Lower bounds for densities of uniformly elliptic non-homogeneous diffusions

Arturo Kohatsu-Higa

Abstract. In this article we interpret heuristically the conditions of the definition of a uniformly elliptic random variable on Wiener space which allow to obtain Aronson type estimates for the density of this random variable. As an example we apply this concept to uniformly elliptic non-homogeneous diffusions.

1. Introduction

In a recent article (see [10]), we have described some minimal conditions that ensure that a random variable F defined on Wiener space has a density with a lower bound of Gaussian type. We called a random variable satisfying such conditions a uniformly elliptic random variable (see Definition 1). This class of random variables is broad enough to be applied to various stochastic equations. In particular, it includes the solution to the uniformly elliptic stochastic heat equation and the uniformly elliptic hyperbolic stochastic partial differential equation, also known as the biparametric diffusion (see [4]). A natural environment for the study of densities is Malliavin Calculus. We refer the reader to any of the well known books available on the matter (see for example, [15], [19], [20]).

The purpose of this article is to clarify the role of each requirement in the definition of uniformly elliptic random variable and show how it can be verified in a non-trivial simple example. Still, the result we obtain seems to be new.

We consider the case of the non-homogeneous uniformly elliptic diffusion. That is, let X be the solution to the following stochastic differential equation

$$X_t = x + \int_0^t b(s, X_s)ds + \int_0^t \sigma_j(s, X_s)dW_s^j$$

where $b : [0,T] \times \mathbb{R}^q \to \mathbb{R}^q$, $\sigma : [0,T] \times \mathbb{R}^q \to \mathbb{R}^q \times \mathbb{R}^k$ and W is a k-dimensional Wiener process. Assume that the coefficients $b(t,\cdot)$, $\sigma(t,\cdot) \in C_b^\infty(\mathbb{R}^q)$ uniformly in time and that the functions are jointly measurable. Under these conditions the

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solution to the above equation exists and is unique. Furthermore assume that the solution process is uniformly elliptic. That is, for any vectors x, $\xi \in \mathbb{R}^q$ and $t \in [0,T]$, there exists a strictly positive constant c so that

$$\xi'\sigma(t,x)\sigma(t,x)'\xi \geq c\left\|\xi\right\|^2.$$

It is natural to expect under this condition that the random variable $F = X_t$ should behave as a Gaussian random variable. In fact, one expects to have that X_t should have a density for $t > 0$, denoted by $p_t(x,\cdot)$, and that there exists two positive constants m and M such that

$$m^{-1}t^{-q/2}\exp\left(-\frac{m\left\|y-x\right\|^2}{t}\right) \leq p_t(x,y) \leq Mt^{-q/2}\exp\left(-\frac{\left\|y-x\right\|^2}{Mt}\right) \tag{1}$$

The fact that the density exists and that it has a Gaussian upper bound can be solved with well known techniques of Malliavin Calculus (see [23]). Also there are very well known established analytical techniques to solve the problem of finding upper and lower bounds for fundamental solutions of homogeneous or non-homogeneous partial differential equations. See for example, [1], [5], [6], [17], [21], [22] and [14].

In the stochastic framework using Malliavin Calculus, the problem of finding a detailed global lower bound for the density of X_t was first studied for the hypoelliptic diffusion case in [13]. Furthermore, the metric in the exponent is more explicit than the one in (1) and is closely related to the Riemmanian metric defined through the coefficients of the stochastic differential equation.

Azencott (see [2]) obtained a Gaussian lower bound estimate for the uniformly elliptic non-homogeneous diffusion in the case the coefficients are smooth in time and space. We generalize this result, via a different approach, weakening the restrictions on the time variable. Furthermore, we want to stress the generality of the definition of uniformly elliptic random variables. In fact, the result presented here for non-homogeneous diffusions can be easily generalized to the case of solutions of uniformly elliptic stochastic differential equations with random coefficients under appropriate continuity and differentiability conditions on the coefficients.

$C_b^\infty(\mathbb{R}^d)$ denotes the space of real bounded functions on $\mathbb{R}^d$ such that they are infinitely differentiable with bounded derivatives. $C_p^\infty(\mathbb{R}^d)$ stands for a similar space but the functions and their derivatives have polynomial growth instead. C, c, m and M denote constants in general that may change from one line to another unless stated otherwise. We also use the double index summation convention. $\|\cdot\|$ without any subindices denotes the usual Euclidean norm in $\mathbb{R}^l$. The dimension l should be clear from the context. For stochastic processes we use indistinctively $X(t)$ or X_t.

2. Preliminaries

Let W be a k-dimensional Wiener process indexed in $[0,T]$. Our base space is a sample space $(\Omega,\mathcal{F},P)$ where the Wiener process is defined (for details see [19], Section 1.1 and [20]). The associated filtration will be defined as $\{\mathcal{F}_t;0\leq t\leq T\}$, where $\mathcal{F}_t$ is the $\sigma-$field generated by the random variables $\{W(s),\ s\in[0,t]\}$. On the sample space $(\Omega,\mathcal{F},P)$ one can define a derivative operator D, associated domains $(\mathbb{D}^{n,p},\|\cdot\|_{n,p})$ where n denotes the order of differentiation and p denotes the $L^p(\Omega)$ space where the derivatives lie. The high order stochastic derivative is denoted by D^v_α for $v\in\{1,...,k\}^n$ and $\alpha\in[0,T]^n$. We say that F is smooth if $F\in\mathbb{D}^\infty=\cap_{n\in\mathbf{N},p>1}\mathbb{D}^{n,p}$.

For a $q-$dimensional random variable $F\in\mathbb{D}^{1,2}$, we denote by ψ_F the Malliavin covariance matrix associated with F. That is, $\psi_F^{i,j}=<DF^i,DF^j>_{L^2[0,T]}$. One says that the random variable is non-degenerate if $F\in\mathbb{D}^\infty$ and the matrix ψ_F is invertible a.s. and $(\det\psi_F)^{-1}\in\cap_{p\geq1}L^p(\Omega)$. In such a case expressions of the type $E(\delta_y(F))$, where δ_y denotes the Dirac delta function, have a well defined meaning through the integration by parts formula.

