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UTILITY MAXIMIZATION IN AN INSIDER INFLUENCED MARKET

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We study a controlled stochastic system whose state is described by a stochastic differential equation with anticipating coefficients. This setting is used to model markets where insiders have some influence on the dynamics of prices. We give a characterization theorem for the optimal logarithmic portfolio of an investor with a different information flow from that of the insider. We provide explicit results in the partial information case that we extend in order to incorporate the enlargement of filtration techniques for markets with insiders. Finally, we consider a market with an insider who influences the drift of the underlying price asset process. This example gives a situation where it makes a difference for a small agent to acknowledge the existence of an insider in the market.

Key Words: anticipating systems, insider market, partial observation control, forward integrals, Malliavin calculus

1. INTRODUCTION

In most of the research in modeling of insiders problems (see Karatzas-Pikovsky 1996, Imkeller 2003, Grorud-Pontier 2001) one postulates the asset price dynamics as given for the small investor. The insider has additional information, for example, in the form of a random variable that depends on future events. The problem is to evaluate the advantage of the insider in the form of additional utility and optimal portfolio. Mathematically, the problem is to determine the semimartingale decomposition of the Wiener process in the filtration enlarged with the additional information of the insider. Then one can express the dynamics of the prices for the informed agent and compute the optimal investment strategy for this informed agent.

In this paper we study this problem from a different point of view. That is, we assume there exists an insider who is also a large trader and therefore influences the prices of the underlying assets with his/her financial behavior. The small investor is a price taker and the dynamics he assigns to these prices may differ from that observed by the insider due to the information difference. We are interested in analyzing the question of the optimal investment strategy of the small investor in such a situation. This point of view was already

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partly studied in Øksendal–Sulem (2003) although the financial consequences and the modeling possibilities for models of markets with insiders were not exploited there.

Mathematically, the asset price is generated by an anticipating stochastic differential equation, since asset prices have coefficients that are not necessarily adapted to the filtration generated by the Brownian motion. We suppose that the investor's portfolio is adapted to a filtration that may be different from the filtration of the insider or the one generated by the Brownian motion, for example, the filtration generated by the underlying asset price. We study a logarithmic utility maximization problem of final wealth at time T in this anticipating market.

We give a characterization theorem (Theorem 4.1) of optimal portfolios. The optimal portfolios can be interpreted as projection formulas of Merton-type solutions plus an extra term (denoted by $a(t)$, see Corollary 4.2), which is interpreted through examples.

In Section 5 we consider and extend the example of partial information. We first consider the typical situation of a small investor who does not have the information of the random drift driving the price process (see Example 5.1). That is, the stochastic differential equation is adapted to the filtration generated by the Brownian motion and the filtration of the small investor is smaller than this filtration. We then extend this situation to the case when the random drift is anticipating Proposition 5.3. This includes all known models of insiders built with an initial enlargement of filtration technique. In this case, the optimal portfolio of the insider coincides with the optimal portfolio of an investor when the coefficients of the price dynamics are adapted to the enlarged filtration (see Example 5.5). In this generalized set-up one can also consider the optimal portfolio of a small investor (see Example 5.6). We will see that in a market where the price dynamics are driven by an insider, using the enlargement of filtration approach, an investor with a filtration smaller than the enlarged filtration becomes only a partially informed agent in an anticipating world. In conclusion the initial enlargement of filtration approach for insiders modeling becomes a particular case of our generalization of partial information with $a(t) = 0$.

On the other hand, it remains to be seen if the general result given in Theorem 4.1 always corresponds to a initial enlargement of filtration setup. A partial negative answer to this question is given in Section 6. It seems to be a fact that $a(t) \neq 0$ is related to the relationship between three filtrations: (i) the natural filtration of the Brownian motion: $\{\mathcal{F}_t\}_{0\le t\le T}$, (ii) the filtration coefficients of the SDE are adapted to $\{\mathcal{G}_t\}_{0\le t\le T}$, and (iii) the information of the investor: $\{\mathcal{H}_t\}_{0\le t\le T}$.

We thus address the issue: Is there a situation where $a(t) \neq 0$ and what is the interpretation of $a(t)$? To answer this question we consider stock dynamics where the drift is influenced by the insider through a smooth (in the sense of stochastic derivatives) random variable and the noise is given by the original Brownian motion (see Section 6). We suppose that a small investor observes the price of the underlying asset and computes his/her optimal portfolio using a logarithmic utility. The results lead to the following conclusion: If the small agent decides that there is no insider in the market, he/she estimates the drift of the underlying with the best estimator (the conditional expectation) with respect to his information and builds a geometric Brownian motion as his/her model to maximize the logarithmic utility. This calculation gives a suboptimal portfolio. The difference between this suboptimal portfolio and the optimal portfolio assuming an anticipating model for the market with insiders is proportional to $a(t)$. Furthermore, the difference in utilities is given by a quantity depending on $a(t)$, which appears due to the anticipating nature of the modeling (see Remark 6.8.2.). Finally, we consider the case where the insider has an effect on the drift through information that is δ units of time ahead. This model seems to lead to some generalizations of insider modeling that may not be tractable by enlargement of filtration techniques.

2. SOME PRELIMINARIES ON FORWARD STOCHASTIC INTEGRALS

We introduce here the forward integral. We change somewhat the definition to fit our goals. We refer to Nualart and Pardoux (1988), and Russo and Vallois (1993, 1995, 2000) for more information about these integrals and to Biagini and Øksendal (2002) for a discussion on the pertinence of the use of forward integrals in insider modeling. Let $B(t)$ be a Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$, and $T > 0$ a fixed horizon.

DEFINITION 2.1. Let $\phi : [0, T] \times \Omega \to \mathbb{R}$ be a measurable process. The forward integral of ϕ with respect $B(.)$ is defined by

$$\int_0^T \phi(t)\, d^- B(t) = \lim_{\epsilon\to 0} \int_0^T \phi(t) \frac{B(t+\epsilon) - B(t)}{\epsilon}\, dt, \tag{2.1}$$

if the limit exists in $L^1(\Omega)$.

Note that if the forward integral exists in this $L^1(\Omega)$-sense, then it also exists in the Russo–Valois sense (convergence of (2.1) in probability).

We state a relation between forward and Skorohod integrals. From now on, δ denotes the Skorohod integral and D denotes the stochastic derivative operator. For details on the notation, see Nualart (1995). We also refer to Proposition 2.3 of Russo and Vallois (1993), for related results.

LEMMA 2.2. *Suppose that* $\phi : [0, T] \times \Omega \to \mathbb{R}$ *belongs to* $\mathbb{L}^{1,2}[0, T]$, *that is* ϕ *satisfies*

$$\mathbb{E}\left(\int_0^T |\phi(t)|^2\, dt + \int_0^T \int_0^T |D_u\phi(t)|^2\, du\, dt \right) < +\infty.$$

Moreover, assume that

$$\lim_{\epsilon\to 0} \frac{1}{\epsilon} \int_{u-\epsilon}^u \phi(t)\, dt = \phi(u) \quad \text{for a.a. } u \in [0, T] \quad \text{in } \mathbb{L}^{1,2}[0, T]$$

and that $D_{t^+}\phi(t) := \lim_{s\to t^+} D_s\phi(t)$ *exists uniformly in* $t \in [0, T]$ *in* $L^1((0, T) \otimes \Omega)$. *Then the forward integral of* ϕ *exists and*

$$\int_0^T \phi(t)\, d^- B(t) = \int_0^T \phi(t)\, \delta B(t) + \int_0^T D_{t^+}\phi(t)\, dt. \tag{2.2}$$

Moreover,

$$\mathbb{E}\left[\int_0^T \phi(t) d^- B(t) \right] = \mathbb{E}\left[\int_0^T D_{t^+}\phi(t)\, dt \right]. \tag{2.3}$$

Proof. We provide a sketch of the proof.

$$\begin{aligned} &\lim_{\epsilon\to 0} \int_0^T \phi(t) \frac{B(t+\epsilon) - B(t)}{\epsilon}\, dt \\ &= \lim_{\epsilon\to 0} \int_0^T \frac{\phi(t)}{\epsilon} \int_t^{t+\epsilon} dB(u)\, dt \end{aligned}$$

$$= \lim_{\epsilon\to 0}\left\{\int_0^T\int_t^{t+\epsilon}\frac{\phi(t)}{\epsilon}\,\delta B(u)\,dt + \int_0^T\int_t^{t+\epsilon}\frac{D_u\phi(t)}{\epsilon}\,du\,dt\right\}\ \text{(see (1.49) in [N])}$$
$$= \lim_{\epsilon\to 0}\left\{\int_\epsilon^T\int_{u-\epsilon}^{u}\frac{\phi(t)}{\epsilon}\,dt\,\delta B(u) + \int_0^\epsilon\int_0^u\frac{\phi(t)}{\epsilon}\,dt\,\delta B(u)\right.$$
$$\left. + \int_\epsilon^T\int_{u-\epsilon}^{u}\frac{D_u\phi(t)}{\epsilon}\,dt\,du + \int_0^\epsilon\int_0^u\frac{D_u\phi(t)}{\epsilon}\,dt\,du\right\}.$$

We can prove that each of these four terms converge when ϵ goes to 0 by straightforward computations. Since Skorohod integrals have expectation 0, we deduce 2.3. □

LEMMA 2.3. *Let ϕ be as in Lemma 2.2 of the form $\phi(t) = \sum_{i=0}^{n-1}\phi(t_i)1_{(t_i,t_{i+1}]}(t)$ for a fixed partition $p := \{0 = t_0 < \cdots < t_n = T\}$. Then*

$$\int_0^T \phi(t)\,d^-B(t) = \sum_{j=0}^{n-1}\phi(t_i)(B(t_{i+1}) - B(t_i)) \quad \text{in } L^1(\Omega).$$

Proof. We have

$$\int_0^T\phi(t)\left(\frac{B(t+\epsilon)-B(t)}{\epsilon}\right)dt = \sum_{i=0}^{n-1}\frac{\phi(t_i)}{\epsilon}\int_{t_i}^{t_{i+1}}\int_t^{t+\epsilon}\delta B(u)\,dt$$
$$= \sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}\int_t^{t+\epsilon}\frac{\phi(t_i)}{\epsilon}\,\delta B(u)\,dt + \int_{t_i}^{t_{i+1}}\int_t^{t+\epsilon}\frac{D_u\phi(t_i)}{\epsilon}du\ dt.$$

Applying the Fubini theorem, we get

$$\int_{t_i}^{t_{i+1}}\int_t^{t+\epsilon}\frac{\phi(t_i)}{\epsilon}\,\delta B(u)\,dt = \int_{t_i}^{t_i+\epsilon}\int_{t_i}^{u}\frac{\phi(t_i)}{\epsilon}\,dt\,\delta B(u) + \int_{t_i}^{t_{i+1}}\int_{u-\epsilon}^{u}\frac{\phi(t_i)}{\epsilon}\,dt\,\delta B(u)$$
$$+ \int_{t_{i+1}}^{t_{i+1}+\epsilon}\int_{u-\epsilon}^{t_{i+1}}\frac{\phi(t_i)}{\epsilon}\,dt\,\delta B(u)$$
$$= \int_{t_i}^{t_i+\epsilon}\frac{\phi(t_i)(u-t_i)}{\epsilon}\,\delta B(u) + \int_{t_i+\epsilon}^{t_{i+1}}\phi(t_i)\,\delta B(u)$$
$$+ \int_{t_{i+1}}^{t_{i+1}+\epsilon}\frac{\phi(t_i)(t_{i+1}-u+\epsilon)}{\epsilon}\,\delta B(u)$$

and

$$\int_{t_i}^{t_{i+1}}\int_t^{t+\epsilon}\frac{D_u\phi(t_i)}{\epsilon}\,du\ dt = \int_{t_i}^{t_i+\epsilon}\frac{D_u\phi(t_i)(u-t_i)}{\epsilon}\,du + \int_{t_i+\epsilon}^{t_i+1}D_u\phi(t_i)\,du$$
$$+\int_{t_{i+1}}^{t_{i+1}+\epsilon}\frac{D_u\phi(t_i)(t_{i+1}-u+\epsilon)}{\epsilon}\,du.$$

