Insider models with finite utility in markets with jumps *

Arturo Kohatsu-Higa[†]and Makoto Yamazato[‡]
Ritsumeikan University (Japan Science and Technology Agency) and University of the Ryukyus

April 5, 2011

Abstract

In this article we consider, under a Lévy process model for the stock price, the utility optimization problem for an insider agent whose additional information is the final price of the stock blurred with an additional independent noise which vanishes as the final time approaches. Our main interest is establishing conditions under which the utility of the insider is finite. Mathematically, the problem entails the study of a "progressive" enlargement of filtration with respect to random measures. We study the jump structure of the process which leads to the conclusion that in most cases the utility of the insider is finite and his optimal portfolio is bounded. This can be explained financially by the high risks involved in models with jumps.
Keywords: Asymmetric markets, markets driven by Lévy processes, enlargement of filtrations.

1 Introduction

The problem of asymmetric markets in continuous time mathematical finance has been considered since Karatzas-Pikovski [21]. They dealt with a financial market where the underlying follows a geometric Brownian motion model. An insider is an agent that has additional information and therefore his portfolio policies are adapted to a filtration which is bigger than the filtration generated by the Wiener process. The additional information is characterized through a random variable. If this random variable is the final price of the asset in some interval $[0, T]$ then the insider makes an infinite amount of money and there is arbitrage in the model. This situation has been corroborated in various other situations by various authors (see e.g. Imkeller [17], Amendiger et. al. [1], Imkeller et. al. [18], Grorud and Pontier [13], Grorud [12]). Furthermore the optimal portfolio of the insider is highly oscillating as it depends on the difference $\frac{W(T)-W(s)}{T-s}$, where W is a Brownian motion, T is the final time and s is the current time.

In the financial economics literature, equilibrium models with insiders have been considered by Back [3] which are mathematically different from the ones previously described. Notably these models lead to a market where lawful insiders obtain a finite optimal utility.

Corcuera et al. [5] introduced a framework to study the behavior of an insider for markets driven by a Wiener process where the information held by the insider changes dynamically through time. Indeed the signal (or information) that the insider receives is biased by the addition of an independent, time deformed Wiener process that disappears at time T. This incorporates an additional realistic feature to the model as information that lawful insiders have is usually not perfect and changes through time. In this case, the optimal portfolio strategy for the insider depends on

$$\frac{W_T - W_t + W'((T-t)^\alpha)}{T - t + (T-t)^\alpha},$$

*The authors acknowledge financial support from grants of the Japanese government and thank the referee for various comments that lead to improvements in the presentation.
†Corresponding author:Arturo Kohatsu-Higa Department of Mathematical Sciences Ritsumeikan University 1-1-1 Nojihigashi, Kusatsu, Shiga, 525-8577, Japan. e-mail: arturokohatsu@gmail.com
‡Makoto Yamazato. Department of Mathematics, Faculty of Science, University of the Ryukyus, Senbaru 1, Nishihara-cho, Okinawa 903-0213, Japan. e-mail: yamazato@math.u-ryukyu.ac.jp

where W' is a Wiener process independent of W. The authors proved that if the rate at which this additional independent Wiener process disappears is slow enough ($\alpha < 1$), then the market does not allow for arbitrage and the optimal utility of the insider is finite in $[0, T]$. Nevertheless one has the undesirable characteristic that $\limsup_{t \to T} \pi^*(t) = +\infty$ and $\liminf_{t \to T} \pi^*(t) = -\infty$. That is, the optimal portfolio for the insider oscillates from $-\infty$ to $+\infty$ due to the law of the iterated logarithm.

In this article, we consider a further extension of the setting considered in [5] that corrects this feature. In fact, the present article is an extension of this problem where models of the asset prices are given by Lévy processes and where additional information is given by the final value of the underlying in the time interval $[0, T]$. This interest is based on two reasons. First, to consider recent extensions of models with jumps for stock prices (e.g. variance gamma models, hyperbolic diffusions etc.). Second, to try to find models where the insider optimal portfolio is bounded a.s. and finally to show that these optimal portfolios lead to finite optimal utility without having to add any additional noise. The possibility of such a result can be understood from a financial point of view as follows. Stochastic models for stock prices with jumps incorporate higher risks than the geometric Brownian motion because of the existence of non-predictable jumps. Therefore, these jumps have an effect in the future evolution of the underlying and are not known by the insider even if he/she knows the final value of the price of the stock. If the size of such a jump is not known, then the only possibility for the insider is to take a non-risky position investing only a percentage between 0% to 100% of his/her wealth in the stock. Therefore the insider is obliged to take a conservative behavior expressed through his/her trading behavior and this will lead to a finite utility after appropriate moment conditions.

Our mathematical discussion starts in Section 2 by setting up the main conditions on the enlarged filtration, the semimartingale representation of the driving process in the enlarged filtration and by rewriting the value process in logarithmic form.

In Section 3 we prove that if the portfolio values are in a bounded set then under enough moment conditions utilities have to be bounded. We also prove that if jumps are all positive or all negative in some random interval determined by stopping times in the enlarged filtration then the utility is infinite. Similarly, if in any interval there are always positive and negative jumps then the utility is finite. In this article, we consider exclusively the case of the logarithmic utility function from Section 4 although some comments on the power utility function appear in Appendix 7.1.

From Section 4 on, we specialize our study to the case that the stock price is driven by a Lévy process Z with Lévy measure ν, the extra information of the insider is given by the final price of the stock in the time interval $[0, T]$ perturbed by a noise $Z'(g(T - \cdot))$ where Z' is independent of Z with the same Lévy measure and g is an increasing continuous function with $g(0) = 0$.

First, we verify that the hypothesis **HI** given in Section 2 is verified. In order to do this we have to enlarge the filtration further as to include all jump sizes and then project into the filtration of the insider. Then we conclude the section with two examples which show that the behaviour of the Lévy measure around zero is crucial in order to determine if the maximal logarithmic utility is finite or not.

In Section 5, we start studying the case where there is Wiener component in the Lévy process. We show that if g^{-1} is an integrable function then the maximal logarithmic utility is finite. Otherwise if $g \equiv 0$ then finite maximal logarithmic utility can be obtained if there are positive and negative jumps in the Lévy process. Otherwise the maximal logarithmic utility is infinite.

In Section 6, we consider the case where there is no Wiener component in the Lévy process. Although this case may be of restricted application, the results in this section help to understand that the inner structure of the jumps (that is, the values in the support of the Lévy measure) in the Lévy process play an important role in characterizing if the maximal logarithmic utility is finite or not. First, we start in subsection 6.1 to study the particular example of a compound Poisson process with positive jumps all of the same size and negative jumps all of the same size. In this particular case, we find that one can characterize the optimal portfolio as the solution of a non-linear equation. This equation can be solved explicitly in this particular case and a detailed analysis of the optimal portfolio and its characteristics can be carried out. Furthermore we see that the determination of finiteness of the logarithmic utility of the insider will be related to the following fundamental algebraic property of the support of the underlying Lévy process: If the insider knows the final value of the Lévy process can he/she deduce that the only way to get this final price is through a sequence of positive or negative jumps? If not, then the utility will be finite and optimal portfolios will be bounded. Otherwise, there are possibilities that the utility will be infinite and the arbitrage opportunity is usually obtained by trading on the jumps that happen throughout the time interval.

In the general case of a pure jump Lévy process the equation characterizing the optimal equation can not be solved in general and we have to study the properties of the possible solutions in other ways. This is done in subsection 6.2 for the case that the Lévy process has positive and negative jumps. In order to better understand the structure of the results we present the statements of the results in subsection 6.2 and the proofs in subsection 6.3. We close the article with the case of Lévy processes with only negative or positive jumps.

From the above arguments, it is clear that many results could be stated in the full generality of Poisson random measures. Nevertheless, we have preferred to state them in the Lévy process case to avoid further technicalities and because this setting is more common in recent financial models.

In this article we will use the notations $\mathbb{R}_+ = (0, \infty)$, $\mathbb{R}_- = (-\infty, 0)$, $\mathbb{R}_{>a} = (a, \infty)$ and $\mathbb{R}_{<a} = (-\infty, a)$. For a set $A \subset \Theta$ and $\omega \in \Theta$, either $\mathbb{I}(A)(\omega)$, $\mathbb{I}_A(\omega)$ or $\mathbb{I}_A$ denote the indicator function of the set A. A^c denotes the complement of the set A. For the value of a stochastic process X at time $t \in [0, T]$, we use the notation $X(t)$ or X_t. $\#(A)$ stands for the cardinality of the set A. For a measure ν, we denote $\nu_+(\cdot) = \nu(\cdot \cap \mathbb{R}_+)$ and $\nu_-(\cdot) = \nu(\cdot \cap \mathbb{R}_-)$. $\mathcal{B}(\mathbb{R})$ denotes the collection of Borel subsets of $\mathbb{R}$. Constants may change from one line to the next although the same symbol may be used.

2 Market model, hypotheses and the wealth process

In this section we give the general set-up for utility optimization of insider agents in a market driven by a Lévy process. The study of utility optimization for markets with Lévy driven asset prices is well known and there are a number of references on the subject. Just to mention some, see Kunita [23], Framstad et al.[9], Corcuera et al. [6] or Goll and Kallsen [10], [11].

