations (28) and (29) coincide.

$$C_2(s,t) = \sigma^{-1} b(bs + \delta)\left((b\delta + 2\sigma)\left(e^{\frac{2bt}{\sigma}} - 1\right) + \sigma\left(\sigma e^{\frac{2bt}{\sigma}} + 1\right)\right)^{-1}$$

$$C_1(s,t) = e^{\frac{2bt}{\sigma}} C_2(s,t).$$

Therefore, we have that

$$E\left(\frac{W(s) - W(t)}{s - t} \Big/ \mathcal{H}_t\right) = \int_0^t \frac{h(s,t,u) - h(t,t,u)}{s - t} dY(u).$$

Then the result follows. ■

Now that we have a proposed solution one can prove that the pertinent hypotheses are all satisfied.

Lemma 80 *The portfolio* π^* *defined by*

$$\pi^*(t) = \frac{\mu - r}{\sigma^2} + bM\left(\sigma + \frac{b(t+\delta)+\delta}{\sigma^2}\right)\int_0^t g(t,u)dY(u).$$

satisfies that $E\int_0^T \pi^*(s)^2 ds < \infty$ *and the vector*

$$\left(\sum_{j=0}^{i-1}\left(\pi(t_j)\left(W(t_{j+1}) - W(t_j)\right)\right), \sum_{j=0}^{i-1}\left(\frac{1}{2}\pi(t_j)\left(1-\pi(t_j)\right)\sigma^2\left(\left(W(t_{j+1}) - W(t_j)\right)^2 - (t_{j+1}-t_j)\right)\right)\right)$$

converges in L^1 *to a random variable* $(X, 0)$ *with*

$$E[X] = b\int_0^t (t-u)g(t,u)du.$$

Proof. Proving that $E\int_0^T \pi^*(s)^2 ds < \infty$ is easy. We only give the sketch of the proof of the L^1-convergence. It is just a matter of separating conveniently the covariance structure between π^* and the Wiener increments. That is,

$$\begin{aligned}
&\int_0^{t_j} g(t_j,u)dY(u)\left(W(t_{j+1}) - W(t_j)\right) \\
&= b\int_0^{t_j} g(t_j,u)\left(\left(W(u+\delta) - W(t_{j+1})\right) + \left(W(t_{j+1}) - W(t_j)\right) + W(t_j)\right)du\left(W(t_{j+1}) - W(t_j)\right) \\
&+ \sigma\int_0^{t_j} g(t_j,u)dW(u)\left(W(t_{j+1}) - W(t_j)\right).
\end{aligned}$$

The sum (for $j = 0, ..., n-1$) of each of the four terms in the above sum converge in L^2. The first to a backward integral, the second to the quadratic variation and the last two to an adapted integrand. In fact, the L^2-limit is

$$\begin{aligned}
&b\int_0^T\int_0^t g(t,u)(W(u+\delta) - W(t))du d^- W(t) + b\int_0^T\int_0^t g(t,u)dudt \\
&+ b\int_0^T\int_0^t g(t,u)duW(t)dW(t) + \sigma\int_0^T\int_0^t g(t,u)dW(u)dW(t).
\end{aligned}$$

Except for the second term all the integrals above have expectation zero. The terms in the second component of the vector are similarly treated (although long to write!). This second term shows that in general expectations of forward integrals are not zero and that in fact their expectations are a result of "trace"terms. ∎

Theorem 81 *Define the class of admissible portfolios as*

$$\mathcal{A} = \left\{\pi; \mathcal{H}\text{-adapted, } E\int_0^T \pi(s)^2 ds < \infty \text{ and } E\left|\int_0^T \pi(s)d^- W(s)\right| < \infty\right\}.$$

Then the optimal portfolio for the logarithmic utility is given by

$$\pi^*(t) = \frac{\mu - r}{\sigma^2} + bM\left(\sigma + \frac{b(t+\delta)+\delta}{\sigma^2}\right)\int_0^t g(t,u)dY(u)$$

and the optimal utility is finite and given by

$$J(t,\pi^*) = \log(V_0) + \frac{\sigma^2}{2}E\left[\int_0^t \pi^*(s)^2 ds\right]$$

.

Proof. First one has that the functional

$$E\left(\int_0^t \left(\pi(s)(\mu - r + bW(s+\delta)) - \frac{1}{2}\sigma^2\pi(s)^2\right) ds + \sigma \int_0^t \pi(s) d^- W(s)\right)$$

is strictly concave. As π^* satisfies the first order condition then a standard argument leads to the optimality of π^*. The utility associated with π^* is finite due to the previous Lemma. To evaluate the utility we use again (26) with $v = \pi^*$ which gives

$$E\left[\int_0^t \left(\pi^*(s)(\mu - r + bW(s+\delta)) - \sigma^2\pi^*(s)^2\right) ds\right] + \sigma E\left[\int_0^t \pi^*(s) d^- W(s)\right] = 0.$$

This replaced in the expression for the logarithmic utility (24) gives the result. ■

Exercise 82 *Prove the convergence part of Lemma 80 using Malliavin Calculus techniques (in particular the duality principle). Hint: Use formula (1.12) in page 130 in Nualart [38].*

Exercise 83 *Prove as in the end of Section 11 that if the small trader makes an inference of his best model in the $\mathcal{H}$ filtration then his expected utility with this model will be smaller than the utility obtained through the "actual" market driving model (23). That is, the model that the small trader proposes is*

$$d\widetilde{S}(t) = E\left[\mu + bW(t+\delta)/\mathcal{H}_t\right]\widetilde{S}(t)dt + \sigma\widetilde{S}(t)dW_{\mathcal{H}}(t)$$

where $W_{\mathcal{H}}$ is a Wiener process in $\mathcal{H}$ (supposing this exists). Find the portfolio $\widetilde{\pi}^*$ that optimizes the logarithmic utility $\widetilde{J}(t,\pi) = E\left[\log\left(\widetilde{V}^\pi(T)\right)\right]$ where $\widetilde{V}^\pi$ denotes the discounted wealth process using the price process $\widetilde{S}$. Prove that $\widetilde{J}(t,\widetilde{\pi}^*) \leq J(t,\widetilde{\pi}^*)$.

A more general situation with other examples is studied in [32]. The case $\delta \leq T$ can also be studied although explicit expressions are difficult to write. There is an important issue that we have not addressed so far: the existence of arbitrage. In fact, given that in principle we are not in a standard set-up one does not know if Girsanov's theorem can be applied and therefore the non-existence of arbitrage is an interesting issue. In fact, W is not adapted to $\mathcal{H}$ and therefore this setup can not be considered as an enlargement of filtration approach.