The integration by parts formula of Malliavin Calculus can be briefly described as follows. Suppose that F is a non-degenerate random variable and $G\in\mathbb{D}^\infty$. Then for any function $g\in C_p^\infty(\mathbb{R}^q)$ and a finite sequence of multi-indexes $\beta\in\cup_{l\geq1}\{1,...,q\}^l$, we have that there exists a random variable $H^\beta(F,G)\in\mathbb{D}^\infty$ so that

$$E(g^\beta(F)G)=E(g(F)H^\beta(F,G))$$

Here g^β denotes the high order derivative of order $l(\beta)$ and whose partial derivatives are taken according the index vector β. This inequality can be obtained following the calculations in Lemma 12 of [16]. In some cases we will consider the above norms and definitions on a conditional form. That is, we will use partial Malliavin Calculus. We will denote this by adding a further time sub-index in the norms. For example, if one completes the space of smooth functionals with the norm

$$\begin{aligned}\|F\|_{2,s}&=(E(\|F\|^2/\mathcal{F}_s))^{1/2}\\ \|F\|_{1,2,s}^2&=\|F\|_{2,s}^2+E(\int_s^T\|D_uF\|^2\,du/\mathcal{F}_s),\end{aligned}$$

we obtain the space $\mathbb{D}_s^{1,2}$.

To simplify the notation we will sometimes denote $E_s(\cdot)=E(\cdot/\mathcal{F}_s)$. Analogously we will write H_s^β and $\psi_F(s)$ when considering integration by parts formula and the Malliavin covariance matrix conditioned on $\mathcal{F}_s$. That is, $\psi_F^{i,j}(s)=<DF^i,DF^j>_{L^2[s,T]}$. Also we say that $F\in\overline{\mathbb{D}}_s^{1,2}$ when $F\in\mathbb{D}_s^{1,2}$ and $\|F\|_{1,2,s}\in\cap_{p\geq1}L^p(\Omega)$. Similarly, we say that F is $s-$conditionally non-degenerate if $F\in\overline{\mathbb{D}}_s^{\infty}$ and $(\det\psi_F(s))^{-1}\in\cap_{p>1}L_s^p(\Omega)$. In such a case, as before, expressions like $E(\delta_y(F)/\mathcal{F}_s)$ have a well defined meaning through the partial integration by parts formula or via an approximation of the delta function.

We will also have to deal with similar situations for sequences F_i that are $\mathcal{F}_{t_i}-$measurable random variables, $i = 1, ..., N$ for a partition $0 = t_0 < t_1 < ... < t_N$. In this case we say that $\{F_i; i = 1, ..., N\} \subseteq \overline{\mathbb{D}}^{\infty}$ uniformly if $F_i \in \overline{\mathbb{D}}^{\infty}_{t_{i-1}}$ for all $i = 1, ..., N$ and for any $l > 1$ one has that there exists a finite positive constant $C(n, p, l)$ such that

$$\sup_N \sup_{i=1,..,N} E \left\| F_i \right\|^l_{n,p,t_{i-1}} \leq C(n, p, l).$$

In what follows we will sometimes expand our basic sample space to include further increments of another independent Wiener process, $\overline{W}$ (usually these increments are denoted by $Z_i = \overline{W}(i+1) - \overline{W}(i) \sim N(0, 1)$) independent of W in such a case we denote the expanded filtration by $\overline{\mathcal{F}}_t = \mathcal{F}_t \vee \sigma(\{\overline{W}(s); s \leq i+1, t_i \leq t\})$. We do this without further mentioning and suppose that all norms and expectations are considered in the extended space. Sometimes we will write $F \in \mathcal{F}_t$ which stands for F is a $\mathcal{F}_t$ -measurable random variable.

We use the notation $I^i_j(h) = \int_{t_{i-1}}^{t_i} h(s) dW^j(s)$ for $j = 1, ..., k$ and $h : \Omega \to L^2([t_{i-1}, t_i]; \mathbb{R}^q)$ a $\mathcal{F}_{t_{i-1}}-$measurable smooth random processes.

3. Some heuristics

In order to motivate the definition of uniformly elliptic random variable we give a brief idea of how to obtain a lower Gaussian estimate for the density of this r.v. (for a complete proof, see [10]). We use the case of non-homogeneous diffusion in order to draw a parallel with some well known concepts. This will also help us guide in the application of the Definition 1 in Section 4. All references to the definition of uniformly elliptic random variable are in Definition 1. The idea for the proof which also appears graphically in Figure 1 can be explained as follows: We say that an estimate for a density $p(x)$ is global if the estimate is valid for all x. In contrast, we say that an estimate is local if it is only valid for points close to x. In order to obtain a global Gaussian type lower bound for a random variable $F = X(t)$ generated by a Wiener process, we first identify a time component in the random variable that will assure adaptedness. In our example this is t. Next, given any sequence of partitions of the time interval $[0, t]$ we assume there exists a sequence of adapted approximations along the time axis. In our case, let $0 = t_0 < t_1 < ... < t_N = t$ be any partition of $[0, t]$. Then we define the sequence of approximations through $F_i = X(t_i)$. Denote by $p(t_i, x_i; t_{i+1}, x_{i+1})$ the conditional density of F_{i+1} at x_{i+1} with respect to $F_i = x_i$.

Idea of the proof: In order to obtain the global Gaussian type lower bound one uses the Chapman-Kolmogorov formula. That is, if one can have a good local lower Gaussian estimate of the density of $F_i = X(t_i)$ conditioned to $\mathcal{F}_{t_{i-1}}$ then one can hope that the global lower estimate for the density of $X(t)$ should be satisfied. In order to obtain this local estimate one needs to consider partitions of small size.

