Enlargement of filtrations with random times for processes with jumps*

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Abstract

We treat an extension of Jacod's theorem for initial enlargement of filtrations with respect to random times. In Jacod's theorem the main condition requires the absolute continuity of the conditional distribution of the random time with respect to a non-random measure. Examples appearing in the theory on insider trading require extensions of this theorem where the reference measure can be random. In this article we consider such an extension which leads to an extra term in the semimartingale decomposition in the enlarged filtration. Furthermore we consider a slightly modified enlargement which allows for the bounded variation part of the semimartingale decomposition to have finite moments depending on the modification considered. Various examples for Lévy processes are treated.

Semimartingale, Lévy processes, Jacod's theorem

1 Introduction

In Corcuera et. al. [3], the authors introduced a framework to study the behavior of an insider for markets driven by a Wiener process where the additional utility obtained is finite and furthermore the market does not allow for arbitrage. To explain this further, suppose that we have a stock market with one asset and two agents, one which has the information contained in the price itself up to the current time. The other agent is an insider.

That is, he/she possesses information regarding future movements of the stock price (such as the value at some time, the maximum value up to some time, the time at which the maximum will be taken, etc.). The goal is then to characterize the dynamics of the underlying for the insider and to quantify his/her advantage.

Mathematical models that deal with this situation in a enlargement of filtrations framework have been considered by Karatzas-Pikovski [13], Imkeller[9], [1], [10], Pontier [6], Grorud [5], Baudoin [2] between others. In all these models, if the information (or sometimes called signal in filtering theory) is a "clear" signal then the extra utility of the insider up to the revelation time of the signal is infinite. This result is on one side, mathematically evident: That is, the extra utility of the insider is infinite due to the degenerative behavior of the semimartingale decomposition of the Wiener process in the enlarged filtration.

On the other hand, this issue restricted the practical interest of these results to the detection of unlawful insiders. The direction taken in Corcuera et. al. is to try to introduce insiders in the market so as to avoid this degenerative behavior but still letting the insider have information about the future. In particular, a model where insiders have dynamical information about an event in future time is provided. In this model the utility of insiders is finite and there is no arbitrage.

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To be more specific, suppose that the information of the insider is composed of the signal plus some independent noise that disappears as the revelation time approaches. Then the filtration is being enlarged continuously as time evolves. Then the authors prove that the semimartingale decomposition of the driving Wiener process is a projection of the semimartingale decomposition in Jacod's theorem. Finally they apply it to the study of the logarithmic utility of the insider. It is proved that if the rate at which the independent noise disappears is slow enough then the market does not allow arbitrage and the logarithmic utility of the insider is finite. This is related to the fact that the Wiener process becomes an integrable semimartingale with a bounded variation part whose Radon-Nikodym derivative (w.r.t. the Lebesgue measure) is square integrable in the enlarged filtration.

There is obviously another practical reason to study this type of models. Insiders usually only have an idea of what the future information is. Therefore this modeling is closer to reality than assuming that the insider knows with probability one the value of a certain random variable. Therefore one needs to study "progressive" enlargement of filtration problems.

We are interested in extending the previous application to random times for jump processes. The problem can then be divided into two parts:

First, is it possible to do the enlargement of filtration for random times with an additional perturbing noise? The answer to this question is that even in the simple case of random times without any perturbation, Jacod's theorem is not applicable. For example, for a simple Poisson process N of parameter λ, with natural filtration $\mathcal{F}$, let T_n denote the time of the n-th jump. Then, we have that

$$P(T_n \geq x/\mathcal{F}_t) = 1\{x \leq T_n \leq t\} + 1\{T_n > t\} \int_{(x-t)\vee 0}^{\infty} \lambda \frac{(\lambda u)^{n-1-N_t} e^{-\lambda u}}{(n-1-N_t)!} du.$$

Therefore the conditional law of the random variable T_n is absolutely continuous with respect to a random measure but not to a fixed one. This case can not be handled by Jacod's Theorem. In the financial application this corresponds to the insider that knows the time of the n-th jump of a stock price of size bigger than a certain number (see example 20). We propose in Theorem 2 a reformulation of Jacod's theorem to deal with perturbed random times in the framework of jump processes.

In the case that a perturbation of the random time is considered then the question is if an appropriate deformation of the information can give a semimartingale in the enlarged filtration where some moments (particularly the second) of the Radon-Nikodym derivative of the bounded variation part of the semimartingale decomposition are finite.

The second part deals with the implications of these semimartingale decompositions on markets with insiders driven by Lévy processes. This will be discussed in another article (but see Remark 15).

The article is structured as follows. First, in Section 2, we give our extension of Jacod's theorem for progressive enlargement of filtrations based on a random time. This result which appears in Theorem 2 is the main result of the paper.

In Section 4, we give some examples of applications and in particular, we analyze the modification of the jump structure of a Lévy process when its filtration is enlarged. We also provide some explicit formulas of the semimartingale decomposition and discuss when the decomposition can be extended to the whole time interval. The important issue to be able to extend the semimartingale property to the whole time interval is the rate of degeneration of the additional drift in the semimartingale decomposition in the enlarged filtration.

Finally in Section 5 we use the results developed in Section 2 and apply it to the case of the progressive enlargement of filtrations based on two random times. One is a stopping time (the time of the n-th jump bigger than a certain size) and a honest time (the last jump bigger than a certain size in a fixed time interval).

We remark here that although the definition of progressive enlargement that can be found in the literature does not include the situation described here, we preferred to keep using this terminology for the situation described here.

2 Expansion of filtration with respect to perturbed random times

Let $Z = \{Z_t, 0 \le t \le T\}$ be a d dimensional semimartingale defined on a complete probability space $(\Omega, \mathcal{F}, P)$. Here, $(\mathcal{F}_t)_{t\in[0,T]} \equiv \left(\mathcal{F}_t^Z\right)_{t\in[0,T]}$ is the filtration generated by the process Z satisfying the usual conditions. We will assume through the article unless stated otherwise that Z satisfies

$$\sup_{t\in[0,T]} E|Z_t| < \infty. \tag{1}$$

Assume that the additional information until time t is given by a family of d dimensional random variables $\{I_s\ , s \le T\}$ which we sometimes call the signal. Suppose that these random variables have the following structure:

$$I_t = G(\tau, Y_t),$$

where $G : \mathbb{R}^{2d} \to \mathbb{R}^d$ is a given measurable function, $\tau = (\tau_1, ..., \tau_d)$ is an $\mathcal{F}_T^Z$-measurable random (time) vector on $\mathbb{R}^d$ and the process $Y = \{Y_t, 0 \le t \le T\}$ is a stochastic process on $\mathbb{R}^d$ adapted to a filtration $\mathcal{H} \supseteq \mathcal{F}$, such that for any integrable random variable $\psi \in \mathcal{F}_T^Y$ and any $s \le T$

$$E(\psi|\mathcal{F}_s^Z \vee \sigma(\tau)) = E(\psi|\tau). \tag{2}$$

Condition (2) states the $\tau-$conditional independence of Y and Z. This is obviously satisfied if the noise process Y is independent of Z. But this noise process can still depend on τ with (2) still satisfied. We assume (2) throughout the article.

We define $\mathcal{G} = (\mathcal{G}_t)_{t\in[0,T]}$ as the smallest filtration, satisfying the usual conditions that contains the filtration $\mathcal{F}_t^Z \vee \sigma(I_s, s \le t)$ (see [19, Section II.67]).

We assume that condition (2) is satisfied throughout the article. We remark that we have assumed without loss of generality that the dimensions of the random vectors τ, Y_t, I_t and Z_t are the same.

For each $t \in [0, T]$, we denote by $P_t(\omega, dx) \equiv P_t^\tau(\omega, dx)$ a regular version of the conditional law of a random variable τ given the σ-field $\mathcal{F}_t$, abbreviating it by $P_t(dx)$ if its nature as a measure is emphasized. We can choose this version in such a way that the following conditions are satisfied:

1. For every Borel set B on $\mathbb{R}^d$, $\{P_t(B), t \in [0, T]\}$ is an $(\mathcal{F}_t)_{t\in[0,T]}$-progressively measurable process.

2. For every $(t, \omega) \in [0, T] \times \Omega$, $P_t(\omega, dx)$ is a probability measure on $\mathbb{R}^d$.

3. For any bounded and $(\mathcal{F}_t)_{t\in[0,T]}$-adapted process $h : \Omega \times [0, T] \to \mathbb{R}$ and for any bounded and measurable function $f : \mathbb{R}^d \to \mathbb{R}$, we have

$$E\left[f(\tau)\int_0^T h_t dt\right] = E\left[\int_0^T \int_{\mathbb{R}^d} f(x)P_t(dx)h_t dt\right].$$

In order to establish the general formula for the compensator, we require the random vector τ to belong to a certain class $\mathcal{L}^*$ to be defined below.

Definition 1 *We say that a random (time) vector τ belongs to the class $\mathcal{L}^*$, denoted by $\tau \in \mathcal{L}^*$, if there exists random kernels $P_t^{(i)}(\omega, dx)$, $i = 1, 2$ and a finite deterministic measure m such that*

1. *For every Borel set B in $\mathbb{R}^d$, $\{P_t^{(i)}(B), t \in [0, T)\}$ is an $(\mathcal{F}_t)_{t\in[0,T)}$-progressively measurable process.*

2. *For every $(t, \omega) \in [0, T) \times \Omega$, $P_t^{(i)}(\omega, dx)$ is a signed measure on $\mathbb{R}^d$.*

3. *For every $t \in [0, T)$, $E\left[\int_0^t \left|P_u^{(i)}\right| m(du)\right] < \infty$.*

4. For any bounded and $(\mathcal{F}_t)_{t\in[0,T]}$-adapted process $h : \Omega \times [0,T] \to \mathbb{R}$, for any bounded and measurable function $f : \mathbb{R}^d \to \mathbb{R}$, and for every $0 \le s < t < T$, we have

$$\begin{aligned} E\left[(Z_t - Z_s)\, 1(\tau_1 < s) f(\tau) h_s\right] &= E\left[\int_s^t \int_{[0,s)\times\mathbb{R}^{d-1}} f(x) P_u^{(1)}(dx) m(du) h_s\right] \\ E\left[(Z_t - Z_s)\, 1(t < \tau_1) f(\tau) h_s\right] &= E\left[\int_s^t \int_{(t,T]\times\mathbb{R}^{d-1}} f(x) P_u^{(2)}(dx) m(du) h_s\right]. \end{aligned}$$

We now give the main theorem of this article.