Let $(\Omega, \mathcal{F}_T, P)$ be a complete probability space with a right continuous increasing family of sub σ-fields $\mathcal{F} = (\mathcal{F}_t)_{t\in[0,T]}$ of $\mathcal{F}_T$ such that all of them contain all the P-null sets. We denote by $N(dx, dt)$ and $\nu(dx)dt$ a stationary Poisson random measure on $\mathbb{R} \times [0, T]$ adapted to this filtration and its compensator, i.e., $EN(A, B) = \nu(A)|B|$ for Borel sets A and B, where $|\cdot|$ denotes the Lebesgue measure. For a Borel set A satisfying $\nu(A) < \infty$, we define $\widetilde{N}(A, [0, t]) = N(A, [0, t]) - \nu(A)t$, which is a martingale. We assume that $\mathrm{supp}(\nu) \neq \emptyset$ and

$$\int_{\mathbb{R}} |x| \vee x^2 \nu(dx) < \infty. \tag{1}$$

In general, the Lévy measure ν satisfies $\int_{\mathbb{R}} 1 \wedge x^2 \nu(dx) < \infty$. The assumption (1) is slightly stronger. Some of our results are valid under the assumption (1), while some others are valid under weaker assumptions. We assume (1) in order to simplify the statements of our results but we retain the following general definition of Lévy process so that the reader may easily realize which results can be generalized. We also remark that $\nu(\{0\}) = 0$.

Let Z be a Lévy process defined by

$$Z_t = cW_t + \int_0^t \int_{|x|\le 1} x\widetilde{N}(dx, ds) + \int_0^t \int_{|x|>1} xN(dx, ds).$$

where $c \geq 0$ and $t \in [0, T]$. The stock price S is given by

$$S_t = S_0 \exp\left(\left(b - \frac{c^2}{2}\right)t + Z_t\right), \tag{2}$$

for $b \in \mathbb{R}$ and $S_0 > 0$. We define $\mu := b - \int_{|x|\le 1} x\nu(dx)$. Note that $|\mu| < \infty$ due to (1).

For more information about Poisson random measures and related details and notation, we refer the reader to Ikeda-Watanabe [14], Chapter II, Section 4 or Applebaum [2], Chapter II, Section 2.7.

The main objective of this article is to compute the optimal portfolio for the insider with information characterized by a filtration $\mathcal{G}' = (\mathcal{G}_t)_{t\in[0,T)}$ satisfying the usual conditions and $\mathcal{G}' \supseteq \mathcal{F}' = (\mathcal{F}_t)_{t\in[0,T)}$. This insider is sometimes called the $\mathcal{G}'$-investor. As a particular case, we will also obtain results on the $\mathcal{F}$-investor. Note that the reason for using the notation $\mathcal{G}'$ is because we are not considering the right end of the interval $[0,T]$. Throughout the article we use the following assumption

Hypothesis I (HI): *We assume that there exist a filtration $\mathcal{H}' = (\mathcal{H}_t)_{t\in[0,T)} \supseteq \mathcal{G}'$ satisfying the usual conditions, such that the following is satisfied*
(i) *There exists an $\mathcal{H}'$-adapted σ-finite random measure $F_t(\cdot,\omega)$, $t \in [0,T)$, $\omega \in \Omega$. That is, for all $t \in [0,T)$ and a.s. $\omega \in \Omega$, $A \longmapsto F_t(A)$ is a σ-finite measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Furthermore, for $A \in \mathcal{B}_0 := \{A \in \mathcal{B}(\mathbb{R}) : E(F_t(A)) < \infty$ for all $t \in [0,T)\}$, $F_\cdot(A)$ is a jointly measurable $\mathcal{H}'$-adapted process with right and left limits.*
(ii) *For $A \in \mathcal{B}_0$, define $M(A,[0,t]) := N(A,[0,t]) - \int_0^t F_s(A)ds$, $t \in [0,T)$. Then $M(A,[0,\cdot])$ is an $\mathcal{H}'$-martingale.*
(iii) *$\int_0^t \int_{|x|\le 1} |x| F_s(dx)ds < \infty$ a.s. for all $t \in [0,T)$.*
(iv) *There exists an $\mathcal{H}'$-adapted process $\{\beta(t); t \in [0,T)\}$ with $\int_0^t |\beta(s)|^2 ds < \infty$ a.s. for all $t \in [0,T)$ and $B_\cdot = W_\cdot - \int_0^\cdot \beta(s)ds$ is an $\mathcal{H}'$-martingale.*

Note that under hypothesis **HI**, S is an $\mathcal{H}'$-semimartingale. We also remark that the number of discontinuities of the process $F(A)$ is at most countable and therefore Lebesgue integrable. Now, we briefly describe the allowed portfolio strategies for the $\mathcal{G}'$-investor. For a general description of portfolio strategies and their associated wealth processes, we refer the reader to part III of Cont-Tankov [4].

Definition 1 *We say that π is an admissible portfolio ($\pi \in \mathcal{A} \equiv \mathcal{A}(\mathcal{H}', \mathcal{G}')$) if π is a $\mathcal{G}'$-predictable real valued process such that there exists a unique solution, $V^\pi \equiv V$, to the wealth equation*

$$V_t^\pi = 1 + \int_0^t \frac{\pi_{s-} V_{s-}^\pi}{S_{s-}} dS_s. \tag{3}$$

which satisfies that for all $t \in [0,T)$, $V_t^\pi > 0$. Furthermore the following quantities are finite a.s. for all $t \in [0,T)$

$$A_1^\pi(t) := \int_0^t |\pi_s|^2 \, ds < \infty \ a.s.,$$

$$A_2^\pi(t) := \int_0^t \int_{\mathbb{R}} |\log(1 + (e^x - 1)\pi_s)| F_s(dx)ds < \infty \ a.s.,$$

$$A_3^\pi(t) := \int_0^t \int_{|x|\le 1} \{\log(1 + (e^x - 1)\pi_s)\}^2 F_s(dx)ds < \infty \ a.s.,$$

and

$$A_4^\pi(t) := \int_0^t \int_{|x|>1} |\log(1 + (e^x - 1)\pi_{s-})| N(dx,ds) < \infty \ a.s.$$

Note that in order that a portfolio $\pi \in \mathcal{A}$ under hypothesis **HI**, it is necessary that

$$P\left(\int_0^{T-} F_s(\{x : (e^x - 1)\pi_s \le -1\})ds = 0\right) = 1$$

and

$$P\left(\int_0^{T-} \int_{\mathbb{R}} \mathbb{I}(\{x : (e^x - 1)\pi_s \le -1\}) N(dx,ds) = 0\right) = 1.$$

The above definition needs some financial interpretation. We briefly give an informal discussion. Let π^0 and π^1 be two $\mathbb{R}$-valued $\mathcal{G}'$-predictable processes which represent the quantities (in shares) invested in the bank account and the asset respectively where negative quantities are interpreted as bank borrowing and short selling respectively. Without loss of generality, suppose that the interest rate on the bank account is 0. Then the value process of the portfolio $\vec{\pi} = (\pi^0, \pi^1)$ is defined as $V_t := \pi_t^0 + \pi_t^1 S_t$. We say that $\vec{\pi}$ is self-financing if $dV_t = \pi_{t-}^1 dS_t$ is satisfied.

Assume that the portfolio $\vec{\pi}$ is such that $V_t > 0$ for all $t \in [0, T)$ and let $\pi_t := \pi_t^1 S_t / V_t$ denote the proportion of total wealth invested in the underlying asset S. Note that π is a process taking values in $\mathbb{R}$ where values bigger than 1 are interpreted as borrowing from the bank and values smaller than 0 are interpreted as short selling.

Then we have that $(1 - \pi_t)V_t$ represents the total amount of money invested in the bank account and $\pi_t V_t$ represents the total amount of money invested in stocks where these quantities may be negative.

In the above definition we have called π the portfolio process rather than $\vec{\pi}$ which will not be used again. This procedure allows the reduction of the number of variables in the portfolio optimization problem.

If the portfolio π is self-financing then we have that the equation describing the wealth process is the one given by (3).

The reason why all the hypotheses are stated in $[0, T)$ will be clear in Section 4. In short, this is done to adapt to the case of insiders whose information is related to an event that is completely revealed at time T. Therefore the utility up to that time may explode which is related to the fact that the above hypotheses are assumed on the interval $[0, T)$.

Next we prove the existence of an alternative expression for $\log(V(t))$ that will be easier to handle. This is obtained after applying Itô's formula (see for example, [20], Chapters I and II) In order to apply this formula we need to use the integral assumptions made in the definition of the set of admissible portfolios $\mathcal{A}$. Its proof is left to the reader.

Lemma 2 *For $\pi \in \mathcal{A}$ we have that $\log(V_t^\pi) = R^\pi(t) \equiv R(t)$, $t \in [0, T)$ where*

$$\begin{aligned} R(t) = &\int_0^t c\pi_s dB_s + \int_0^t \int_{\mathbb{R}} \log\left(1 + (e^x - 1)\pi_{s-}\right) M(dx, ds) \\ &+ \int_0^t \left\{ (\mu + c\beta(s))\pi_s - \frac{c^2}{2}\pi_s^2 \right\} ds + \int_0^t \int_{\mathbb{R}} \log\left((1 + (e^x - 1)\pi_s\right) F_s(dx) ds. \end{aligned}$$

This expression gives the semimartingale decomposition of the so-called "return process" R in the filtration $\mathcal{H}'$. This will be useful when calculating the utility.

In order to define the utility associated with admissible portfolios, we need to consider the following sequence of localizing stopping times.

$$\tau_n = \inf\{t; \max\{A_i^\pi(t); i = 1, ..., 4\} > n\}.$$

It is standard to prove that if $\pi \in \mathcal{A}$, then $\tau_n \to T$ as $n \to \infty$.