Theorem 84 *If the logarithmic utility is finite then there is no arbitrage.*

Proof. We have that the filtration $\mathcal{H}$ is generated by the process Y. Now we compute the semimartingale decomposition of the process Y. That is,

$$\begin{aligned} E\left[Y(t) - Y(s)/\mathcal{H}_s\right] &= b\left(\int_t^{t+\delta}\int_0^s h(r,s,u)dY(u)dr - \int_s^{s+\delta}\int_0^s h(r,s,u)dY(u)dr\right) \\ &\quad + \sigma\int_0^s \left(h(t,s,u) - h(s,s,u)\right)dY(u) =: A(t) \end{aligned}$$

where h is given in (31). It is not difficult to prove that A is differentiable and therefore this gives the semimartingale decomposition of the process Y. For this, define

$$B(t) = \int_0^t\int_0^r h(r+\delta,r,u)dY(u)dr - \int_0^t\int_0^r h(r,r,u)dY(u)dr + \sigma\int_0^t\int_0^r D_1h(r,r,u)dY(u)dr.$$

Here D_1 denotes the derivative with respect to the first variable in h. In fact, $Y - B$ is a continuous $\mathcal{H}$-martingale. To prove this is enough to prove that given a sequence of partitions $s = t_0 < ... < t_n = t$ whose norm is tending to zero, we have that

$$\lim_{n\to\infty} E\left[\sum_{i=0}^{n-1} E\left[Y(t_{i+1}) - Y(t_i) - B(t_{i+1}) + B(t_i)/\mathcal{H}_{t_i}\right]\mathcal{H}_s\right] = 0.$$

Furthermore $\langle Y - B \rangle_t = t\sigma^2$. Therefore by Lévy's characterization theorem (see Example 16) we have that there exists a $\mathcal{H}$- Wiener process $W_{\mathcal{H}}$ such that $Y - A = \sigma W_{\mathcal{H}}$. Therefore the price process becomes

$$S(t) = S(0) \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right) t + A(t) + \sigma W_{\mathcal{H}}(t)\right).$$

Therefore the classical theory of no-arbitrage applies. ■

This proof may lead to the misconception that the above explicit calculations are not necessary because everything becomes a consequence of the previous theorem. In general, this is not so when the calculations are not so explicit. In fact, we have the following exercise.

Exercise 85 *Prove, without using the explicit optimal portfolio, that if there is an optimal portfolio leading to a finite logarithmic utility that satisfies (27) then there is no arbitrage. Hint: follow the same structure of proof as above without the explicit calculations.*

Exercise 86 *For $\mu \geq r$ prove that*

$$E \int_0^T \pi^*(s)^2 ds \leq E \left(\int_0^T \pi^*(s) d^- W(s) \right)^2 .$$

Interpret this result as a risk issue of the small trader in an insider influenced model. Link this risk with the existence of the "trace" terms as explained at the end of the proof of Lemma 80.

Exercise 87 *Set $\delta = T/2$. Using the ideas of Girsanov's theorem in an anticipating setting to prove that the model*

$$dS(t) = (\mu + bW(t + \delta)) S(t) + \sigma S(t) d^- W(t)$$

does not allow for arbitrage strategies for the small trader in the interval $[0, T]$, who uses the filtration $\mathcal{H}_t = \sigma(S(s); s \leq t)$, inside a certain class of portfolio strategies.

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13 Solutions and hints to the proposed exercises

Solution of Exercise 2. Suppose by contradiction that $p_1(t_i) < 0$ for some $i = 0, ..., n-1$. Then $P(V(t_i) + p_1(t_i)S(t) + p_0(t_i) < 0 \; for \; some \; t \in [t_i, t_{i+1}]) > 0$.

Solution of Exercise 5. Define the measure

$$\frac{dR}{dP} = \exp\left(-\int_0^t \theta(s)dW(s) - \frac{1}{2}\int_0^t \theta(s)^2 ds\right),$$

where $\theta(s) = \sigma(s)^{-1}\left(\mu(s) - r + \lambda(E(e^X) - 1)\right)$. Under R the dynamics of $\hat{S}$ are given by

$$d\log(\hat{S})(t) = -\frac{1}{2}\sigma^2(t)dt + \sigma(t)d\tilde{W}(t) - \lambda(E(e^X) - 1)dt + dZ(t),$$

where $\tilde{W}$ denotes a Wiener process on the space $(\Omega, \mathcal{F}, R)$. As before, consider for $s < t$

$$\begin{aligned}
&E_R\left[\hat{S}(t)\Big/\mathcal{F}_s\right] \\
&= \hat{S}(s)E_R\left[\exp\left(-\frac{1}{2}\int_s^t \sigma^2(u)du + \int_s^t \sigma(u)d\tilde{W}(u) - \lambda(E(e^X) - 1)(t-s) + Z(t) - Z(s)\Big/\mathcal{F}_s\right)\right] \\
&= \hat{S}(s).
\end{aligned}$$

Here we have used Itô's formula for jump type processes.

Solution of Exercise 6. For $n \in \mathbb{N}$, define the stopping time $T_n = \inf\{t_i; |p_0(t_i)| \wedge |p_1(t_i)| > n\}$. Then $E[|p_0(t_i \wedge T_n)| + |p_1(t_i \wedge T_n)|] \leq n$. Therefore the argument in the proof can applied to obtain that $\hat{V}$ is a discrete time martingale.

$$E_Q\left[\hat{V}(t_{i+1} \wedge T_n)\Big/\mathcal{F}_{t_i}\right] = \hat{V}(t_i \wedge T_n)).$$

As before one obtains that $E_Q[\hat{V}(T_n)] = V_0$. Then the contradiction follows after an application of Fatou's lemma. In fact, $\hat{V}$ is a supermartingale.

Solution of Exercise 7. In fact, one can also perform a change of measure on the compound Poisson process. Suppose that X has a density given by f and let g be another density such that f/g is well defined. Then the Girsanov's theorem in this setting can be applied to obtain the following equation in θ, λ_1 and g:

$$\mu - r - \sigma(s)\theta(s) + \lambda_1\left(\int e^x g(x)dx - 1\right) = 0$$

Obviously this equation has an infinite number of solutions except for trivial cases (such as $\lambda = 0$). Then the change of measure is given by

$$\frac{dR}{dP} = \exp\left(-\int_0^t \theta(s)^2 ds - \int_0^t \theta(s)dW(s)\right)\exp\left(-(\lambda - \lambda_1)t + \log(\lambda/\lambda_1)N(t)\right)\prod_{i=1}^{N(t)}\frac{f}{g}(X_i)$$

Solution of Exercise 8. We give the idea of the solution. First, we define $\eta(t) = \sup\{t_i; t_i \leq t\}$ and write equation (5) in differential form as

$$\hat{V}^n(t) = V_0 + \int_0^t \frac{\pi(\eta(s))\hat{V}^n(\eta(s))}{\hat{S}(\eta(s))}d\hat{S}(s). \tag{32}$$

$\hat{V}^n$ is a continuous time extension of $\hat{V}$ defined in equation (5). The idea of the proof is to take the difference between equations (6) and (32) taking into account that

$$\begin{aligned}
\int_0^t \frac{u(s-)}{\hat{S}(s-)}d\hat{S}(s) = &\int_0^t u(s-)\left(\mu(s-) - r - \frac{1}{2}\sigma^2(s-)\right)ds + \int_0^t u(s-)\sigma(s-)dW(s) \\
&+ \sum_{s \leq t} u(s-)\left(e^{\Delta Z(s)} - 1\right).
\end{aligned}$$

Here $\Delta Z(s) = Z(s) - Z(s-)$. It has to be proven that the last sum above is well defined. The final estimates are carried through $L^2(\Omega, \mathcal{F}, P)$ estimates of the differences assuming that π is integrable enough. The general case is carried out through a classical stopping time argument.