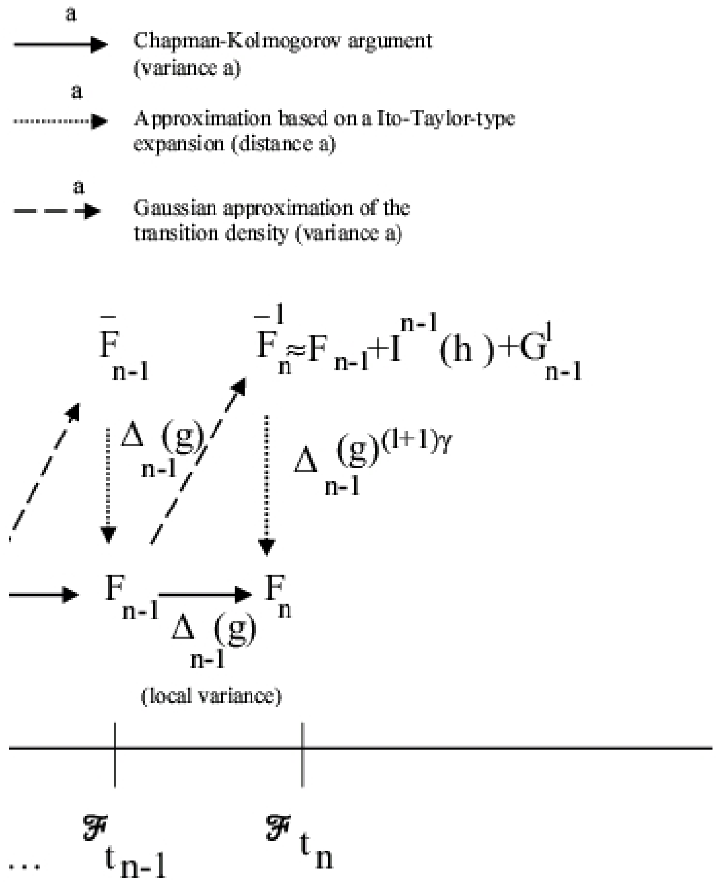

FIGURE 1. Idea of the proof to obtain Gaussian lower bounds for F_n. Here $I^{n-1}(h)$ is conditionally Gaussian and its local variance is of the order $\Delta_{n-1}(g)$. G^l_{n-1} is a residue in the sense that is of smaller order than $I^{n-1}(h)$. As l becomes bigger $\overline{F}^l_n$ approaches F_n.

The main idea of the argument is to use Chapman-Kolmogorov formula to obtain that

$$p(0,x;t,y) = \int_{\mathbb{R}^q} \ldots \int_{\mathbb{R}^q} p(0,x;t_1,x_1) \ldots p(t_{n-1},x_{n-1};t_n,y) dx_1 \ldots dx_{n-1}.$$

Here $\pi = \{t_i;\ i = 0,...,N\}$ is a partition of $[0,t]$ with $0 = t_0 < ... < t_n = t$. This is noted in Figure 1 by the solid arrows. Next we localize the previous estimation by considering balls $B_i = B(x_i, r_i)$ for some appropiate radius r_i and $x_n = y$. Then the above can be bounded by below giving that

$$p(0,x;t,y) \geq \int_{B_n} \dots \int_{B_2} p(0,x;t_1,x_1)\dots p(t_{n-1},x_{n-1};t_n,y)dx_1\dots dx_{n-1}.$$

Therefore one needs to find a local Gaussian type lower bound for $p(t_i, x_i; t_{i+1}, x_{i+1})$. Nevertheless, still, there are various points to this argument:

1. It is difficult to see how to obtain the local Gaussian type lower bound for the density of a general random variable $F_i = X(t_i)$ conditioned to $\mathcal{F}_{t_{i-1}}$. First one needs that the conditional density exists. This is the condition in **(H2b)** in Definition 1.

2. In order to obtain a Gaussian type lower bound for the conditional density of $F_i = X(t_i)$ with respect to $\mathcal{F}_{t_{i-1}}$, one expands $X(t_i)$, conditioned on $\mathcal{F}_{t_{i-1}}$, using a Itô-Taylor expansion (this is (2)) so that the main term in this expansion, $\sum_{j=1}^{k} I_j^i(h_j)$, is a Gaussian random variable. This Gaussian random variable should be non-degenerated and have the same local variance as the conditional density of $F_i = X(t_i)$ conditioned to $\mathcal{F}_{t_{i-1}}$ which we will denote by $\Delta_{i-1}(g) = \int_{t_{i-1}}^{t_i} \|g(s)\|^2 ds$. Note that g can change with time therefore it effectively measures a local variance. In order for the estimate to lead to a global lower bound we need that this local variance should not vanish or be unbounded. This is embodied in **(H2c)**. This is the core of the uniform elliptic condition.

3. The argument in 2. is not sufficient as we also need that the density of the newly defined random variable has to be close to the conditional density of F_i. In conclusion, instead of finding directly a local lower estimate for the conditional density of $F_i = X(t_i)$ with respect to $\mathcal{F}_{t_{i-1}}$ we find a local lower estimate for the conditional density of the further approximation $\overline{F}_i^l \equiv \overline{F}_i$ with respect to $\mathcal{F}_{t_{i-1}}$. We call this new approximation the truncated approximation, the order of the approximation being determined by the parameter l. In order to find the local estimate for the conditional density of the truncated approximation one needs that the Gaussian term $\sum_{j=1}^{k} I_j^i(h_j)$ has to be the dominant one in the series expansion. That is, one needs to prove that the higher order terms, $\Delta_{i-1}(g)^{(l+1)\gamma} Z_i + G_i^l$ do not contribute significantly to the Gaussian estimate (this is condition **(H2d)**). That is, the variance of the higher order terms is smaller than the variance of the Gaussian term. This step is noted by the dashed arrows in Figure 1.

4. Next one needs to have that the conditional density of the truncated approximation, $\overline{F}_i$, with respect to $\mathcal{F}_{t_{i-1}}$ has to be close to the conditional density of $F_i = X(t_i)$ with respect to $\mathcal{F}_{t_{i-1}}$. For this, one needs to use a high number of terms in the Itô-Taylor series expansion so as to obtain that the conditional density of the truncated series and the conditional density of $X(t_{i+1})$ are close enough. This is embodied in condition **(H2a)**. This step is noted by the dotted arrows in Figure 1.

5. Finally the estimates have to be uniform in the sense that the constants appearing in the estimate should not depend "too much" on the partition or the

sigma field $\mathcal{F}_{t_{i-1}}$. For this reason we need to require that various estimates are uniform such as **(H1)** and that all constants appearing do not depend on $\omega \in \Omega$.

The definition of a uniformly elliptic random variable is as follows

Definition 1. *Let $F \in \mathcal{F}_t$. Suppose that there exists:*
1. $\epsilon > 0$ such that for any sequence of partitions $\pi_N = \{0 = t_0 < t_1 < ... < t_N = t\}$ whose norm is smaller than ϵ and $|\pi_N| = \max\{|t_{i+1} - t_i|; i = 0, ..., N-1\} \to 0$ as $N \to \infty$ there exists a sequence $F_i \in L^2(\Omega; \mathbb{R}^q)$, $i = 1, ..., N$ such that $F_N = F$. F_i is a $\mathcal{F}_{t_i}$-measurable random variable and is a t_{i-1}-conditionally non-degenerate random variable.
2. a function $g : [0,T] \to \mathbb{R}_{>0}$ (the local variance function) and a positive constant $C(T)$ such that $\|g\|_{L^2([0,T])} \leq C(T)$. We denote the local variance $\Delta_{i-1}(g) = \int_{t_{i-1}}^{t_i} \|g(t)\|^2 dt$. This quantity measures approximately the variance in the local Gaussian approximation to the density of F_i conditioned to $\mathcal{F}_{t_{i-1}}$.
3. Suppose that for each F_i and each $l \in \mathbb{N}$ there exists a sequence $\overline{F}_i \equiv \overline{F}_i^l$ such that

$$\overline{F}_i = \Delta_{i-1}(g)^{(l+1)\gamma} Z_i + F_{i-1} + \sum_{j=1}^{k} I_j^i(h_j) + G_i^l. \tag{2}$$