The main objective of this article is to maximize $u(t, \pi) = \sup_n u_n(t, \pi)$ where $u_n(t, \pi) = E\left[U(V_{t \wedge \tau_n -})\right]$, $U(x) = \log(x)$ and $\pi \in \mathcal{A}$ and to determine whether the optimal logarithmic utility is finite or infinite. Nevertheless many of the results can be expressed in a larger generality.

3 Finite and infinite utility

The goal of this section is to state under certain generality, conditions which ensure that the maximal logarithmic utility is finite or infinite. In this section, the Lévy characteristics of the process Z are not essential and only the support of the compensator $F_s(dx)$ is used in order to characterize if the maximal logarithmic utility is finite or infinite. Our first result states that if there exists the possibility that the insider will view the price dynamics in a predictable random time interval as the result of jumps of only positive or negative type, then there is a portfolio which uses this fact that leads to a utility which may be as big as desired. Therefore the maximal utility will be infinite. This basic principle appears recurrently in our proofs. The discussion on finite utility will appear in and after Corollary 5.

For this, we need to introduce the following hypothesis $\mathbf{HII_k}$ for $k=0,1$.

Hypothesis $\mathbf{HII_k}$: There exist $\mathcal{G}'$-stopping times $0 \le \tau^1 < \tau^2 < T$ and a $\mathcal{G}'$-predictable non-negative bounded process Υ^B_s, $s \in [0,T)$ so that

$$E\Big[\int_{\tau^1}^{\tau^2}\int_{\mathbb{R}} \Upsilon^B_s \mathbb{I}\{(-1)^k x < 0\} F_s(dx)ds\Big] = 0,$$
$$E\Big[\int_{\tau^1}^{\tau^2}\int_{\mathbb{R}} \Upsilon^B_s \mathbb{I}\{(-1)^k x > 0\} F_s(dx)ds\Big] > 0.$$

Proposition 3 *Assume* **HI** *and $c=0$. Let $U(x)=\log(x)$ or $U(x)=x^\alpha$ for $\alpha>0$. Then, we have the following two results.*
(1) Assume $\mathbf{HII_0}$ and that for every nonnegative constant $\bar\pi$, the portfolio $\pi(s)=\bar\pi\mathbb{I}_{(\tau_1,\tau_2]}(s)\Upsilon^B_s$ is admissible with $E[U(V^\pi_T)]<\infty$. If $\mu \ge 0$ then the maximal utility of the $\mathcal{G}'$-investor is infinite.
(2) Similarly, assume $\mathbf{HII_1}$ and that for every nonpositive constant portfolio $\bar\pi$, the portfolio $\pi(s) = \bar\pi\mathbb{I}_{(\tau^1,\tau^2]}(s)\Upsilon^B_s$ is admissible with $E[U(V^\pi_T)]<\infty$. If $\mu \le 0$ then the maximal utility of the $\mathcal{G}'$-investor is infinite.

Proof We will only do the proof of (1) as the proof of (2) is similar. First, we remark that in this case there are no negative jumps in the interval $[\tau^1,\tau^2]$. In fact, $\int_{t\wedge\tau^1}^{t\wedge\tau^2}\int \mathbb{I}_{\mathbb{R}_-}(x)\Upsilon^B_s M(dx,ds) = \int_{t\wedge\tau^1}^{t\wedge\tau^2}\int \mathbb{I}_{\mathbb{R}_-}(x)\Upsilon^B_s N(dx,ds)$ is an increasing martingale, therefore equal to zero for almost all ω. Note that, as $c=0$, the return process can be written as

$$R^{\bar\pi}(t) = \int_{t\wedge\tau^1}^{t\wedge\tau^2}\Bigg(\bar\pi\Upsilon^B_s\mu ds + \int_{\mathbb{R}_+}\log(1+(e^x-1)\bar\pi\Upsilon^B_s)M(dx,ds) + \int_{\mathbb{R}_+}\log(1+(e^x-1)\bar\pi\Upsilon^B_s)F_s(dx)ds\Bigg) \ge 0.$$

From here, it is clear that the return process $R^{\bar\pi}$ is increasing in $\bar\pi$ and $R^{\bar\pi}_t \uparrow \infty$ as $\bar\pi\uparrow\infty$ for $t\in(\tau^1,\tau^2]$ with positive probability. We have by Itô's formula that

$$\begin{aligned}
&u(T,\pi)\\
&= E\left[U\left(\exp(R^{\bar\pi}(T))\right)\right]\\
&= U(1) + E\Bigg[\int_{\tau^1}^{\tau^2} U'(\exp(R^{\bar\pi}(s)))\exp(R^{\bar\pi}(s))\bar\pi\Upsilon^B_s\mu ds\\
&+ \int_{\tau^1}^{\tau^2}\int_{\mathbb{R}}\Big(U\left(\left(1+(e^x-1)\bar\pi\Upsilon^B_s\right)\exp(R^{\bar\pi}(s))\right) - U\left(\exp(R^{\bar\pi}(s))\right)\Big)F_s(dx)ds\Bigg].
\end{aligned}$$

Note that each of the above integrands is non-negative. Letting $\bar\pi\to\infty$ then for $\Upsilon^B_s>0$, $x>0$ and $U(x)=\log(x)$ or $U(x)=x^\alpha$ for $\alpha>0$ we have that as $\pi\to\infty$ then with positive probability

$$U\left(\left(1+(e^x-1)\bar\pi\Upsilon^B_s\right)\exp(R^{\bar\pi}(s))\right) - U\left(\exp(R^{\bar\pi}(s))\right)\to\infty.$$

Finally using the hypothesis $\mathbf{HII_0}$, we obtain that $u(T,\pi)\to\infty$ by Fatou's lemma. □

It is clear (by straighforward application of the definition of FLVR, see [7]) that the above proof also proves the existence of a free lunch with vanishing risk.

We now give conditions which ensure that bounded portfolios have finite logarithmic utility. We introduce the following hypothesis :

Hypothesis HIII : If $c\neq 0$, then we assume that $E\left[\int_0^T|\beta(s)|ds\right]<\infty$.

Proposition 4 *Assume that* **HIII** *holds. Let* $U(x) = \log(x)$ *and* $\pi \in \mathcal{A}$ *be an admissible portfolio such that there exists a positive constant* M *with* $|\pi(s)| \le M$ *for almost all* $(s, \omega) \in [0, T] \times \Omega$. *Then the logarithmic utility* $u(t, \pi) < C_M$ *for all* $t \le T$ *and some positive constant* C_M.

Proof First note that for the $\mathcal{G}$-stopping time $\sigma_n = \inf \left\{ t; \int_0^t \int_{|x|\le 1} |x| F_s(dx) ds > n \right\}$,

$$E\left[\int_0^{T\wedge\sigma_n} \int_{|x|\le 1} |x| F_s(dx) ds \right] = E\left[\int_0^{T\wedge\sigma_n} \int_{|x|\le 1} |x| N(dx, ds) \right] < \infty.$$

Taking monotone limits with respect to $n \uparrow \infty$ we have that

$$E\left[\int_0^{T} \int_{|x|\le 1} |x| F_s(dx) ds \right] = \int_0^{T} \int_{|x|\le 1} |x| \nu(dx) ds < \infty.$$

Therefore, we also obtain that

$$E\left[\int_0^{T} \int_{|x|\le 1} |x|^p F_s(dx) ds \right] < \infty \text{ for any } p \ge 1.$$

In the following arguments we will use the inequalities

$$\log(1 + (e^x - 1)y) \le xy$$

for $y \in (-\infty, 0] \cup [1, \infty)$ and $1 + (e^x - 1)y > 0$, and

$$|\log(1 + (e^x - 1)y)| \le |x| \tag{4}$$

for $y \in [0, 1]$ and $1 + (e^x - 1)y > 0$. To prove the first inequality it is enough to find an upper bound to the function $f(x) = 1 + (e^x - 1)y - e^{xy}$ and for the second, one uses the functions $f_i(x) = 1 + (e^x - 1)y - e^{(-1)^i |x|}$ for $i = 1, 2$.

Using **HI** and Definition 1, we have that $E\left[\int_0^{t\wedge\tau_n} \int_{\mathbb{R}} \log\left(1 + (e^x - 1)\pi_s\right) M(dx, ds) \right] = 0$. Therefore we bound the logarithmic utility, using Lemma 2, as follows,

$$\begin{aligned} u_n(t, \pi) &= E\left[\int_0^{t\wedge\tau_n} \left((\mu + c\beta(s))\pi_s - \frac{c^2}{2}\pi_s^2 \right) ds \right. \\ &\quad \left. + \int_0^{t\wedge\tau_n} \int_{\mathbb{R}} \left\{ \log\left(1 + (e^x - 1)\pi_s\right) \right\} F_s(dx) ds \right] \qquad (5) \\ &\le E\left[\int_0^{t\wedge\tau_n} \left\{ \left((\mu + c\beta(s))\pi_s - \frac{c^2}{2}\pi_s^2 \right) ds \right. \right. \\ &\quad \left. \left. + \int_{|x|>1} (x\pi_s) \vee |x| F_s(dx) ds + \int_{|x|\le 1} (e^x - 1)\pi_s F_s(dx) ds \right\} \right] \qquad (6) \\ &\le C_M E\left[\int_0^t \left((|\mu| + c|\beta(s)|) ds + \int_{\mathbb{R}} |x| F_s(dx) ds \right) \right] < \infty \end{aligned}$$

for all $t \in [0, T]$ where C_M is a positive constant depending on M. The above sequence of inequalities follow from (4), $|e^x - 1| \le c|x|$ for $|x| \le 1$, the hypothesis **HIII** and (1). Finally one takes the supremum with respect to n. □

We will frequently use this Proposition in order to prove that the maximal logarithmic utility for the $\mathcal{G}'$-investor is finite. In fact, if all admissible portfolios are bounded by a uniform bound M, then as the utility is bounded by C_M then the optimal logarithmic utility is bounded by C_M and therefore finite.