Similarly,

$$\phi(t_i)(B(t_{i+1}) - B(t_i)) = \int_{t_i}^{t_{i+1}}\phi(t_i)\delta B(u) + \int_{t_i}^{t_{i+1}}D_u\phi(t_i)\,du.$$

Therefore,

$$\int_0^T \phi(t)\left(\frac{B(t+\epsilon)-B(t)}{\epsilon}\right) dt - \sum_{j=0}^{n-1} \phi(t_i)(B(t_{i+1})-B(t_i))$$
$$= \sum_{j=0}^{n-1} \int_{t_i}^{t_i+\epsilon} \frac{\phi(t_i)(u-t_i)}{\epsilon}\,\delta B(u) - \int_{t_i}^{t_i+\epsilon} \phi(t_i)\,\delta B(u) + \int_{t_{i+1}}^{t_{i+1}+\epsilon} \frac{\phi(t_i)(t_{i+1}-u+\epsilon)}{\epsilon}\,\delta B(u)$$
$$+ \int_{t_i}^{t_i+\epsilon} \frac{D_u\phi(t_i)(u-t_i)}{\epsilon}\,du - \int_{t_i}^{t_i+\epsilon} D_u\phi(t_i)\,du + \int_{t_{i+1}}^{t_{i+1}+\epsilon} \frac{D_u\phi(t_i)(t_{i+1}-u+\epsilon)}{\epsilon}\,du.$$

Now we prove that each term goes to 0 in $L^1(\Omega)$ as $\epsilon \to 0$. We have

$$\mathbb{E}\left|\int_{t_i}^{t_i+\epsilon} \frac{\phi(t_i)(u-t_i)}{\epsilon}\,\delta B(u)\right| \le \left\|\phi(t_i)\frac{(\cdot - t_i)}{\epsilon}\right\|_{\mathbb{L}^{1,2}[t_i,t_i+\epsilon]} \to 0 \quad \text{as } \epsilon \to 0,$$
$$\mathbb{E}\left|\int_{t_i}^{t_i+\epsilon} \frac{D_u\phi(t_i)(u-t_i)}{\epsilon}\,du\right| \le \left\|\phi(t_i)\frac{(\cdot - t_i)}{\epsilon}\right\|_{\mathbb{L}^{1,2}[t_i,t_i+\epsilon]} \to 0 \quad \text{as } \epsilon \to 0$$

and similarly for all other terms. □

LEMMA 2.4. *Suppose that ϕ satisfies the conditions of Lemma 2.2. For any sequence of partitions $p_n = \{t_0 = 0 < t_1 < \cdots < t_n = T\}$ such that $\Delta_n := \sup_{i=0\ldots N-1}(t_{i+1}-t_i)$ goes to 0 when $n \to +\infty$, define $\phi_n(t) := \phi(t_i)$ for $t_i < t \le t_{i+1}$. Suppose*

$$\|\phi - \phi_n\|_{\mathbb{L}^{1,2}[0,T]} + \mathbb{E}\int_0^T |D_+(\phi-\phi_n)(u)|\,du \to 0 \quad \text{as } n \to +\infty, \tag{2.4}$$

then

$$\int_0^T \phi(t)\,d^- B(t) = \lim_{n\to+\infty} \sum_{i=0}^{n-1} \phi(t_i)(B(t_{i+1})-B(t_i)). \tag{2.5}$$

Proof. By Lemma 2.3, we have that

$$\int_0^T \phi_n(t)\,d^- B(t) = \sum_{i=0}^{n-1} \phi(t_i)(B(t_{i+1})-B(t_i)).$$

Furthermore,

$$\int_0^T (\phi-\phi_n)(t)\,d^- B(t)$$
$$= \lim_{\epsilon\to 0} \int_0^T (\phi-\phi_n)(t)\left(\frac{B(t+\epsilon)-B(t)}{\epsilon}\right) dt$$
$$= \lim_{\epsilon\to 0}\left\{\frac{1}{\epsilon}\int_0^T\int_t^{t+\epsilon} (\phi-\phi_n)(t)\,\delta B(u)\,dt + \frac{1}{\epsilon}\int_0^T\int_t^{t+\epsilon} D_u(\phi-\phi_n)(t)\,du\,dt\right\}$$
$$= \lim_{\epsilon\to 0}\left\{\frac{1}{\epsilon}\int_\epsilon^T\int_{u-\epsilon}^{u} (\phi-\phi_n)(t)\,dt\,\delta B(u) + \frac{1}{\epsilon}\int_\epsilon^T\int_{u-\epsilon}^{u} D_u(\phi-\phi_n)(t)\,dt\,du\right.$$
$$+ \frac{1}{\epsilon}\int_0^\epsilon\int_0^u (\phi-\phi_n)(t)\,dt\,\delta B(u) + \frac{1}{\epsilon}\int_0^\epsilon\int_0^u D_u(\phi-\phi_n)(t)\,dt\,du$$
$$\left. + \frac{1}{\epsilon}\int_T^{T+\epsilon}\int_{u-\epsilon}^{T} (\phi-\phi_n)(t)\,dt\,\delta B(u) + \frac{1}{\epsilon}\int_T^{T+\epsilon}\int_{u-\epsilon}^{T} D_u(\phi-\phi_n)(t)\,dt\,du\right\}.$$

Now we prove that each term goes to 0 in $L^1(\Omega)$ as $n \to +\infty$. We have

$$\mathbb{E}\left|\frac{1}{\epsilon}\int_\epsilon^T \int_{u-\epsilon}^u (\phi - \phi_n)(t)\,dt\,\delta B(u)\right| \le \frac{1}{\epsilon}\left\| \int_{\cdot-\epsilon}^{\cdot} (\phi - \phi_n)(t)\,dt\right\|_{\mathbb{L}^{1,2}[0,T]}.$$

Consider first

$$\mathbb{E}\int_0^T \left|\int_{u-\epsilon}^u (\phi - \phi_n)(t)\,dt\right|^2 du = \mathbb{E}\int_0^T \left|\int_0^T (\phi - \phi_n)(t) 1_{\{t-u\in[-\epsilon,0]\}}\,dt\right|^2 du$$
$$\le \epsilon\,\mathbb{E}\int_0^T (\phi - \phi_n)^2(t)\,dt$$

by Young's inequality for convolutions. Similarly,

$$\mathbb{E}\int_0^T\int_0^T \left|\int_{u-\epsilon}^u D_s(\phi - \phi_n)(t)\,dt\right|^2 ds\,du = \mathbb{E}\int_0^T\int_0^T \left|\int_0^T D_s(\phi - \phi_n)(t) 1_{\{t-u\in[-\epsilon,0]\}} dt\right|^2 dsdu$$
$$\le \epsilon\,\mathbb{E}\int_0^T\int_0^T |D_s(\phi - \phi_n)(t)|^2\,ds\,dt.$$

Then by Fatou's lemma

$$\mathbb{E}\lim_{\epsilon\to 0}\left|\frac{1}{\epsilon}\int_\epsilon^T \int_{u-\epsilon}^u (\phi - \phi_n)(t)\,dt\,\delta B(u)\right| \le C\|\phi - \phi_n\|_{\mathbb{L}^{1,2}[0,T]} \to 0 \quad \text{as } n \to +\infty.$$

The other terms follow similarly. We have

$$\mathbb{E}\lim_{\epsilon\to 0}\frac{1}{\epsilon}\left|\int_0^\epsilon \int_0^u D_u(\phi - \phi_n)(t)\,dt\,du\right| = \frac{1}{\epsilon}\mathbb{E}\left|\int_0^\epsilon \int_0^u D_u(\phi - \phi_n)(t)\,dt\,du\right|$$
$$\le C\|\phi - \phi_n\|_{\mathbb{L}^{1,2}[0,\epsilon]} \to 0 \quad \text{as } n \to +\infty.$$

Moreover,

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_\epsilon^T \int_{u-\epsilon}^u D_u(\phi - \phi_n)(t)\,dtdu = \int_0^T D_{u^+}(\phi - \phi_n)(u)\,du \quad \text{a.s.}$$

and

$$\mathbb{E}\int_0^T |D_{u^+}(\phi - \phi_n)(u)|\,du \to 0 \quad \text{as } n \to +\infty.$$

Consequently,

$$\mathbb{E}\left(\left|\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_\epsilon^T \int_{u-\epsilon}^u D_u(\phi - \phi_n)(t)\,dt\,du\right|\right) \to 0 \quad \text{as } n \to +\infty. \qquad \square$$

See Biagini and Øksendal (2002) for a related result.

REMARK 2.5. Condition (2.4) is a continuity-type condition of the forward integral. That is, if for any sequence (u_n) satisfying the conditions of Lemma 2.2, we have

$$\|\phi - u_n\|_{\mathbb{L}^{1,2}} + \mathbb{E}\int_0^T |D_+(\phi - u_n)(u)|\,du \to 0 \quad \text{as } n \to +\infty,$$

then

$$\int_0^T u_n(t)\,d^- B(t) \to \int_0^T \phi(t)\,d^- B(t) \quad \text{in } L^1(\Omega). \tag{2.6}$$

3. FORMULATION OF THE UTILITY MAXIMIZATION PROBLEM

Let $\{\mathcal{G}_t\}_{t\geq 0}$ be a filtration such that

$$\mathcal{F}_t \subseteq \mathcal{G}_t \subseteq \mathcal{F} \quad \text{for all } t \geq 0. \tag{3.1}$$

Consider a financial market with one risk-free investment, with price S_0 given by

$$dS_0(t) = \rho(t)S_0(t)\,dt; \quad S_0(0) = 1$$

and one risky investment, whose price $S(t)$ at time t is described by

$$dS(t) = S(t)[\mu(t)\,dt + \sigma(t)\,d^- B(t)], \quad S(0) > 0, \tag{3.2}$$

where $\rho(t) = \rho(t,\omega)$, $\mu(t) = \mu(t,\omega)$, and $\sigma(t) = \sigma(t,\omega) \geq 0$ are $\mathcal{G}_t$-adapted real-valued processes. We assume $\mathbb{E}\int_0^t(|\rho(s)| + |\mu(s)| + \sigma(s)^2)ds < +\infty$ for all t and σ satisfies the conditions of Lemma 2.4. Since $B(t)$ need not be a semimartingale with respect to $\{\mathcal{G}_t\}_{t\geq 0}$, the last integral in (3.2) is an *anticipating* stochastic integral that we interpret as a *forward* integral.

Moreover, we consider another filtration $\{\mathcal{H}_t\}_{t\geq 0}$ for modeling the information of the investor but no assumption is made on the relation between $\{\mathcal{H}_t\}_{t\geq 0}$ and $\{\mathcal{F}_t\}_{t\geq 0}$ or $\{\mathcal{G}_t\}_{t\geq 0}$.

We introduce the set of admissible strategies defined as $\mathcal{H}_t$-adapted processes $p(t) = (p_0(t), p_1(t))$ giving the numbers of shares held in each asset, such that $p_1\sigma$ satisfies the conditions of Lemma 2.4. The associated wealth process is given by

$$W^{(p)}(t) = p_0(t)S_0(t) + p_1(t)S(t).$$

We assume that the portfolio p is self-financing, that is,

$$dW^{(p)}(t) = p_0(t)dS_0(t) + p_1(t)d^- S(t).$$

Note that this definition of "self-financing strategy" with forward integrals corresponds to the usual one.

We restrict ourselves to tame portfolios, that is, to portfolios p such that $W^{(p)}(t) > 0$ for all $t \in [0, T]$. We can thus parameterize our problem by using the fraction of wealth invested in the risky asset $\pi(t) = \pi(t,\omega) = p_1(t)S(t)/W^{(p)}(t)$ for all $t \in [0, T]$.