Theorem 2 *Suppose that τ is a random vector in the class $\mathcal{L}^*$ and Z is a semimartingale satisfying (1) such that $E\left|\Delta Z(\tau_1)\right| < \infty$. Assume that for almost all (t,ω), the signed measures $P_t^{(i)}(dx)$, $i = 1,2$ are absolutely continuous with respect to $P_t(dx)$, and set*

$$\alpha_t^{(i)}(x) = \frac{dP_t^{(i)}}{dP_t}(x).$$

We can choose a version of $\alpha_t^{(i)}(x)$ which is $\mathcal{P}\otimes\mathcal{B}(\mathbb{R}^d)$-measurable. Set the progressively measurable version of the compensator

$$\beta(u) = \alpha_u^{(1)}(\tau) 1(u > \tau_1) + \alpha_u^{(2)}(\tau) 1(u < \tau_1).$$

Then

$$Z(t) - \int_0^t E\left[\beta(u) | \mathcal{G}_u\right] m(du) - E\left[\Delta Z(\tau_1) 1(t \ge \tau_1) | \mathcal{G}_t\right]$$

is a martingale with respect to the filtration $(\mathcal{G}_t)_{t\in[0,T)}$.

Proof. First note that due to property 3 in definition 1, we have that $E\left[\int_0^t |\beta_s| m(ds)\right] < \infty$ for all $t \in [0,T)$. We can choose a version of $\alpha_t(x)$ which is $\mathcal{P}\otimes\mathcal{B}(\mathbb{R}^d)$-measurable and adapted process therefore it has a version that is progressively measurable, see Meyer [16], page 68.

Let h be a measurable $\mathcal{F}$- adapted bounded process and $f : \mathbb{R}^{dn} \to \mathbb{R}$ a bounded measurable function. Let $0 \le s_1 < ... < s_n \le s < t < T$ and consider $F = f(I_{s_1}, \ldots, I_{s_n})$. Denote $Y = (Y_{s_1}, ..., Y_{s_n})$ and $P(y_{s_1}, ..., y_{s_n}|\tau)$ the conditional probability measure of $Y = (y_{s_1}, ..., y_{s_n})$ conditioned to $\sigma(\tau)$. Using (2), we have that $P(y_{s_1}, ..., y_{s_n}|\mathcal{F}_s \vee \sigma(\tau)) = P(y_{s_1}, ..., y_{s_n}|\tau)$ for any $s \in [0,T)$. Then we have

$$\begin{aligned} & E\left[(Z_t - Z_s) F 1(t < \tau_1) h_s\right] \\ = \; & E\left[(Z_t - Z_s) f(G(\tau, Y(s_1)), \ldots, G(\tau, Y(s_n))) 1(t < \tau_1) h_s\right] \\ = \; & E\left[(Z_t - Z_s) \int_{\mathbb{R}^{dn}} f(G(\tau, y_{s_1})), \ldots, G(\tau, y_{s_n})) dP(y_{s_1}, ..., y_{s_n}|\tau) 1(t < \tau_1) h_s\right] \\ = \; & E\left[\int_s^t \int_{(t,T]\times\mathbb{R}^{d-1}} \int_{\mathbb{R}^{dn}} f(G(x, y_{s_1}), \ldots, G(x, y_{s_n})) dP(y_{s_1}, ..., y_{s_n}|\tau_1 = x_1) \alpha_u^{(2)}(x) P_u(dx) m(du) h_s\right] \\ = \; & E\left[\int_s^t f(G(\tau, Y(s_1)), \ldots, G(\tau, Y(s_n))) \alpha_u^{(2)}(\tau) m(du) 1(t < \tau_1) h_s\right] \\ = \; & E\left[\int_s^t 1(t < \tau_1) \alpha_u^{(2)}(\tau) m(du) F h_s\right]. \end{aligned}$$

Similarly, one obtains that

$$E\left[(Z_t - Z_s) 1(\tau_1 < s) F h_s\right] = E\left[\int_s^t 1(\tau_1 < s) \alpha_u^{(1)}(\tau) m(du) F h_s\right].$$

To finish the proof we consider the general case. Let $\pi = \{t_0 < s = t_1 < ... < t_{n-1} = t < t_n\}$ be a partition with $|\pi| = \max\{t_k - t_{k-1}; 1 \le k \le n\}$.

$$
\begin{aligned}
& E\left[(Z_t - Z_s)Fh_s\right] \\
= \; & E\left[\left(1(\tau_1 \le t_0)\int_s^t \alpha_u^{(1)}(\tau)m(du) + 1(t_n < \tau_1)\int_s^t \alpha_u^{(2)}(\tau)m(du)\right)Fh_s\right] \\
& + \sum_{j=1}^{n-2} E\Big[(Z_{t_{j+1}} - Z_{t_j})\Big(1(t_0 < \tau_1 \le t_j) + 1(t_j < \tau_1 \le t_{j+1}) + 1(t_{j+1} < \tau_1 \le t_n)\Big)Fh_s\Big].
\end{aligned}
$$

The second term of the right-hand side can be rewritten as follows:

$$
\begin{aligned}
E\left[\sum_{j=1}^{n-2}(Z_{t_{j+1}} - Z_{t_j})1(t_0 < \tau_1 \le t_j)Fh_s\right] &= E\left[\sum_{j=1}^{n-2}\int_{t_j}^{t_{j+1}} \alpha_u^{(1)}(\tau)m(du)1(t_0 < \tau_1 \le t_j)Fh_s\right] \\
&\to E\left[\int_s^t 1(s < \tau_1 < u)\alpha_u^{(1)}(\tau)m(du)Fh_s\right], \\
\sum_{k=1}^{n-2} E\left[(Z_{t_{k+1}} - Z_{t_k})1(t_k < \tau_1 \le t_{k+1})Fh_s\right] &\to E\left[\Delta Z(\tau_1)1(s < \tau_1 \le t)Fh_s\right]
\end{aligned}
$$

and

$$
\begin{aligned}
E\left[\sum_{j=1}^{n-2}(Z_{t_{j+1}} - Z_{t_j})1(t_{j+1} < \tau_1 \le t_n)Fh_s\right] &= E\left[\sum_{j=1}^{n-2}\int_{t_j}^{t_{j+1}} \alpha_u^{(2)}(\tau)m(du)1(t_{j+1} < \tau_1 \le t_n)Fh_s\right] \\
&\to E\left[\int_s^t 1(u < \tau_1 \le t)\alpha_u^{(2)}(\tau)m(du)Fh_s\right],
\end{aligned}
$$

as $|\pi| \downarrow 0$. Therefore $Z_t - \int_0^t E\left[\beta(u)|\,\mathcal{G}_u\right]m(du) - E\left[\Delta Z(\tau_1)1(t \ge \tau_1)|\,\mathcal{G}_t\right]$ is a martingale in the filtration $(\mathcal{G}_t)_{t\in[0,T)}$. ■

Remark 3 *The condition $P_t^{(i)}(dx)$ is absolutely continuous with respect to $P_t(dx)$ in Definition 1 replaces the condition $P_u << P_0^{\tau}$ which appears in Jacod's theorem (see [11]). This introduces some advantages as the reference (deterministic) measure in Jacod's theorem (usually the Lebesgue measure) is not used. In fact, example 20 (n-th jump of the driving process of size bigger than a) shows an example where Theorem 2 is applicable and therefore the semimartingale decomposition can be obtained.*

We now give some corollaries of our main result.

Definition 4 *We say that an $\mathcal{F}_T$-measurable, $\mathbb{R}^d$-valued random variable X belongs to the class $\mathcal{L}_1$ if there exists a random kernel $P_t^{(1)}(\omega, dx, dz)$ and some deterministic finite measure m such that*

1. *Properties 1, 2 and 3 of Definition 1 are satisfied for $P_t^{(1)}(\omega, dx)$.*
2. *For any bounded and $(\mathcal{F}_t)_{t\in[0,T]}$-adapted process $h : \Omega \times [0,T] \to \mathbb{R}$, for any bounded and measurable function $f : \mathbb{R}^d \to \mathbb{R}$, and for every $0 < s < t < T$, we have*

$$
E\left[(Z_t - Z_s)\,f(X)h_s\right] = E\left[\int_s^t \int_{\mathbb{R}^d} f(x)P_u^{(1)}(dx)m(du)h_s\right].
$$

Theorem 5 *Suppose that Z is a semimartingale satisfying (1) and X is an $\mathcal{F}_T$-measurable $\mathbb{R}^d$ valued random vector in the class $\mathcal{L}_1$ satisfying condition (2) with $\tau = X$. Assume that for almost all $(t,\omega) \in [0,T) \times \Omega$, the signed measure $P_t^{(1)}(dx)$ is absolutely continuous with respect to $P_t(dx)$, and set*

$$\begin{aligned} \alpha_t(x) &= \frac{dP_t^{(1)}}{dP_t}(x), \\ \beta_t &= E[\alpha_t(X)|\mathcal{G}_t], \end{aligned}$$

where $\alpha(x)$ and β are chosen to be progressively measurable. That is, $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d)$ and $\mathcal{P}$-measurable, where $\mathcal{P}$ denotes the $\mathcal{F}_t$-progressive σ-field.

Then $Z_t - \int_0^t \beta_s m(ds)$ is a martingale with respect to the filtration $(\mathcal{G}_t)_{t\in[0,T)}$.

Proof. To obtain the proof is enough to note that due to Property 2 in Definition 4

$$\begin{aligned} \sum_{k=1}^{n-2} E\left[(Z_{t_{k+1}} - Z_{t_k})1(t_k < X_1 \le t_{k+1})Fh_s\right] &= \sum_{k=1}^{n-2} E\left[\int_{t_k}^{t_{k+1}} \alpha_u(X)m(du)1(t_k < X_1 \le t_{k+1})Fh_s\right] \\ &\to E\left[m(\{X_1\})\alpha_{X_1}(X)Fh_s\right]. \end{aligned}$$

From here the result follows. ■

From the above proof, it is clear that the jump term $E\left[\Delta Z(\tau_1)1(t \ge \tau_1)|\,\mathcal{G}_t\right]$ in Theorem 2 becomes part of the integral compensator in Theorem 5.