Corollary 5 *Assume* **HIII**. *Moreover, assume that there exists a constant $a > 0$ such that for all $s \in [0, T)$*

$$E\left[\int_{\mathbb{R}} \mathbb{I}\{(-1)^k x > a)\} F_s(dx) \Big/ \mathcal{G}_s\right] > 0 \tag{7}$$

a.s. for $k = 0, 1$. Then the maximal logarithmic utility of the $\mathcal{G}'$-investor is finite. That is, $\sup_{\pi \in \mathcal{A}(\mathcal{H}', \mathcal{G}')} E\left[\log(V_T^\pi)\right] < \infty$.

Proof Here is enough to note that under the above assumption (7), we have that any admissible portfolio is bounded by a universal constant. In fact, for any $\pi \in \mathcal{A}$, we have that $E\left[A_2^\pi(t \wedge \tau_n)\right] < \infty$ for all $t \in [0, T)$ and assuming without loss of generality that for all $s \in [0, T)$ and any $\varepsilon > 0$,

$$E\left[\int_{\mathbb{R}} \mathbb{I}\{a + \varepsilon > (-1)^k x > a)\} F_s(dx) \Big/ \mathcal{G}_s\right] > 0,$$

we obtain that $(1 - e^a)^{-1} \le \pi_s \le (1 - e^{-a})^{-1}$ for almost all (s, ω). In fact, suppose by contradiction that $E\left[\int_0^t \mathbb{I}\{\pi_s > (1 - e^{-a})^{-1} - r\} ds\right] > 0$ for some $t \in [0, T)$ and consider for any sufficiently small $r > 0$

$$\begin{aligned} &E\left[\int_0^{t \wedge \tau_n} \mathbb{I}\{\pi_s > (1 - e^{-a})^{-1} - r\} \int_{\mathbb{R}} |\log(1 + (e^x - 1)\pi_s)| \mathbb{I}\{a + \varepsilon > -x > a\} F_s(dx) ds\right] \\ &\le E\left[\int_0^{t \wedge \tau_n} \int_{\mathbb{R}} |\log(1 + (e^x - 1)\pi_s)| F_s(dx) ds\right] \le n \ . \end{aligned}$$

On the other hand, on the set $\{\pi_s > (1 - e^{-a})^{-1} - r\} \cap \{a + \varepsilon > -x > a\}$

$$|\log(1 + (e^x - 1)\pi_s)| \ge |\log((1 - e^{-a})r)|$$

goes to infinity as $r \to 0$. Therefore

$$E\left[\int_0^{t \wedge \tau_n} \mathbb{I}\{\pi_s > (1 - e^{-a})^{-1} - r\} ds\right] = 0$$

which clearly goes to leads to a contradiction. Letting $n \to \infty$ and $r \to 0$, we obtain that $\pi_s \le (1 - e^{-a})^{-1}$ for almost all (s, ω). The other inequality is obtained in a similar fashion. Therefore the result follows from Proposition 4. □

In particular, note that the above implies in particular that $\sup_{\pi \in \mathcal{A}(\mathcal{G}', \mathcal{G}')} E\left[\log(V_T^\pi)\right] < \infty$. We will see later in Theorem 13 an extension of Corollary 5 when a is replaced by a stochastic process (or even a function of time). An example where a depends on time and the logarithmic utility is infinite for the case that Z is an additive process exists[1]. For more results of the above type for power utility functions, see Appendix 7.1. Now we state the above results in the particular case of the small investor.

Corollary 6 *For the small investor, i.e. $\mathcal{G} = \mathcal{F}$, the following assertions hold.*
a) Assume that $c = 0$ and $\emptyset \neq \mathrm{supp}(\nu) \subseteq \mathbb{R}_+$. Then the maximal logarithmic utility for the $\mathcal{F}$-investor is either infinite or finite according to $\mu \ge 0$ or $\mu < 0$.
b) Assume that $c = 0$ and $\emptyset \neq \mathrm{supp}(\nu) \subseteq \mathbb{R}_-$. Then the maximal logarithmic utility for the $\mathcal{F}$-investor is either infinite or finite according to $\mu \le 0$ or $0 < \mu \le \infty$.
c) If $\mathrm{supp}(\nu_+) \neq \emptyset$ and $\mathrm{supp}(\nu_-) \neq \emptyset$ then the maximal logarithmic utility for the $\mathcal{F}$-investor is finite.

Proof Since $F_s = \nu$ and **HIII** is trivially satisfied then *c*) follows from Corollary 5. Now, we prove *a*). Note that any constant π is admissible, therefore, by Proposition 3 (1) the maximal logarithmic utility is infinite if $\mu \ge 0$.

[1]Please contact the authors if interested.

Now, assume that $\mu < 0$. Then the logarithmic utility for small investor is given by

$$u_n(t,\pi) = E\left[\int_0^{t\wedge\tau_n}\left(\mu\pi_s ds + \int_{\mathbb{R}}(\log(1+(e^x-1)\pi_s)\nu(dx)ds\right)\right].$$

We define the function $f : [(1-e^{a_+})^{-1},\infty) \to [-\infty,\infty)$ by

$$f(y) = \mu y + \int_{\mathbb{R}} \log(1+(e^x-1)y)\nu(dx).$$

Then f is twice differentiable and we have

$$f'(y) = \mu + \int_{\mathbb{R}} \frac{e^x-1}{1+(e^x-1)y}\nu(dx) \tag{8}$$

$$f''(y) = -\int_{\mathbb{R}}\left(\frac{e^x-1}{1+(e^x-1)y}\right)^2\nu(dx) < 0$$

The last strict inequality comes from the assumption $\text{supp}(\nu) \neq \emptyset$. Hence f' is strictly decreasing. Denote by $a_+ = \sup\{\text{supp}(\nu_+)\}$.

Note that the above integrals in (8) are well defined in the case that $a_+ < \infty$ and $y \in ((1-e^{a_+})^{-1},\infty)$. In the case that $a_+ = \infty$ these integrals are also well defined as $\left|\frac{e^x-1}{1+(e^x-1)y}\right| \leq y^{-1}$ for $|x| \geq 1$ and $y > 0$. Therefore the domain of definition of f and its derivatives include $((1-e^{a_+})^{-1},\infty)$. At the point $y = (1-e^{a_+})^{-1}$ there are two possibilities. Either the function is $-\infty$ or it is finite. Therefore, we interpret the domain of f as the domain of the extended function. A similar remark applies to the derivatives.

Then the domain of f is given by $(1-e^{a_+})^{-1} \leq y < \infty$. As $\mu < 0$, then $\lim_{y\uparrow\infty} f'(y) = \mu < 0$. Hence, $f(y)$ has a maximum in $[(1-e^{a_+})^{-1},\infty)$, which is independent of ω. Hence the optimal logarithmic utility is finite and equals $\max\{f(y);\ (1-e^{a_+})^{-1} \leq y\}$. The proof of b) is similar to the proof of a). □

4 A larger filtration and finite utility with blurred information

In this section, we proceed to show explicit situations where the results in the previous section apply. In particular, we will consider the case where a perturbation of the final value of the stock price is the additional information of the insider. We consider the maximization problem under the logarithmic utility function.

Throughout this section we assume that Z' is a Lévy process of the form

$$Z'_t = cW'_t + \int_0^t\int_{|x|\leq 1} x\widetilde{N'}(dx,ds) + \int_0^t\int_{|x|>1} xN'(dx,ds)$$

independent of Z, where N' is a stationary Poisson random measure with compensator $\nu(dx)ds$, $\widetilde{N'}(dx,ds) = N'(dx,ds) - \nu(dx)ds$ and W' is a Wiener process. Furthermore, we assume that W, W', N and N' are mutually independent.

The above setting still allows the application of our results to most of the jump type models such as the variance gamma model among others. Hence to avoid cumbersome statements we have not pursued the greatest generality in some of the results to follow.

From now on, the insider at time t knows a perturbation of the value $S(T)$ or more explicitly, $Z(T) + Z'(g(T-s)); s \leq t$. Therefore we define

$$\mathcal{G}_t = \mathcal{F}_t \vee \sigma(Z(T) + Z'(g(T-s)); s \leq t)$$

where g is a positive continuous increasing function with $g(0) = 0$.

The filtration $\mathcal{G}$ is further enlarged using all the jump structure of the Poisson random measure in $Z(T) + Z'(g(T-s))$ in the following way,

$$\begin{aligned}\mathcal{H}_t &= \mathcal{F}_t \vee \sigma(W(T) + W'(g(T-u)); u \le t) \\ &\quad \vee \sigma(N(A, [0,T]) + N'(A, [0, g(T-u)]); A \in \mathcal{B}(\mathbb{R}), d(A, 0) > 0, u \le t).\end{aligned}$$

Here, $d(A, 0)$ is a distance between the set A and the origin 0. The reason for this further enlargement is clear from the following Lemma where we verify that Hypothesis **HI** is satisfied. That is, we will prove that Z is an $\mathcal{H}'$-semimartingale and therefore by Stricker's Theorem Z is a $\mathcal{G}'$-semimartingale and its semimartingale decomposition will be a projection of the corresponding one in $\mathcal{H}'$.