Solution of Exercise 9. Evaluating π^* in the logarithmic utility we have

$$E\left[\log(\hat{V}^*(t))\right] = \log(V_0) + E\left[\int_0^t \frac{(\mu(s)-r)^2}{\sigma^2(s)} ds\right].$$

Here $\hat{V}^*$ denotes the wealth associated with the optimal portfolio π^*.

Solution of Exercise 10. Let $\pi \in \mathcal{A}(t)$ then the stochastic integral $\int_0^\cdot \sigma(s)\pi(s)dW(s)$ is well defined and using Itô's formula we have due to the strict concavity of f_s that

$$\begin{aligned} E\left[\log(\hat{V}(t))\right] &= \log(V_0) + E\left(\int_0^t \left((\mu(s)-r)\pi(s) - \frac{1}{2}\sigma^2(s)\pi(s)^2\right) ds\right) \\ &\leq E\left[\log(\hat{V}^*(t))\right]. \end{aligned}$$

Solution of Exercise 11. Consider the stopping time $\tau_n = \inf\{t \geq 0; \int_0^t \pi(s)^2 ds \leq n\}$ then as before $\int_0^{\cdot\wedge\tau_n} \pi(s)dW(s)$ is well defined and is a martingale. Therefore as before

$$E\left[\log(\hat{V}(t\wedge\tau_n))\right] \leq E\left[\log(\hat{V}^*(t\wedge\tau_n))\right] \leq E\left[\log(\hat{V}^*(t))\right] < \infty.$$

By taking limits the result follows.

Solution of Exercise 12. Define the class

$$\mathcal{A}_\theta(t) = \{\pi : \pi \text{ is } \mathcal{F}-\text{adapted}, \int_0^t \pi(s)^2 ds < \infty \text{ a.s. and } E\hat{V}(t)^\theta < \infty\}.$$

As before consider $\tau_n = \inf\{r \geq 0; \int_0^r \pi(s)^2 ds \leq n\}$ First note that for any portfolio π we have that

$$\begin{aligned} E\left[\hat{V}(t\wedge\tau_n)^\theta\right] &= V_0^\theta E\left[\exp\left(\int_0^{t\wedge\tau_n} \theta\left((\mu(s)-r)\pi(s) - \frac{1}{2}\sigma^2(s)\pi(s)^2\right) ds + \theta\int_0^{t\wedge\tau_n} \sigma(s)\pi(s)dW(s)\right)\right]. \\ &= V_0^\theta E^Q\left[\exp\left(\theta\int_0^{t\wedge\tau_n} \left((\mu(s)-r)\pi(s) - \frac{1}{2}(1-\theta)\sigma(s)^2\pi(s)^2\right) ds\right)\right] \end{aligned}$$

where

$$\frac{dQ^n}{dP} = \exp\left(-\frac{1}{2}\int_0^{t\wedge\tau_n} \sigma^2(s)\theta^2\pi(s)^2 ds + \theta\int_0^{t\wedge\tau_n} \sigma(s)\pi(s)dW(s)\right).$$

As before the function $f_s(\pi) = \theta(\mu(s)-r)\pi - \frac{1}{2}\theta(1-\theta)\sigma^2(s)\pi^2$ is a strictly concave function for $\theta \in (0,1)$ and its maximal value is attained by $\pi^* = \frac{(\mu(s)-r)}{(1-\theta)\sigma^2(s)}$. Then $E\hat{V}(t\wedge\tau_n)^\theta \leq E\hat{V}^*(t\wedge\tau_n)^\theta$ where

$$\begin{aligned} E\left[\hat{V}^*(t\wedge\tau_n)^\theta\right] &= V_0^\theta E^{Q^n}\left[\exp\left(\int_0^{t\wedge\tau_n} \frac{(\theta-1/2)(\mu(s)-r)^2}{2(1-\theta)\sigma^2(s)}\right)\right] \\ &\leq V_0^\theta E^{Q^n}\left[\exp\left(\int_0^t \frac{|\theta-1/2|(\mu(s)-r)^2}{2(1-\theta)\sigma^2(s)} ds\right)\right]. \end{aligned}$$

We can therefore take limits to obtain that π^* is the optimal portfolio. The optimal wealth is then given by $E\left[\hat{V}^*(t)^\theta\right] = V_0^\theta \exp\left(\frac{\theta(\mu-r)^2}{2(1-\theta)\sigma^2} t\right)$ in the case that μ and σ are constant.

Solution of Exercise 16. Applying the Itô's formula to $\exp(i\theta(M_t - M_s))$ we obtain

$$\exp(i\theta(M_t - M_s)) = 1 + \int_s^t i\theta\exp(i\theta(M_u - M_s))\, dM_u - \frac{\theta^2}{2}\int_s^t \exp(i\theta(M_u - M_s))\, du.$$

Taking conditional expectations we obtain

$$E\left[\exp(i\theta(M_t - M_s))/\mathcal{F}_s\right] = 1 - \frac{\theta^2}{2}\int_s^t E\left[\exp(i\theta(M_u - M_s))/\mathcal{F}_s\right] du.$$

Solving this equation we obtain the result.

Solution of Exercise 17. First we have that $\mathcal{F}_{t+\varepsilon} \cup \sigma(I) \supseteq \mathcal{F}_t \cup \sigma(I)$ for all $\varepsilon > 0$ therefore $\mathcal{F}_t \vee \sigma(I) \subset \cap_{\varepsilon \geq 0} \sigma\left(\mathcal{F}_{t+\varepsilon} \cup \sigma(I)\right)$.

Solution of Exercise 20. It is enough to note first that $E\left[|W(T) - W(u)|^r\right] = (T-u)^{r/2} C_r$ where $C_r = \int_0^\infty \sqrt{\frac{2}{\pi}} x^r e^{-\frac{x^2}{2}} dx$. Therefore if $r \in [0,2)$

$$E\left[\int_0^T \left|\frac{W(T) - W(u)}{T-u}\right|^r du\right] = C_r \int_0^T (T-u)^{-r/2} du = \frac{C_r}{1 - r/2} T^{1-r/2}.$$

If $r \geq 2$ then the above integral diverges and if $r < 0$ then the expectation is infinite. For the second part we use Hölder inequality with r a positive integer to obtain that

$$E\left[\left|\int_0^T \frac{W(T) - W(u)}{T-u} du\right|^r\right] \leq C_r \int_{[0,T]^r} \prod_{i=1}^r (T-u_i)^{-1/2} du_i < \infty.$$

Solution of Exercise 21. As $\hat{W}$ is a $\mathcal{G}$ Wiener process and $\mathcal{G}_0 = \sigma(W(T))$ then the independence follows.

Solution of Exercise 22. Define the measure

$$\frac{dQ}{dP} = \frac{dP_0}{dP_t},$$

then for two measurable bounded functions f and g

$$\begin{aligned} E^Q\left[f(W(t)) g(W(T))\right] &= E\left[f(W(t)) \int g(x) \frac{dP_0}{dP_t}(x) dP_t(x)\right] \\ &= E\left[f(W(t))\right] E(\left[g(W(T))\right]. \end{aligned}$$

Taking g a constant one obtains that $E^Q\left[f(W(t))\right] = E\left[f(W(t))\right]$ and similarly for g. From here the conclusion follows.