We define the set $\mathcal{A}_{\mathcal{H}}$ of admissible portfolios as follows:

DEFINITION 3.1. The space $\mathcal{A}_{\mathcal{H}}$ consists of all $\mathcal{H}_t$-adapted processes π such that $\pi\sigma$ satisfies the conditions of Lemma 2.4 and

$$\mathbb{E}\left[\int_0^T (|\mu(t) - \rho(t)|.|\pi(t)| + \sigma^2(t)\pi^2(t))\,dt\right] < \infty.$$

The dynamics of the discounted wealth process

$$X(t) = X^{(\pi)}(t) = \exp\left(-\int_0^t \rho(s)\,ds\right) W^{(\pi)}(t)$$

corresponding to the portfolio π is then

$$dX(t) = X(t)[(\mu(t) - \rho(t))\pi(t)\,dt + \pi(t)\sigma(t)\,d^- B(t)] \quad X(0) = x > 0. \tag{3.3}$$

This equation is justified by using Itô's formula for forward integrals (see Russo and Vallois (2000)) and has the solution

$$X^{(\pi)}(T) = x\exp\left\{\int_0^T\left((\mu(t) - \rho(t))\pi(t) - \frac{1}{2}\pi^2(t)\sigma^2(t)\right)dt + \int_0^T \pi(t)\sigma(t)d^- B(t)\right\}. \tag{3.4}$$

We consider the following performance criterion:

$$
\begin{aligned}
(3.5)\ J(\pi) &:= \mathbb{E}[\ln X^{(\pi)}(T)] - \ln x \\
&= \mathbb{E}\left[\int_0^T \left((\mu(t)-\rho(t))\pi(t) - \frac{1}{2}\pi^2(t)\sigma^2(t)\right) dt + \int_0^T \pi(t)\sigma(t)d^- B(t)\right].
\end{aligned}
$$

The goal is to find the optimal portfolio $\pi^* \in \mathcal{A}_{\mathcal{H}}$ for the logarithmic utility portfolio problem

$$
\sup_{\pi \in \mathcal{A}_{\mathcal{H}}} J(\pi) = J(\pi^*). \tag{3.6}
$$

The case when $\mathcal{H}_t \subseteq \mathcal{F}_t \subseteq \mathcal{G}_t \subseteq \mathcal{F}$ is considered in Øksendal and Sulum (2003) and $\mathcal{F}_t \subseteq \mathcal{G}_t \subseteq \mathcal{H}_t$ in Biagini and Øksendal (2002).

4. CHARACTERISATION OF THE OPTIMAL PORTFOLIO

We give a theorem that characterizes optimal portfolios. We suppose that the optimal utility is finite (see Remark 4.5).

THEOREM 4.1. *The following assertions are equivalent:*

(i) *There exists an optimal portfolio $\pi^* \in \mathcal{A}_{\mathcal{H}}$ for Problem (3.6).*

(ii) *There exists $\pi^* \in \mathcal{A}_{\mathcal{H}}$ such that the process*

$$
M_{\pi^*}(t) := \mathbb{E}\left[\int_0^t (\mu(s)-\rho(s)-\sigma^2(s)\pi^*(s))\,ds + \int_0^t \sigma(s)d^- B(s) \,|\, \mathcal{H}_t\right] \tag{4.1}
$$

is an $\mathcal{H}$-martingale.

(iii) *the function*

$$
s \mapsto \mathbb{E}\left[\int_0^s \sigma(u)d^- B(u) \,|\, \mathcal{H}_t\right]; \quad s > t
$$

is absolutely continuous and there exists $\pi^* \in \mathcal{A}_{\mathcal{H}}$ such that for a.a. t, ω,

$$
\begin{aligned}
&\frac{d}{ds}\mathbb{E}\left[\int_0^s \sigma(u)d^- B(u) \,|\, \mathcal{H}_t\right] \\
&\quad = -\mathbb{E}[\mu(s)-\rho(s)-\sigma^2(s)\pi^*(s) \,|\, \mathcal{H}_t]; \quad \text{a.a. } s > t
\end{aligned} \tag{4.2}
$$

Proof.

(i) $\Rightarrow$ (ii): Suppose (i) holds. Since $\pi^* \in \mathcal{A}_{\mathcal{H}}$ is optimal, we have

$$
J(\pi^*) \geq J(\pi^* + r\beta)
$$

for all $\beta \in \mathcal{A}_{\mathcal{H}}$ and $r \in \mathbb{R}$. Therefore,

$$
\left.\frac{d}{dr} J(\pi^* + r\beta)\right|_{r=0} = 0.
$$

This gives

$$
\mathbb{E}\left[\int_0^T \{\mu(t)-\rho(t)-\sigma^2(t)\pi^*(t)\}\beta(t)\,dt + \int_0^T \beta(t)(\sigma(t)d^- B(t))\right] = 0 \tag{4.3}
$$

for all $\beta \in \mathcal{A}_{\mathcal{H}}$. In particular, applying this to

$$\beta(u) = \beta_0(t) 1_{[t,s]}(u)$$

for $0 \le t < s \le T$, $u \in [t, s]$, where $\beta_0(t)$ is $\mathcal{H}_t$-measurable and bounded, we obtain

(4.4)
$$\mathbb{E}\left[\left(\int_t^s \{\mu(u) - \rho(u) - \sigma^2(u)\pi^*(u)\}\,du + \int_t^s \sigma(u) d^- B(u)\right) \beta_0(t)\right] = 0.$$

Since this holds for all such $\beta_0(t)$ we conclude that

(4.5)
$$\mathbb{E}\left[\left(\int_t^s \{\mu(u) - \rho(u) - \sigma^2(u)\pi^*(u)\}\,du + \int_t^s \sigma(u)\, d^- B(u)\right) \Big| \mathcal{H}_t\right] = 0.$$

This is equivalent to saying that the process

$$K_{\pi^*}(t) := \int_0^t \{\mu(u) - \rho(u) - \sigma^2(u)\pi^*(u)\}\,du + \int_0^t \sigma(u)\, d^- B(u)$$

satisfies

$$\mathbb{E}[K_{\pi^*}(s) \,|\, \mathcal{H}_t] = \mathbb{E}[K_{\pi^*}(t) \,|\, \mathcal{H}_t] \quad \text{for all } s \ge t. \tag{4.6}$$

From this we get, for $s \ge t$

$$\begin{aligned} \mathbb{E}[M_{\pi^*}(s) \,|\, \mathcal{H}_t] &= \mathbb{E}[\mathbb{E}[K_{\pi^*}(s) \,|\, \mathcal{H}_s] \,|\, \mathcal{H}_t] = \mathbb{E}[K_{\pi^*}(s) \,|\, \mathcal{H}_t] \\ &= \mathbb{E}[K_{\pi^*}(t) \,|\, \mathcal{H}_t] = M_{\pi^*}(t), \end{aligned}$$

which is (ii).

(ii) $\Rightarrow$ (iii): Suppose (ii) holds. Then, for $s \ge t$,

$$\begin{aligned} \mathbb{E}[K_{\pi^*}(s) \,|\, \mathcal{H}_t] &= \mathbb{E}[\mathbb{E}[K_{\pi^*}(s) \,|\, \mathcal{H}_s] \,|\, \mathcal{H}_t] = \mathbb{E}[M_{\pi^*}(s) \,|\, \mathcal{H}_t] = M_{\pi^*}(t) \\ &= \mathbb{E}[K_{\pi^*}(t) \,|\, \mathcal{H}_t]. \end{aligned}$$

Hence (4.6)—and then also (4.5)—holds. And (4.5) clearly implies (iii).

(iii) $\Rightarrow$ (i): Suppose (iii) holds.

Then integrating (4.2), we get (4.5), which again implies (4.4). By taking a linear combination of (4.4), we obtain that (4.3) holds for all $\beta = \beta^\Delta \in \mathcal{A}_\mathcal{H}$ of the form

$$\beta^\Delta(u) = \Sigma_{i=1}^N \beta_i(t_i) 1_{(t_i, t_{i+1}]}(u)$$

where $0 = t_0 < t_1 < \cdots < t_{N+1} = T$, $\Delta = \sup_{i=0\ldots N-1}(t_{i+1} - t_i)$, and $\beta_i(t_i)$ is $\mathcal{H}_{t_i}$-measurable and bounded. Moreover, for all $\beta \in \mathcal{A}_\mathcal{H}$ such that

$$\|\beta^\Delta\sigma - \beta\sigma\|_{\mathbb{L}^{1,2}[0,T]} + \mathbb{E}\int_0^T \left|D_t((\beta^\Delta\sigma - \beta\sigma))(u)\right| du \to 0,$$

we have by Remark 2.5 that

$$\int_0^T \beta(t)\sigma(t) d^- B(t) = \lim_{\Delta\to 0} \int_0^T \beta^\Delta(t)\sigma(t) d^- B(t) = \lim_{\Delta\to 0} \Sigma_{i=0}^{N-1} \beta(t_i) \int_{t_i}^{t_{i+1}} \sigma(s)\, d^- B(s)$$

in $L^1(\Omega)$. Hence, by a density argument, (4.3) holds for all $\beta \in \mathcal{A}_\mathcal{H}$.

This means that the directional derivative of J at π^* with respect to the direction β, denoted by $D_\beta J(\pi^*)$ is 0, i.e.,

$$D_\beta J(\pi^*) := \lim_{r\to 0} \frac{J(\pi^* + r\beta) - J(\pi^*)}{r} = 0; \quad \beta \in \mathcal{A}_{\mathcal{H}}. \tag{4.7}$$

Note that $J : \mathcal{A}_{\mathcal{H}} \to \mathbb{R}$ is concave, in the sense that

$$J(\lambda\alpha + (1-\lambda)\beta) \geq \lambda J(\alpha) + (1-\lambda)J(\beta); \quad \lambda \in [0,1], \alpha, \beta \in \mathcal{A}_{\mathcal{H}}.$$

Therefore, for all $\alpha, \beta \in \mathcal{A}_{\mathcal{H}}$ and $\varepsilon \in (0,1)$, we have

$$\begin{aligned} J(\alpha + \varepsilon\beta) - J(\alpha) &= J\left((1-\varepsilon)\frac{\alpha}{1-\varepsilon} + \varepsilon\beta\right) - J(\alpha) \\ &\geq (1-\varepsilon)J\left(\frac{\alpha}{1-\varepsilon}\right) + \varepsilon J(\beta) - J(\alpha) \\ &= J\left(\frac{\alpha}{1-\varepsilon}\right) - J(\alpha) + \varepsilon\left(J(\beta) - J\left(\frac{\alpha}{1-\varepsilon}\right)\right). \end{aligned} \tag{4.8}$$

Now, with $\frac{1}{1-\varepsilon} = 1 + \eta$ we have

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\left(J\left(\frac{\alpha}{1-\varepsilon}\right) - J(\alpha)\right) = \lim_{\eta\to 0} \frac{1+\eta}{\eta}(J(\alpha + \eta\alpha) - J(\alpha)) = D_\alpha J(\alpha).$$

Combining this with (4.8) we get

$$D_\beta J(\alpha) = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}(J(\alpha + \varepsilon\beta) - J(\alpha)) \geq D_\alpha J(\alpha) + J(\beta) - J(\alpha).$$

We conclude that

$$J(\beta) - J(\alpha) \leq D_\beta J(\alpha) - D_\alpha J(\alpha); \quad \alpha, \beta \in \mathcal{A}_{\mathcal{H}}.$$

In particular, applying this to $\alpha = \pi^*$ and using that $D_\beta J(\pi^*) = 0$ by (4.7), we get

$$J(\beta) - J(\pi^*) \leq 0 \quad \text{for all } \beta \in \mathcal{A}_{\mathcal{H}},$$

which proves that π^* is optimal. □

This characterization theorem provides a closed formula for the optimal strategy π^*.