Lemma 7 *Hypothesis* **HI** *is satisfied with*

$$\beta_t = \frac{W(T) - W(t) + W'(g(T-t))}{T - t + g(T-t)}, \tag{9}$$

$$M(dx, dt) = N(dx, dt) - F_t(dx)dt,$$

$$F_t(dx) = \frac{\int_t^T N(dx, du) + \int_0^{g(T-t)} N'(dx, du)}{T - t + g(T-t)}. \tag{10}$$

Proof The verification of **HI(iv)** for W can be found in Corcuera et al. [5], Example 1 (see also Remark 9 below). Although the result for Z', which is not necessarily identical in law with Z, is given in [22], we give the proof here in this simple case for the reader's convenience.

F_t clearly satisfies (i) and (iii) (see Remark 8 below). We only need to check (ii). Let $s_i = \frac{si}{n}$ for $i = 0, ..., n$ and $\{A_j\}_{j=1}^n$ with

$$A_i \cap A_j = \emptyset \text{ for } i \neq j, \text{ and } d(A_j, 0) > 0 \text{ for } j = 1, ..., n$$

and define

$$\begin{aligned}X &= (X_j)_{j=1}^n = (N(A_j, (0,T]))_{j=1}^n, \\ Y &= (Y_{s_j})_{j=1}^n = (N'(A_j, (0, g(T - s_j)]))_{j=1}^n.\end{aligned}$$

Let $\phi(x_1, \ldots, x_n) = \prod_{j=1}^n e^{i\theta_j x_j}$, where $\theta_j \in \mathbb{R}$ for $j = 1, \ldots, n$ and $i = \sqrt{-1}$. We have for $s \le u < t < T$, $C \in \mathcal{B}(\mathbb{R})$ with $d(C, 0) > 0$ and any bounded $\mathcal{F}_s$-measurable random variable h_s,

$$\begin{aligned}&E[\phi(X+Y)h_s N(C, (u,t])] \\ &= E[h_s \prod_{j=1}^n \exp\left\{i\theta_j \left(N(A_j, (0,u] \cup (t,T]) + Y_{s_j} + N(C^c \cap A_j, (u,t])\right)\right\}] \\ &\times E\left[\left(\sum_{j=1}^n N(C \cap A_j, (u,t]) + N(C \cap (\cup_{k=1}^n A_k)^c, (u,t])\right)\right. \\ &\left.\times \prod_{j=1}^n \exp\{i\theta_j N(C \cap A_j, (u,t])\}\right].\end{aligned} \tag{11}$$

Using the Lévy-Khintchine formula and its derivative we obtain

$$\begin{aligned}&E[\phi(X+Y)h_s N(C, (u,t])] \\ &= (t-u)\left(\sum_{j=1}^n e^{i\theta_j}\nu(C \cap A_j) + \nu(C \cap (\cup_{k=1}^n A_k)^c)\right) E[\phi(X+Y)h_s],\end{aligned}$$

and similarly

$$
\begin{aligned}
&E\left[\phi(X+Y)h_s N'(C,[0,g(T-u)])\right]\\
&= g(T-u)\left(\sum_{j=1}^{n} e^{i\theta_j}\nu(C\cap A_j)+\nu(C\cap(\cup_{k=1}^{n}A_k)^c)\right)E[\phi(X+Y)h_s].
\end{aligned}
$$

Letting $t=T$ in (11) and adding the two previous equations, we have

$$
\begin{aligned}
&E\left[\left(N\left(C,(u,T]\right)+N'\left(C,[0,g(T-u)]\right)\right)\phi(X+Y)h_s\right]\\
&= (T-u+g(T-u))\left(\sum_{j=1}^{n} e^{i\theta_j}\nu(C\cap A_j)+\nu(C\cap(\cup A_j)^c))\right)E[\phi(X+Y)h_s].
\end{aligned}
$$

Hence we have

$$
E[\phi(X+Y)h_s N(C,(s,t])] = E\left[\int_s^t \frac{N(C,(u,T])+N'(C,[0,g(T-u)])}{T-u+g(T-u)}du\phi(X+Y)h_s\right].
$$

Therefore, we have that

$$
E[N(C,(s,t])/\mathcal{H}_s] = \int_s^t E\left[\frac{N(C,(u,T])+N'(C,[0,g(T-u)])}{T-u+g(T-u)}\Big/\mathcal{H}_s\right]du.
$$

Finally this proves that

$$
N(C,[0,t]) - \int_0^t \frac{N(C,(u,T])+N'(C,[0,g(T-u)])}{T-u+g(T-u)}du
$$

is an $\mathcal{H}'$ martingale for all $C\in\mathcal{B}(\mathbb{R})$. □

Note that the measure $F_s(dx)$ besides being a compensator also behaves like a jump measure on $\mathcal{F}$. This point is stressed in the following remarks.

Remark 8 *For a given $\mathcal{F}$-predictable process h, we have that for the positive increasing function g with $g(0)=0$, introduced at the beginning of Section 4 and any $t\in[0,T)$*
a) If $E\left[\int_0^t\int_{\mathbb{R}}|h(x,s)|\,\nu(dx)ds\right]<\infty$, then

$$
E\left[\int_0^t\int_{\mathbb{R}} h(x,s)(F_s-\nu)(dx)ds\right]=0
$$

and

$$
E\left[\left|\int_0^t\int_{\mathbb{R}} h(x,s)(F_s-\nu)(dx)ds\right|\right] \le 2E\left[\int_0^t\int_{\mathbb{R}}|h(x,s)|\,\nu(dx)ds\right].
$$

b) If $E\left[\int_0^t\int_{\mathbb{R}}|h(x,s)|^2\nu(dx)ds\right]<\infty$, then

$$
E\left[\int_0^t\left(\int_{\mathbb{R}} h(x,s)(F_s-\nu)(dx)\right)^2 ds\right] = E\left[\int_0^t\int_{\mathbb{R}}\frac{|h(x,s)|^2}{T-s+g(T-s)}\nu(dx)ds\right],
$$

and

$$
\begin{aligned}
&E\left[\int_0^t\left(\int_{\mathbb{R}} h(x,s)F_s(dx)\right)^2 ds\right]\\
&= E\left[\int_0^t\left\{\int_{\mathbb{R}}\frac{h(x,s)^2}{T-s+g(T-s)}\nu(dx)ds+\left(\int_{\mathbb{R}} h(x,s)\nu(dx)\right)^2\right\}ds\right]. \qquad (12)
\end{aligned}
$$

Remark 9 *β in Lemma 7 satisfies* **HIII** *since*

$$E\left[\int_0^T |\beta_s|^p ds\right] = C\int_0^T (T-s+g(T-s))^{-p/2} ds < \infty$$

for $0<p<2$.

5 Logarithmic utility in the case $c \neq 0$ (Lévy process with non-zero Wiener component)

The next result is a general theorem which shows that the logarithmic utility is finite regardless of the jump structure (i.e. $\mathrm{supp}(\nu)$) if the speed at which the blurring noise dissapears is slow enough.

Theorem 10 *Assume that $c \neq 0$. If $\int_0^T g(T-s)^{-1}ds < \infty$, then the maximal logarithmic utility for the $\mathcal{H}'$-investor is finite.*

Proof As in Proposition 4, we define $\tau_n = \inf\{t; \max\{A_i^\pi(t); i=1,...,4\} > n\}$. Then, using the inequality $\log(1+(e^x-1)y) \leq x(1+y)$ for $x \geq 0$ and $y > -(e^x-1)^{-1}$ and as in (5), we have that for a $\mathcal{H}'$-adapted admissible portfolio π the following inequality is satisfied.

$$\begin{aligned}
u_n(t,\pi) \leq{}& E\left[\int_0^{t\wedge\tau_n}\Big((b+c\beta(s))\pi_s - \frac{c^2}{2}\pi_s^2\Big)ds\right] - E\left[\int_0^{t\wedge\tau_n}\pi_s\int_{|x|\leq 1} x\nu(dx)ds\right] \\
&+ E\left[\int_0^{t\wedge\tau_n}\pi_s\int_{x\in(-\infty,0)\cup(0,1]}(e^x-1)F_s(dx)ds\right] + E\left[\int_0^{t\wedge\tau_n}(1+\pi_s)\int_{x>1} xF_s(dx)ds\right] =: u_n^1(t,\pi).
\end{aligned}$$

$u_n^1(t,\pi)$ is a value function for $\mathcal{H}'$-adapted portfolios which can be maximized explicitly for $c \neq 0$. In fact, the optimal portfolio for the utility function $u_n^1(t,\pi)$ satisfies

$$\begin{aligned}
\pi_t^o = \frac{1}{c^2}\Big\{&(b+c\beta(t)) - \int_{|x|\leq 1} x\nu(dx) \\
&+ \int_{x\in(-\infty,0)\cup(0,1]}(e^x-1)F_t(dx) + \int_{x>1} xF_t(dx)\Big\}.
\end{aligned} \tag{13}$$

Now to prove that the utility $\sup_n u_n^1(t,\pi^o)$ is finite, we use the moment hypothesis in our statement. In fact, replacing (13) in u_n^1, we have

$$\sup_n u_n^1(T,\pi^o) \leq \frac{c^2}{2}E\left[\int_0^T (\pi_s^o)^2 ds\right] + E\left[\int_0^T\int_{x>1} xF_s(dx)ds\right]. \tag{14}$$

For the last term in (14), using Remark 8, we have that

$$\left|E\left[\int_0^T\int_{x>1} xF_s(dx)ds\right]\right| \leq T\int_{|x|>1}|x|\nu(dx) < \infty.$$

Note that $E\left[\int_0^T \beta(s)^2 ds\right] < \infty$ by the assumption $\int_0^T g(T-s)^{-1}ds < \infty$. Note also that

$$\begin{aligned}
&E\left[\int_0^T\Big(\int_{\mathbb{R}} xF_s(dx)\Big)^2 ds\right] \\
&\leq \int_0^T g(T-s)^{-1}ds\int_{\mathbb{R}} x^2\nu(dx) + \Big(T\int_{\mathbb{R}}|x|\nu(dx)\Big)^2 < \infty
\end{aligned}$$

because of (12). These estimates guarantee the finiteness of the first term in (14). □

In the case that $\int_0^T g(T-s)^{-1}ds = \infty$, both finite and infinite maximal utility can happen. The following two theorems treat each case. The next theorem also contains the case $c = 0$ but we include it here.