Remarks 6 *1. If $P_t^{(1)}(dx)$ has a Radon-Nikodym derivative $\alpha_t(x)$ with respect to $P_t(dx)$, then*

$$E[Z_t - Z_s|\mathcal{F}_s \vee \sigma(X)] = E[\int_s^t \alpha_u(X)m(du)|\mathcal{F}_s \vee \sigma(X)].$$

This implies that $Z_t - \int_0^t \alpha_u(X)m(du)$ is an $\mathcal{F} \vee \sigma(X)$-martingale.
2. Define the $\mathcal{F}$-martingale $M^f(u) = \int f(x)P_u(dx)$ and suppose that $Z = N + A$ where, N is a square integrable martingale and A is an integrable bounded variation process with $dA(u) << m(du)$ and $d\langle M^f, N\rangle(u) << m(du)$, then $X \in \mathcal{L}_1$ with

$$\int f(x)P_u^{(1)}(dx) = M^f(u)\frac{dA}{dm}(u) + \frac{d\langle M^f, N\rangle}{dm}(u).$$

Although this result shows the nature of $P_u^{(1)}$, it is not easy to apply in the examples given in this paper. This is because computing $P_u^{(1)}$ from $\langle M^f, Z\rangle$ is not straightforward.
3. Similarly, if $E\int_0^T |\beta_s|m(ds) < \infty$ then $Z_t - \int_0^t \beta_s m(ds)$ is a martingale in the filtration $(\mathcal{G}_t)_{t\in[0,T]}$. This will be used in some of the examples. In most examples, we will show that all the above conditions are satisfied, compute β and therefore giving a $\mathcal{G}$- martingale in $[0,T)$. Then we may finally discuss if we can close the martingale in the interval $[0,T]$ or if the local martingale property is satisfied without assuming the integrability condition (1).

3 Explicit formulas for the compensator

In the next theorem we give a formula for β in the case $G(x,y) = x+y$, τ_1 is a $[0,T]$-valued random time and $Y_t = Z'_{\tau_1 - t}$ where $\{Z'_t;\ t \in \mathbb{R}\}$ is a one dimensional process independent of $\{Z_t;\ t \ge 0\}$ such that $Z'_0 = 0$ and that $\{Z'_t,\ t \ge 0\}$ and $\{Z'_t,\ t \le 0\}$ are additive processes (that is, processes with independent increments but they are not necessarily stationary) . Note that in this case Y is not independent of Z. Before obtaining explicit formulas for the compensator, we introduce some further notation related with conditional expectations.

As before, let $P_u(dt)$ be the regular conditional probability of τ given $\mathcal{F}_u$. We assume that the laws of Z'_t and Z'_{-t} are equal and denote it by Q_t. Let $\gamma_t(x,\omega)$ be $\mathcal{P}\otimes\mathcal{B}(\mathbb{R}^d)$- measurable process such that $\int_{\mathbb{R}^d}|\gamma_t(x,\omega)|P_0(dx)<\infty$ for a.a. ω, $0\le t<T$ and let Q_t be the law of Z'_t.

For a measurable set A in $\mathbb{R}^d$ with $P_t(A)>0$, define the following measures

$$\mu^A(B,t,\omega)=\int_A\int_{\mathbb{R}}1_B(x_1+y)Q_{x_1-t}(dy)P_t(dx)$$

and

$$\mu^A_\gamma(B,t,\omega)=\int_A\int_{\mathbb{R}}1_B(x_1+y)\gamma_t(x)Q_{x_1-t}(dy)P_t(dx)$$

for $B\in\mathcal{B}(\mathbb{R})$. The random measures μ^A and μ^A_γ are $(\mathcal{F}_t)_{t\in[0,T)}$ progressively measurable for fixed B and μ^A_γ is absolutely continuous with respect to μ^A for all (t,ω). We define $\rho_t^{\gamma,A}(\omega,x)$ as a progressively measurable version of the Radon-Nikodym derivative $\frac{d\mu^A_\gamma}{d\mu^A}(x,t,\omega)$ which therefore satisfies

$$\int_A\int_{\mathbb{R}}1_B(x_1+y)\rho_t^{\gamma,A}(x_1+y)Q_{x_1-t}(dy)P_t(dx)=\int_A\int_{\mathbb{R}}1_B(x_1+y)\gamma_t(x)Q_{x_1-t}(dy)P_t(dx),\qquad(3)$$

for any $B\in\mathcal{B}(\mathbb{R})$.

First we give a Lemma.

Lemma 7 *Let $I_t=\tau_1+Y_t$. Then*

$$E(\gamma_t(\tau)1(\tau\in A)|\mathcal{G}_t)=\rho_t^{\gamma,A}(I_t)P(\tau\in A|\mathcal{G}_t)\ a.e.$$

Proof. As (3) holds, then for any $\mathcal{B}(\mathbb{R}^2)$ measurable bounded function f and $A\in\mathcal{B}(\mathbb{R}^d)$, we have

$$\begin{aligned}\int_A\gamma_u(t)E[f(t_1,t_1+Z'(t_1-u))]P_u(dt) &= \int_A\int_{\mathbb{R}}\gamma_u(t)f(t_1,t_1+y)Q_{t_1-u}(dy)P_u(dt)\\ &= \int_A\int_{\mathbb{R}}\rho_u^{\gamma,A}(t_1+y)f(t_1,t_1+y)Q_{t-u}(dy)P_u(dt).\quad(4)\end{aligned}$$

For $0\le s_1<s_2<\cdots<s_n=u$, h an $\mathcal{F}$-adapted bounded process and bounded measurable functions f_1, f_2, define $F_1(t)=f_1(t+Z'(t-u))$ and $F_2(t)=f_2(Z'(t-s_{n-1})-Z'(t-u),\ldots,Z'(t-s_1)-Z'(t-s_2))$. We have, by Fubini's theorem, the independence of $\mathcal{F}_T$ and Z', the independent increment property of Z' and property (4) that

$$\begin{aligned}E[1(\tau\in A)\gamma_u(\tau)F_1(\tau_1)F_2(\tau_1)h_u] &= E[\int_A\gamma_u(t)E[F_1(t_1)F_2(t_1)]P_u(dt)h_u]\\ &= E[\int_A\gamma_u(t)E[F_1(t_1)]E[F_2(t_1)]P_u(dt)h_u]\\ &= E[\int_A\int_{\mathbb{R}}\rho_u^{\gamma,A}(t_1+y)f_1(t_1+y)Q_{t_1-u}(dy)E[F_2(t_1)]P_u(dt)h_u]\\ &= E[\int_A\rho_u^{\gamma,A}(t_1+Z'(t_1-u))F_1(t_1)E[F_2(t_1)]P_u(dt)h_u]\\ &= E[\rho_u^{\gamma,A}(I_u)F_1(\tau_1)F_2(\tau_1)h_u1(\tau\in A)]\end{aligned}$$

Hence, $\rho_u^{\gamma,A}(I_u)P(\tau\in A|\mathcal{G}_u)=E\left[\gamma_u(\tau)1(\tau\in A)|\mathcal{G}_u\right]$. ■

In the next result we restrict our attention to the case of one dimensional random times $d=1$. Define two (signed) measures for $i=1,2$

$$\mu^{(i)}(B,u,\omega)=\mu^{A^i}(B,u,\omega)$$

and

$$\mu_\alpha^{(i)}(B,u,\omega) = \mu_{\alpha^{(i)}}^{A^i}(B,u,\omega)$$

for $B \in \mathcal{B}(\mathbb{R})$, $A^1 = (0,u)$ and $A^2 = (u,T)$, $0 \le u \le T$. Let $\rho_u^{\alpha,(i)}(y,\omega)$ denote the Radon-Nikodym derivative $\frac{d\mu_\alpha^{(i)}}{d\mu^{(i)}}(y,u,\omega)$ which satisfies (3).

Theorem 8 *Assume that $d = 1$ and the same conditions as in Theorem 2 and assume that $\{Z'_t; t \in \mathbb{R}\}$ is an d-dimensional additive process on $\mathbb{R}$. Let $I_t = \tau + Z'(\tau - t)$ and $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(I_u; u \le t)$. Then,*

$$E[\beta_u|\mathcal{G}_u] = \rho_u^{\alpha,(1)}(I_u)P(u > \tau|\mathcal{G}_u) + \rho_u^{\alpha,(2)}(I_u)P(u < \tau|\mathcal{G}_u) \ a.s.$$

Moreover, if Q_{t-u} has a density q_{t-u}, then

$$E[\beta_u|\mathcal{G}_u] = \sum_{i=1}^{2} \frac{\int_{A^i} \alpha_u^{(i)}(t) q_{t-u}(I_u - t)P_u(dt)}{\int_{A^i} q_{t-u}(I_u - t)P_u(dt)} P(\tau \in A^i|\mathcal{G}_u) \ a.s. \tag{5}$$

Proof. The first part is a direct application of Theorem 2 and Lemma 7. Now, assume that Q_{t-u} has a density q_{t-u}. Then

$$\rho_u^{(i)}(y) = \frac{\int_{A^{(i)}} \alpha_u^{(i)}(t) q_{t-u}(y-t)P_u(dt)}{\int_{A^{(i)}} q_{t-u}(y-t)P_u(dt)}$$

satisfies (3). Hence we have (5) after applying Theorem 2. ■

Similarly , we can also give a formula for β in the case $G(x,y) = x + y$, $\tau = X$ and $Y_t = Z'_{T-t}$ where Z' is an additive process independent of $\{Z_t\}$.

Theorem 9 *Suppose that the assumptions of Theorem 5 hold and $I_t = X + Y_t$ for $t \in [0,T)$. Then $Z_t - \int_0^t \beta_u du$ is a $\mathcal{G}$-martingale in $[0,T)$ where β_t is a progressively measurable version of $\rho_t^\alpha(I_t,\omega)$ given in Lemma 7. Furthermore,*

1. *If Q_{T-t} has a density q_{T-t} then we have for $t \in [0,T)$*

$$\beta_t = \frac{\int_{\mathbb{R}^d} \alpha_t(x)\, q_{T-t}(I_t - x)\, P_t(dx)}{\int_{\mathbb{R}^d} q_{T-t}(I_t - x)\, P_t(dx)}. \tag{6}$$

2. *If both Q_{T-t} and $P_t(\cdot)$ are discrete distributions with probability functions $q_{T-t}(y)$ and $p_t(x)$, then $P_t^{(1)}(dx)$ is discrete with probability function $p_t^{(1)}(x) = P_t^{(1)}(\{x\}) = \alpha_t(x)p_t(x)$ and*

$$\beta_t = \frac{\sum \alpha_t(x)\, q_{T-t}(I_t - x)\, p_t(x)}{\sum q_{T-t}(I_t - x)\, p_t(x)}.$$

The reason for the general formulation introduced so far is that we believe it to be more general than the following result which is a slight extension of Proposition 1 of [3].