Solution of Exercise 23. The above property called the harness property is closely tied with the enlargement of filtrations for Lévy processes. For more on this see [12] and [35]. To prove this property we consider

$$W_T - W_b = \hat{W}_T - \hat{W}_b + \int_b^T \frac{W_T - W_u}{T-u} du.$$

Taking conditional expectations we obtain the equation

$$\frac{d}{db} E\left[W_T - W_b / \mathcal{G}_s\right] = -\frac{E\left[W_T - W_b / \mathcal{G}_s\right]}{T-b},$$

on $[s,T]$ with initial condition $E\left(W_T - W_s / \mathcal{G}_s\right) = W_T - W_s$. The solution is

$$E\left[W_T - W_b / \mathcal{G}_s\right] = \frac{W_T - W_s}{T-s}(T-b).$$

To finish, one only needs to note that

$$E\left[W_a - W_b / \mathcal{G}_s\right] = \int_b^a \frac{E\left[W_T - W_u / \mathcal{G}_s\right]}{T-u} du.$$

From here the formula follows.

Solution of Exercise 24. By the integration by parts formula, we have that $I = \int_0^T h(r) W_r dr = \int_0^T a(r) dW_r$, where we denote $a(t) = \int_t^T h(s) ds$. Then we have for $A(t) = \int_t^T a(r)^2 dr$

$$\frac{dP_t(x)}{dx} = \sqrt{\frac{1}{2\pi A(t)}} \exp\left(-\frac{\left(x - \int_0^t a(r) dW_r\right)^2}{2A(t)}\right).$$

Applying the same sequence of ideas as before we have that

$$E\left[(W(t)-W(s))f(I)h_s\right]=E\left[\int_s^t \frac{\int_u^T a(r)dW_r}{A(u)}a(u)duf(I)h_s\right].$$

Therefore as the process $\int_\cdot^T a(r)dW_r$ is $\mathcal{G}$ adapted we finally have that

$$W_t=\hat{W}_t+\int_0^t \frac{\int_u^T a(r)dW_r}{A(u)}a(u)du$$

where $\hat{W}$ is a $\mathcal{G}$ Wiener process in $[0,T)$. In order to prove that the definition is valid in the closed interval we have

$$E\left[\int_0^T \frac{\left|\int_u^T a(r)dW_r\right|}{A(u)}\left|a(u)\right|du\right]=\sqrt{\frac{2}{\pi}}\int_0^T \frac{|a(u)|}{\sqrt{A(u)}}du<\infty.$$

Therefore the needed condition is $\int_0^T \frac{|a(u)|}{\sqrt{A(u)}}du<\infty$. For example, if $h(r)=(T-r)^\theta$ with $\theta>-1/2$, this condition is satisfied.

Solution of Exercise 25. As in the proof of Theorem 19 consider

$$\begin{aligned}E\left[(W(t)-W(s))f(X(T))h_s\right]&=E\left[(W(t)-W(s))\int f(x)dP_t(x)h_s\right]\\&=E\left[(W(t)-W(s))\int f(x)p_{T-t}(X_t,x)dxh_s\right].\end{aligned}$$

Applying Itô's formula to $(W(t)-W(s))p_{T-t}(W_t,x)$ in the interval $[s,t]$, and using that p_{T-t} solves the parabolic equation (11), we have

$$\begin{aligned}&E\left[(W(t)-W(s))f(X(T))h_s\right]\\&=E\left[\int f(x)\int_s^t \partial_y p_{T-u}(X(u),x)dudxh_s\right]\\&=E\left[\int_s^t\int f(x)\partial_y\log(p_{T-u}(X(u),x))p_u(X(u),x)dxduh_s\right]\\&=E\left[f(W(T))\int_s^t \partial_y\log(p_{T-u}(X(u),X(T)))duh_s\right].\end{aligned}$$

Therefore by a density argument, one has

$$E\left[W(t)-W(s)-\int_s^t \partial_y\log(p_{T-u}(X(u),X(T)))du\Big/\mathcal{G}_s\right]=0.$$

As $\partial_y\log(p_{T-u}(X(u),X(T)))\in\mathcal{G}_u$ then $\hat{W}(t)=W(t)-\int_0^t\partial_y\log(p_{T-u}(X(u),X(T))du$ is a $\mathcal{G}$-continuous martingale with $\langle\hat{W}\rangle_t=\langle W\rangle_t=t$ and therefore by Lévy's theorem one has that $\hat{W}$ is a $\mathcal{G}$-Wiener process in $[0,T)$. We then define $\hat{W}(T)=\lim_{t\to T}\hat{W}(t)$ and all above properties follow for the closed interval $[0,T]$ as $E\left[\int_0^T|\partial_y\log(p_{T-u}(X(u),X(T))|\,du\right]<\infty$

Solution of Exercise 28. To simplify the notation define $\alpha_s=\frac{W_T-W_s}{T-s}$. Define $\mathcal{I}=\{\pi:\pi$ is $\mathcal{G}$-adapted, $\int_0^t\pi(s)^2\left(\alpha_s^2+1\right)ds<\infty$ a.s. and $E[\log(\hat{V}(t))]<\infty\}$. Then, as before, we define the $\mathcal{G}$

stopping times $\tau_n = \inf\{t \geq 0; \int_0^t \pi(s)^2 ds \leq n\}$. First note that for any portfolio π we have that

$$\begin{aligned}
& E\left[\log\left(\hat{V}(t\wedge\tau_n)\right)\right] \\
&= \log V_0 + E\left[\left(\int_0^{t\wedge\tau_n}\left((\mu(s)-r+\sigma(s)\alpha_s)\pi(s) - \frac{1}{2}\sigma^2(s)\pi(s)^2\right)ds + \int_0^{t\wedge\tau_n}\sigma(s)\pi(s)d\widehat{W}(s)\right)\right]. \\
&\leq \log V_0 + E\left[\int_0^{t\wedge\tau_n}\left((\mu(s)-r+\sigma(s)\alpha_s)\widehat{\pi}(s) - \frac{1}{2}\sigma^2(s)\hat{\pi}(s)^2\right)ds\right] \\
&= \log V_0 + E\left[\int_0^{t\wedge\tau_n}\frac{(\mu(s)-r+\sigma(s)\alpha_s)^2}{2\sigma^2(s)}ds\right] \\
&\leq \log V_0 + E\left[\int_0^t \frac{(\mu(s)-r)^2}{2\sigma^2(s)}ds\right] + E\left[\int_0^t \frac{\alpha_s^2}{2}ds\right] \\
&= \log V_0 + E\left[\int_0^t \frac{(\mu(s)-r)^2}{2\sigma^2(s)}ds\right] + \frac{1}{2}\log\left(\frac{T-t}{T}\right).
\end{aligned}$$

From here the result follows.