Here G_i^l are $\mathcal{F}_{t_i} \cap \overline{\mathbb{D}}_{t_{i-1}}^{\infty}$ random variables and $h_j \equiv h_j|_{[t_{i-1},t_i]} : \Omega \to L^2([t_{i-1}, t_i]; \mathbb{R}^q)$ is a collection of $\mathcal{F}_{t_{i-1}}-$measurable smooth random processes which satisfies for almost all $\omega \in \Omega$:
(H1) *There exists a constant $C(n,p,T)$ such that*

$$\|F_i\|_{n,p} + \sup_{\omega \in \Omega} \|h_j\|_{L^2([t_{i-1},t_i])}(\omega) \leq C(n,p,T)$$

for any $j = 1, ..., k$, $i = 0, ..., N$ and n, $p \in \mathbb{N}$.
Furthermore the following four conditions are satisfied for the approximation sequence $\overline{F}_i$ and any $i = 1, ..., N$ and almost all $\omega \in \Omega$
(H2a) *There exists a constant $\gamma > 0$, such that for any n, p, $l \in \mathbb{N}$, $\left\|F_i - \overline{F}_i\right\|_{n,p,t_{i-1}} \leq C(n,p,T)\Delta_{i-1}(g)^{(l+1)\gamma}$.*
(H2b) *There exists a constant $C(p,T) > 0$ such that for any $p > 1$*

$$\left\|\det \psi_{F_i}^{-1}(t_{i-1})\right\|_{p,t_{i-1}} \leq C(p,T)\Delta_{i-1}(g)^{-q}.$$

(H2c) *Define*

$$A = \Delta_{i-1}(g)^{-1} \begin{pmatrix} \int_{t_{i-1}}^{t_i} \left\langle h^1(s), h^1(s)\right\rangle ds & ... & \int_{t_{i-1}}^{t_i} \left\langle h^1(s), h^q(s)\right\rangle ds \\ : & . & : \\ \int_{t_{i-1}}^{t_i} \left\langle h^q(s), h^1(s)\right\rangle ds & ... & \int_{t_{i-1}}^{t_i} \left\langle h^q(s), h^q(s)\right\rangle ds \end{pmatrix}.$$

We assume that there exists strictly positive constants $C_1(T)$ and $C_2(T)$, such that for all $\xi \in \mathbb{R}^q$,

$$C_1(T)\xi'\xi \geq \xi' A \xi \geq C_2(T)\xi'\xi.$$

(H2d) *There exist constants $\varepsilon > 0$ and $C(n,p,l,T)$ such that*

$$\left\| G_i^l \right\|_{n,p,t_{i-1}} \leq C(n,p,l,T)\Delta_{i-1}(g)^{\frac{1}{2}+\varepsilon}.$$

In the previous definition γ is a constant that may change depending on the characteristics of how the underlying noise appears in the structure of F and the quality of the approximation sequence $\overline{F}_i$. For example in the case of the non-homogeneous diffusion we will use for $\overline{F}_i$ a high order Itô Taylor type of approximation and in such a case $\gamma = 1/2$.

In this setting we try to give conditions for the sequence as close as possible to the general set-up of stochastic differential equations and requiring the least amount of conditions so that the lower bound for the density of the approximative random variable can be obtained. Note that in this definition, $\overline{F}_i$ is measurable with respect to the expanded filtration $\overline{\mathcal{F}}_{t_i}$ as we are adding the variables Z_i to its definition. In particular the norm appearing in condition **(H2a)** is the norm in the extended space.

Also we remark that the random variables $\overline{F}_i$, considered in this Theorem will not necessarily be non-degenerate unless one adds the independent random variable $\Delta_{i-1}(g)^{(l+1)\gamma} Z_i$. Then the main result obtained in [10] is

Theorem 2. *Let F be a uniformly elliptic random variable. Then there exists a constant $M > 0$ that depends on all other constants in the definition 1 such that*

$$p_F(y) \geq \frac{\exp\left(-M \frac{\|y - F_0\|^2}{\|g\|^2_{L^2([0,t])}} \right)}{M \left\| g \right\|_{L^2([0,t])}{}^{q/2}}.$$

4. Lower bound density estimates for non-homogeneous diffusions

Throughout this section we assume that the following hypotheses are satisfied:

(H) The measurable coefficients b and σ satisfy for any $j = 0, 1, ...$

$$\begin{aligned} \sup_{t\in[0,T]} \left(\sup_{x\in\mathbb{R}^q} \left\| b^{(j)}(t,x) \right\| + \sup_{x\in\mathbb{R}^q} \left\| \sigma^{(j)}(t,\cdot) \right\| \right) &< \infty, \\ \xi'\sigma(t,x)\sigma(t,x)'\xi &\geq c \left\| \xi \right\|^2 \end{aligned}$$

for some positive constant c and any vectors $x, \xi \in \mathbb{R}^q$ and $t \in [0,T]$.

First we start with a preparatory Lemma that describes the smoothness of the random variables $F_i = X(t_i)$.

Lemma 3. *Assume condition (H). Then $F_i \in \overline{\mathbb{D}}^{\infty}_{t_{i-1}} \cap \mathcal{F}_{t_i}$-measurable r.v. and is a t_{i-1}-conditionally non-degenerate r.v. for all $i = 1, ..., N$*

The proof of this statement is done through the usual techniques of stochastic differentiation. The technique is similar as in the homogeneous case, see for example, [3], [19]. The non-homogeneous case is treated in [11]. Here we only need to do a small modification to their argument in order to incorporate the conditioning.