Theorem 11 *Define $L(s) = Z(T) - Z(s) + Z'(g(T-s))$. Assume that $\nu \neq 0$ and that for any $s \in [0, T)$, $L(s)$ has a strictly positive density in $\mathbb{R}$ then the optimal logarithmic utility of the $\mathcal{G}'$-investor is finite for any g.*

As the proof of the above theorem is quite related with the notation and ideas of Section 6.2, we give its proof in section 6.3.

Theorem 12 *Suppose that $c \neq 0$, $g \equiv 0$. Furthermore assume that either*
(1) $\text{supp}(\nu_-) = \emptyset$ *or*
(2) $\text{supp}(\nu_+) = \emptyset$.
Then the maximal logarithmic utility of the $\mathcal{G}'$-investor is infinite

Proof (1) Let $A_s^n = \{\omega : 0 \leq \mu + \frac{Z(T)-Z(s)}{T-s} \leq n\}$. Define the following portfolio,

$$\begin{aligned}\pi_s^n &= \gamma\left(\mu + \frac{Z(T)-Z(s)}{T-s}\right)\mathbb{I}_{A_s^n}\\ &= \gamma\left(\mu + c\beta(s) + \xi(s)\right)\mathbb{I}_{A_s^n}\end{aligned}$$

with $0 < \gamma < \left(c^2 + A\right)^{-1}$, $A := \int_{\mathbb{R}} x^2\nu(dx) \geq 0$ and $\xi(s) = \frac{Z(T)-Z(s)}{T-s} - c\beta(s)$. Then π^n is non-negative and $\mathcal{G}'$-adapted. Furthermore as there are no negative jumps and $0 \leq \pi_s^n \leq \gamma n$ then $\pi^n \in \mathcal{A}$. Furthermore, since $\int_0^1 x\nu(dx) < \infty$, the utility (see (5)) can be written as

$$u(t, \pi^n) = \int_0^t \left\{E\left[(\mu + c\beta(s))\pi_s^n - \frac{c^2}{2}\left(\pi_s^n\right)^2\right] + E\left[\int_{\mathbb{R}} \log(1 + (e^x - 1)\pi_s^n)F_s(dx)\right]\right\}ds. \quad (15)$$

Using the inequality $\log(1 + (e^x - 1)y) \leq x(1 + y)$ for $x,\ y > 0$ (see (4)), we have that for fixed $n, u(t, \pi^n) < \infty$ for $0 \leq t < T$. Also note that the last term in (15) is always positive.

Now we prove that $\lim_{t\uparrow T}\lim_{n\to\infty} u(t, \pi^n) = +\infty$. In fact, note first that

$$\sup_n u(t, \pi^n) \geq E\left[\int_0^t \gamma(\mu + c\beta(s))^2\mathbb{I}_{\{\mu+c\beta(s)\geq 0\}} - \gamma|\mu + c\beta(s)|\,|\xi(s)| - \frac{c^2\gamma^2}{2}\Big(\mu + c\beta(s) + \xi(s)\Big)^2 ds\right].$$

Now it is enough to note that

$$\lim_{s\uparrow T}(T-s)E\left[(\mu + c\beta(s))^2\mathbb{I}_{\{\mu+c\beta(s)\geq 0\}}\right] = \frac{c^2}{2} > 0. \quad (16)$$

The above follows as $\beta(s)$ is a Gaussian random variable with mean zero and variance $(T-s)^{-1}$. Next, we have that for any $\varepsilon > 0$,

$$\lim_{s\uparrow T}(T-s)^{\frac{1}{2}+\varepsilon}E[|\mu + c\beta(s)|\,|\xi(s)|] = 0 \quad (17)$$

This follows using the independence of the Brownian motion and the jump part of the Lévy process and an explicit calculation with Gaussian densities and Remark 8. Similarly,

$$\lim_{s\uparrow T}(T-s)E\left[\Big(\mu + c\beta(s) + \xi(s)\Big)^2\right] = c^2 + A,$$

In conclusion, we get that there exists $\varepsilon > 0$ so that

$$\lim_{s\uparrow T}\sup_n u(t, \pi^n) \geq C + \int_{T-\varepsilon}^T \frac{\gamma c^2}{2}\frac{\left(1 - \gamma\left(c^2 + A\right)\right)}{T-s}ds = \infty.$$

Hence the conclusion follows. The proof of (2) is similar. □

In the next Theorem, we weaken the hypothesis of Corollary 5, to conclude that the insider's logarithmic utility is finite. In this extension, a is a bounded random process satisfying certain moment properties. Although the result does not use the fact that $c \neq 0$, we include it here.

Theorem 13 *Assume that for some $\epsilon \in (0, 1/2)$ and $k = 0, 1$ and for any $s \in [0, T)$*

$$E\left[\int_{\mathbb{R}} \mathbb{I}\{(-1)^k x > a_s (T-s)^{1/2-\epsilon}\} F_s(dx) \Big/ \mathcal{G}_s \right] > 0 \ a.s. \tag{18}$$

Here a is a bounded strictly positive $\mathcal{G}$-predictable process satisfying $\sup_{s\in[0,T]} E[a_s^{-(2+\delta)}] < \infty$ for some $\delta \in (0, 2\epsilon(\frac{1}{2} - \epsilon)^{-1})$. Then the optimal logarithmic utility of the $\mathcal{G}'$-investor is finite for any function g.

Proof As in Corollary 5, we have that for any admissible portfolio process

$$(1 - e^{a_s(T-s)^{1/2-\epsilon}})^{-1} \leq \pi(s) \leq (1 - e^{-a_s(T-s)^{1/2-\epsilon}})^{-1}, \tag{19}$$

for almost all (s, ω). Therefore using the inequalities $|1 - e^{-x}|^{-1} \leq Cx^{-1}$ for $x \in (0, C_1)$ (here $C > 1$ and $e^{-C_1} - C^{-1} > 0$) and $(e^x - 1)^{-1} \leq x^{-1}$ for $x > 0$ and assuming without loss of generality that $a_s(T-s)^{1/2-\epsilon} \leq C_1$, we have that there exists a positive constant C_T such that

$$E\left[\int_0^T |\pi(s)|^{2+\delta} ds \right] \leq C_T \sup_{s\in[0,T]} E[a_s^{-(2+\delta)}] \int_0^T (T-s)^{(2+\delta)(\epsilon - \frac{1}{2})} ds < \infty \tag{20}$$

and

$$E\left[\int_0^T |\beta_s \pi(s)| ds \right] \leq \left(E\Big[\int_0^T |\beta_s|^p ds \Big] \right)^{1/p} \left(E\Big[\int_0^T |\pi_s|^q ds \Big] \right)^{1/q}$$

for $p = \frac{2+\delta}{1+\delta}$, $q = 2 + \delta$. We see that due to Remark 9 and (20), the above quantity is finite as $p < 2$. To prove that the utility is finite we prove that each term in the expression (6) is finite. Consider for example the last term in (6). This gives, after using Remark 8 and the inequality $|e^x - 1| \leq (e-1)|x|$ for $|x| \leq 1$ that there exists a positive constant C_0 such that

$$\begin{aligned}
& E\left[\left| \int_0^T \int_{|x|\leq 1} (e^x - 1) \pi_s F_s(dx) ds \right| \right] \\
& \leq (e-1) \int_0^T E\left[(a_s(T-s)^{1/2-\epsilon})^{-1} \int_{|x|\leq 1} |x| F_s(dx) \right] ds \\
& \leq C_0 \left(\sup_{s\in[0,T]} E\left[a_s^{-2}\right] \int_0^T (T-s)^{2\epsilon' - 1} ds E\left[\int_0^T \left((T-s)^{\epsilon - \epsilon'} \int_{|x|\leq 1} |x| F_s(dx) \right)^2 ds \right] \right)^{1/2} \\
& < \infty
\end{aligned}$$

where, $0 < \epsilon' < \epsilon$. The previous to the last term in (6) is treated using the inequality $(x\pi_s) \vee |x| \leq |x\pi_s| + |x|$ and the first term is treated using (20). □

Note that if $\min\{\nu(0, \varepsilon), \nu(-\varepsilon, 0)\} > 0$ for all $\varepsilon > 0$ then the condition (18) is satisfied (see the proof of Theorem 17-1).