Proposition 10 *Let Z be an adapted process in a subfiltration $\mathcal{B} \subseteq \mathcal{A}$. If Z is a semimartingale in $\mathcal{A}$ with a Doob-Meyer decomposition*

$$Z_t = M_t + \int_0^t \alpha_s m(ds)$$

with M a local martingale and m a finite signed measure then Z is also a $\mathcal{B}$-semimartingale with a Doob-Meyer decomposition

$$Z_t = M'_t + \int_0^t E(\alpha_s|\mathcal{B}_s)m(ds).$$

Note that in order to apply Proposition 10, it is necessary to know the semimartingale decomposition of Z in a bigger filtration $\mathcal{A}$ in order to obtain the respective decomposition in the smaller filtration. From this point of view, it is obvious that our previous results are more general than this proposition.

An example where the above proposition may not be applicable is $Y_t = f(Z_{t+\epsilon})Z'_{t+\epsilon}$ with $\epsilon < T/2$. For a related example in the Brownian setting, see [15].

4 Examples for $\tau = X = Z(T)$ and Z is a Lévy process

In our first example of application we consider the case where Z is a Lévy process. The following is an extension of an example of enlargement of filtrations with respect to Lévy processes known as Kurtz theorem (although this example was known since Itô, see Jacod and Protter [12] , Chaumont and Yor [4] where the concept of harness is stressed). Let $\{Z_t; t \in [0,T]\}$ be an $\mathbb{R}^d$-valued Lévy process with characteristic function $E[e^{i\langle\theta,Z_t\rangle}] = e^{t\psi(\theta)}$ where

$$\psi(\theta) = i\langle b,\theta\rangle - \langle c\theta, c\theta\rangle/2 + \int_{\mathbb{R}^d} (e^{i\langle\theta,x\rangle} - 1 - i\langle\theta,x\rangle 1_{\{|x|\le 1\}}(x))\nu(dx).$$

Here, $b \in \mathbb{R}^d$, c is a nonnegative definite $d \times d$ matrix and ν is a measure on $\mathbb{R}^d\backslash\{0\}$ satisfying $\int |x| \wedge |x|^2 \nu(dx) < \infty$. Note that under these hypotheses, (1) is satisfied. Let $\{Z'_t; t \in [0,T]\}$ be an $\mathbb{R}^d$-valued additive process independent of $\{Z_t; t \in [0,T]\}$. Furthermore, let $Y_t = Z'_{T-t}$, $G(x,y) = x + y$ and $X = Z_T$.

Let $R_t(dx) = P(Z_t \in dx)$ and let h_s be an $\mathcal{F}_s$-measurable bounded random variable. For $s \le u < t \le T$, we have

$$\begin{aligned}
& E[(Z_t - Z_u)e^{i\langle\theta,Z_T\rangle}h_s] \\
= \; & E[\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \exp\{i\langle\theta, x+y+z+Z_s\rangle\} y R_{T-t}(dx)R_{t-u}(dy)R_{u-s}(dz)h_s] \\
= \; & E[\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^d} e^{i\langle\theta,y\rangle} y R_{t-u}(dy)\Big) \exp\{i\langle\theta, x+z+Z_s\rangle\} R_{T-t}(dx)R_{u-s}(dz)h_s] \\
= \; & \frac{1}{i}(t-u)\nabla\psi(\theta)\exp\{(T-s)\psi(\theta)\}E[e^{i\langle\theta,Z_s\rangle}h_s].
\end{aligned}$$

Hence,

$$E[\frac{Z_t - Z_u}{t-u}e^{i\langle\theta,Z_T\rangle}h_s] = \frac{1}{i}\nabla\psi(\theta)\exp\{(T-s)\psi(\theta)\}E[e^{i\langle\theta,Z_s\rangle}h_s].$$

Letting $t = T$, and integrating the both sides with respect to du, we have

$$\int_s^t E[\frac{Z_T - Z_u}{T-u}e^{i\langle\theta,Z_T\rangle}h_s]du = \frac{1}{i}(t-s)\nabla\psi(\theta)\exp\{(T-s)\psi(\theta)\}E[e^{i\langle\theta,Z_s\rangle}h_s].$$

Therefore

$$E[(Z_t - Z_s)e^{i\langle\theta,Z_T\rangle}h_s] = E[\int_s^t \frac{Z_T - Z_u}{T-u}du e^{i\langle\theta,Z_T\rangle}h_s].$$

In conclusion, we have that $Z_T \in \mathcal{L}_1$ with $m(du) = du$, $P_u^{(1)}(dx) = \frac{x - Z_u}{T-u}P_u(dx)$, where $P_t(dx) = R_{T-t}(dx - Z_t)$ is the regular conditional law of Z_T given $\mathcal{F}_t$ and

$$\alpha_t(x) = \frac{x - Z_t}{T-t}.$$

Next, note that $E\left[\int_0^T \left|P_u^{(1)}\right| du\right] = \int_0^T E\left|\frac{Z_T - Z_u}{T-u}\right| du < \infty$ (see the proof of 1 in Proposition 16). Therefore by 1 of Remark 6 with $X = Z_T$, we have that $Z_t - \int_0^t \alpha_u(Z_T)du$ is a $\mathcal{F} \vee \sigma(Z_T)$-martingale in $[0,T]$. Note that $E\int_0^T \left(\frac{Z_T - Z_u}{T-u}\right)^2 du = \infty$. This quantity is of importance when considering the logarithmic utility of the insider in mathematical finance (see Remark 15).

In particular, if the previous integral were finite it would imply that the Radon-Nikodym derivative of the bounded variation part of the semimartingale decomposition of Z in the enlarged filtration is square integrable.

One way to solve the previous problem is to consider the filtration $\mathcal{G}$. Therefore, our goal now is to compute β as explicitly as possible using Theorem 9. In the present example, we have $I_t = Z_T + Z'_{T-t}$, $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(I_s, s \le t)$ and let $L_t = Z_T - Z_t + Z'_{T-t}$. U_{T-t} will denote the law of L_t and $\widetilde{R}_{T-t}(dx) = xR_{T-t}(dx)$. As before, Q_t denotes the law of Z'_t.

Theorem 11 *The signed vector measure $\widetilde{R}_{T-t}$ is a finite measure and $Q_{T-t} * \widetilde{R}_{T-t}$ is absolutely continuous with respect to U_{T-t}. Furthermore, $Z_t - \int_0^t \beta_u du$ is a $\mathcal{G}$- martingale in $[0,T)$, where*

$$\beta_t = \frac{1}{T-t} \frac{d(Q_{T-t} * \widetilde{R}_{T-t})}{dU_{T-t}} (L_t).$$

In the next three cases, β can be rewritten as follows:
1. If Q_{T-t} has a density q_{T-t}, then U_{T-t} has a density u_{T-t} and

$$\beta_t = \frac{\int_{\mathbb{R}^d} x q_{T-t}(L_t - x) R_{T-t}(dx)}{(T-t) u_{T-t}(L_t)}.$$

2. If R_{T-t} has a density r_{T-t} and $\int_{\mathbb{R}^d} |x| Q_{T-t}(dx) < \infty$, then U_{T-t} has a density u_{T-t} and

$$\beta_t = \frac{L_t}{T-t} - \frac{\int_{\mathbb{R}^d} x r_{T-t}(L_t - x) Q_{T-t}(dx)}{(T-t) u_{T-t}(L_t)}.$$

3. If both Q_{T-t} and R_{T-t} are discrete with probability functions q_{T-t} and r_{T-t}, then U_{T-t} is discrete with probability function u_{T-t} and

$$\beta_t = \frac{\sum_x x q_{T-t}(L_t - x) r_{T-t}(x)}{(T-t) u_{T-t}(L_t)} = \frac{L_t}{T-t} - \frac{\sum_x x r_{T-t}(L_t - x) q_{T-t}(x)}{(T-t) u_{T-t}(L_t)},$$

where we assume $\sum |x| q_{T-t}(x) < \infty$ for the second expression of β.

Proof. In order to compute β in Theorem 9, it is enough to note that

$$\begin{aligned} \mu_\alpha(B,t) &= \int_{\mathbb{R}^d} \alpha_t(x) Q_{T-t}(B-x) P_t(dx) \\ &= \int_{\mathbb{R}^d} \alpha_t(z+Z_t) Q_{T-t}(B-z-Z_t) R_{T-t}(dz) \\ &= \frac{1}{T-t} \int_{\mathbb{R}^d} Q_{T-t}(B-z-Z_t) z R_{T-t}(dz) \\ &= \frac{1}{T-t} Q_{T-t} * \widetilde{R}(B-Z_t) \end{aligned} \tag{7}$$

and

$$\begin{aligned} \mu(B,t) &= \int_{\mathbb{R}^d} Q_{T-t}(B-x) P_t(dx) \\ &= \int_{\mathbb{R}^d} Q_{T-t}(B-z-Z_t) R_{T-t}(dz) \\ &= U_{T-t}(B-Z_t). \end{aligned}$$

■

In the following, we will use the stochastic representation for the Lévy process Z. That is, there exists a d-dimensional Wiener process W_t and a Poisson random measure $N(dx,ds)$ on $(\mathbb{R}^d \backslash \{0\}) \times [0,T]$ with compensator $\overline{N}(dx,ds) = \nu(dx)ds$ such that

$$Z_t = bt + cW_t + \int_0^t \int_{|x|\le 1} x \widetilde{N}(dx,ds) + \int_0^t \int_{|x|>1} x N(dx,ds). \tag{8}$$

Here, $\widetilde{N}(dx,ds) = N(dx,ds) - \overline{N}(dx,ds)$ denotes the compensated martingale measure. For the additive process Z', there exists a continuous $\mathbb{R}^d$-valued deterministic function b'_t, a d-dimensional

Gaussian additive process G' with a covariance matrix $c'_\cdot$, and a Poisson random measure N' on $(\mathbb{R}^d\backslash\{0\}) \times [0,T]$ such that

$$Z'(t) = b'_t + G'_t + \int_0^t \int_{|x|\le 1} x\widetilde{N}'(dxds) + \int_0^t \int_{|x|>1} xN'(dxds)$$

where $\widetilde{N}'$ is the compensated martingale measure. Let $\overline{N}'(dxds) = \nu'(dx,ds)$ be the compensator of N'.