Solution of Exercise 30. We consider the maximization of $E(V(t)^\theta)$ for $t < T$. As in Exercise 12, we have

$$\mathcal{A}_\theta(t) = \{\pi : \pi \text{ is } \mathcal{G}-\text{adapted}, \int_0^t \pi(s)^2 ds < \infty, \int_0^t |\pi(s)\alpha(s)|\, ds < \infty \text{ and } E\hat{V}(t)^\theta < \infty\}.$$

As before, consider $\tau_n = \inf\{r \geq 0; \int_0^r \pi(s)^2 ds + \int_0^r |\pi(s)\alpha(s)|\, ds \leq n\}$ First note that for any portfolio π we have that

$$E\left[\hat{V}(t\wedge\tau_n)^\theta\right] = V_0^\alpha E^{Q_n}\left[\exp\left(\theta\int_0^{t\wedge\tau_n}\left((\mu(s)-r+\sigma(s)\alpha(s))\pi(s) - \frac{1}{2}(1-\theta)\sigma^2(s)\pi(s)^2\right)ds\right)\right]$$

where

$$\frac{dQ^n}{dP} = \exp\left(-\frac{1}{2}\int_0^{t\wedge\tau_n}\sigma^2(s)\theta^2\pi(s)^2 ds + \theta\int_0^{t\wedge\tau_n}\sigma(s)\pi(s)d\widehat{W}(s)\right).$$

The function $f_s(\pi) = \theta(\mu(s)-r+\sigma(s)\alpha(s))\pi - \frac{1}{2}\theta(1-\theta)\sigma^2(s)\pi^2$ is a strictly concave function for $\theta \in (0,1)$ and its optimum portfolio process is given by $\pi^*(s) = \frac{\mu(s)-r+\sigma(s)\alpha(s)}{(1-\theta)\sigma^2(s)}$.

Solution of Exercise 32. Considering $S(T) = x$ is equivalent to $W(T) = \sigma^{-1}(\log(x/S_0) - \mu T)$. Therefore without loss of generality we consider $W(T) = x$. Repeating the same calculations as before we have that the conditional expectation of the logarithmic wealth is

$$\begin{aligned}
& E\left[\log(\hat{V}(t))\Big/ W(T) = x\right] \\
&= \log(V_0) + \int_0^t E\left[\left(\mu(s) - r + \sigma(s)\frac{x-W_s}{T-s}\right)\pi(s) - \frac{1}{2}\sigma^2(s)\pi(s)^2 \Big/ W(T) = x\right]ds.
\end{aligned}$$

The optimal portfolio is also $\hat{\pi}(s) = \frac{1}{\sigma^2(s)}\left(\mu(s) - r + \sigma(s)\frac{x-W_s}{T-s}\right)$. The optimal portfolio value is

$$\log(V_0) + E\left[\int_0^t \frac{(\mu(s)-r)^2}{2\sigma^2(s)}ds\right] + \int_0^t E\left[(\mu(s)-r)\frac{x-W_s}{\sigma(s)(T-s)} + \frac{1}{2}\left(\frac{x-W_s}{T-s}\right)^2 \Big/ W(T) = x\right]ds.$$

Using conditional expectation of Gaussian random vectors we have that $E[W(T) - W_s/W(T)] = W(T)(T-s)/T$ and $E\left[(W(T)-W_s)^2\Big/ W(T)\right] = (T-s)s/T + (W(T)(T-s)/T)^2$. Therefore the optimal utility is

$$\log(V_0) + E\left[\int_0^t \frac{(\mu(s)-r)^2}{2\sigma^2(s)} + \frac{x(\mu(s)-r)}{\sigma(s)T}ds\right] + \frac{x^2 t}{2T^2} - \frac{t}{2T} + \frac{1}{2}\log\left(\frac{T}{T-t}\right).$$

Solution of Exercise 33. The equation $u^{\mathcal{F}}(t, V_0) = u^{\mathcal{G}}(t, V_0 - \rho_t(V_0, I))$ can be rewritten as

$$\log(V_0) = \log\left(V_0 - \rho_t(V_0, I)\right) + \frac{1}{2}\log\left(\frac{T}{T-t}\right).$$

Therefore

$$V_0\left(1 - \sqrt{\frac{T-t}{T}}\right) = \rho_t(V_0, I).$$

Note that as $t \to T$ then the value of the information I for the insider is V_0 as he can perform arbitrage. That is, the insider will be willing to exchange his/her information only if offered all the money of the other market players is transferred to him/her.

Solution of Exercise 35.

$$\begin{aligned}
E\left[(W(t) - W(s))f(I)h_s\right] &= E\left[(W(t) - W(s))\int f(x)\frac{dP_t}{d\eta}(x)d\eta(x)h_s\right] \\
&= E\left[\int_s^t \int f(x)\frac{d}{du}\left\langle W, \frac{dP_\cdot}{d\eta}(x)\right\rangle_u d\eta(x)duh_s\right] \\
&= E\left[\int_s^t \int f(x)\beta(u,x)dP_u(x)duh_s\right],
\end{aligned}$$

where $\beta(u, I) = \alpha(u)$.

Solution of Exercise 36. First we use the same arguments as before to obtain that

$$E\left[\log(\hat{V}(t))\right] = \log(V_0) + E\left[\int_0^t \left((\mu(s) - r + \sigma(s)\alpha_s)\pi(s) - \frac{1}{2}\sigma^2(s)\pi(s)^2\right) ds\right].$$

The optimal value of the strictly concave function $f_s(\pi) = (\mu(s) - r + \sigma(s)\alpha_s)\pi - \frac{1}{2}\sigma^2(s)\pi^2$ is $\hat{\pi}(s) = \frac{\mu(s)-r}{\sigma^2(s)} + \frac{\alpha_s}{\sigma(s)}$. Next, we note that $E(\alpha_s) = 0$ which follows from the semimartingale decomposition. That is, we have that for any $\mathcal{F}_s$ measurable r.v. h_s

$$0 = E[(W_t - W_s)h_s] = E[(\widehat{W}_t - \widehat{W}_s)h_s] + \int_s^t E[h_s\alpha_u]du.$$

Therefore $E\left[\alpha(s)/\mathcal{F}_s\right] = 0$. Now we compute the optimal utility as

$$E\left[\log(\hat{V}(t))\right] = \log(V_0) + E\left[\int_0^t \left(\frac{1}{2\sigma^2(s)}(\mu(s) - r + \sigma(s)\alpha_s)^2\right) ds\right],$$

and the result follows.

Solution of Exercise 41. The compensator is given by

$$\alpha(u) = \frac{W(u)(\sigma^2 - T) + \mu T}{(\sigma^2 - T)(u) + T^2}.$$

The optimal logarithmic utility in the interval $[0, T]$ is finite. Note that in this approach the knowledge of the insider given by μ and σ^2 are always constant throughout the interval $[0, T]$. Compare with Section 8.2.