The results obtained in this section for the case of logarithmic utility for Lévy processes with Wiener components are briefly summarized in the following table:

support of ν	$g \equiv 0$	$g(s) = s^\alpha$	small investor
$\text{supp}(\nu) \subset \mathbb{R}_+$	$= \infty$	$< \infty$	$< \infty$
$\text{supp}(\nu) \subset \mathbb{R}_-$	$= \infty$	$< \infty$	$< \infty$
$\text{supp}(\nu) \cap \mathbb{R}_+ \neq \emptyset$ and $\text{supp}(\nu) \cap \mathbb{R}_- \neq \emptyset$	$< \infty$	$< \infty$	$< \infty$

The last column corresponds to the optimal logarithmic utility of the small investor. The first two results are obtained through an analysis of (6) in this particular case. The last result in the last column is a particular case of Theorem 11. The previous two correspond to the optimal logarithmic utility of the insider first without blurring ($g \equiv 0$) and then with blurring. In this table one supposes that there is a non-zero Wiener component in the Lévy process (i.e. $c \neq 0$) and $\alpha < 1$. The case of Lévy processes with no Wiener component is summarized in a table at the end of Subsection 6.2.

6 Logarithmic utility in the case $c = 0$ (Lévy process with no Wiener component)

In this section, we assume that $c = 0$. Before going into the main results of this section, we will describe the illustrative example of a compound Poisson process with two types of jumps. In this case the calculation is explicit and shows that the values of the possible jumps (the so called "effect of the jumps structure") is an important issue to determine if the logarithmic utility is finite or not.

6.1 The example of a stock price driven by a compound Poisson process

Let us suppose that we are given two independent compound Poisson processes Z and Z' which have only two types of jumps. One of size $a_+ > 0$ and the other of size $a_- < 0$.

Denote the intensity parameters for each type of jump by $\lambda_+ > 0$ and $\lambda_- > 0$, respectively. As before, recall that N and N' denote the Poisson random measures associated with Z and Z' respectively. So,

$$Z_t = \sum_{p \in \{+,-\}} \int_0^t a_p N(\{a_p\}, ds), \;\; Z'_t = \sum_{p \in \{+,-\}} \int_0^t a_p N'(\{a_p\}, ds).$$

Then the stock price model is $S(t) = S_0 \exp(bt + Z_t)$. There is an insider in the market who has information, at time t, about the final value of the stock in the form of $Z_T + Z'_{g(T-t)}$ where $g : [0,T] \to [0, g(T)]$ is a continuous strictly increasing function with $g(0) = 0$.

Mathematically, this means that the insider has an additional information flow of the form $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(Z(T) + Z'(g(T-s)); s \leq t)$. Then we define, as before

$$\mathcal{H}_t = \mathcal{F}_t \vee \sigma(N(\{a_p\}, [0,T]) + N'_p(\{a_p\}, [0, g(T-s)]); s \leq t, p \in \{+,-\}).$$

Note that $\mathcal{G}_t \subseteq \mathcal{H}_t$.

The goal of this example is to show that if the insider has information about the number of jumps left to happen in the future of the stock price then he can create an arbitrage in the market. Otherwise, the optimal utility is finite and the optimal portfolio is bounded. In this section we provide sketch of the proofs and no detailed calculations[2]. Furthermore given that there is no need for compensation of Z or Z' the set-up is slightly different from other sections. The trivial modification of replacing μ in the previous results by b in the present model with $c = 0$ (therefore hypothesis **HIII** is not needed) yield the corresponding results for the model in this section.

We start with the following result which is satisfied when $a_+ a_-^{-1} \in \mathbb{Q}$.

Result 14 *The filtration $\mathcal{H}$ satisfies hypothesis* **HI**. *Furthermore assume that there exists $k_1, k_2 \in \mathbb{N}$ such that $k_1 a_+ + k_2 a_- = 0$. Then the optimal logarithmic utility of the $\mathcal{G}'$-investor is finite. In the particular case when $a_- = \log(2 - e^{a_+})$ with $a_+ \in (0, \log 2)$, then the optimal portfolio is given by*

$$\pi^*(s) = \begin{cases} y_+(s) & \text{if } b > 0, \\ \frac{B_+ - B_-}{B_+ + B_-}(s)(e^{a_+} - 1)^{-1} & \text{if } b = 0, \\ y_-(s) & \text{if } b < 0 \end{cases}$$

[2]Please contact the authors if interested on detailed calculations.

where for $p \in \{+,-\}$

$$y_p = G_p(B_+(s), B_-(s)) \ ,$$

$$G_p(x,y) = -\frac{x+y}{2b} + p\sqrt{\left(\frac{x+y}{2b}\right)^2 + \frac{x-y}{b(e^{a_+}-1)} + \frac{1}{(e^{a_+}-1)^2}} \ ,$$

$$B_p \equiv B_p(s) = E\left(\frac{\int_s^T N(\{a_p\}, du) + \int_0^{g(T-s)} N'(\{a_p\}, du)}{T-s+g(T-s)} \Bigg/ \mathcal{G}_s \right).$$

Proof By Lemma 7 one has that Hypothesis **HI** is verified. That is, for

$$N(\{x\},[0,t]) - \int_0^t F_s(\{x\})ds$$

is an $\mathcal{H}$-martingale for $x = a_+, \ a_-$ where

$$F_s(\{x\}) = \frac{\int_s^T N(\{x\}, du) + \int_0^{g(T-s)} N'(\{x\}, du)}{T-s+g(T-s)}.$$

Next, we will prove that for any $x \in \mathbb{N}a_+ + \mathbb{N}a_-$ such that $P(Z(T) = x) > 0$, we have

$$B_+(s) > 0 \text{ and } B_-(s) > 0 \text{ for all } s \in [0,T) \text{ a.s.}$$

In fact, the above follows from two assertions. The first is that if $P(Z(T) = x) > 0$ then it means that there exists k_+,$k_- \in \mathbb{N}$ such that $k_+a_+ + k_-a_- = x$ (in fact, due to the hypothesis there exists an infinite number of such pairs of natural numbers). Clearly, for the same k_+, k_- we have that as the support of simple Poisson random variables is $\mathbb{N} \cup \{0\}$ then

$$P\left(Z(T) + Z'(g(T-s)) = x\right) \geq \prod_{p\in\{+,-\}} P\left(N\left(\{a_p\},[0,T]\right) + N'\left(\{a_p\},[0,g(T-s)]\right) = k_p\right) > 0.$$

Actually, one also sees that the support of $Z(T)$ and the support of $Z(T) + Z'(g(T-s))$ are the same for all $s \in [0,T)$. The second assertion which has a similar proof, states that for all $k \leq k_p$ we have

$$P\left(N\left(\{a_p\},(s,T]\right) + N'\left(\{a_p\},[0,g(T-s)]\right) = k; Z(T) + Z'(g(T-s)) = x\right) > 0.$$

Therefore for all $s \in [0,T)$, we have

$$E\left[N\left(\{a_p\},(s,T]\right) + N'\left(\{a_p\},[0,g(T-s)]\right) / \, Z(T) + Z'(g(T-s))\right] > 0.$$

Therefore, $B_+(s) > 0$ and $B_-(s) > 0$ for all $s \in [0,T)$ a.s.

Therefore by Corollary 5 (note that here $c = 0$, therefore **HIII** is not required), the maximal logarithmic utility of the $\mathcal{G}'$-investor is finite.

Finally to characterize the optimal portfolio one considers the expression for R in Lemma 2, which conveniently modified for our case leads to the study of the function

$$f_s(\pi) = b\pi + \int_{\mathbb{R}} \log\left(1 + (e^x - 1)\,\pi\right) E\left[F_s(dx)/\mathcal{G}_s\right].$$

Here

$$E\left[F_s(dx)/\mathcal{G}_s\right] = B_+(s)\delta_{\{a_+\}}(dx) + B_-(s)\delta_{\{a_-\}}(dx),$$

where $\delta_{\{y\}}(dx)$ denotes the point mass measure. Therefore

$$f_s(\pi) = b\pi + \sum_{p\in\{+,-\}} \log\left(1 + (e^{a_p} - 1)\pi\right) E\left[F_s(\{a_p\})/\mathcal{G}_s\right].$$

This function f satisfies that $\lim_{\pi \to \pm(e^{a_+}-1)^{-1}\mp} f_s(\pi) = -\infty$, therefore the optimal portfolio value is the solution of $f_s'(\pi) = 0$. Then the equation characterizing the optimal portfolio is

$$b + B_+(s)\frac{(e^{a_+}-1)}{1+(e^{a_+}-1)y} + B_-(s)\frac{(1-e^{a_+})}{1+(1-e^{a_+})y} = 0. \tag{21}$$

This equation reduces to a quadratic equation for $b \neq 0$ which has two solutions given by $y_+(s)$ and $y_-(s)$. The restriction $\frac{1}{e^{a_+}-1} > y > -\frac{1}{e^{a_+}-1}$ determines the optimal portfolio. The case $b = 0$ follows directly as the optimal equation (21) becomes a linear equation in y. □

There are various conclusions that one can directly obtain from the above result. We briefly summarize them here without giving all the details.

1. Note that the existence of a_+ such that there exists $k_1,\ k_2 \in \mathbb{N}$ with $k_1 a_+ + k_2 a_- = 0$ is assured by the continuity of the function $h(a) = -a^{-1}\log(2-e^a)$ for $a \in (0, \log 2)$.

2. The optimal logarithmic utility for the small trader is finite as the portfolio values are bounded (which can be obtained from Proposition 4). The optimal logarithmic utility for the small trader is given by

   $$(b\pi^* + \lambda_+ \log(1+(e^{a_+}-1)\pi^*) + \lambda_- \log(1+(1-e^{a_+})\pi^*))T.$$

   This result is the analogous result of the classical Merton problem.