The objective of the next results leading to Theorem 14 is to find how the jump structure of the Lévy process is modified by the progressive enlargement of the filtration.

First we discuss a special case of Theorem 11. That is, let $Z_t = N(B,(0,t])$, $Z'_t = N'(B,(0,t])$ where $B \in \mathcal{B}(\mathbb{R}^d)$ such that $d(B,0) := \inf_{y\in B}|y| > 0$. Let $Y_t = N'(B,(0,T-t])$ and

$$\mathcal{G}^B_t = \mathcal{F}_t \vee \sigma\Big(N(B,(0,T]) + N'(B,(0,T-u]); u \le t\Big).$$

Theorem 12 *Let*

$$\beta_s = \frac{\nu(B)\{N(B,(s,T]) + N'(B,(0,T-s])\}}{\nu'(B,(0,T-s]) + \nu(B)(T-s)}.$$

Then $Z_t - \int_0^t \beta_s ds$ *is a* $\mathcal{G}^B$*-martingale.*

Proof. We have

$$Q_{T-t}(\{k\}) = \exp[-\nu'(B,(0,T-t])]\frac{\{\nu'(B,(0,T-t])\}^k}{k!}$$

and

$$R_{T-t}(\{k\}) = \exp[-\nu(B)(T-t)]\frac{\{\nu(B)(T-t)\}^k}{k!}$$

for $k = 0,1,2,\dots$. Therefore

$$\begin{aligned} & Q_{T-t} * \widetilde{R}_{T-t}(\{n\}) \\ = \; & \exp[-\nu(B)(T-t) - \nu'(B,(0,T-t])]\nu(B)(T-t)\frac{\{\nu'(B,(0,T-t]) + \nu(B)(T-t)\}^{n-1}}{(n-1)!}. \end{aligned}$$

Similarly, we have

$$\begin{aligned} & U_{T-t}(\{n\}) \\ = \; & \exp[-\nu(B)(T-t) - \nu'(B,(0,T-t])]\frac{\{\nu'(B,(0,T-t]) + \nu(B)(T-t)\}^{n}}{n!}. \end{aligned}$$

Hence, by Theorem 11, we have that

$$\beta_t = \frac{\nu(B)L_t}{\nu'(B,(0,T-t]) + \nu(B)(T-t)}.$$

■

Theorem 12 suggests that the explicit form of the compensator of N w.r.t. a given filtration $\mathcal{G}$ is not simple in general.

If $\mathcal{G}$ is included in $\mathcal{G}^B$, then we can obtain an explicit form of the compensator using Proposition 10, otherwise one uses Theorem 5.

In order to obtain the formula for the compensator in greater generality, we will use the following result.

Lemma 13 *For bounded measurable functions f, $g : \mathbb{R}^d \times [0,T] \to \mathbb{R}$ vanishing in a neighborhood of the origin, we have*

$$
\begin{aligned}
& E\Big[\int_{\mathbb{R}^d\times[0,T]} f(x,t)N(dx,dt)\exp\{i\theta\int_{\mathbb{R}^d\times[0,T]} g(x,t)N(dx,dt)\}\Big] \\
= & \int_{\mathbb{R}^d\times[0,T]} f(x,t)\exp\{i\theta g(x,t)\}\nu(dx)dtE\Big[\exp\{i\theta\int_{\mathbb{R}^d\times[0,T]} g(x,t)N(dx,dt)\}\Big] \qquad (9)\\
= & \int_{\mathbb{R}^d\times[0,T]} f(x,t)\exp\{i\theta g(x,t)\}\nu(dx)dt\exp\{\int_{\mathbb{R}^d\times[0,T]}(e^{i\theta g(x,t)}-1)\nu(dx)dt\}.
\end{aligned}
$$

Proof. For mutually disjoint $A_1,\dots,A_m \in \mathcal{B}(\mathbb{R}^d)$ satisfying $d(A_j,0) > 0$, for $1 \le j \le m$, mutually disjoint $B_1,\dots,B_n \in [0,\infty)$ and $a_{jk}, b_{jk} \in \mathbb{R}$ $1 \le j \le m$, $1 \le k \le n$, consider

$$
\begin{aligned}
& E\Big[\sum_{j=1}^m\sum_{k=1}^n a_{jk}N(A_j,B_k)\exp\{i\theta\sum_{j=1}^m\sum_{k=1}^n b_{jk}N(A_j,B_k)\}\Big] \\
= & \sum_{j=1}^m\sum_{k=1}^n a_{jk}e^{i\theta b_{jk}}\nu(A_j)|B_k|E[\exp\{i\theta\sum_{j=1}^m\sum_{k=1}^n b_{jk}N(A_j,B_k)\}] \\
= & \sum_{j=1}^m\sum_{k=1}^n a_{jk}e^{i\theta b_{jk}}\nu(A_j)|B_k|\exp\{\sum_{j=1}^m\sum_{k=1}^n(e^{i\theta b_{jk}}-1)\nu(A_j)|B_k|\}).
\end{aligned}
$$

Here, $|B|$ denotes the Lebesgue measure of B. Using a limit argument, we have the conclusion. ■

Now we consider the following enlargement of filtration which generalizes the result in Theorem 12. Define,

$$
\begin{aligned}
\mathcal{H}_t &= \mathcal{F}_t \vee \sigma\Big(cW_T + G'_{T-s}; 0 \le s \le t\Big) \\
&\quad \vee\sigma\Big(N(B,(0,T]) + N'(B,(0,T-s]); B \in \mathcal{B}(\mathbb{R}^d), d(B,0) > 0, 0 \le s \le t\Big).
\end{aligned}
$$

Decompose ν' as $\nu'(dx,ds) = \gamma(x,s)\nu(dx)ds + \nu'^s(dx,ds)$ where γ is the Radon-Nikodym derivative of the absolutely continuous part and ν'^s is the singular part of ν' w.r.t. ν.

Theorem 14 *Let*

$$
\begin{aligned}
B_t &= cW_t - \int_0^t c\eta_s ds, \\
M(dx,ds) &= N(dx,ds) - F_s(dx)ds,
\end{aligned}
$$

with

$$
\begin{aligned}
\eta_s &= \Big((T-s)c + c'_{T-s}\Big)^{-1}\Big(c(W_T - W_s) + G'_{T-s}\Big) \\
F_s(dx) &= \frac{1_{\mathrm{supp}(\nu)}(x)}{\int_0^{T-u}(1+\gamma(x,v))dv}\Big(N(dx,(u,T]) + N'(dx,(0,T-u])\Big).
\end{aligned}
$$

Here, $\Big((T-s)c + {c'}_{T-s}\Big)^{-1}$ denotes the inverse of the restriction of $(T-s)c + {c'}_{T-s}$ to its own range. Then B_t and $M(dx,dt)$ are $\mathcal{H}$-martingales.

Proof. Let $0 \le s_1 < \cdots < s_n \le s$ and let $X_k = N(A_k,(0,T])$, $Y_{s_k} = N'(A_k,(0,T-s_k])$ for $k = 1,\dots,n$ with $A_i \cap A_j = \emptyset$ for $i \ne j$ and $d(A_k,0) > 0$ for $k = 1,\dots,n$. Let $\phi(x_1,\dots,x_n) = \prod_{k=1}^n e^{i\theta_k x_k}$ for $\theta = (\theta_1,\dots,\theta_n) \in \mathbb{R}^n$ and let $X = (X_k)_{k=1}^n$, $Y = (Y_{s_k})_{k=1}^n$. Let f be a bounded measurable function vanishing in a neighborhood of the origin. We have, for $s \le u < t \le T$ and h_s a bounded $\mathcal{F}_s$-measurable function that

$$\begin{aligned} & E[\phi(X+Y)h_s \int_{\mathbb{R}^d} f(x)N(dx,(u,t])] \\ = \ & E\Big[h_s \prod_{k=1}^n \exp\Big(i\theta_k\{N(A_k,(0,u]\cup(t,T])+Y_{s_j}\}\Big)\Big] \\ & \quad \times E\Big[\int_{\mathbb{R}^d} f(x)N(dx,(u,t])\exp\{i\sum_{k=1}^n \theta_k N(A_k,(u,t])\}\Big]. \end{aligned}$$

We have by (9) that

$$\begin{aligned} & E\Big[\int_{\mathbb{R}^d} f(x)N(dx,(u,t])\exp\{i\sum_{k=1}^n \theta_k N(A_k,(u,t])\}\Big] \\ = \ & (t-u)\Big(\int_{\mathbb{R}^d} f(x)\exp\{i\sum_{k=1}^n \theta_k 1_{A_k}(x)\}\nu(dx)\Big)E\Big[\exp\{i\sum_{k=1}^n \theta_k N(A_k,(u,t])\}\Big]. \end{aligned}$$

Hence

$$\begin{aligned} & E\Big[\phi(X+Y)h_s \int_{\mathbb{R}^d} f(x)N(dx,(u,t])\Big] \\ = \ & (t-u)\Big(\int_{\mathbb{R}^d} f(x)\exp\{i\sum_{k=1}^n \theta_k 1_{A_k}(x)\}\nu(dx)\Big)E[\phi(X+Y)h_s]. \end{aligned} \tag{10}$$

By letting $t = T$ and integrating w.r.t. u, we have

$$\begin{aligned} & \int_s^t E\Big[\phi(X+Y)h_s \int_{\mathbb{R}^d} f(x)\frac{N(dx,(u,T])}{T-u}\Big]du \\ = \ & (t-s)\Big(\int_{\mathbb{R}^d} f(x)\exp\{i\sum_{k=1}^n \theta_k 1_{A_k}(x)\}\nu(dx)\Big)E[\phi(X+Y)h_s]. \end{aligned} \tag{11}$$

By an argument similar to (10), we have

$$\begin{aligned} & E\Big[\phi(X+Y)h_s \int_{\mathbb{R}^d} f(x)N'(dx,(0,T-u])\Big] \\ = \ & \Big(\int_{\mathbb{R}^d} f(x)\exp\{i\sum_{k=1}^n \theta_k 1_{A_k}(x)\}\nu'(dx,(0,T-u])\Big)E[\phi(X+Y)h_s]. \end{aligned}$$