Solution of Exercise 42. Note that the numerator of the compensator is given by

$$\int \frac{x}{T-u}\frac{d\nu}{dP^{W(T)}}(x + W(u))\frac{\exp\left(-\frac{x^2}{2(T-u)}\right)}{\sqrt{2\pi(T-u)}}dx.$$

Integrating by parts with respect to x and taking in consideration that the denominator normalizes the measure, we have that

$$E^{\nu}\left[\alpha(u)^2\right] \leq E^{\nu}\left[\left(\frac{\partial}{\partial x}\log\left(\frac{d\nu}{dP^{W(T)}}\right)\right)^2 (W(T))\right].$$

Solution of Exercise 45. First, we compute the compensator of W^2 in the enlarged filtration:

$$\begin{aligned}&E\left[\left(W^2(t)-W^2(s)\right)f(W^1(T))h_s\right]\\&=E\left[\int_s^t E\left[\frac{W^2(T)-W^2(u)}{T-u}\Big/ W^1(T)-W^1(u)\right]du f(W^1(T))h_s\right].\end{aligned}$$

Since $E\left(W^2(T)-W^2(u)/\, W^1(T)-W^1(u)\right)=\rho\left(W^1(T)-W^1(u)\right)$, repeating the same sequel of calculations as in Exercise 28, the optimal portfolio is

$$\widehat{\pi}(s)=\frac{1}{\sigma^2(s)}\left(\mu(s)-r+\sigma(s)\rho\frac{W_T^1-W_s^1}{T-s}\right).$$

The expected logarithmic utility in $[0,T]$ is infinite for any $\rho>0$.
In the second case the calculation is similar, except that the compensator will be

$$E\left[\frac{W^2(T)-W^2(u)}{T-u}\Big/ \mathcal{F}_u^2\vee\sigma\left(W^1(T)\right)\right]=\rho\frac{W^1(T)-\rho W^2(u)}{T-\rho^2 u}.$$

Therefore

$$\widehat{\pi}(s)=\frac{1}{\sigma^2(s)}\left(\mu(s)-r+\sigma(s)\rho\frac{W^1(T)-\rho W^2(s)}{T-\rho^2 s}\right)$$

and the optimal logarithmic utility is finite and given by

$$\log(V_0)+E\left[\int_0^t\frac{(\mu(s)-r)^2}{2\sigma^2(s)}ds\right]-\rho^2\log\left(1-\rho^2\right).$$

The first model corresponds to two correlated assets where the insider has information about the first but he/she is restricted to trade only on the second asset while observing the evolution of the first. In the second model the insider has related information about the first asset which is not traded in the market (for example, the volatility of the asset).

Solution of Exercise 46. The argument is the same as in Section 3. In fact the optimal solution is just the projection on the interval $[0,1]$ of the solution without constraints.

Solution of Exercise 47. Following the result of exercise 24 and 36 we have to compute

$$E\left[\int_0^T\left(\frac{\int_u^T a(r)dW_r}{A(u)}a(u)\right)^2du\right]=-\left[\log(A(u))\right]_0^T=+\infty.$$

Solution of Exercise 48. For case 1 we use Theorem 43. Due to this theorem we have that the extra utility of the insider is given by $E[\log p_T(I)]$. First we have that

$$p_t(x)=\frac{P(W(T)\geq a/\,\mathcal{F}_t)}{P(W(T)\geq a)}1(x=1)+\frac{P(W(T)<a/\,\mathcal{F}_t)}{P(W(T)<a)}1(x=0).$$

Therefore

$$\begin{aligned}&E[\log p_t(I)]\\&=E\left[P(W(T)\geq a/\,\mathcal{F}_t)\log\left(P(W(T)\geq a/\,\mathcal{F}_t)\right)\right]\\&+E\left[P(W(T)<a/\,\mathcal{F}_t)\log\left(P(W(T)<a/\,\mathcal{F}_t)\right)\right]\\&-\left(P(W(T)\geq a)\log\left(P(W(T)\geq a)\right)+P(W(T)<a)\log\left(P(W(T)<a)\right)\right).\end{aligned}$$

In particular

$$E[\log p_T(I)] = -\left(P(W(T) \geq a)\log\left(P(W(T) \geq a)\right) + P(W(T) < a)\log\left(P(W(T) < a)\right)\right)$$

which is obviously finite. For the second case, we obtain that

$$P_t^I(dx) = \frac{e^{-\frac{(x-W_t)^2}{2(T-t+1)}}}{\sqrt{2\pi(T-t+1)}}dx.$$

Applying Jacod's theorem (see Theorem 34) with $\eta(dx) = dx$ we have that

$$W_t = \hat{W}_t + \int_0^t \frac{W_T + \varepsilon - W_u}{T-t+1}du$$

and the optimal logarithmic utility is given by

$$\log(V_0) + E\left[\int_0^T \frac{(\mu(s)-r)^2}{2\sigma^2(s)}ds\right] + \log\left(\frac{T+1}{T-t+1}\right).$$

Note that this is finite even for $t = T$ but in this model $\mathcal{G}_t$ gives always the same deformed information, $W(T) + \varepsilon$, to the insider even when t is close to T.

Solution of Exercise 50. We leave the first part to the reader. Suppose that Z and Z' are two Lévy processes with the same characteristics. Then we want to compute for $X(s) = Z(T) + Z'((T-s)^\theta)$ the quantity

$$E\left[(Z_t - Z_u)\, f(Z_T - Z_s + Z'((T-s)^\theta))\right].$$

Due to the independence of the increments and the invariance of the law of the increments (which only depend on the size of the interval) we have that if $s < t < u < T$ with $t - u = \frac{T-s}{2}$ then

$$2E\left[(Z_t - Z_u)\, f(Z_T - Z_s + Z'((T-s)^\theta))\right] = E\left[(Z_T - Z_s)\, f(Z_T - Z_s + Z'((T-s)^\theta))\right].$$

Then by continuity of the expectation in the time variables we have that

$$E\left[(Z_t - Z_u)\, f(Z_T - Z_s + Z'((T-s)^\theta))\right] = \frac{t-u}{T-s}E\left[(Z_T - Z_s)\, f(Z_T - Z_s + Z'((T-s)^\theta))\right].$$

A similar argument also gives that

$$\begin{aligned}&E\left[(Z_t - Z_u)\, f(Z_T - Z_s + Z'((T-s)^\theta))\right]\\ &= \frac{t-u}{T-s+(T-s)^\theta}E\left[\left(Z_T - Z_s + Z'((T-s)^\theta)\right) f(Z_T - Z_s + Z'((T-s)^\theta))\right].\end{aligned}$$

From here the result follows.

Solution of Exercise 54. First note that $\mathcal{F}_{s,T} = \mathcal{F}_s \vee \sigma(Z_r; r \geq T) = \mathcal{F}_s \vee \sigma(Z_T) \vee \sigma(Z_T - Z_t; r \geq t)$, where the last sigma algebra is independent of the other two. Therefore we have using Theorem 53 that

$$E\left[Z_t - Z_s / \mathcal{F}_s \vee \sigma(Z_T)\right] = E\left[\widehat{Z}_t - \widehat{Z}_s \Big/ \mathcal{F}_s \vee \sigma(Z_T)\right] + \int_s^t \frac{E\left[Z_T - Z_u / \mathcal{F}_s \vee \sigma(Z_T)\right]}{T-u}du.$$

Then setting $\phi(t) = E\left[Z_t - Z_s / \mathcal{F}_s \vee \sigma(Z_T)\right]$, we have that the following ordinary differential equation is satisfied

$$\phi(t) = \int_s^t \frac{Z_T - Z_s}{T-u}du - \int_s^t \frac{\phi(u)}{T-u}du,$$

whose unique solution is

$$E\left[Z_t - Z_s / \mathcal{F}_s \vee \sigma(Z_T)\right] = \frac{Z_T - Z_s}{T-s}(t-s).$$