COROLLARY 4.2. *Suppose that an optimal portfolio $\pi^* \in \mathcal{A}_{\mathcal{H}}$ for Problem (3.6) exists. Then it must satisfy*

$$\pi^*(t)\mathbb{E}\big[\sigma^2(t)\big|\mathcal{H}_t\big] = \mathbb{E}[(\mu(t) - \rho(t))\,|\,\mathcal{H}_t] + a(t). \tag{4.9}$$

where

$$a(t) := \lim_{h\to 0^+} \frac{1}{h}\mathbb{E}\left[\int_t^{t+h} \sigma(s)\, d^- B(s)\,|\,\mathcal{H}_t\right]. \tag{4.10}$$

Note that the optimal portfolio has a similar form as the solution of the Merton problem. Here the rate of appreciation and volatility are replaced by their best estimators, the conditional expectations. There is an extra term $a(t)$ that appears due to the anticipative nature of the original equation. An interpretation of this term is given in Section 6.

REMARK 4.3. If $\mathcal{G}_{t+\delta} \subseteq \mathcal{H}_t$, $\delta > 0$, then in most cases $a(t)$ does not exist, because then

$$\frac{1}{h}\mathbb{E}\left[\int_t^{t+h} \sigma(s)\, d^- B(s) \,|\, \mathcal{H}_t\right] = \frac{1}{h}\int_t^{t+h} \sigma(s)\, d^- B(s) \quad \text{for } h \le \delta.$$

Similarly, if $\mathcal{H}_t = \mathcal{F}_{t+\delta}$, $a(t)$ does not exist. This is also related to the fact that such insiders obtain an infinite amount of wealth and that the market admits arbitrage by the insider.

We compute now the value function when the optimal portfolio exists.

THEOREM 4.4. *Suppose that* $\sigma(t) \neq 0$ *for a.a.* (t, ω). *Suppose there exists an optimal portfolio* $\pi^* \in \mathcal{A}_{\mathcal{H}}$ *for Problem (3.6). The optimal utility is then given by*

$$J(\pi^*) = \mathbb{E}\left[\int_0^T \left\{\frac{1}{2}\frac{\mathbb{E}[\mu(s) - \rho(s) \,|\, \mathcal{H}_s]^2}{\mathbb{E}\big[\sigma^2(s) \,\big|\, \mathcal{H}_s\big]} - \frac{1}{2}\frac{a(s)^2}{\mathbb{E}\big[\sigma^2(s) \,\big|\, \mathcal{H}_s\big]} + D_{s^+}\left(\sigma(s)\frac{\mathbb{E}[\mu(s) - \rho(s) \,|\, \mathcal{H}_s] + a(s)}{\mathbb{E}\big[\sigma^2(s) \,\big|\, \mathcal{H}_s\big]}\right)\right\} ds\right]. \tag{4.11}$$

Proof. From (4.9) we have

$$\pi^*(t) = \frac{\mathbb{E}[\nu(t) \,|\, \mathcal{H}_t] + a(t)}{\mathbb{E}\big[\sigma^2(t) \,|\, \mathcal{H}_t\big]} \tag{4.12}$$

where we have set $\nu(s) = \mu(s) - \rho(s)$. Plugging (4.12) into (3.5) we obtain

$$J(\pi^*) = \mathbb{E}\left[\int_0^T \left\{\nu(s)\left(\frac{\mathbb{E}[\nu(s) \,|\, \mathcal{H}_s] + a(s)}{\mathbb{E}\big[\sigma^2(s) \,\big|\, \mathcal{H}_s\big]}\right) - \frac{\sigma^2(s)}{2}\left[\frac{\mathbb{E}[\nu(s) \,|\, \mathcal{H}_s] + a(s)}{\mathbb{E}\big[\sigma^2(s) \,\big|\, \mathcal{H}_s\big]}\right]^2\right\} ds + \int_0^T \sigma(s)\left[\frac{\mathbb{E}[\nu(s) \,|\, \mathcal{H}_s] + a(s)}{\mathbb{E}\big[\sigma^2(s) \,\big|\, \mathcal{H}_s\big]}\right] d^- B(s)\right].$$

Now we use that

$$\mathbb{E}[\nu(s)\mathbb{E}[\nu(s) \,|\, \mathcal{H}_s]] = \mathbb{E}\big[\mathbb{E}[\nu(s) \,|\, \mathcal{H}_s]^2\big],$$

and $a(s)$ is $\mathcal{H}_s$-measurable $0 \le s \le T$, so that

$$\mathbb{E}\left[\nu(s)a(s)\right] = \mathbb{E}\left[\nu(s)\mathbb{E}[a(s) \,|\, \mathcal{H}_s]\right] = \mathbb{E}\left[\mathbb{E}[\nu(s) \,|\, \mathcal{H}_s]a(s)\right].$$

Moreover,

$$\mathbb{E}\left[\frac{\sigma^2(s)}{\mathbb{E}\big[\sigma^2(s) \,\big|\, \mathcal{H}_s\big]}\right] = 1$$

and by Lemma 2.2

$$\mathbb{E}\left[\int_0^T \sigma(s)\frac{\mathbb{E}[\nu(s) \,|\, \mathcal{H}_s] + a(s)}{E\big[\sigma(s)^2 \,\big|\, \mathcal{H}_s\big]}\, d^- B(s)\right] = \mathbb{E}\left[\int_0^T D_{s^+}\left(\sigma(s)\frac{\mathbb{E}[\nu(s) \,|\, \mathcal{H}_s] + a(s)}{E\big[\sigma(s)^2 \,\big|\, \mathcal{H}_s\big]}\right) ds\right].$$

The conclusion follows. □

REMARK 4.5. Note that the performance $\pi \mapsto J(\pi)$ given in (3.5) is strictly concave. Consequently, if $a(t)$ exists, the candidate π^* given by (4.9) is indeed an optimal control if $J(\pi^*)$ is finite. If $J(\pi^*)$ is infinite, then the optimal control problem has no solution.

5. AN EXTENSION OF THE PARTIAL INFORMATION FRAMEWORK

In this section we consider a generalization of the partial observation control problem that includes most known cases of utility maximization for markets with insiders where enlargement of filtration techniques are used.

EXAMPLE 5.1 [Partial observation case]. Suppose $\mathcal{H}_t \subseteq \mathcal{F}_t$ and $\mathcal{F}_t = \mathcal{G}_t$. Then, we have

$$\frac{d}{ds}\mathbb{E}\left[\int_0^s \sigma(u)\,dB(u)\,|\,\mathcal{H}_t\right] = 0, \quad s > t.$$

That is, $a(t) = 0$ and the optimal portfolio π^* is thus given by

$$\pi^*(t) = \frac{\mathbb{E}[\mu(t) - \rho(t)\,|\,\mathcal{H}_t]}{\mathbb{E}\big[\sigma^2(t)\,|\,\mathcal{H}_t\big]},$$

if the right-hand side is well defined as an element in $\mathcal{A}_{\mathcal{H}}$. Furthermore, the optimal utility is

$$J(\pi^*) = \frac{1}{2}\mathbb{E}\left[\int_0^T \frac{\mathbb{E}[\mu(s) - \rho(s)\,|\,\mathcal{H}_s]^2}{\mathbb{E}\big[\sigma^2(s)\,|\,\mathcal{H}_s\big]}\,ds\right].$$

This result follows directly from Theorem 4.1 (iii). One set of conditions that assures that $\pi^* \in \mathcal{A}_{\mathcal{H}}$ is that μ and ρ are uniformly bounded and $|\sigma(t)| \geq c > 0$ for all (t, ω). Similar existence conditions can also be found for the following examples.

REMARK 5.2. Note that the uniform ellipticity condition $\sigma(t) \geq c > 0$ guarantees the existence of an equivalent martingale measure that precludes the existence of an arbitrage in this case.

We consider now a more general situation:

PROPOSITION 5.3 [Partial observation in an anticipative market]. *Suppose* $\mathcal{H}_t \subseteq \mathcal{F}_t \subseteq \mathcal{G}_t$. *Moreover, suppose that* σ *satisfies the conditions of Lemma 2.4. Then,*

$$\pi^*(t) = \frac{\mathbb{E}[\mu(t) - \rho(t) + D_{t^+}\sigma(t)\,|\,\mathcal{H}_t]}{\mathbb{E}\big[\sigma^2(t)\,\big|\,\mathcal{H}_t\big]}$$

provided that the right-hand side is a well-defined element of $\mathcal{A}_{\mathcal{H}}$. *Furthermore, if the conditions of Theorem 4.4 are satisfied then*

$$J(\pi^*) = \frac{1}{2}\mathbb{E}\left[\int_0^T \left\{\frac{\mathbb{E}[\mu(s) - \rho(s)\,|\,\mathcal{H}_s]^2}{\mathbb{E}\big[\sigma^2(s)\,|\,\mathcal{H}_s\big]} - \frac{1}{2}\frac{a(s)^2}{\mathbb{E}\big[\sigma^2(s)\big|\mathcal{H}_s\big]}\right\} ds\right]$$

where $a(s) = \mathbb{E}[D_{s^+}\sigma(s)\,|\,\mathcal{H}_s]$.

Proof. Let M be a smooth $\mathcal{H}_t$-measurable random variable. Then

$$\begin{aligned}
\mathbb{E}\left[M\int_t^{t+h} \sigma(s)d^- B(s)\right] &= \mathbb{E}\left[\int_t^{t+h} M\sigma(s)\,d^- B(s)\right] \\
&= \mathbb{E}\left[\int_t^{t+h} D_{s^+}(M\sigma(s))\,ds\right] \\
&= \mathbb{E}\left[\int_t^{t+h} MD_{s^+}\sigma(s)\,ds\right] \\
&= \mathbb{E}\left[M\int_t^{t+h} D_{s^+}\sigma(s)\,ds\right].
\end{aligned}$$

This proves that

$$\mathbb{E}\left[\int_t^{t+h} \sigma(s)\, d^- B(s) \mid \mathcal{H}_t\right] = \mathbb{E}\left[\int_t^{t+h} D_{s^+}\sigma(s)\, ds \mid \mathcal{H}_t\right].$$

Hence, by Lemma 2.4

$$a(t) \equiv \lim_{h\to 0^+} \frac{1}{h}\mathbb{E}\left[\int_t^{t+h} \sigma(s)\, d^- B(s) \mid \mathcal{H}_t\right] = \mathbb{E}[D_{t^+}\sigma(t) \mid \mathcal{H}_t].$$

We conclude by using Theorems 4.1 and 4.4. □

REMARK 5.4. If B is a $\mathcal{G}$-semimartingale and $\mathcal{H}_t \subset \mathcal{G}_t$, then it is clear by the Girsanov theorem that there is no arbitrage.

Next we want to show that Proposition 5.3, which generalizes the partial information framework also includes the case of financial markets with insiders modeled through enlargement of filtrations. To this purpose, let us first recall the classical setup for models of markets with insiders through enlargement of filtrations in a simple case.

Let $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(B(T))$ and $\mathcal{H}_t = \mathcal{G}_t$. Consider an insider who can influence the asset prices in the following way:

$$dS(t) = \left(\mu + \sigma \frac{B(T) - B(t)}{T - t}\right) S(t)\, dt + \sigma S(t)\, d\tilde{B}(t), \quad t \in [0, T'],\ T' < T$$

where $\tilde{B}(t) = B(t) - \int_0^t \frac{B(T)-B(t)}{T-t} dt$ is a $\mathcal{G}_t$-Brownian motion, μ and σ are constants, and $B(t)$ is a $\mathcal{F}_t$-Brownian motion.

Note that in this case $\mathcal{H}_t \not\subseteq \mathcal{F}_t$ and $\tilde{B}$ is not a $\mathcal{F}_t$-Brownian motion and thus it may seem that Proposition 5.3 cannot be applied here. Therefore, instead of continuing in this way, we now modify the above formulation in order that the enlargement of filtration approach fits this proposition.