By (10), we have

$$\begin{aligned} & E\Big[\phi(X+Y)h_s \int_{\mathbb{R}^d} \frac{f(x)1_{\mathrm{supp}(\nu)}(x)}{\int_0^{T-u}(1+\gamma(x,v))dv}N(dx,(u,T])\Big] \\ = \ & \Big(\int_{\mathbb{R}^d} \frac{f(x)(T-u)}{\int_0^{T-u}(1+\gamma(x,v))dv}\exp\{i\sum_{k=1}^n \theta_k 1_{A_k}(x)\}\nu(dx)\Big)E[\phi(X+Y)h_s]. \end{aligned}$$

Similarly, we have

$$\begin{aligned} & E\Big[\phi(X+Y)h_s \int_{\mathbb{R}^d} \frac{f(x)1_{\mathrm{supp}(\nu)}(x)}{\int_0^{T-u}(1+\gamma(x,v))dv}N'(dx,(0,T-u])\Big] \\ = \ & \Big(\int_{\mathbb{R}^d} \frac{f(x)\int_0^{T-u}\gamma(x,v)dv}{\int_0^{T-u}(1+\gamma(x,v))dv}\exp\{i\sum_{k=1}^n \theta_k 1_{A_k}(x)\}\nu(dx)\Big)E[\phi(X+Y)h_s]. \end{aligned}$$

Therefore, we have

$$\begin{aligned}
& E\Big[\phi(X+Y)h_s\int_{\mathbb{R}^d}\frac{f(x)1_{\mathrm{supp}(\nu)}(x)}{\int_0^{T-u}(1+\gamma(x,v))dv}\Big(N(dx,(u,T])+N'(dx,(0,T-u])\Big)\Big] \\
= & \Big(\int_{\mathbb{R}^d} f(x)\exp\{i\sum_{k=1}^n \theta_k 1_{A_k}(x)\}\nu(dx)\Big)E[\phi(X+Y)h_s] \\
= & E\Big[\phi(X+Y)h_s\int_{\mathbb{R}^d} f(x)\frac{N(dx,(u,T])}{T-u}\Big]
\end{aligned}$$

by (10). Integrating both sides of the above equality w.r.t. u in $[s,t]$, we have that

$$N(B,(0,t])-\int_0^t\int_B\frac{1_{\mathrm{supp}(\nu)}(x)}{\int_0^{T-u}(1+\gamma(x,v))dv}\Big\{N'(dx,(u,T])+N(dx,(0,T-u])\Big\}du$$

is an $\mathcal{H}$-martingale for all $B\in\mathcal{B}(\mathbb{R}^d)$ satisfying $d(B,0)>0$ by (11). The proof for cW_t is essentially the same and easier. We remark that for two nonnegative definite symmetric matrices A and B, $\mathrm{Ker}(A+B)=\mathrm{Ker}A\cap\mathrm{Ker}B$. Hence, $Ax=A(A+B)^{-1}(A+B)x$ where the inverse is understood as the inverse of the restriction of $A+B$ to the range of $A+B$. ■

Remark 15 *Without giving details (see [14]), we remark that the logarithmic utility of an insider can be characterized as*

$$\begin{aligned}
& u(t,\pi)=E\left[\int_0^t\left((b+c\eta(s))\pi_s-\frac{c^2}{2}\pi_s^2\right)ds\right] \\
& +E\left[\int_0^t\int_{\mathbb{R}} x\pi_s(F_s(dx)-\nu(dx))ds\right] \\
& +E\left[\int_0^t\int_{\mathbb{R}}\{\log(1+(e^x-1)\pi_s)-x1(|x|\le 1)\pi_s\}F_s(dx)ds\right].
\end{aligned}$$

Here $c>0$, $d=1$ and π is a process satisfying enough integrability conditions and models the portfolio process of the insider. One can write the equation characterizing the optimal portfolio but no explicit expression is available. Instead one proves that

$$u(t,\pi)\le\frac{c^2}{2}E\left[\int_0^T(\pi_s^o)^2ds\right]+E\left[\int_0^T\int_{\{x>1\}}xF_s(dx)ds\right].$$

where

$$\begin{aligned}
\pi_t^o \quad = \quad & \frac{1}{c^2}\Big\{(b+c\eta(t))+\int_{|x|\le 1}x(F_t(dx)-\nu(dx)) \\
& +\int_{\{x<1\}}(e^x-1-x1(|x|\le 1))F_t(dx) \\
& +\int_{\{x>1\}}xF_t(dx)\Big\}.
\end{aligned}$$

From here one sees that if $E\left[\int_0^T\eta(s)^2ds\right]<\infty$ and $E\left[\int_0^T\left(\int_A xF_s(dx)\right)^2ds\right]<\infty$ for $A=\{|x|\le 1\}$, $\{x>1\}$ and $\{x<1\}$ then the logarithmic utility of the insider is finite.

We now study when we can define the compensator up to T. That is, we specify when β is locally integrable. We will not make any assumptions on the moment properties for Z'. In particular, we will not assume (1) a priori.

Proposition 16 *1. If $\int_{|x|>1}|x|\nu(dx)<\infty$, then $Z_t-\int_0^t\beta_s ds$ is a $\mathcal{G}$-martingale in $[0,T]$.*
2. Without the assumption $\int_{|x|>1}|x|\nu(dx)<\infty$, Z_t is a $\mathcal{G}$-semimartingale in $[0,T]$.

Proof. Proof of 1.

$$\begin{aligned} & E(|\beta_t|) \\ \le\ & \frac{1}{T-t}E(|Z_T - Z_t|) \\ \le\ & \frac{|b|(T-t)+|c|\sqrt{T-t}+\sqrt{T-t}\int_{\{|z|\le 1\}}|z|^2\nu(dz)+(T-t)\int_{\{|z|>1\}}|z|\nu(dz)}{T-t} \\ =\ & |b| + \int_{\{|z|>1\}}|z|\nu(dz) + \frac{1}{\sqrt{T-t}}\Big(|c| + \int_{\{|z|\le 1\}}|z|^2\nu(dz)\Big) \end{aligned}$$

Hence

$$\int_0^T E(|\beta_t|)dt < \infty.$$

Therefore we have proved that $Z_t - \int_0^t \beta_s ds$ is a $\mathcal{G}$- martingale in $[0,T]$.

Proof of 2. Define $Z_t^2 = \sum_{s\le t}\Delta Z_s 1(|\Delta Z_s| > 1) + bt$ and $Z_t^1 = Z_t - Z_t^2$. Note that Z^1 and Z^2 are independent Lévy processes with $E|Z^1| < \infty$. By Theorem 14,

$$Z_t^1 - \int_0^t c\Big((T-s)c + c'_{T-s}\Big)^{-1}\Big(c(W_T - W_s) + G'_{T-s}\Big)ds + \int_{(0,t]}\int_{|x|\le 1} x\Big(F_s(dx)ds - \overline{N}(dx,ds)\Big)$$

is an $\mathcal{H}$-martingale. Obviously the second term and the third term are processes of bounded variation.

Therefore, taking conditional expectations, we have that Z^1 is a semimartingale in the filtration $\mathcal{G}$. As Z^2 is adapted to the filtration $\mathcal{G}$ and it is a process of bounded variation, then $Z_t = Z_t^1 + Z_t^2$ is a $\mathcal{G}$-semimartingale and therefore the conclusion follows. ∎

Next, we give alternative expressions of β in the case that Z' is a Lévy process with characteristic function $E(e^{i\langle\theta, Z'_t\rangle}) = e^{t\widetilde{\psi}(\theta)}$, where

$$\widetilde{\psi}(\theta) = i\langle\widetilde{b},\theta\rangle - \frac{1}{2}\langle c'\theta, c'\theta\rangle + \int_{\mathbb{R}^d}(e^{i\langle\theta,x\rangle} - 1 - i\langle\theta,x\rangle 1_{\{|x|\le 1\}}(x))\nu'(dx)$$

where $\widetilde{b}\in\mathbb{R}^d$, c' is a nonnegative definite $d\times d$-matrix and $\int_{\mathbb{R}^d} 1\wedge|x|^2\nu'(dx) < \infty$.

Theorem 17 *Let g be a continuous increasing function on $[0,T]$. If the law of Z' is identical with the law of Z and $Y_t = Z'_{g(T-t)}$, then $Z_t - \int_0^t \beta_u du$ is a $\mathcal{G}$-martingale in $[0,T)$, where*

$$\beta_t = \frac{L_t}{T-t+g(T-t)}.$$

Proof. We calculate the Fourier transform of the measure μ_α defined in (7).

$$\begin{aligned} \int_{\mathbb{R}^d} e^{i\langle\theta,u\rangle}\mu_\alpha(du,t) &= \int_{\mathbb{R}^d} e^{i\langle\theta,u\rangle}\frac{1}{T-t}\int_{\mathbb{R}^d} Q_{T-t}(du - z - Z_t)zR_{T-t}(dz) \\ &= \frac{1}{i}\psi'(\theta)e^{i\langle\theta,Z_t\rangle}E(e^{i\langle\theta,L_t\rangle}) \\ &= \frac{1}{T-t+g(T-t)}\int_{\mathbb{R}^d} e^{i\langle\theta,u\rangle}(u-Z_t)U_{T-t}(du - Z_t) \\ &= \int_{\mathbb{R}^d} e^{i\langle\theta,u\rangle}\frac{u-Z_t}{T-t+g(T-t)}\int_{\mathbb{R}^d} Q_{T-t}(du-x)P_t(dx). \end{aligned} \qquad (12)$$

Hence $\rho_t^\alpha(\omega,x) = \frac{x-Z_t(\omega)}{T-t+g(T-t)}$. Therefore by Theorem 9 , we have

$$\beta_t = \frac{L_t}{T-t+g(T-t)}.$$

■

Note that in Theorem 17, densities do not need to exist and the process L is not necessarily additive in time as the function g is not necessarily linear. Also, note that in general, L is not a Lévy process. As explained before, note that $E\int_0^T |\beta_s|^2 ds < \infty$ if $g(T-t) = O((T-t)^a)$ for $a < 1$ under the assumptions of Theorem 17. Therefore adding the Lévy process Z' is justified if we want to obtain the properties required in Remark 15.