Therefore

$$E\left[\frac{Z_t - Z_s}{t-s} \Big/ \mathcal{F}_{s,T}\right] = \frac{Z_T - Z_s}{T-s}.$$

Furthermore for $u \in (s,t)$

$$\begin{aligned} E\left[\frac{Z_t - Z_u}{t-u} \Big/ \mathcal{F}_{s,T}\right] &= E\left[E\left[\frac{Z_t - Z_u}{t-u} \Big/ \mathcal{F}_{u,T}\right] \Big/ \mathcal{F}_{s,T}\right] \\ &= E\left[\frac{Z_T - Z_u}{T-u} \Big/ \mathcal{F}_{s,T}\right] \\ &= \frac{Z_T - Z_s}{T-u} + E\left[\frac{Z_s - Z_u}{T-u} \Big/ \mathcal{F}_{s,T}\right] \\ &= \frac{Z_T - Z_s}{T-s}. \end{aligned}$$

Solution of Exercise 55. First prove that for $s < t_1 < t_2 < t$

$$\begin{aligned} E\left[(Z_{t_2} - Z_{t_1})^2 \Big/ \mathcal{G}_s\right] &= E\left[(Z_{t_2} - Z_{t_1})^2 \Big/ Z_T - Z_s\right] \\ &= (Z_T - Z_s)(Z_T - Z_s - 1)\left(\frac{t_2 - t_1}{T-s}\right)^2 + (Z_T - Z_s)\frac{t_2 - t_1}{T-s}. \end{aligned}$$

Consider for the partition $s = t_0 < t_1 < \ldots < t_n = t$ then the quantity

$$\begin{aligned} &E\left[\sum_{i=0}^{n-1}\left\{(Z_{t_{i+1}} - Z_{t_i})^2 - \frac{Z_T - Z_{t_i}}{T - t_i}(t_{i+1} - t_i)\right\} \Big/ \mathcal{G}_s\right] \\ &= E\left[\sum_{i=0}^{n-1}(Z_T - Z_{t_i})(Z_T - Z_{t_i} - 1)\left(\frac{t_{i+1} - t_i}{T - t_i}\right)^2 \Big/ \mathcal{G}_s\right] \end{aligned}$$

goes to zero a.s. and therefore taking limits with respect to the norm of the partition we have that

$$E\left[[Z]_t - [Z]_s - \int_s^t \frac{Z_T - Z_u}{T-u} du \Big/ \mathcal{G}_s\right] = 0.$$

From here the result follows. Similarly be taking approximations of pure jump Lévy processes one can also prove that

$$E\left[[Z]_t - [Z]_s - \int_s^t \frac{[Z]_T - [Z]_u}{T-u} du \Big/ \mathcal{G}_s\right] = 0.$$

A similar calculation also leads to the fact that the formula is valid for general square integrable Lévy processes.

Solution of Exercise 57. If there are only positive jumps and $\mu - r \geq 0$ then there is arbitrage and in fact any investment on the underlying will provide a positive return. On the contrary if $\mu - r < 0$ then the function f becomes

$$f(\pi) = (\mu - r)\pi + \lambda^+ \log\left(1 + (e^a - 1)\pi\right).$$

This strictly concave function has its optimum at $\pi^* = -\frac{1}{e^a - 1} - \frac{\lambda^+}{\mu - r}$ and the optimal logarithmic utility is

$$\log(V_0) - \left(\lambda^+ + \frac{\mu - r}{e^a - 1} - \lambda^+ \log\left(-\frac{\lambda^+(e^a - 1)}{\mu - r}\right)\right)T$$

Solution of Exercise 58. Let x such that $P(N(T) = x) > 0$, then there exists a unique pair (l_1, l_2) of natural numbers such that $l_1 a + l_2 \ln(2 - e^a) = x$. Then $N(T) = x$ implies that $N^+(T) = l_1$ and $N^-(T) = l_2$ and therefore the two filtrations coincide.

Solution of Exercise 62. It is enough to note that the conditional logarithmic utility can be written as

$$E\left[\log(\hat{V}_t)\Big/ N(T)\right]$$
$$= \log(V_0) + E\left[\int_0^t (\mu - r)\pi(s)ds \Big/ N(T)\right] + \sum_{i=-}^{+} E\left[\int_0^t \log(1+\pi(s)(e^{a_i}-1))B_i(s)ds \Big/ N(T)\right].$$

Solution of Exercise 63. In such a case, we have that $B_i(s) = \frac{N_i(T)-N_i(s)}{T-s}$ then an arbitrage is to wait until $B_i(s) = 0$ and invest all resources in the asset if $i = -$ or borrow the asset if $i = +$.

Solution of Exercise 64. Try the following portfolio:

$$\pi_s = \theta(a\frac{W(T)-W(s)}{T-s} + b\frac{N_T - N_s}{T-s}) \vee 0.$$

Find a set of constants a and b such that the utility is infinite.

Solution of Exercise 68. We only give the main idea of the solution. First obtain that on the set $1(T_n > t)$ we have that

$$E\left[\frac{N_t^+}{T_n - t} - \frac{N_s^+}{T_n - s}\Big/\mathcal{G}_s\right] = \frac{(n-1)(t-s)}{(T_n - t)(T_n - s)}.$$

Finally prove that $\frac{N_{T_n-}^+ - N_u^+}{T_n - u}$ is a $\mathcal{G}_u$ martingale on $T_n > u$ to conclude.

Solution of Exercise 69. This exercise follows as in the proof of Theorem 60 and at the end is necessary to compute the joint law of (T_{n-1}, T_n) to prove that $E(\int_0^{+\infty} \frac{1(T_{n-1}>u)}{T_n - u}du) < \infty$. In fact, in the interval $[0, T_{n-1}]$ we will have that the wealth is smaller than

$$\log(V_0) + \lambda^- \log(2) E\left[\int_0^\infty 1(s < T_{n-1})ds\right] + \log(2)E\left[\int_0^\infty 1(s < T_{n-1})\frac{n-1-N_s^+}{T_n - s}ds\right].$$

The last expectation above is bounded by

$$\int_{t_1<\ldots<t_n} \frac{e^{-\lambda^+ t_n} 1(t_{n-1} > s)}{t_n - s} \left(\lambda^+\right)^n dt_1 \ldots dt_n = \left(\lambda^+\right)^n \int_s^\infty \frac{e^{-\lambda^+ t_n}\left(t_n^{n-1} - s^{n-1}\right)}{(n-1)!\,(t_n - s)} dt_n < \infty.$$

Solution of Exercise 70. The analysis in the interval $[T_{n-1}, T_n)$ follows as in Case II(b) analyzed previously. For $\mu \leq r$ is enough to note that there is a positive probability that there is going to be a negative jump in the interval $[T_{n-1}, T_n)$ which gives an infinite utility. It is also interesting that in the case that the interval goes beyond T_n then there is no optimum for the problem in the case $\mu > r$.

Solution of Exercise 72. Note that

$$E\left[f(t, W(T))(W(t_{i+1}) - W(t_i))\right] = (t_{i+1} - t_i)E\left[\frac{\partial}{\partial x} f(t, W(T))\right].$$

Solution of Exercise 76. Find the solution of equation (21) as explicitly as possible and prove that it generates the same filtration as the one generated by $Y - \sigma(T-t)^\theta \int_0^s (T-r)^\theta dW(r)$, $s \leq t$ and conclude.