Proof. We only briefly sketch the main points of this proof. We will prove by induction that for $f \in C_p^\infty(\mathbb{R}^q, \mathbb{R})$, any $n \in \mathbb{N}$, $p > 0$, $T > t > t_{i-1}$, $\alpha \in [t_{i-1}, t]^n$, $\upsilon \in \{1, ..., k\}^n$ there exists a positive non-random constant $C(n, p, T)$ such that

$$\sup_{\alpha \in [t_{i-1}, t]^n} E_{t_{i-1}} \left(D_\alpha^\upsilon f(X(t)) \right)^p \le C(n, p, T). \tag{3}$$

Note that the constant $C(n, p, T)$ depends on all the other constants in the problem but is independent of N, ω and the partition. First, let us prove the assertion for $n = 1$, $\alpha \in [t_{i-1}, t]$ and $i \le k$ then we have that by the chain rule for the stochastic derivative

$$D_\alpha^i f(X(t)) = f'(X(t)) D_\alpha^i X(t).$$

Next one obtains using the Picard iteration method that $X(t) \in \mathbb{D}^\infty$ and that

$$D_\alpha^i X(t) = \sigma_i(\alpha, X(\alpha)) + \int_\alpha^t b'(s, X(s)) D_\alpha^i X(s) ds + \int_\alpha^t \sigma_j'(s, X(s)) D_\alpha^i X(s) dW^j(s),$$

then given that the coefficients are bounded with bounded derivatives, it follows that

$$E_{t_{i-1}} \left\| D_\alpha^i X(t) \right\|^p \le C(t) \left(1 + \int_\alpha^t E_{t_{i-1}} \left\| D_\alpha^i X(s) \right\|^p ds \right).$$

for any $p \ge 2$ and $C(t)$ is a positive constant increasing in t and independent of α. Then the conclusion follows applying Gronwall's lemma. Now assume that the assertion is true for $n - 1$. The proof for n follows along the same lines as before. That is, let $\alpha = (s_1, ..., s_n)$, $\upsilon = (j_1, ..., j_n)$ and denote by $\alpha- = (s_1, ..., s_{n-1})$ and $\upsilon- = (j_1, ..., j_{n-1})$ then as before using a Picard approximation method, one can prove that $X(t) \in \mathbb{D}_{t_{i-1}}^{n,p}$ for any $n \ge 1$, $p \ge 1$ and $t \ge t_{i-1}$. As a consequence one also obtains the following equation for $D_\alpha^\upsilon X(t)$

$$\begin{aligned}
D_{\alpha-}^{\upsilon-} D_{s_n}^{j_n} X(t) &= D_{\alpha-}^{\upsilon-} \sigma_{j_n}(s_n, X(s_n)) + \int_{s_n}^t D_{\alpha-}^{\upsilon-} \left(b'(s, X(s)) D_{s_n}^{j_n} X(s) \right) ds \\
&\quad + D_{\alpha-}^{\upsilon-} \left(\int_{s_n}^t \sigma_j'(s, X(s)) D_{s_n}^{j_n} X(s) dW^j(s) \right), \\
&= D_{\alpha-}^{\upsilon-} \sigma_{j_n}(s_n, X(s_n)) + \int_{s_n}^t \sum_{\upsilon_1} D_{\alpha_1}^{\upsilon_1} b'(s, X(s)) D_{\alpha_2}^{\upsilon_2} D_{s_n}^{j_n} X(s) ds \\
&\quad + \sum_{k=1}^{n-1} D_{\alpha(k)}^{\upsilon(k)} \left(\sigma_{j_k}'(s, X(s)) D_{s_n}^{j_n} X(s) \right) \\
&\quad + \int_{s_n}^t \sum_{\upsilon_1} D_{\alpha_1}^{\upsilon_1} \sigma_j'(s, X(s)) D_{\alpha_2}^{\upsilon_2} D_{s_n}^{j_n} X(s) dW^j(s).
\end{aligned}$$

The first sum index is composed of splitting the indexes $\upsilon-$ and $\alpha-$ into two disjoint subsets υ_1 and υ_2 for $\upsilon-$ and $\alpha_1$and α_2 for $\alpha-$. Similarly, $\upsilon(k)$ and $\alpha(k)$ denote the indices without the j_k and s_k component respectively. The final result follows as before by using the chain rule and the Gronwall lemma together with the inductive hypothesis.

The assertion on the conditional non-degeneracy of F_i follows a similar argument as in Theorem 3.5 of [11]. □

In the particular case that σ and b do not depend on t, Kusuoka and Stroock proved in [13] under hypoelliptic conditions that if $b = \sum_{i=1}^{r} a_i \sigma_i$ for some functions $a_i \in C_b^\infty$ then a Gaussian type lower bound is satisfied by the density of X_t.

Here we improve their result in the sense that this extra condition is not required, the diffusion is non-homogeneous and the conditions on the coefficients with respect to the time parameters are minimal. This is obtained by applying the definition of uniformly elliptic random variable. To apply this definition we need to define all its ingredients. That is, define $F = X(t)$ and $g(s) \equiv 1$ for all $s \in [0, T]$. For any partition $0 = t_0 < ... < t_n = t$, let $F_i = X(t_i)$. In order to define $\overline{F}_i$ we have to explicitly write an Itô-Taylor expansion for the case of non-homogeneous diffusions. This is obtained considering the difference $F_i - F_{i-1}$. To introduce the Itô-Taylor expansion we will deal with indices $\beta \in \cup_{n \geq 1} \{0, 1, ..., k\}^n \cup \{\nu\}$, ν denotes the empty index. In such a case, $l(\beta)$ denotes the length of the multi-index β and $n(\beta)$ the quantity of zeros in β, $l(\nu) = 0$. $\beta(i)$ denotes the i-th component in β, $-\beta$ denotes the index β without its first component. Similarly, one defines $\beta-$.

We also define the following operators for a smooth function $f : [t_{i-1}, t_i]^{l(\beta-)} \times \mathbb{R}^q \to \mathbb{R}^q$, (here we adopt the double index summation notation)

$$
\begin{aligned}
L^r f(u_\beta, x) &= \frac{\partial f}{\partial x_i}(u_{\beta-}, x)\sigma_{ir}(u_{l(\beta)}, x) \\
L^0 f(u_\beta, x) &= \frac{1}{2}\frac{\partial^2 f}{\partial x_i \partial x_j}(u_{\beta-}, x)\sigma_{ir}\sigma_{jr}(u_{l(\beta)}, x) + \frac{\partial f}{\partial x_i}(u_{\beta-}, x) b_i(u_{l(\beta)}, x)
\end{aligned}
$$

where $u_\beta = (u_1, ..., u_{l(\beta)})$ and similarly $u_{\beta-} = (u_1, ..., u_{l(\beta)-1})$. Then we define inductively

$$
\begin{aligned}
f_\beta(u_\beta, x) &= \left(L^{\beta(l(\beta))} f_{\beta-}\right)(u_\beta, x) \\
f_\nu(u, x) &= f(x).
\end{aligned}
$$