Proposition 18 *Assume that $d = 1$. In any of the following cases, $Z_t - \int_0^t \beta_u du$ is a $\mathcal{G}$-martingale in $[0, T)$.*
1. If either
1a. $c^2 + c'^2 > 0$,
1b. $c^2 + c'^2 = 0$ and $\liminf_{r\downarrow 0} r^{\alpha-2}\int_{[-r,r]} z^2(\nu+\nu')(dz) > 0$ for some $0 < \alpha < 2$ or
1c. $c^2 + c'^2 = 0$ and, $\nu(dx)$ and $\nu'(dx)$ have respective densities $n(x)$ and $\widetilde{n}(x)$ such that

$$\lim_{x\downarrow 0} |x| \{n(x) + n(-x) + \widetilde{n}(x) + \widetilde{n}(-x)\} = \infty,$$

then U_{T-t} has a bounded density u_{T-t} with bounded derivative u'_{T-t} and

$$\beta_t = \frac{\int_{\mathbb{R}}\{u_{T-t}(L_t - z) - u_{T-t}(L_t)1_{\{|z|\le 1\}}(z)\}z\nu(dz) - c^2 u'_{T-t}(L_t)}{u_{T-t}(L_t)} + b. \tag{13}$$

2. If $c^2 + c'^2 = 0$, $\nu(dx) + \nu'(dx)$ is absolutely continuous, $\int_{\{|x|\le 1\}}(\nu(dx) + \nu'(dx)) = \infty$ and $\int_{\{|x|\le 1\}} |x|(\nu(dx) + \nu'(dx)) < \infty$, then U_{T-t} has a density and β satisfies (13) with $c = 0$.
3. If $c^2 + c'^2 = 0$, $\int_{\mathbb{R}}(\nu(dx) + \nu'(dx)) < \infty$ and both ν and ν' are discrete, then U_{T-t} is discrete. Let u_{T-t} be its probability function. Then β satisfies (13) with $c = 0$.
Furthermore, suppose that $b = \int_{|x|\le 1} x\nu(dx)$ holds. Then in either case 2 or 3, we have that

$$\beta_t = \frac{\int_{\mathbb{R}} u_{T-t}(L_t - z)z\nu(dz)}{u_{T-t}(L_t)}.$$

Proof. We start by proving that in any of the cases considered in 1, U_{T-t} has a smooth density with bounded derivatives. In Case 1a, $|E(e^{i\theta L_t})| \le e^{-((T-t)c^2 + g(T-t)c'^2)\theta^2/2}$. Hence U_{T-t} has a C_b^∞ density (that is, the density is an infinitely differentiable bounded function with bounded derivatives).
Case 1b. If $\nu + \nu'$ satisfies the assumption of Case 1b, then $(T-t)\nu + g(T-t)\nu'$ also satisfies the assumption. Hence

$$\begin{aligned} |E[e^{i\theta L_t}]| &\le \exp\Big[\int_{-1/|\theta|}^{1/|\theta|} (\cos(\theta z) - 1)\{(T-t)\nu + g(T-t)\nu'\}(dz)\Big] \\ &\le \exp\Big[-\frac{1}{8}\int_{-1/|\theta|}^{1/|\theta|} (\theta^2 z^2)\{(T-t)\nu + g(T-t)\nu'\}(dz)\Big] \\ &\le e^{-C|\theta|^\alpha} \quad \textit{for large } |\theta|, \end{aligned}$$

where C is a positive constant. Then $U_{T-t}(dx)$ has a C_b^∞ density u_{T-t} (see Orey [17]).
Case 1c. Let $k \ge 0$. For each $t \in [0, T)$ and $M > k + 1$, there is $\delta > 0$ such that $(T-t)\{n(z) + n(-z)\} + g(T-t)\{\widetilde{n}(z) + \widetilde{n}(-z)\} > M/z$ for $0 < z < \delta$. Hence

$$\begin{aligned} |E[\theta^k e^{i\theta L_t}]| &\le |\theta|^k \exp\{\int_{1/|\theta|}^{\delta} (\cos(|\theta|z) - 1)\frac{M}{z} dz\} \\ &\le |\theta|^{k-M} \exp\{M(-\log\delta + \int_1^{|\theta|\delta} \frac{\cos z}{z} dz)\} \\ &\le C|\theta|^{k-M} \quad \textit{for large } \theta, \end{aligned}$$

where C is a positive constant independent of θ. Hence $U_{T-t}(dx)$ has a C_b^∞ density u_{T-t}.

Since u_{T-t} is bounded and has a bounded derivative in all these cases, we have that

$$\int_{\{|z|\leq 1\}} \{u_{T-t}(x-z) - u_{T-t}(x)\}|z|\nu(dz) + \int_{\{|z|>1\}} u_{T-t}(x-z)|z|\nu(dz) < \infty \tag{14}$$

for each $x \in \mathbb{R}$. Therefore, one can easily compute the Fourier transform of

$$\int_{\mathbb{R}} \Big\{u_{T-t}(x - Z_t - z) - u_{T-t}(x - Z_t)1_{\{|z|\leq 1\}}(z)\Big\} z\nu(dz) - c^2 u'_{T-t}(x - Z_t) + b u_{T-t}(x - Z_t).$$

This gives (12), and therefore the proof of (13) is finished.

In Case 2, U_{T-t} has a density u_{T-t} (see [20] Theorem 27.7). Since u_{T-t} and $zn(z)$ are integrable, then the condition (14) holds and we have (13) with $c = 0$. In Case 3, u_{T-t} becomes the probability function of L_t, the integral becomes a summation and the argument of the proof follows the same reasoning as in the previous cases. ∎

Remark 19 *Assume $\nu = \nu'$ in any of the cases of Proposition 18. Then we have*

$$\begin{aligned} x u_{T-t}(x) \quad &= \quad \{T - t + g(T-t)\} \int_{\mathbb{R}} \{u_{T-t}(x-z) - u_{T-t}(z)1_{|z|\leq 1}(z)\} z\nu(dz) \\ &\quad + \{(T-t)b + g(T-t)\widetilde{b}\} u_{T-t}(x) - \{(T-t)c^2 + g(T-t){c'}^2\} u'_{T-t}(x) \end{aligned}$$

for x a.e. Applying the above formula to (13), we obtain that

$$\beta_t = \frac{1}{T - t + g(T-t)} \Big[L_t - (\widetilde{b} - b) g(T-t) + ({c'}^2 - c^2) g(T-t) \frac{u'_{T-t}(L_t)}{u_{T-t}(L_t)}\Big].$$

5 Examples of enlargements with respect to random times

In this section, we consider some simple examples of applications of Theorem 2. One corresponds to a stopping time and the other to a honest time. The first example treats the situation where the filtration is enlarged by the time of the n-th jump of size bigger than $a > 0$ in absolute value. In these examples we use the representation of a Lévy process using Poisson random measures as explained in Section 3.

First, we consider a setup for initial enlargement of filtrations. That is, $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(\tau)$. After that we consider the case where the filtration is enlarged progressively with I_t.

Example 20 *(time of the n-th jump of absolute size bigger than a) Let $N_t = \int_0^t \int_{|x|>a} N(dx, ds)$ and let T_n be the n-th jump time of N_t. In this example we have that $\mathcal{F}_t = \sigma(Z_u; u \leq t)$ and $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(T_n)$. Further define $X_t = \int_0^t \int_{|x|>a} x N(dx, ds)$ and $Y_t = Z_t - X_t$. Let $\mathcal{F}^1_t = \sigma(N_u; u \leq t)$, $\mathcal{G}^1_t = \mathcal{F}^1_t \vee \sigma(T_n)$, $\mathcal{G}^2_t = \sigma(X_u; u \leq t) \vee \sigma(T_n)$ and $\mathcal{G}^3_t = \sigma(Y_u; u \leq t)$. To avoid studying many different cases we assume that $n \geq 2$ and $a \geq 1$.*

Also, as the time of the $n-$th jump of size bigger than a has a range in $(0, \infty)$ we use the extension of the previous theory to this time interval without any further comment. By the independence of X and Y, we have

$$E[N_t|\mathcal{G}_s] = E[N_t|\mathcal{G}^1_s],\ E[X_t|\mathcal{G}_s] = E[X_t|\mathcal{G}^2_s],\ E[Y_t|\mathcal{G}_s] = E[Y_t|\mathcal{G}^3_s].$$

Let $\lambda = E[N_1]$. For $s < t$,

$$E[1(T_n \leq t)|\mathcal{F}_s] = P[N_t \geq n|\mathcal{F}_s] = \sum_{k=(n-N_s)\vee 0}^{\infty} \frac{\lambda^k (t-s)^k}{k!} e^{-\lambda(t-s)}.$$

Hence for bounded measurable function f, we have

$$E[f(T_n)|\mathcal{F}_s] = \begin{cases} \int_s^{+\infty} \lambda^{n-N_s} \frac{(u-s)^{n-1-N_s}}{(n-1-N_s)!} e^{-\lambda(u-s)} f(u)du & if\ T_n > s, \\ f(T_n) & if\ T_n \leq s. \end{cases}$$

Therefore

$$P_s(dx) = 1(x \leq s)\delta_{T_n}(dx) + 1(x > s)\lambda^{n-N_s} \frac{(x-s)^{n-1-N_s}}{(n-1-N_s)!} e^{-\lambda(x-s)} dx.$$

First we will compute the measure $P^{(2)}$. For this, consider h an $\mathcal{F}$-adapted bounded process

$$\begin{aligned} & E\left[(Z_t - Z_s) f(T_n) 1(t < T_n) h_s\right] \\ = \ & E\left[(X_t - X_s) f(T_n) 1(t < T_n) h_s\right] + E\left[(Y_t - Y_s) f(T_n) 1(t < T_n) h_s\right] \end{aligned} \tag{15}$$

Here, one obtains that

$$E\left[(Y_t - Y_s) f(T_n) 1(t < T_n) h_s\right] = E[Y_1](t-s) E\left[f(T_n) 1(t < T_n) h_s\right].$$