Solution of Exercise 78. Consider the definition of semimartingale as given in Protter page 52. If W is a $(\mathcal{F}_{t+\delta})$-semimartingale, then for any partition whose norm tends to zero and always smaller than δ, consider the process

$$H(t) = \sum_{i=0}^{n-1} (W(t_{i+1}) - W(t_i)) 1_{(t_i, t_{i+1}]}(t).$$

This process is then $(\mathcal{F}_{t+\delta})$-adapted and converges uniformly to zero but its stochastic integral converges to the quadratic variation of W leading to a contradiction.

Solution of Exercise 83. The model proposed by the small trader is an adapted model with random drift. Therefore the analysis to obtain the optimal portfolio for the logarithmic utility follows the same lines as in Section 2. The optimal porfolio is given by

$$\widetilde{\pi}^*(t) = \frac{\mu - r}{\sigma^2} + \frac{bE\left[W(t+\delta)/\mathcal{H}_t\right]}{\sigma^2}.$$

This gives

$$\widetilde{J}(t, \widetilde{\pi}^*) = \log(V_0) + \frac{1}{2\sigma^2} E\left[\int_0^t \left(\mu - r + bE\left[W(s+\delta)/\mathcal{H}_s\right]\right)^2 ds\right],$$

while

$$J(t, \widetilde{\pi}^*) = \widetilde{J}(t, \widetilde{\pi}^*) + \sigma E\left[\int_0^t \widetilde{\pi}^*(s) d^- W(s)\right].$$

Finally as in Exercise 72, one proves that

$$E\left[\int_0^t \widetilde{\pi}^*(s) d^- W(s)\right] = \int_0^t D_{s+}\widetilde{\pi}^*(s) ds,$$

where

$$D_{s+}\widetilde{\pi}^*(s) = bM\left(\sigma + \frac{b(s+\delta)+\delta}{\sigma^2}\right)\int_{t-\delta}^t bg(t,u)du \geq 0.$$

For more details on the notation D_{s+} see Kohatsu-Sulem.

Solution of Exercise 85. As in the proof of the Theorem 84 $\mathcal{H}$ is generated by Y. Using equation (27) we have that

$$E\left[Y(t) - Y(s)/\mathcal{H}_s\right] = \sigma^2 E\left[\int_s^t \pi^*(s) ds \Big/ \mathcal{H}_s\right] - (\mu - r)(t - s).$$

Define $B(t) = \sigma^2 \int_0^t \pi^*(s) ds - (\mu - r)t$ and finish the proof as in the proof of the Theorem 84.

Solution of Exercise 86. The solution of this exercise is long and requires some knowledge of anticipating calculus. The main steps are as follows. First rewrite

$$\int_0^T \pi^*(s) d^- W(s) = \int_0^T \phi_1(t) dW(t) + \int_0^T \phi_2(t) d_- W(t),$$

for some specific stochastic processes ϕ_1 and ϕ_2 where ϕ_1 is adapted and ϕ_2 is adapted to the backward filtration. The notation d_- denotes the backward Itô integral. Then prove that

$$E\left[\int_0^T \phi_1(t) dW(t) \int_0^T \phi_2(t) d_- W(t)\right] = E\left[\int_0^T \phi_1(t)\phi_2(t) dt + \int_0^T \int_0^s D_s\phi_2(u) D_u\phi_1(s) du ds\right]$$

and that the expectation of the last term above is strictly positive for $\mu \geq r$.

Solution of Exercise 86. We only sketch the solution: In this case note that for the small trader we will have that

$$\mathcal{H}_t = \sigma(S(s); s \leq t) = \sigma(\int_0^s bW(\theta + \delta) d\theta + \sigma W(s); s \leq t)$$

We will now deduce an anticipating Girsanov's theorem that will allow us to apply it to the above model.

For this, set $I(t) = \mu + bW(t + \delta)$. As the portfolios of the small trader have to be adapted to $\mathcal{H}$, let us suppose that

$$\pi(t_i) = \pi(\sum_{k=0}^{j-1} I(t_k)\Delta + \sigma W(t_j), j \leq i - 1)$$
$$I(t_j) = I(t_j, W(t_{i+1}) - W(t_i); i = 0, ..., n - 1)$$

where $t_{j+1} - t_j = \Delta = T/n$. Now consider the following expression

$$E^{Q^n} \sum_{j=0}^{n-1} \{\pi(t_j)I(t_j)\Delta + \sigma\pi(t_j)(W(t_{j+1}) - W(t_j))\}$$

$$= \int_{\mathbb{R}^{2n}} \sum_{j=0}^{n-1} \pi(t_j) \{I(t_j)\Delta + \sigma z_j\} \frac{\exp\left(-\frac{|z|^2}{2\Delta}\right)}{(2\pi\Delta)^{n/2}} \frac{dQ^n}{dP} dz,$$

where

$$\frac{dQ^n}{dP} = \exp\left(-\sum_{i=0}^{n-1} \frac{z_i\sigma^{-1}I(t_i)\Delta + \left(\sigma^{-1}I(t_i)\Delta\right)^2}{2\Delta}\right).$$

Now we will perform the following change of variables for $j = 0, ..., n-1$

$$w_i = \sigma^{-1}I(t_i)\Delta + z_i.$$

To compute the inverse of the jacobian $J_n = \left(\frac{\partial w_i}{\partial z_j}\right)$, we need to compute

$$\frac{\partial w_i}{\partial z_j} = \sigma^{-1} \frac{\partial I(t_i)}{\partial z_j} \Delta + I_{ij}.$$

Now we consider the particular case that $I(t) = \mu + bW(t + \delta)$ with fixed $\delta = T/2$. Then we can rewrite

$$\frac{\partial w_i}{\partial z_j} = \sigma^{-1}bI(t_{j+1} \leq t_i + \delta)\Delta + I_{ij}.$$

After some heavy algebraic manipulation with the jacobian matrix J one finds that

$$\det(J_n) = \left\{1 + \sigma^{-1}b \sum_{j=0}^{j_0} \left(1 + \frac{\sigma^{-1}b\Delta j}{\sigma^{-1}b\Delta + 1}\right)^{-1} \Delta\right\} \left(1 + \sigma^{-1}b\Delta\left(j_0 + 2\right)\right),$$

if $b > 0$ then the above quantity is strictly positive and the change of variables is allowed. Therefore we have that

$$E^{Q^n} \left[\sum_{j=0}^{n-1} \{\pi(t_j)I(t_j)\Delta + \sigma\pi(t_j)(W(t_{j+1}) - W(t_j))\}\right]$$

$$= E^{Q^n} \left[\sum_{j=0}^{n-1} \left\{\pi(t_j)(\widehat{W}(t_{j+1}) - \widehat{W}(t_j))\right\}\right] \det(J_n)^{-1},$$

where $\widehat{W}(t_k) = W(t_k) + \sum_{j=0}^{j=k-1} I(t_j)\sigma^{-1}\Delta$ is a Wiener process in the filtration $\mathcal{H}$. Therefore by taking limits we have that there exists an equivalent martingale measure therefore not allowing for arbitrage.