Consider the following model:

$$dS(t) = \left(\mu + \sigma \frac{B(T) - B(t)}{T - t}\right) S(t)\, dt + \sigma S(t)\, d\tilde{B}(t), \tag{5.1}$$

where now $\mathcal{F}_t := \mathcal{F}_t^B \vee \sigma(B(T))$, $\mathcal{F}^B$ stands for the filtration generated by the Brownian motion B and $\tilde{B}(t)$ is an $\mathcal{F}_t$-Brownian motion. Furthermore, let $\mathcal{G}_t = \mathcal{F}_t^B \vee \sigma(B(T))$. We consider two examples:

EXAMPLE 5.5 [The insider strategy]. Let $\mathcal{H}_t = \mathcal{F}_t^B \vee \sigma(B(T))$ and consider model (5.1). We are in the case of Example 5.1 with $\mathcal{F}_t = \mathcal{G}_t = \mathcal{H}_t$. We have

$$a(t) = \lim_{h\to 0^+} \frac{1}{h}\sigma\mathbb{E}[\tilde{B}_{t+h} - \tilde{B}(t) \mid \mathcal{H}_t] = 0.$$

The optimal policy for the insider is

$$\pi^*(t) = \frac{1}{\sigma^2}\left(\mu - \rho(t) + \sigma \frac{B(T) - B(t)}{T - t}\right)$$

and the optimal utility is

$$\begin{aligned}\mathbb{E}\ln(X^{\pi^*}(T')) - \ln x &= \frac{1}{2\sigma^2}\mathbb{E}\int_0^{T'} \left(\mu - \rho(t) + \sigma \frac{B(T) - B(t)}{T - t}\right)^2 dt \\ &\sim \ln\sqrt{\frac{1}{T - T'}} \ \text{ when } T' \to T.\end{aligned}$$

Consequently, the optimal utility is infinite:

$$\lim_{T'\to T} \mathbb{E} \ln(X^{\pi^*}(T')) = \infty.$$

This is the well-known result of Karatzas–Pikovsky (1996). The case $\mathcal{H}_t \subset \mathcal{G}_t$ can be considered similarly.

EXAMPLE 5.6 [The small investor strategy]. Let $\mathcal{H}_t = \mathcal{F}_t^B$ and consider model (5.1). Then

$$a(t) = \lim_{h\to 0^+} \frac{1}{h}\sigma \mathbb{E}\big[\tilde{B}_{t+h} - \tilde{B}(t) \,\big|\, \mathcal{F}_t^B\big] = 0.$$

Consequently, if $\rho(t) \equiv \rho$, then

$$\pi^*(t) = \frac{\mu - \rho}{\sigma^2}$$

and the optimal utility is

$$J(\pi^*) = \frac{(\mu - \rho)^2 T}{2\sigma^2} \quad \text{(Merton problem)}.$$

One can generalize model (5.1) as follows.

COROLLARY 5.7. *Let S be described as the unique solution of*

$$dS(t) = (\mu + X(t))S(t)\,dt + \sigma S(t)\,d^- B(t),$$

where $(X(t), t \geq 0)$ is a $\mathcal{F}_T$-measurable process and $B(t)$ is a $\mathcal{F}_t$-Brownian motion. Suppose $\mathcal{H}_t \subset \mathcal{F}_t$. Then $a(t) = 0$, and the optimal portfolio is

$$\pi^*(t) = \frac{\mathbb{E}[\mu + X(t) - \rho(t) \,|\, \mathcal{H}_t]}{\sigma^2}. \tag{5.2}$$

provided it is an element of $\mathcal{A}_{\mathcal{H}}$.

An extension of this model is studied in Section 6.4.

A further generalization to any enlargement of filtration is the following.

PROPOSITION 5.8. *Consider the following model:*

$$dS(t) = \mu(t)S(t)\,dt + \sigma S(t)\,d^- B(t)$$

where σ is constant, $\mu(t)$ is $\mathcal{G}_t$-adapted, $B(t)$ is a $\mathcal{F}_t$-Brownian motion, $\mathcal{F}_t \subseteq \mathcal{G}_t$ for all t, and $\{\mathcal{H}_t\}_{t\geq 0}$ is a general filtration. If $B(t) = \tilde{B}(t) + \int_0^t \beta(s)ds$ where $\tilde{B}(t)$ is a $\mathcal{H}_t$-Brownian motion and β is an $\mathcal{H}$-adapted càdlàg process with $\int_0^T |\beta(s)|ds < \infty$, then $a(t)$ defined in (4.10) exists and we have $\frac{a(t)}{\sigma} = \beta(t)$.

6. THE CASE OF PORTFOLIOS ADAPTED TO THE FILTRATION OF THE PRICE PROCESS

In this section we consider examples where $\mathcal{H}_t \nsubseteq \mathcal{F}_t$ and thus do not fit the framework of Proposition 5.3. We consider a small investor acting in a market influenced by an insider. We suppose that this investor can observe neither the Brownian motion B nor the drift μ, but only the stock price process S given by (3.2), that is, his portfolio is adapted to

$$\mathcal{H}_t = \sigma(S(s), 0 \le s \le t), \tag{6.1}$$

the filtration generated by the price process S. The quadratic variation process of S is given by (see Russo and Vallois 2000)

$$<S, S>_t = \int_0^t \sigma(s)^2 S(s)^2\, ds, \quad 0 \le t \le T.$$

It follows that the process $(\sigma(t), 0 \le t \le T)$ is $\mathcal{H}_t$-adapted and $\mathbb{E}[\sigma^2(t) \mid \mathcal{H}_t] = \sigma^2(t)$. The optimal portfolio if it exists, must then satisfy (see Corollary 4.2)

$$\pi^*(t)\sigma^2(t) = \mathbb{E}[(\mu(t) - \rho(t)) \mid \mathcal{H}_t] + \lim_{h\to 0} \frac{1}{h}\mathbb{E}\left[\int_t^{t+h} \sigma(s)\, d^- B(s) \mid \mathcal{H}_t\right]. \tag{6.2}$$

6.1. The Arbitrage Issue

THEOREM 6.1. *Suppose that $\rho(t) \in \mathcal{H}_t$ for all t and $\mathcal{H}_t$ is given by (6.1). Suppose that there exists an optimal portfolio π^* in $\mathcal{A}_{\mathcal{H}}$ leading to a finite utility. Then there exists an equivalent martingale measure in this anticipative market and therefore there is no arbitrage.*

Proof. Note first that

$$\mathcal{H}_t = \sigma(S(s), 0 \le s \le t) = \sigma\left(\int_0^s \left(\mu(u) - \frac{1}{2}\sigma^2(u)\right) du + \int_0^s \sigma(u)\, d^- B(u), s \le t\right).$$

Now we compute

$$\begin{aligned}
&\mathbb{E}\left(\int_s^t \left(\mu(u) - \frac{1}{2}\sigma^2(u)\right) du + \int_s^t \sigma(u)\, d^- B(u) \mid \mathcal{H}_s\right) \\
&\quad = \mathbb{E}\left(\int_s^t (\mu(u) - \rho(u) - \sigma^2(u)\pi^*(u))\, du \mid \mathcal{H}_s\right) \\
&\qquad + \mathbb{E}\left(\int_s^t \rho(u) + \sigma^2(u)\left(\pi^*(u) - \frac{1}{2}\right) du \mid \mathcal{H}_s\right) + \mathbb{E}\left(\int_s^t \sigma(u)\, d^- B(u) \mid \mathcal{H}_s\right) \\
&\quad = 0 + \mathbb{E}\left(\int_s^t \rho(u) + \sigma^2(u)\left(\pi^*(u) - \frac{1}{2}\right) du \mid \mathcal{H}_s\right)
\end{aligned}$$

by using (6.2). Using $\int_s^t (\mu(u) - \frac{1}{2}\sigma^2(u))\, du + \int_s^t \sigma(u)\, d^- B(u)$ that is $\mathcal{H}_t$-adapted, we have that

$$\begin{aligned}
N_{\pi^*}(t) := \int_0^t \left(\mu(u) - \frac{1}{2}\sigma^2(u)\right) du - \int_0^t \mathbb{E}\left(\rho(u) + \sigma^2(u)\left(\pi^*(u) - \frac{1}{2}\right) \,\middle|\, \mathcal{H}_u\right) du \\
+ \int_0^t \sigma(u)\, d^- B(u)
\end{aligned} \tag{6.3}$$

is an $\mathcal{H}$-martingale. Actually, $N_{\pi^*}(t) \equiv M_{\pi^*}(t)$ as defined in (4.1).

Furthermore, the quadratic variation of N_{π^*} is (see Russo and Vallois 2002)

$$<N_{\pi^*}, N_{\pi^*}>_t = \int_0^t \sigma(s)^2\, ds$$

and the process $(\sigma(t), t \geq 0)$ is $\mathcal{H}_t$-adapted. Consequently, there exists an $\mathcal{H}$-Brownian motion, say $\hat{B}_{\pi^*}$ such that

$$N_{\pi^*}(t) = \int_0^t \sigma(u)\, d\hat{B}_{\pi^*}(u).$$

For any $\mathcal{H}$-adapted portfolio π, the wealth equation can thus be rewritten as

$$\begin{aligned} X^{(\pi)}(t) &= x \exp\left\{ \int_0^t \left(((\mu(s) - \rho(s))\pi(s) - \frac{1}{2}\pi^2(s)\sigma^2(s) \right) ds + \int_0^t \pi(s)\sigma(s)\, d\hat{B}_{\pi^*}(t) \right. \\ &\quad \left. - \int_0^t \left(\mu(s) - \frac{1}{2}\sigma^2(s) \right) \pi(s)\, ds + \int_0^t \pi(s) \left(\rho(s) + \sigma^2(s) \left(\pi^*(s) - \frac{1}{2} \right) \right) du \right\} \\ &= x \exp\left\{ \int_0^t \frac{1}{2}\sigma^2(s)\pi(s)(-\pi(s) + 2\pi^*(s))\, ds + \int_0^t \pi(s)\sigma(s)\, d\hat{B}_{\pi^*}(t)(s) \right\}, \end{aligned}$$

where $\frac{1}{2}\sigma^2(s)\pi(s)(-\pi(s) + 2\pi^*(s))$ is $\mathcal{H}_s$-adapted. Therefore the usual Girsanov theorem in $\mathcal{H}$ implies that there is no arbitrage in this market. □

We can express $a(t)$ in terms of π^* by using (6.3),

$$\begin{aligned} a(t) &= \lim_{h \to 0} \frac{1}{h} \mathbb{E}\left[\int_t^{t+h} \sigma(s)\, d^- B(s) \mid \mathcal{H}_t \right] \\ &= \lim_{h \to 0} \frac{1}{h} \mathbb{E}\left[N_{\pi^*}(t+h) - N_{\pi^*}(t) - \int_t^{t+h} \left(\mu(u) - \frac{1}{2}\sigma^2(u) \right) du \right. \\ &\quad \left. + \int_t^{t+h} \mathbb{E}\left(\rho(u) + \sigma^2(u) \left(\pi^*(u) - \frac{1}{2} \right) \Big| \mathcal{H}_u \right) du \mid \mathcal{H}_t \right] \\ &= \mathbb{E}[-\mu(t) + \rho(t) + \sigma^2(t)\pi^*(t) \mid \mathcal{H}_t]. \end{aligned}$$

We have

$$\mathbb{E}\left[\int_0^t \sigma(s)\, d^- B(s) \mid \mathcal{H}_t \right] - \int_0^t a(s)\, ds = \int_0^t \sigma(s)\, d\hat{B}_{\pi^*}(s),$$

so that $a(t)$ can be interpreted as the projection on $\mathcal{H}_t$ of the compensator of $\int_0^t \sigma(s) d^- B(s)$.

REMARK 6.2. The fact that there exists an equivalent martingale measure in this model does not lead easily to the optimal portfolio π^* since this measure depends itself on π^*.