The first term in (15) can be rewritten as

$$\begin{aligned} & E\left[(X_t - X_s) f(T_n) 1(t < T_n) h_s\right] \\ = \ & E[(X_t - X_s]\, E\left[f(T_n)|\, \mathcal{F}_t\right) 1(t < T_n) h_s] \\ = \ & E\left[(X_t - X_s) \int_t^\infty \lambda^{n-N_t} \frac{(u-t)^{n-1-N_t}}{(n-1-N_t)!} e^{-\lambda(u-t)} f(u) du 1(t < T_n) h_s\right] \\ = \ & \frac{E[X_1]}{\lambda} E\left[(N_t - N_s) \int_t^\infty \lambda^{n-N_t} \frac{(u-t)^{n-1-N_t}}{(n-1-N_t)!} e^{-\lambda(u-t)} f(u) du 1(t < T_n) h_s\right] \\ = \ & \frac{E[X_1]}{\lambda} E\left[\sum_{l=1}^{n-N_s-1} \int_t^\infty \lambda^{n-N_s} \frac{(u-t)^{n-1-l-N_s}}{(n-1-l-N_s)!} \frac{(t-s)^l}{(l-1)!} e^{-\lambda(u-s)} f(u) du h_s\right] \\ = \ & \frac{E[X_1]}{\lambda}(t-s) E\left[\sum_{l=0}^{n-N_s-2} \int_t^\infty \lambda^{n-N_s} \frac{(u-t)^{n-2-l-N_s}}{(n-2-l-N_s)!} \frac{(t-s)^l}{l!} e^{-\lambda(u-s)} f(u) du h_s\right]. \end{aligned}$$

Similarly, we calculate $E\left[\frac{n-1-N_t}{T_n-t} f(T_n) 1(t < T_n) h_s\right]$,

$$\frac{E[X_1]}{\lambda}(t-s) E\left[\frac{n-1-N_t}{T_n - t} f(T_n) 1(t < T_n) h_s\right] = E\left[(X_t - X_s) f(T_n) 1(t < T_n) h_s\right].$$

In particular, supposing that X is a Poisson process, we have that

$$E\left[(N_t - N_s)|\, \mathcal{G}_s\right] 1(t < T_n) = (t-s) E\left[\left.\frac{n-1-N_t}{T_n - t}\right|\mathcal{G}_s\right] 1(t < T_n).$$

Subtracting the right-hand side from the left-hand side, we obtain that on the set $\{t < T_n\}$

$$E\left[\left.\frac{N_t}{T_n - t} - \frac{N_s}{T_n - s}\right|\mathcal{G}_s\right] = \frac{(n-1)(t-s)}{(T_n - s)(T_n - t)} = \frac{n-1}{T_n - t} - \frac{n-1}{T_n - s}.$$

By this equality, we have that $\frac{n-1-N_u}{T_n-u}$ is a $\mathcal{G}$-martingale on the set $\{u < T_n\}$. Through a similar calculation one also obtains that

$$E\left[f(T_n) 1(t < T_n) \int_s^t \frac{n-1-N_u}{T_n - u} du h_s\right] = (t-s) E\left[f(T_n) 1(t < T_n) \frac{n-1-N_t}{T_n - t} h_s\right].$$

Therefore we have that $m(du) = du$ and

$$P_u^{(2)}(dx) = 1(u < x)\left(\frac{E[X_1]}{\lambda} \frac{n-1-N_u}{x-u} + E[Y_1]\right) P_u(dx)$$

with $E\int_0^T \left|P_u^{(2)}\right| du < \infty$. *Computing* $P^{(1)}$ *is easier since*

$$E\left[f(T_n)1(T_n < s)\left(Z_t - Z_s\right)h_s\right] = E\left[f(T_n)1(T_n < s)h_s\right]E\left[Z_t - Z_s\right],$$

therefore

$$P_u^{(1)}(dx) = E[Z_1]1(x \le u)\delta_{T_n}(dx) = E[Z_1]P_u(dx).$$

From here we see that the conditions of Theorem 2 are satisfied and

$$Z(t) - \int_0^t \left(\left(\frac{E\left[X_1\right]}{\lambda}\frac{n-1-N_u}{T_n-u} + E[Y_1]\right)1(u < T_n) + E[Z_1]1(u \ge T_n)\right)du - \Delta Z(T_n)1(t \ge T_n)$$

is a $\mathcal{G}$ *martingale in* $[0,T]$ *due to the integrability of the compensator. With some further calculation one also obtains that* $E\left[\int_0^T |\beta_s|\, ds\right] < \infty$ *and* $E\left[\int_0^T |\beta_s|^2\, ds\right] = \infty$.

Now, suppose that the time is perturbed as $I_t = T_n + Z'(T_n - t)$ *where* Z' *is a Lévy process with density function* q. *Then using Theorem 2, we have*

$$Z(t) - \int_0^t \left(\frac{E\left[X_1\right]}{\lambda}E\left[1(u < T_n)\frac{n-1-N_u}{T_n-u}\Big|\mathcal{G}_u\right] + E[Y_1]E\left[1(u < T_n)|\,\mathcal{G}_u\right]\right.$$
$$\left.+E[Z_1]E\left[1(u \ge T_n)\Big|\mathcal{G}_u\right]\right)du - E\left[\Delta Z(T_n)1(t \ge T_n)\Big|\mathcal{G}_u\right].$$

Furthermore, using Theorem 8, we have that

$$E\left[1(u < T_n)\frac{n-1-N_u}{T_n-u}\Bigg|\,\mathcal{G}_u\right] = \frac{\lambda P(u, I_u, n-1)}{P(u, I_u, n)}$$

where

$$P(u,z,n) = \int_u^\infty \frac{(y-u)^{n-1-N_u}}{(n-1-N_u)!}e^{-\lambda(y-u)}q_{y-u}(z-y)dy.$$

The result is similar if Z' *has a discrete distribution.*

The following is a classical example of a random time which is not a stopping time.

Example 21 (*the last jump of absolute size bigger than a before T*) *Let* X_t, Y_t, N_t *and* T_n *be the same as Example 20. Let* τ *be the last jump time of* N_t *before* T. *Let* $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(\tau)$ *and* $\mathcal{G}_t^1 = \sigma(N_u; u \le t) \vee \sigma(\tau)$. *Let*

$$\tau_t = \inf\{s > 0 : N_t - N_s = 0\}.$$

Then $\tau_T = \tau$ *and*

$$P(\tau_t \le s) = P(N_t - N_s = 0) = e^{-\lambda(t-s)}.$$

Using the Markov property, we have for $t \le v \le T$,

$$\begin{aligned} E[f(\tau_T)1(\tau_T > v)|\mathcal{F}_t] &= E_{N_t}[f(\tau_T)1(\tau_T > v)] \\ &= E[f(\tau_{T-t}+t)1(\tau_{T-t} > v-t)] \\ &= \lambda\int_v^T e^{-\lambda(T-y)}f(y)dy =: g(v). \end{aligned}$$

Hence, we obtain that

$$P_t(dx) = 1(x \le t)e^{-\lambda(T-t)}\delta_{\tau_t}(dx) + 1(x > t)\lambda e^{-\lambda(T-x)}dx$$

To compute $P^{(2)}$, *we consider for* $0 < s < t < u$ *and* h *an* $\mathcal{F}$*-adapted bounded process,*

$$\begin{aligned} E[(Z_t - Z_s)f(\tau)1(\tau > u)h_s] &= E[(Z_t - Z_s)h_s E[f(\tau)1(\tau > u)|\mathcal{F}_u]] \\ &= E[(Z_t - Z_s)h_s g(u)] \\ &= E[Z_t - Z_s]E[h_s]g(u) \\ &= E[(t-s)E(Z_1)f(\tau)1(\tau > u)h_s]. \end{aligned}$$

Hence $P_u^{(2)}(dx) = E(Z_1)P_u(dx)$*. Next, in order to compute* $P^{(1)}$*, consider* $0 < u < s < t$*,*

$$E[(X_t - X_s)f(\tau)1(\tau < u)h_s] = 0$$

and

$$\begin{aligned} E[(Y_t - Y_s)f(\tau)1(\tau \leq u)h_s] &= E[f(\tau)1(\tau \leq u)h_s E[(Y_t - Y_s)]] \\ &= E[(t-s)E(Y_1)f(\tau)1(\tau \leq u)h_s]. \end{aligned}$$

Therefore $P_u^{(1)}(dx) = E(Y_1)P_u(dx)$*. Finally we have that*

$$Z_t - E[Z_1]\,(t \wedge \tau) - E[Y_1]((t \vee \tau) - \tau) - \Delta Z(\tau)1(t \geq \tau)$$

is a $\mathcal{G}$ *martingale in* $[0, T]$*.*

We remark here that the conclusion of Theorem 2 is still valid for $G(\tau, Z'(T-t))$ *instead of* $G(\tau, Z'(\tau - t))$*. This remark is made because for* $\tau = \tau_T$ *it is difficult to give a financial interpretation to a model of the type* $I_t = \tau_T + Z'(\tau_T - t)$*. For this reason, we prefer to consider the model* $I_t = \tau_T + Z'(T-t)$*.*

Therefore the compensator becomes

$$\int_0^t \left\{ E[Z_1]P\left(u < \tau_T \middle| \mathcal{G}_u\right) + E[Y_1]P\left(u \geq \tau_T \middle| \mathcal{G}_u\right) \right\} du + E\left[\Delta Z(\tau_T)1(t \geq \tau_T) \middle| \mathcal{G}_t\right].$$

In this situation we obtain that

$$\begin{aligned} & P\left(u < \tau_T \middle| \mathcal{G}_u\right) \\ =\; & \frac{\int_u^T P_u(dy) q_{T-u}(I_u - y)}{\int_0^T P_u(dy) q_{T-u}(I_u - y)} \\ =\; & \frac{\lambda \int_u^T e^{-\lambda(T-y)} q_{T-u}(I_u - y) dy}{e^{-\lambda(T-u)} q_{T-u}(I_u - \tau_u) + \lambda \int_u^T e^{-\lambda(T-y)} q_{T-u}(I_u - y) dy}. \end{aligned}$$

Since $\Delta Z(\tau_T)1(t \in [\tau_T, T]) = \Delta Z(\tau_t)1(t \in [\tau_T, T])$ *and* $\Delta Z(\tau_t)$ *is* $\mathcal{G}_t$ *adapted, one obtains for* $t \leq T$ *that*

$$E\left[\Delta Z(\tau_T)1(t \geq \tau_T) \middle| \mathcal{G}_t\right] = \Delta Z(\tau_t) P\left(t \geq \tau_T \middle| \mathcal{G}_t\right).$$

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