We will frequently use the above formulas for $f(x) = x$ in such a case we note that $\|f_\beta\|_\infty \leq C(\beta)$ for any $\beta \in \cup_{n \geq 1}\{0, 1, ..., k\}^n$. I_β denotes the multiple stochastic integral where the indexes of the integral are determined by the set of indices β. That is, for an adapted process Y we define

$$
I_\beta \left[f_\beta(\cdot, Y(\cdot))\right]_{t_{i-1}, t} = \int_{t_{i-1}}^{t} I_{\beta-}\left[f_\beta((\cdot, u_{l(\beta)}), Y(\cdot))\right]_{t_{i-1}, u_{l(\beta)}} dW^{\beta(l(\beta))}(u_{l(\beta)}).
$$

Note that these formulae are not exactly the same as the usual Itô-Taylor expansion (see for example, [9]). In fact, for $\beta=(0,1)$ then for $f(x)=x$, we have

$$I_\beta\left[f_\beta(\cdot,Y(\cdot))\right]_{t_{i-1},t}=\int_{t_{i-1}}^{t}\int_{t_{i-1}}^{u_2}\frac{\partial b}{\partial x_i}(u_1,Y(u_1))\sigma_{i1}(u_2,Y(u_1))du_1dW^1(u_2).$$

In particular note that the integrand depends on both integrating variables u_1 and u_2 which is not the common in the usual Itô-Taylor formula. In this way one can weaken some restrictive conditions on the coefficients. Then we have the following Itô-Taylor formula for X.

Lemma 4. *Let $f\in C_p^\infty(\mathbb{R}^q)$ and $\mathcal{A}_l=\{\beta\in\cup_{n\geq1}\{0,1,...,k\}^n;1\leq l(\beta)+n(\beta)\leq l\}$, $\mathcal{B}_l=\{\beta\in\cup_{n\geq1}\{0,1,...,k\}^n;-\beta\in\mathcal{A}_l,\beta\notin\mathcal{A}_l\}$ then*

$$(4)\quad f(X(t))-f(X(s))=\sum_{\beta\in\mathcal{A}_l}I_\beta[f_\beta(\cdot,X_{t_{i-1}})]_{t_{i-1},t_i}+\sum_{\beta\in\mathcal{B}_l}I_\beta[f_\beta(\cdot,X(\cdot))]_{t_{i-1},t_i}.$$

In the case that $f(x)=x$ we have the following estimate for $\beta\in\cup_{n\geq1}\{0,1,...,k\}^n$

$$(5)\qquad \left\|I_\beta\left[f_\beta(\cdot,X(\cdot))\right]_{t_{i-1},t}\right\|_{n,p,t_{i-1}}\leq C(n,p,T)\left(t-t_{i-1}\right)^{\frac{n\vee l(\beta)}{2}}.$$

Proof. First we prove the first statement by induction. Obviously the result is true for $l=0$. Now suppose that the result is true for l. To prove that the expansion is true for $l+1$, one has to take every term of the type $I_\beta[f_\beta(\cdot,X(\cdot))]_{t_{i-1},t_i}$ for $\beta\in\mathcal{B}_l$ with $n(\beta)+l(\beta)=l+1$ and consider the difference

$$I_\beta[f_\beta(\cdot,X(\cdot))]_{t_{i-1},t_i}-I_\beta[f_\beta(\cdot,X_{t_{i-1}})]_{t_{i-1},t_i}=I_\beta\left[f_\beta(\cdot,X(\cdot))-f_\beta(\cdot,X_{t_{i-1}})\right]_{t_{i-1},t_i}$$

Then we apply Itô's formula for $f_\beta(u_\beta,X(u_{l(\beta)}))-f_\beta(u_\beta,X_{t_{i-1}})$ fixing the time component to obtain that

$$\begin{aligned}f_\beta(u_\beta,X(u_{l(\beta)}))-f_\beta(u_\beta,X_{t_{i-1}}) \quad &= \quad \int_{t_{i-1}}^{u_{l(\beta)}}L^rf_\beta(u_{(\beta,r)},X_{u_{l(\beta)+1}})dW^r(u_{l(\beta)+1})\\ &\quad +\int_{t_{i-1}}^{u_{l(\beta)}}L^0f_\beta(u_{(\beta,0)},X_{u_{l(\beta)+1}})du_{l(\beta)+1}\end{aligned}$$

Therefore

$$I_\beta[f_\beta(\cdot,X(\cdot))]_{t_{i-1},t_i}=I_\beta[f_\beta(\cdot,X_{t_{i-1}})]_{t_{i-1},t_i}+\sum_{r=0}^{k}I_{(\beta,r)}[f_{(\beta,r)}(\cdot,X(\cdot))]_{t_{i-1},t_i}$$

and $(\beta,r)\in\mathcal{B}_{l+1}$. To finish the proof of (4) one has to prove that:
1. $\mathcal{A}_{l+1}=\mathcal{A}_l\cup\{\beta\in\mathcal{B}_l;n(\beta)+l(\beta)=l+1\}$.
2. $\mathcal{B}_{l+1}=(\mathcal{B}_l-\mathcal{A}_{l+1})\cup\{(\beta,r);r=0,...,k,\beta\in\mathcal{B}_l,n(\beta)+l(\beta)=l+1\}$.
To prove property 1., take $\beta\in\mathcal{A}_{l+1}-\mathcal{A}_l$ then obviously $n(-\beta)+l(-\beta)\leq l$ and therefore $-\beta\in\mathcal{A}_l$. Therefore the inclusion $\subseteq$ follows. The other inclusion is trivial. For property 2., the inclusion $\supseteq$ is trivial. For the other, suppose that there exists $\beta'\in\mathcal{B}_{l+1}$ such that $\beta'\notin(\mathcal{B}_l-\mathcal{A}_{l+1})\cup\{(\beta,r);r=0,...,k,\beta\in\mathcal{B}_l,n(\beta)+l(\beta)=$

$l+1\}$. Then $l(\beta')+n(\beta')=l+2$, $l(-\beta')+n(-\beta')=l$. Therefore $\beta'\in\mathcal{B}_l-\mathcal{A}_{l+1}$ which is a contradiction.