REMARK 6.3. There is no arbitrage when $\mathcal{H}$ is any subfiltration of $\sigma(S(s), 0 \leq s \leq t)$, but there is arbitrage if $\mathcal{H}_t = \sigma(S(s + \delta), 0 \leq s \leq t)$ with $\delta > 0$.

6.2. The Case When Prices are Affected by Large Investors

We consider a market with an insider who is also a large investor.

To motivate what follows, we first consider the case where the portfolio choices of the large investor affect the instantaneous expected returns on the traded assets (see Cuoco and Cvitanić 1998). We denote by $\mathcal{G}$ the filtration modeling the large investor's information. We suppose that $\rho(t) = \rho$ and the prices of the underlying risky asset is modeled by

$$dS(t) = (\mu + b\pi(t))\, S(t)\, dt + \sigma\, S(t)\, d^- B(t) \tag{6.4}$$

where π represents the strategy of the large investor, which is adapted to $\mathcal{G}$ and $0 < b < \sigma^2/2$. Suppose that B is a $\mathcal{G}$-semimartingale with the decomposition $B(t) = \hat{B}(t) + \int_0^t \alpha(s)ds$ where α is a $\mathcal{G}$-adapted process and $\hat{B}$ is a $\mathcal{G}$-Brownian motion. Here anticipative calculus is not necessary and direct computations lead to the optimal portfolio for the insider:

$$\hat{\pi} = \frac{\mu - \rho + \sigma\alpha(s)}{\sigma^2 - 2b}.$$

Consider a small investor who has only access to the filtration $\mathcal{H}_t := \sigma(S(s); s \leq t)$ and models the price process as

$$dS(t) = \mathbb{E}(\mu + b\hat{\pi}(t) \,|\, \mathcal{H}_t) S(t)\, dt + \sigma\, S(t)\, d\tilde{B}(t),$$

where $\tilde{B}$ is an $\mathcal{H}$-Brownian motion. His optimal portfolio is $\frac{\mu-\rho}{\sigma^2} + \frac{b}{\sigma^2}\mathbb{E}\,(\hat{\pi}(s) \,|\, \mathcal{H}_s)$.

In the particular case when $\mathcal{G}_\tau = \mathcal{F}_t \vee \sigma(B(T))$ then $\alpha(t) = \frac{B(T)-B(t)}{T-t}$. This model gives us a hint of how to introduce anticipations due to insiders. It suggests use of an anticipative drift in the dynamics of the price process.

6.3. The Particular Case: $\mu(t) = \mu + bB(T)$

We consider the case when the dynamics of the prices are given by

$$dS(t) = S(t)(\mu + b\, B(T))\, dt + \sigma\, S(t) d^- B(t), \tag{6.5}$$

where μ and b are real numbers, $\sigma > 0$. We suppose, moreover, that $\rho(t) = \rho =$ constant. The interpretation of this model when $b \geq 0$ is that the insider introduces a higher appreciation rate in the stock price if $B(T) > 0$. Given the linearity of the equation of S this indicates that the higher the final stock price the bigger the value of the drift of the equation driving S. Some cases of negative values for b can also be studied but the practical interpretation of such a study is dubious.

Although this model may be studied by using the enlargement of filtration techniques, we use here the approach we developped in Section 4 in order to provide an interpretation of $a(t)$.

LEMMA 6.4. *Suppose that $S(t)$ satisfies (6.5) and $\mathcal{H}_t$ is given by (6.1). Then the quantity $a(t)$ defined in (4.10) is explicitly given by*

$$a(t) \equiv \lim_{h \to 0^+} \frac{1}{h} \mathbb{E}[\sigma(B(t+h) - B(t)) \,|\, \mathcal{H}_t] = \frac{\sigma b(bB(T)t + \sigma B(t))}{(b^2 T + 2b\sigma)t + \sigma^2}. \tag{6.6}$$

Proof. Integrating equation (6.5), we obtain

$$S(t) = S_0 \exp\left(\mu t + bt\, B(T) - \frac{1}{2}\sigma^2 t + \sigma\, B(t)\right).$$

Consequently,

$$\begin{aligned}\mathcal{H}_t &= \sigma\left(\mu s - \frac{1}{2}\sigma^2 s + bsB(T) + \sigma\, B(s), 0 \leq s \leq t\right) \\ &= \sigma(bsB(T) + \sigma\, B(s), 0 \leq s \leq t)\end{aligned}$$

and

$$\sigma\mathbb{E}[B(t+h) - B(t)|\mathcal{H}_t] = \sigma\mathbb{E}[B(t+h) - B(t)|bs\, B(T) + \sigma\, B(s), 0 \leq s \leq t)$$

Consider the following partition:

$$0 = s_0 < s_1 < \cdots < s_n = t \quad \text{with time interval } \Delta = s_{i+1} - s_i.$$

and denote $\mathcal{H}_t^n$ as the σ-algebra generated by $\{bs_i B(T) + \sigma B(s_i), i = 0 \dots n\}$.

Since $(bs_i B(T) + \sigma B(s_i), i = 0 \dots n)$ is a Gaussian vector, the conditional expectation can be expressed as

$$\begin{aligned}
&\sigma \mathbb{E}[B(t+h) - B(t) \,|\, bs_i B(T) + \sigma B(s_i), i = 0, \dots n] \\
&\quad = \sum_{i=0}^{n-1} \alpha_i (bB(T)(s_{i+1} - s_i) + \sigma(B(s_{i+1}) - B(s_i))),
\end{aligned}$$

where the constant coefficients α_i have to be determined by using the correlations of each term with $bB(T)(s_{j+1} - s_j) + \sigma(B(s_{j+1}) - B(s_j))$. Doing this calculation, one gets

$$\sigma bh\Delta = \sum_{i=0, i\neq j}^{n-1} \alpha_i (b^2 T\Delta^2 + 2b\sigma\Delta^2) + \alpha_j (b^2 T\Delta^2 + 2b\sigma\Delta^2 + \sigma^2\Delta).$$

In matrix form this gives

$$\sigma bh \mathbf{1}_{n\times 1} = \big((b^2 T + 2b\sigma)\Delta \mathbf{1}_{n\times n} + \sigma^2 I_{n\times n}\big)\alpha,$$

where $\mathbf{1}_{a\times b}$ denotes the matrix of order $a \times b$ with all entries equal to 1, $I_{a\times a}$ denotes the identity matrix of order $a \times a$, and $\alpha = (\alpha_0, \dots, \alpha_{n-1})^T$. By linear combinations of these equations we get

$$\begin{aligned}
\alpha_0 &= \alpha_1 = \cdots \alpha_{n-1} \equiv \alpha \\
\sigma bh &= \alpha(b^2 T + 2b\sigma)\Delta(n-1) + \alpha(b^2 T\Delta + 2b\sigma\Delta + \sigma^2),
\end{aligned}$$

which gives

$$\alpha = \frac{\sigma bh}{(b^2 T + 2b\sigma)\Delta n + \sigma^2}.$$

We thus get

$$\sigma \mathbb{E}\big[B(t+h) - B(t) \,\big|\, \mathcal{H}_t^n\big] = \frac{\sigma bh}{(b^2 T + 2b\sigma)\Delta n + \sigma^2}(bB(T)n\Delta + \sigma B(t)).$$

Since $n\Delta = t$ the above expression is independent of n and

$$\sigma \mathbb{E}[B(t+h) - B(t) \,|\, \mathcal{H}_t] = \frac{\sigma bh}{(b^2 T + 2b\sigma)t + \sigma^2}(bB(T)t + \sigma B(t)). \tag{6.7}$$

Consequently,

$$a(t) = \lim_{h\to 0^+} \frac{1}{h} \sigma \mathbb{E}[B(t+h) - B(t) \,|\, \mathcal{H}_t] = \frac{\sigma b}{(b^2 T + 2b\sigma)t + \sigma^2}(bB(T)t + \sigma B(t)). \tag{6.8}$$

□

Note that this is an example where $a(t) \neq 0$. Furthermore, as $\mathcal{H}_t = \sigma(bsB(T) + \sigma B(s), 0 \leq s \leq t)$, the small investor cannot determine $B(T)$ out of the observed $B(T) + \sigma B(s)$, $s \leq t$. But as $s \to T$ the knowledge of the small investor about B_T improves. This is in the spirit of a continuous enlargement of filtration setting introduced in Corcuera et al. (2004).

LEMMA 6.5.

$$\mathbb{E}[B(T)\,|\,\mathcal{H}_s] = \frac{(bT+\sigma)}{(b^2T+2b\sigma)s+\sigma^2}(bB(T)s+\sigma B(s)). \tag{6.9}$$

Proof. We proceed as before. Let $0 = s_0 < s_1 < \cdots < s_n = t$ and $\Delta = s_{i+1} - s_i$.

$$\begin{aligned}\mathbb{E}\big(B(T)\,\big|\,\mathcal{H}_t^n\big) &= \mathbb{E}(B(T)\,|\,bs_i B(T) + \sigma B(s_i), 0 \le i \le n)\\ &= \sum_{i=0}^{n-1}\alpha_i(bB(T)\Delta + \sigma(B(s_{i+1}) - B(s_i))).\end{aligned}$$

By computing the correlation with $bB(T)\Delta + \sigma(B(s_{j+1}) - B(s_j))$ we get

$$bT\Delta + \sigma\Delta = \sum_{i=0, i\neq j}^{n-1}\alpha_i(b^2\Delta^2T + 2\sigma b\Delta^2) + \alpha_j(b^2\Delta^2T + 2\sigma b\Delta^2 + \sigma^2\Delta).$$

In matrix form this leads to

$$(bT+\sigma)\mathbf{1}_{n\times 1} = ((bT+2\sigma)b\Delta\mathbf{1}_{n\times n} + \sigma^2 I_{n\times n})\alpha.$$

As before, this gives

$$\begin{aligned}\alpha_0 &= \alpha_1 = \cdots \alpha_{n-1} \equiv \alpha\\ \alpha &= \frac{bT+\sigma}{(b^2T+2b\sigma)t+\sigma^2},\end{aligned}$$

which implies (6.9). □

THEOREM 6.6. *Suppose that $S(t)$ is given by (6.5) with $b \ge 0$ and $\mathcal{H}_t$ is given by (6.1). Then*

(i) The optimal portfolio for Problem (3.6) exists and is given by

$$\pi^*(t) = \frac{\mathbb{E}[\mu(t)\,|\,\mathcal{H}_t] - \rho}{\sigma^2} + \frac{b(bB(T)t + \sigma B(t))}{\sigma((b^2T+2b\sigma)t+\sigma^2)} \tag{6.10}$$

which can be rewritten as

$$\pi^*(t) = \frac{\mu-\rho}{\sigma^2} + \frac{b(bB(T)t+\sigma B(t))(bT+\sigma+\sigma^{-1})}{\sigma^2((b^2T+2b\sigma)t+\sigma^2)}.$$

(ii) The optimal utility is finite and is given by

$$J(\pi^*) = \frac{(\mu-\rho)^2T}{2\sigma^2} + \frac{1}{2\gamma}\left(1 - \gamma\ln\left(1+\frac{1}{\gamma}\right)\right), \tag{6.11}$$

where we have set

$$\gamma \equiv \frac{\sigma^2}{bT(bT+2\sigma)}. \tag{6.12}$$

REMARK 6.7. If $\rho(t)$ is not constant, then the optimal portfolio and utility are, respectively, given by

$$\pi^*(t) = \frac{\mu - \mathbb{E}[\rho(t)\,|\,\mathcal{H}_t]}{\sigma^2} + \frac{b(bB(T)t+\sigma B(t))(bT+\sigma+\sigma^{-1})}{\sigma^2((b^2T+2b\sigma)t+\sigma^2)},$$

$$J(\pi^*) = \frac{1}{2\sigma^2}\mathbb{E}\left[\int_0^T \mathbb{E}[\mu(s)-\rho(s)\,|\,\mathcal{H}_s]^2\,ds\right] - \frac{1}{\sigma}\int_0^T \mathbb{E}\left[D_{s^+}\mathbb{E}[\rho(s)\,|\,\mathcal{H}_s]\right]ds$$
$$+ \left(\frac{bT}{\sigma}+\frac{3}{2}\right)\left(\frac{(bT)^2\gamma}{\sigma^2}\right)\left(1-\gamma\ln\left(1+\frac{1}{\gamma}\right)\right),$$

provided sufficient hypotheses are assumed on $\rho(t)$ in order that $\pi^* \in \mathcal{A}_{\mathcal{H}}$ and the conditions of Theorem 4.4 are satisfied.

Proof. The expression (6.10) is obtained using (6.8) and Corollary 4.2. To check that the candidate π^* given by (6.10) is indeed an optimal portfolio, we have to prove that $M_{\pi^*}(t)$ is an $\mathcal{H}$-martingale. One can verify easily that $\pi^* \in \mathcal{A}_{\mathcal{H}}$. Plugging (6.10) into (4.1), we get

$$\begin{aligned}
M_{\pi^*}(t) &= \mathbb{E}\left[\int_0^t (\mu(s)-\rho(s)-\mathbb{E}(\mu(s)-\rho(s)\,|\,\mathcal{H}_s))\,ds\,|\,\mathcal{H}_t\right]\\
&\quad - \mathbb{E}\left[\int_0^t \sigma b(bB(T)s+\sigma B(s))((b^2T+2b\sigma)s+\sigma^2)^{-1}ds\,|\,\mathcal{H}_t\right] + \sigma\mathbb{E}[B(t)\,|\,\mathcal{H}_t]\\
&\equiv M^1_{\pi^*}(t) + M^2_{\pi^*}(t) + M^3_{\pi^*}(t).
\end{aligned}$$

Let $u < t$. We want to prove

$$\mathbb{E}[M_{\pi^*}(t) - M_{\pi^*}(u)\,|\,\mathcal{H}_u] = 0.$$

First, we show that $M^1_{\pi^*}$ satisfies the martingale property. For $u < t$,

$$\begin{aligned}
\mathbb{E}\big[M^1_{\pi^*}(t) - M^1_{\pi^*}(u)\,|\,\mathcal{H}_u\big] &= \mathbb{E}\left[\mathbb{E}\left[\int_0^t (\mu(s)-\rho(s)-\mathbb{E}(\mu(s)-\rho(s)\,|\,\mathcal{H}_s))\,ds\,|\,\mathcal{H}_t\right]\middle|\mathcal{H}_u\right]\\
&\quad - \mathbb{E}\left[\int_0^u (\mu(s)-\rho(s)-\mathbb{E}(\mu(s)-\rho(s)\,|\,\mathcal{H}_s))\,ds\,|\,\mathcal{H}_u\right]\\
&= \mathbb{E}\left[\mathbb{E}\left[\int_0^u (\mu(s)-\rho(s)-\mathbb{E}(\mu(s)-\rho(s)\,|\,\mathcal{H}_s))\,ds\,|\,\mathcal{H}_t\right]\middle|\mathcal{H}_u\right]\\
&\quad + \int_u^t (\mathbb{E}[\mu(s)-\rho(s)\,|\,\mathcal{H}_u] - \mathbb{E}[\mu(s)-\rho(s)\,|\,\mathcal{H}_u])\,ds\\
&\quad - \int_0^u (\mathbb{E}[\mu(s)-\rho(s)\,|\,\mathcal{H}_u] - \mathbb{E}[\mu(s)-\rho(s)\,|\,\mathcal{H}_u])\,ds\\
&= 0.
\end{aligned}$$

Next, we prove that $M^2_{\pi^*} + M^3_{\pi^*}$ is an $\mathcal{H}$-martingale. We have using Lemmas 6.4 and 6.5,

$$\begin{aligned}
&-\mathbb{E}\big[M^2_{\pi^*}(t) - M^2_{\pi^*}(u)\,\big|\,\mathcal{H}_u\big]\\
&\quad= \mathbb{E}\left[\mathbb{E}\left[\int_u^t \sigma b(bB(T)s+\sigma B(s))((b^2T+2b\sigma)s+\sigma^2)^{-1}\,ds\,|\,\mathcal{H}_t\right]\middle|\mathcal{H}_u\right]\\
&\quad= \mathbb{E}\left[\int_0^u \sigma b(bB(T)s+\sigma B(s))((b^2T+2b\sigma)s+\sigma^2)^{-1}\,ds\,|\,\mathcal{H}_u\right]
\end{aligned}$$

and

$$\mathbb{E}\big[M^3_{\pi^*}(t) - M^3_{\pi^*}(u)\,\big|\,\mathcal{H}_u\big] = \sigma\mathbb{E}[\mathbb{E}(B(t)\,|\,\mathcal{H}_t) - B(u)\,|\,\mathcal{H}_u] = \sigma\mathbb{E}[B(t) - B(u)\,|\,\mathcal{H}_u].$$

Let $u = t_0 < t_1 < \cdots < t_n = t$ be a partition of $[u, t]$ with time interval $\Delta = s_{i+1} - s_i$. We have

(6.13)

$$\begin{aligned}\sigma\mathbb{E}[B(t) - B(u)\,|\,\mathcal{H}_u] &= \sigma\mathbb{E}\left[\sum_{i=0}^{n-1}(B(t_{i+1}) - B(t_i))\,\middle|\,\mathcal{H}_u\right]\\ &= \sigma\sum_{i=0}^{n-1}\mathbb{E}[B(t_{i+1}) - B(t_i)\,|\,\mathcal{H}_u]\\ &= \sigma\sum_{i=0}^{n-1}\mathbb{E}\big[\mathbb{E}\big(B(t_{i+1}) - B(t_i)\,\big|\,\mathcal{H}_{t_i}\big)\,\big|\,\mathcal{H}_u\big]\\ &= \sigma\sum_{i=0}^{n-1}\mathbb{E}\big[b\Delta(bB(T)t_i + \sigma B(t_i))\big((b^2T + 2b\sigma)t_i + \sigma^2\big)^{-1}\,\big|\,\mathcal{H}_u\big]\end{aligned}$$

by using (6.9), and this last expression converges to

$$\mathbb{E}\left[\int_u^t \sigma b(bB(T)s + \sigma B(s))((b^2T + 2b\sigma)s + \sigma^2)^{-1}\,ds\,|\,\mathcal{H}_u\right]$$

when $n \to \infty$. Consequently

$$\mathbb{E}\big[M^2_{\pi^*}(t) + M^3_{\pi^*}(t) - M^2_{\pi^*}(u) - M^3_{\pi^*}(u)\,\big|\,\mathcal{H}_u\big] = 0$$

and M_{π^*} is an $\mathcal{H}$-martingale.

We compute now the value function. We use (4.11) together with equalities (6.6) and (6.9). We have

$$\begin{aligned}-\frac{1}{2\sigma^2}\mathbb{E}\int_0^T a(s)^2\,ds &= -\frac{b^2}{2}\int_0^T \frac{1}{((b^2T + 2\sigma b)s + \sigma^2)^2}\\ &\quad\times\mathbb{E}(b^2s^2B(T)^2 + \sigma^2B(s)^2 + 2b\sigma s\,B(s)B(T))ds\\ &= -\frac{b^2}{2}\int_0^T \frac{b^2s^2T + \sigma^2 s + 2b\sigma s^2}{((b^2T + 2\sigma b)s + \sigma^2)^2}\,ds\\ &= -\frac{b^2}{2}\int_0^T \frac{s}{(b^2T + 2b\sigma)s + \sigma^2}ds.\end{aligned}$$

Moreover, we have

$$D_{s^+}\mathbb{E}[B(T)\,|\,\mathcal{H}_s] = \frac{(bT + \sigma)bs}{(b^2T + 2b\sigma)s + \sigma^2}$$

$$D_{s^+}a(s) = \frac{\sigma b^2 s}{(b^2T + 2b\sigma)s + \sigma^2}$$

so that

$$D_{s^+}\left[\frac{\mathbb{E}[\mu(s)-\rho\,|\,\mathcal{H}_s]+a(s)}{\sigma}\right]=\frac{1}{\sigma}\left(\frac{b^2s(bT+\sigma)}{(b^2T+2\sigma b)s+\sigma^2}+\frac{\sigma b^2 s}{(b^2T+2\sigma b)s+\sigma^2}\right)$$
$$=\frac{b^2s(bT+2\sigma)}{\sigma((b^2T+2\sigma b)s+\sigma^2)}.$$

We thus get

$$J(\pi^*)=\frac{1}{2\sigma^2}\mathbb{E}\int_0^T\mathbb{E}[\mu(s)-\rho\,|\,\mathcal{H}_s]^2\,ds-\frac{b^2}{2}\int_0^T\frac{s}{(b^2T+2b\sigma)s+\sigma^2}\,ds$$
$$+\frac{b^2}{\sigma}(bT+2\sigma)\int_0^T\frac{s}{(b^2T+2b\sigma)s+\sigma^2}\,ds$$
$$=\frac{1}{2\sigma^2}\mathbb{E}\int_0^T\mathbb{E}[\mu(s)-\rho\,|\,\mathcal{H}_s]^2\,ds+b^2\left(\frac{bT}{\sigma}+\frac{3}{2}\right)\int_0^T\frac{s}{(b^2T+2b\sigma)s+\sigma^2}\,ds.$$

We now use that $b\geq 0$ and by integration we have

$$\int_0^T\frac{s}{(b^2T+2b\sigma)s+\sigma^2}ds=\frac{T}{b^2T+2b\sigma}\left(1-\frac{\sigma^2}{(b^2T+2b\sigma)T}\ln\left(1+\frac{b^2T+2b\sigma}{\sigma^2}T\right)\right),$$

which can also be written as

$$\frac{T^2\gamma}{\sigma^2}\left(1-\gamma\ln\left(1+\frac{1}{\gamma}\right)\right),$$

which is positive. Similarly, one computes $\frac{b^2}{2\sigma^2}\int_0^T\mathbb{E}[\mathbb{E}[B(T)\,|\,\mathcal{H}_s]^2]\,ds$. We thus get (6.11). □

REMARK 6.8.

(1) The coefficient $\frac{1}{\gamma}$ (see (6.12)) can be interpreted as the insider effect on the utility of the $\mathcal{H}$-investor. When $\gamma\to+\infty$ (which is implied by $b\to 0$, that is the insider effect vanishes) the utility of the $\mathcal{H}$-investor is closer to the optimal utility in the classical Merton problem. A similar interpretation can be applied for $\gamma\to 0$. From (6.11), we obtain

$$\lim_{b\to 0}J(\pi^*)=\frac{(\mu-\rho)^2T}{2\sigma^2}\text{ (Merton problem)}$$
$$\lim_{b\to\infty}J(\pi^*)=+\infty\text{ (Strong drift problem).}$$

(2) Consider an investor who estimates the appreciation rate of the prices by using the best linear estimate given by $\mathbb{E}[\mu(t)\,|\,\mathcal{H}_t]$ and builds his price model as

$$d\tilde{S}_t=\mathbb{E}[\mu(t)\,|\,\mathcal{H}_t]\tilde{S}_t\,dt+\sigma\tilde{S}_t\,d\tilde{B}_t,\tag{6.14}$$

where $\tilde{B}$ is an $\mathcal{H}$-Brownian motion. He faces the following optimization problem:

$$J_0(\pi_0^*)=\max_{\pi\in\mathcal{H}}J_0(\pi),$$

where

$$J_0(\pi)=\mathbb{E}(\ln(\tilde{X}(T)))$$

and

$$d\tilde{X}(t) = \tilde{X}(t)[(\mathbb{E}[\mu(t) \,|\, \mathcal{H}_t] - \rho)\pi(t)\,dt + \pi(t)\sigma d\tilde{B}(t)] \quad \tilde{X}(0) = x > 0.$$

The solution of this problem is similar to the "classical" Merton case. The optimal portfolio is

$$\pi_0^*(t) = \frac{\mathbb{E}[\mu(t) - \rho \,|\, \mathcal{H}_t]}{\sigma^2},$$

which is different from (6.10) and the optimal utility for this investor is

$$\begin{aligned} (6.15) \qquad J_0(\pi_0^*) &= \frac{1}{2\sigma^2}\int_0^T \mathbb{E}\big[\mathbb{E}[\mu(s) - \rho \,|\, \mathcal{H}_s]^2\big]\,ds \\ &= \frac{(\mu-\rho)^2 T}{2\sigma^2} + \frac{(bT+\sigma)^2}{2(bT+2\sigma)^2\gamma}\left(1 - \gamma\ln\left(1+\frac{1}{\gamma}\right)\right) \\ &< J(\pi^*) \text{ (given by (6.11) under the model (6.5)).} \end{aligned}$$

The utility generated by the portfolio π_0^* in the "real" model (6.5), $J(\pi_0^*)$, is different from $J_0(\pi_0^*)$ obtained in (6.15). The quantity $J(\pi_0^*) - J_0(\pi_0^*) = \frac{\sigma(bT+\sigma)}{(bT+2\sigma)^2\gamma}(1 - \gamma\ln(1+\frac{1}{\gamma}))$ represents the difference between the actual earnings of the policy π_0^* under model (6.5) and the earnings expected by the small investor using model (6.14). Notably this quantity is positive.