Next we prove the norm estimate (5) by induction. For this, we first have that $I_\beta[f_\beta(\cdot,X(\cdot))]_{t_{i-1},t}\in\mathbb{D}^{n,p}_{t_{i-1}}$ for any $n\geq 1$, $p\geq 1$ and any $t\geq t_{i-1}$ due to Lemma 3. Next, suppose that for $\beta\in\cup_{n\geq 1}\{0,1,...,k\}^n$ we have $\upsilon\subseteq\beta$, then we define $\beta(\upsilon)$ as the set of indices that are in β but not in υ. We will prove that there exits a positive constant $C(j,a,n,p,T)$ such that for $\upsilon=(i_1,...,i_a)\in\{1,...,k\}^a$ with $\upsilon\subseteq\beta$, $\upsilon_1\in\{1,...,k\}^n$, $\upsilon_2\in\{1,...,k\}^j$, $\alpha=(s_1,...,s_a)$, $\alpha_1\in[t_{i-1},t]^n$, $\alpha_2\in[t_{i-1},t]^j$, $j\geq 0, n\geq 0$ and $a\geq 0$, we have

$$E_{t_{i-1}}\left(\int_{[t_{i-1},t]^{n+j+a}}\left\|D^{\upsilon_1}_{\alpha_1}I_{\beta(\upsilon)}[\prod_{i=1}^a 1(u_i=s_i)D^{\upsilon_2}_{\alpha_2}f_\beta(\cdot,X_\cdot)]_{t_{i-1},t}\right\|^2 d\alpha_1 d\alpha_2 d\alpha\right)^p$$
$$\leq C(j,a,n,p,T)(t-t_{i-1})^{(n\vee l(\beta(\upsilon))+j+a)p}. \tag{6}$$

Here we denote by $u_\beta=(u_1,...,u_{l(\beta)})$ the variables of integration in I_β where the first a components correspond to the indices in υ. To simplify notation we will denote $\Pi_\alpha=\prod_{i=1}^a 1(u_i=s_i)$. The induction is performed in n. So let us suppose first that $n=0$. Then we have

$$\begin{aligned}&E_{t_{i-1}}\left(\int_{[t_{i-1},t]^{j+a}}\left\|I_{\beta(\upsilon)}[\Pi_\alpha D^{\upsilon_2}_{\alpha_2}f_\beta(\cdot,X_\cdot)]_{t_{i-1},t}\right\|^2 d\alpha_2 d\alpha\right)^p\\
\leq\ &(t-t_{i-1})^{(j+a)(p-1)}\int_{[t_{i-1},t]^{j+a}}E_{t_{i-1}}\left\|I_{\beta(\upsilon)}[\Pi_\alpha D^{\upsilon_2}_{\alpha_2}f_\beta(\cdot,X_\cdot)]_{t_{i-1},t}\right\|^{2p}d\alpha_2 d\alpha\\
\leq\ &C(T)(t-t_{i-1})^{(j+a+l(\beta(\upsilon)))(p-1)}\int_{[t_{i-1},t]^{j+l(\beta)}}E_{t_{i-1}}\left\|D^{\upsilon_2}_{\alpha_2}f_\beta(u_\beta,X_{u_1})\right\|^{2p}du_\beta d\alpha_2.\end{aligned}$$

This step finishes by noting that $E_{t_{i-1}}\left\|D^{\upsilon_2}_{\alpha_2}f_\beta(u_\beta,X_{u_1})\right\|^{2p}\leq C(j,l(\beta),p,T)$ where the positive constant C is independent of α_2, υ_2 and β (see (3)). Next suppose that the assertion is valid for $n-1$. Then let $\upsilon_1=(j_1,...,j_n)$ and $\alpha_1=(s'_1,...,s'_n)$. We consider two cases: First suppose that $j_n\notin\beta(\upsilon)$, then we have that

$$D^{\upsilon_1}_{\alpha_1}I_{\beta(\upsilon)}[\Pi_\alpha D^{\upsilon_2}_{\alpha_2}f_\beta(\cdot,X_\cdot)]_{t_{i-1},t}=D^{\upsilon_1-}_{\alpha_1-}I_{\beta(\upsilon)}[\Pi_\alpha D^{j_n}_{s'_n}D^{\upsilon_2}_{\alpha_2}f_\beta(\cdot,X_\cdot)]_{t_{i-1},t}.$$

Then it is clear that the assertion follows from the inductive hypothesis.

Next suppose that $j_n\in\beta(\upsilon)=(i_1,...,i_{l(\beta(\upsilon))})$ and that the indices $i_1,...,i_h$, $h\leq l(\beta(\upsilon))$, are all the indices in $\beta(\upsilon)$ equal to j_n. That is , $j_n\notin\{i_{h+1},...,i_{l(\beta(\upsilon))}\}$. Then

$$\begin{aligned}&D^{\upsilon_1}_{\alpha_1}I_{\beta(\upsilon)}[\Pi_\alpha D^{\upsilon_2}_{\alpha_2}f_\beta(\cdot,X_\cdot)]_{t_{i-1},t}=D^{\upsilon_1-}_{\alpha_1-}I_{\beta(\upsilon)}[\Pi_\alpha D^{j_n}_{s'_n}D^{\upsilon_2}_{\alpha_2}f_\beta(\cdot,X_\cdot)]_{t_{i-1},t}\\
&+\sum_{r=1}^h D^{\upsilon_1-}_{\alpha_1-}I_{\beta(\upsilon,i_r)}[1(u_{l(\upsilon)+r}=s'_n)\Pi_\alpha D^{\upsilon_2}_{\alpha_2}f_\beta(\cdot,X_\cdot)]_{t_{i-1},t}.\end{aligned}$$

Again it is clear then that the assertion follows from the inductive hypothesis. From (6) one obtains (5). □

It should be clear that the estimate (6) is not optimal. In fact, the estimate improves as the number of zeros in β increase. Nevertheless this is sufficient for our purposes.

Now we can define the approximation random variables $\overline{F}_i = \overline{X}(t_i)$ as the Itô-Taylor approximation of order l of X based on time t_{i-1}. That is, $\overline{X}$ is defined as follows

$$\begin{aligned}
\overline{X}_{t_i} &= (t_i - t_{i-1})^{\frac{l+1}{2}} Z_i + X_{t_{i-1}} + \sum_{\beta \in \mathcal{A}_1} I_\beta[f_\beta(\cdot, X_{t_{i-1}})]_{t_{i-1},t_i} + G_i^l \\
G_i^l &= \sum_{2 \le l(\beta)+n(\beta) \le l} I_\beta[f_\beta(\cdot, X_{t_{i-1}})]_{t_{i-1},t_i} \\
h_j(u) &= f_{(j)}(u, X_{t_{i-1}}),\ u \in [t_{i-1}, t_i] \\
g(u) &= 1,\ u \in [0, T]
\end{aligned}$$

where $f(x) = x$, $Z_i = \overline{W}(i+1) - \overline{W}(i)$ is a q-dimensional $N(0, I)$ random variable independent of the Wiener process W and Z_j for $j \neq i$.

Theorem 5. *Assume (H) and $t > 0$. Let X be the unique solution of the non-homogeneous diffusion equation. Then $X(t)$ has a smooth density, denoted by $p(t, x, y)$ that satisfies*

$$\frac{m \exp(-\frac{\|x-y\|^2}{mt})}{t^{q/2}} \ge p(t, x, y) \ge \frac{\exp(-M\frac{\|x-y\|^2}{t})}{M t^{q/2}}$$

for two constants $m,\ M \in [1, +\infty)$.

Proof. We verify each hypothesis in the Definition 1:
Preliminaries: First, $F_i = X(t_i)$ is t_{i-1}-conditionally non-degenerate by Lemma 3, therefore 1. follows. Next, $\|g\|^2_{L^2[0,T]} = T$, therefore 2. follows. $G_i^l \in \overline{\mathbb{D}}^\infty_{t_{i-1}} \cap \mathcal{F}_{t_i}$ due to Lemma 4, $\|f_\beta\|_\infty \le C(\beta)$ and classical estimates for the $\mathbb{D}^{n,p}$-norms of stochastic integrals. h_j is obviously smooth and adapted to $\mathcal{F}_{t_{i-1}}$. Therefore 3. follows and all the conditions in the preliminaries of the definition of uniformly elliptic random variable are satisfied.
(H1): $\|F_i\|_{n,p} = \|X(t_i)\|_{n,p} \le C(n,p)$ due to Theorem 3.5 in [11]. $\sup_\omega \|h_j\|_{L^2([t_{i-1},t_i])}(\omega) \le C(n,p)$ is satisfied due to the boundeness of h_j.
(H2a): Using (4) and (5), we obtain that there exists a universal positive constant independent of the initial point,

$$\begin{aligned}
\left\|X_{t_i} - \overline{X}_{t_i}\right\|_{n,p,t_{i-1}} &= \left\|(t_i - t_{i-1})^{\frac{l+1}{2}} Z_i + \sum_{\beta \in \mathcal{B}_l} I[f_\beta(\cdot, X_\cdot)]_{t_{i-1},t_i}\right\|_{n,p,t_{i-1}} \\
&\le (t_i - t_{i-1})^{\frac{l+1}{2}} \|Z_i\|_{n,p,t_{i-1}} + \sum_{\beta \in \mathcal{B}_l} \left\|I[f_\beta(\cdot, X_\cdot)]_{t_{i-1},t_i}\right\|_{n,p,t_{i-1}} \\
&\le C\,(t_i - t_{i-1})^{\frac{l+1}{2}}.
\end{aligned}$$

Therefore $\gamma = 1/2$ and **(H2a)** follows.
(H2b): It is also known (see Theorem 3.5 in [11]) that

$$\left\| \det \psi^{-1}_{X(t_i)}(t_{i-1}) \right\|_{p,t_{i-1}} \leq C\left(t_i - t_{i-1}\right)^{-q}$$

for a positive constant C independent of t_{i-1}.

(H2c): In this case we have that

$$A = (t_i - t_{i-1})^{-1} \int_{t_{i-1}}^{t_i} \sigma(s, X(s))\sigma'(s, X(s))ds.$$

Then due to the hypothesis (H), we have uniform ellipticity and uniform boundedness of σ and therefore we have that **(H2c)** is satisfied.
(H2d):$\left\|G_i^l\right\|_{n,p,t_{i-1}} \leq \sum_{2\leq l(\beta)+n(\beta)\leq l} \left\|I_\beta[f_\beta(\cdot, X_{t_{i-1}})]_{t_{i-1},t_i}\right\|_{n,p,t_{i-1}} \leq C(n,p,T)(t_i - t_{i-1})$. To obtain this estimate one can either use known estimates for the $\mathbb{D}^{n,p}$-norms of stochastic integrals (see [19]) or (5) for $l(\beta) \geq 2$ and compute separately the case $\beta = (0)$. This finishes the proof that the hypothesis in Definition 1 are satisfied and therefore by Theorem 2 the lower bound follows. The upper bound follows using the same technique used for the Itô-Taylor expansion together with classical techniques (see [23], Section 3). □

5. Comments and Applications

Looking at the proof of Theorem 5 one sees that although the definition of uniformly elliptic random variable may seem to be too complicated all its conditions are naturally satisfied. Furthermore most of the properties required in Definition 1 are usually proved when studying the existence and smoothness of densities in the framework of Malliavin Calculus. This result besides giving an explicit Gaussian lower bound estimate for various equations also characterizes the support of an uniformly elliptic random variable as the whole space, $\mathbb{R}^q$. In very loose terms the above estimate also means that the behavior of the non-homogeneous uniformly elliptic diffusion is the same as the Wiener process itself. This has various applications in different areas. Just to show this point let us introduce the following application to potential theory.

Theorem 6. *Assume the same conditions as in the previous theorem with $d > 2$. Define $k(x) = \|x\|^{2-d}$ and let $Cap_{d-2}(\cdot)$ denote the $d-2$ Newtonian capacity associated with the kernel k. Then one has that for any set $A \subseteq B(0,R)$ and any time interval $[a,b] \subseteq [0,T]$ there exist two positive constants $C_1(a,b,R)$ and $C_2(a,b,R)$ such that*

$$C_1 Cap_{d-2}(A) \leq P(X_t \in A, \text{ for some } t \in [a,b]) \leq C_2 Cap_{d-2}(A).$$

Furthermore the Haussdorf dimension of the random set $\{X(t), 0 \leq t\}$ is 2 if $d > 2$.

This result follows by combining the result in Theorem 5 in conditional form together with Theorem 2.4 in [4]. One can also use Theorem 5 to improve the results in [24] (quantile estimation) and [7] (statistics of diffusions).

By looking at the proof of Theorem 2 one may be lead to believe that some of the hypotheses are redundant. For example, **(H2b)** and **(H2c)** are obviously related. Similarly, **(H2a)** and **(H2d)** are also related. Nevertheless, for each case, the generality of the truncated approximation sequence in the definition of uniformly elliptic random variable allows for counterexamples. One open problem is how to improve this definition so as to minimize its requirements. However, from the practical point of view, verifying one of these hypotheses is not very different (but not the same) as verifying the other.

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Universitat Pompeu Fabra, Department of Economics and Business, Ramón Trias Fargas 25-27, 08005 Barcelona, Spain
E-mail address: `kohatsu@upf.es`