Moreover, $J(\pi^*) - J(\pi_0^*) = \frac{\sigma^2}{2(bT+2\sigma)^2\gamma}(1 - \gamma\ln(1+\frac{1}{\gamma}))$ represents the difference between the optimal earnings if the small investor uses π^* acknowledging an anticipating model (6.11) and the actual earnings of the small investor that use portfolio π_0^* taking (6.14) as model for the underlying prices. This difference comes from considering $a \equiv 0$ or not in Theorem 4.4. The difference in utility is obviously positive due to the optimal property of the portfolio with $a(t) \neq 0$.

6.4. A More General Case: $\mu(t) = \mu + bX, \quad X \in \mathcal{F}_T$

We consider a generalization of the previous section to the case when $\mu(t) = \mu + bX$, where X is a general smooth $\mathcal{F}_T$-measurable random variable. The dynamics of the prices is

$$dS(t) = S(t)(\mu + bX)\,dt + \sigma S(t)\,d^- B(t),$$

where μ and b are real numbers, $\sigma > 0$. The goal here is just to show that $a(t)$ exists in other situations provided $J(\pi^*)$ is finite. We will not write down here the long and tedious expressions for the optimal portfolio and optimal utility.

LEMMA 6.9. *The quantity $a(t)$ defined in (4.10) is given by*

$$a(t) \equiv \lim_{h\to 0^+}\frac{1}{h}\mathbb{E}[\sigma(B(t+h) - B(t)) \,|\, \mathcal{H}_t] = \sigma\mathbb{E}\left[\int_t^T \frac{D_v X D_t X}{\int_t^T (D_r X)^2\,dr}\delta B(v) \,|\, \mathcal{H}_t\right],$$

if the right-hand side above is well defined and right-continuous in t.

Proof. Consider the following partition:

$$0 = s_0 < s_1 < \cdots < s_n = t \quad \text{with time interval } \Delta = s_{i+1} - s_i$$

and denote $\mathcal{H}_t^n$ as the σ-algebra generated by $\{bs_iX + \sigma B(s_i), i = 0 \dots n\}$. We have for a smooth bounded function f

$$\begin{aligned}&\mathbb{E}[B(t+h) - B(t) \,|\, bs_i X + \sigma B(s_i), i = 0, \dots n] \\ &\quad = \mathbb{E}[(B(t+h) - B(t)) f(bX(s_n - s_{n-1}) + \sigma(B(s_n) - B(s_{n-1})), \dots, bXs_1 + \sigma B(s_1))].\end{aligned}$$

Denote

$$Z = (bX(s_n - s_{n-1}) + \sigma(B(s_n) - B(s_{n-1})), \dots, bXs_1 + \sigma B(s_1)).$$

By the duality formula and the Fubini theorem, we can write

$$\mathbb{E}[(B(t+h) - B(t)) f(Z)] = \int_t^{t+h} \mathbb{E}\left[\sum_{i=1}^n \frac{\partial f}{\partial x_i}(Z) b(s_i - s_{i-1}) D_u X\right] du. \tag{6.16}$$

Now, we have for $\alpha_2 > \alpha_1 \geq t$

$$\int_{\alpha_1}^{\alpha_2} D_v X D_v f(Z)\, dv = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(Z) b(s_i - s_{i-1}) \int_{\alpha_1}^{\alpha_2} (D_v X)^2\, dv.$$

Multiplying both sides by $\frac{D_u X}{\int_{\alpha_1}^{\alpha_2} (D_u X)^2 du}$ and using the duality principle in the Malliavin calculus, we get

$$\mathbb{E} \int_{\alpha_1}^{\alpha_2} \frac{D_v X D_u X}{\int_{\alpha_1}^{\alpha_2} (D_v X)^2\, dv} D_v f(Z)\, dv = \mathbb{E}\left[f(Z) \int_{\alpha_1}^{\alpha_2} \frac{D_v X D_u X}{\int_{\alpha_1}^{\alpha_2} (D_v X)^2\, dv} \delta B(v)\right]. \tag{6.17}$$

Combining (6.16) and (6.17), we get

$$\mathbb{E}((B(t+h) - B(t)) f(Z)) = \int_t^{t+h} \mathbb{E}\left[f(Z) \mathbb{E}\left[\int_{\alpha_1}^{\alpha_2} \frac{D_v X D_u X}{\int_{\alpha_1}^{\alpha_2} (D_v X)^2\, dv} \delta B(v) \,\middle|\, \mathcal{H}_u \right]\right] du$$

since $f(Z)$ is $\mathcal{H}_t$-measurable. The process

$$\tilde{B}_t \equiv \mathbb{E}[B(t) \,|\, \mathcal{H}_t] - \int_0^t \mathbb{E}\left[\int_{\alpha_1}^{\alpha_2} \frac{D_v X D_u X}{\int_{\alpha_1}^{\alpha_2} (D_v X)^2\, dv} \delta B(v) \,\middle|\, \mathcal{H}_u \right] du$$

is an $\mathcal{H}$-martingale. We deduce under the continuity hypothesis that for any $t \leq \alpha_1 < \alpha_2 \leq T$

$$\lim_{h \to 0^+} \frac{1}{h} \mathbb{E}[B(t+h) - B(t) \,|\, \mathcal{H}_t] = \mathbb{E}\left[\int_{\alpha_1}^{\alpha_2} \frac{D_v X D_t X}{\int_{\alpha_1}^{\alpha_2} (D_v X)^2\, dv} \delta B(v) \,\middle|\, \mathcal{H}_t \right]. \tag{6.18}$$

We can take in the above formulas $\alpha_1 = t$ and $\alpha_2 = T$. $\square$

6.5. Continuous Stream of Information

Consider now a model where the insider has an effect on the drift through information that is δ units of time ahead:

$$S(t) = S(0) + \int_0^t (\mu + bB(s+\delta))\, S(s)\, ds + \int_0^t \sigma S(s)\, dB(s).$$

We assume for simplicity $\delta \geq T$ fixed. We are interested in computing the optimal policy of the small investor with filtration $\mathcal{H}_t = \sigma(S_s; s \leq t)$. We have

$$S(t) = S(0) \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right) t + b \int_\delta^{t+\delta} B(s)\, ds + \sigma B(t)\right)$$

and therefore $\mathcal{H}_t = \sigma(b \int_\delta^{s+\delta} B(r)\, dr + \sigma B(s); s \leq t)$.

THEOREM 6.10. *Define* $Y(t) = b \int_\delta^{t+\delta} B(r)\, dr + \sigma B(t)$. *Then, for* $\delta \geq T$

$$\frac{a(t)}{\sigma} = \lim_{h \to 0^+} \mathbb{E}\left(\frac{B(t+h) - B(t)}{h} \middle| \mathcal{H}_t\right) = bM \int_0^t g(t,u)\, dY(u), \tag{6.19}$$

$$\mathbb{E}(B(t+\delta)|\mathcal{H}_t) = (b(t+\delta) + \sigma) M \int_0^t g(t,u)\, dY(u),$$

where

$$M \equiv M_t = \sigma^{-1}\big((b\delta + 2\sigma)\big(e^{\frac{2bt}{\sigma}} - 1\big) + \sigma\big(e^{\frac{2bt}{\sigma}} + 1\big)\big)^{-1},$$
$$g(t,u) = e^{\frac{b}{\sigma}(2t-u)} + e^{\frac{b}{\sigma}u}.$$

Furthermore,

$$\pi^*(t) = \frac{\mu - \rho}{\sigma^2} + \frac{b}{\sigma^2} M(b(t+\delta) + 2\sigma) \int_0^t g(t,u)\, dY(u).$$

Proof. First note that Y is a Gaussian process. Therefore, $\mathbb{E}(B(s)\,|\,\mathcal{H}_t) = \int_0^t h(s,t,u)\, dY(u)$ where h is some deterministic function. To determine h, we compute the covariances between $B(s)$ and the stochastic integral and $Y(v)$ for some $v \leq t$. We have

$$\mathbb{E}(B(s) Y(v)) = bsv + \sigma(s \wedge v)$$

and

$$\mathbb{E}\left(\int_0^t h(s,t,u)\, dY(u) Y(v)\right) = b^2 \int_0^t \int_0^v h(s,t,\theta_1)(\theta_1 \wedge \theta_2 + \delta)\, d\theta_2 d\theta_1$$
$$+ 2b\sigma v \int_0^t h(s,t,\theta)\, d\theta + \sigma^2 \int_0^v h(s,t,\theta)\, d\theta.$$

The above two expressions have to be equal. Differentiating w.r.t. $v \leq t$ three times, we obtain

$$-b^2 h(s,t,u) + \sigma^2 \frac{\partial^2 h}{\partial u^2}(s,t,u) = 0$$

with the initial conditions $\frac{\partial h}{\partial u}(s, t, t) = 0$ and $bs + \sigma = b(b\delta + 2\sigma)\int_0^t h(s, t, \theta)d\theta + \sigma^2 h(s, t, 0)$. Solving this differential equation gives

$$h(s, t, u) = C_1(s, t)e^{-\frac{b}{\sigma}u} + C_2(s, t)e^{\frac{b}{\sigma}u},$$

with

$$C_2(s, t) = \sigma^{-1}(bs + \sigma)\big((b\delta + 2\sigma)(e^{\frac{2bt}{\sigma}} - 1) + \sigma(e^{\frac{2bt}{\sigma}} + 1)\big)^{-1}$$
$$C_1(s, t) = e^{\frac{2bt}{\sigma}} C_2(s, t).$$

Therefore, we have that

$$\mathbb{E}\left(\frac{B(s) - B(t)}{s - t}\middle|\mathcal{H}_t\right) = \int_0^t \frac{h(s, t, u) - h(t, t, u)}{s - t}\, dY(u)$$

and (6.19) holds. We deduce the expression of π^* after verifing that $\pi^* \in \mathcal{A}_{\mathcal{H}}$. □

The case $\delta \leq T$ can also be studied although explicit expressions are long to write. The case $\delta \leq \frac{T}{2}$ is especially interesting because it involves a “continuous” stream of information into the market, preserving still finite utility. This problem cannot in general be approached through enlargement of filtration techniques.

7. CONCLUDING REMARKS

In this paper we have studied markets where insiders are also large traders and therefore have an influence on the drift of the price dynamics. This leads naturally to the study of the optimization problems in an anticipative framework. We believe that this formalism goes beyond the classical formulation of markets with insiders using initial enlargement of the filtration approach.

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