3. The optimal portfolio proportion invested in stocks is constant as long as the values of (B_+, B_-) lie on the line $B_+ = mB_- + c$ for some given constants m and c that depend on a_+ and b. That is, the value of the optimal portfolio is determined by the linear relation between expected future positive and negative jumps. Furthermore, the portfolio value is an increasing function of the slope m and decreasing or increasing in c depending on the sign of b. We remark that in the particular case that $b = 0$ then $c = 0$.

4. In the classical Merton portfolio optimization problem the proportion of wealth invested in the stock grows linearly with respect to the return parameter. In the jump case considered here, the proportion of wealth invested in stocks is influenced by the effect of the risks represented in the two limits $\pi_+^* = (e^{a_+}-1)^{-1}$ and $\pi_-^* = -(e^{a_+}-1)^{-1}$. In fact, as the risks decrease $(a_+ \to 0)$, the distance between these two constants increases. Also as the return parameter b increases to ∞ the optimal portfolio proportion approaches the value π_+^*. Similar statement is valid when $b \to -\infty$.

   Furthermore note that if $b = 0$ then $\lim_{B_p \to +\infty} \pi^* = \pi_p^*$ for $p \in \{+,-\}$. That is, as the number of jumps of one type increases and the other remains constant the optimal portfolio tends to the opposite risk jump values. This is natural because that risk will tend to disappear when most of the jumps become only positive or negative. For other values of b a similar conclusion follows.

   Note also that if $B_+ > B_-$ then the intersection of the Merton line with the π axis is positive revealing again that there is less risk of negative jumps. .

5. The above analysis is valid as long as $B_+(s) > 0$ and $B_-(s) > 0$. Otherwise, if $B_-(s) = 0$ as noted in Proposition 3 the optimal utility is infinite if $b \geq 0$. The case where $a_+ a_-^{-1} \notin \mathbb{Q}$ (that is, the insider can count the jumps in order to know when to use his advantage optimally) is related to this case and leads to infinite logarithmic utility and therefore generate arbitrage in the model as it is shown in the proof of the next Result.

Result 15 *Suppose that $g(t) = 0$ and that there is no $k_1,\ k_2 \in \mathbb{N}$ such that $k_1 a_+ + k_2 a_- = 0$ then the maximal logarithmic utility of the $\mathcal{G}'$-investor is infinite.*

Proof Let x be such that $P(Z(T)=x)>0$. Then by the assumption $a_+a_-^{-1}\notin\mathbb{Q}$, there is a unique pair $k_+(x),k_-(x)\in\mathbb{N}$ such that $k_+(x)a_++k_-(x)a_-=x$. Therefore if $Z(T)=x$, we have for $p\in\{+,-\}$ that $N(\{a_p\},[0,T])=k_p(x)$ and

$$B_p(s)=\frac{k_p(x)-N(\{a_p\},[0,s])}{T-s}.$$

Now, we note that the hypotheses $\mathbf{HII_0}$ and $\mathbf{HII_1}$ are satisfied. In fact, note that

$$E\Big[\int_0^T\int_{\mathbb{R}}\mathbb{I}\{N(\{a_p\},[0,s))=k_p(Z(T))\}\mathbb{I}\{px>0\}F_s(dx)ds\Big]=0,$$
$$E\Big[\int_0^T\int_{\mathbb{R}}\mathbb{I}\{N(\{a_p\},[0,s))=k_p(Z(T))\}\mathbb{I}\{px<0\}F_s(dx)ds\Big]>0.$$

The first equality above is clear. As the jumps of one type are exhausted there are no jumps left and therefore the expectation becomes zero. The second inequality follows because the event that once after the jumps of one type are exhausted there still remains jumps of the other type has positive probability.

Therefore no matter what is the sign of b by Proposition 3 it follows that the utility is infinite.

Alternatively, one can also repeat the proof of Proposition 3 using the portfolio

$$\pi(s)=c_1\mathbb{I}\{N(\{a_-\},[0,s))=k_-(Z(T))\}-c_2\mathbb{I}\{N(\{a_+\},[0,s))=k_+(Z(T))\}$$

with $c_1,c_2>0$. □

Remark 16 *1. We recall that in the Wiener case considered in Corcuera et al. with $g(t)=t^\alpha$ played an important role in order to obtain finite logarithmic utility (if $\alpha<1$). In the case that the price is driven by a Poisson process, it is clear that the extra noise N' does not play the same role as in the Wiener case. Nevertheless, Theorem 10 also shows that there are cases where this addition is still meaningful.*

2. *Studying the case where Z is a simple compound Poisson process of the type described above shows why and how one needs to use a bigger filtration $\mathcal{H}'$ in hypothesis* **HI**. *First, by doing so, one obtains an explicit expression for optimal portfolios. Second, the projection on the smaller filtration $\mathcal{G}'$, is necessary in order to obtain finite utilities. In fact, one can prove that the logarithmic utility of the $\mathcal{H}'$-insider is infinite and leads to arbitrage in most cases. This result follows because if the information flow of the insider is $\mathcal{H}'$, then this agent knows at any time how many positive and negative jumps are left in the rest of the time interval.*

3. *Note that the support of the measure $F_s(dx)$ and the support of the conditioned measure $E[F_s(dx)/\mathcal{G}_s]$ do not necessarily lead to the same conclusions. In fact, the first is the set $\{a_+,a_-\}$ while in the proof of Result 15 we see that the support of the second may be concentrated in one of the two points a_+ or a_-.*

4. *In this article we have decided to concentrate on the case where the information of the insider is given by the final value of S. One can also do other examples such as the case of insider's filtrations with information about random times (see Kohatsu-Yamazato [22]).*

In the next subsection we generalize these results to general Lévy processes satisfying (1).

6.2 General Lévy process without Wiener component

In this subsection we study the optimal logarithmic utility in the case where Z is composed of positive and negative jumps without Wiener component (i.e. $c=0$). As various quantities repeat throughout the calculations we need to introduce some notations to simplify expressions.

Definitions and Notations

1. For $a > 0$ we define the measures $\nu_a(\cdot) := \nu(\cdot \cap [a, \infty))$ and $\nu_{-a}(\cdot) := \nu(\cdot \cap (-\infty, -a])$. Similarly, recall that we defined $\nu_+(\cdot) := \nu(\cdot \cap (0, \infty))$ and $\nu_-(\cdot) := \nu(\cdot \cap (-\infty, 0))$, respectively.

2. For $p \in \{+, -\}$, $\lambda_{pa} = \nu_{pa}(\mathbb{R})$ and $\lambda_p = \nu_p(\mathbb{R})$. Furthermore, we let $\lambda := \lambda_+ + \lambda_-$.

3.
$$\begin{aligned} a^* &:= a_+^* \wedge a_-^*, \\ a_+^* &:= \inf \{\text{supp}\,(\nu_+)\}, \\ a_-^* &:= -\sup \{\text{supp}\,(\nu_-)\}. \end{aligned}$$

   We remark that $a^* > 0$ implies that $0 \notin$ supp (ν) which further implies that $\lambda < \infty$.

4. We denote for $A \in \mathcal{B}(\mathbb{R})$ (when they can be defined)
$$N_t(A) := \int_t^T N(A, ds) + \int_0^{g(T-t)} N'(A, ds)$$
   where N' is the Poisson random measure associated with the process Z'.

5. Let $X(t) := \int_{\mathbb{R}} x N_t(dx)$. The process X is called the jump part of the process $L(t) := Z(T) - Z(t) + Z'(g(T-t))$.

6. We denote the distribution of a random variable Y by P^Y. Let ν^{n*} denote the n-th convolution of the measure ν provided that $\lambda < \infty$. Note that supp$(P^{X(s)}) = \mathcal{S}$ where $\mathcal{S} := \overline{\cup_{n=0}^{\infty}\text{supp}(\nu^{n*})}$ if $\lambda < \infty$ where $\nu^{0*} = \delta_{\{0\}}$.

7. Set $\mathcal{S}_+ = \mathcal{S} \cap (0, \infty)$ and $\mathcal{S}_- = \mathcal{S} \cap (-\infty, 0)$. Hence, $\mathcal{S} = \mathcal{S}_- \cup \{0\} \cup \mathcal{S}_+$.

8. Define the following collection of Borel sets by
$$E_p = \{A \in \mathcal{B}(\mathbb{R}) : A \subset \mathcal{S}_p, \forall n \geq 1, \nu^{n*}(A) = \nu_p^{n*}(A), \exists n \geq 1, \nu^{n*}(A) > 0\}$$
   for $p \in \{+, -\}$.

Most of the above definitions have a clear meaning. We only comment that intuitively, E_p is the family of sets of points that can be reached only through a sequence of positive ($p = +$) or negative ($p = -$) jumps. For example, E_p is the family of all nonempty subsets of $\{na_p : n = 0, 1, 2, \cdots\}$ for $p \in \{+, -\}$ in the setting of Section 6.1.

The following two theorems study when the logarithmic utility in the case $c = 0$ is finite or infinite. These results show that in this case, the existence of blurring noise ($g \neq 0$) does not affect to the finiteness of the maximal logarithmic utility. Their proofs are given in subsection 6.3.

Theorem 17 *Suppose that $c = 0$, supp$(\nu_+) \neq \emptyset$, supp$(\nu_-) \neq \emptyset$ and $0 \notin$ supp(ν). Furthermore, suppose that one of the following two conditions is satisfied.*
1) There exists $n \geq 2$ such that $\nu^{n*}(\{0\}) > 0$ and either
(a) $\mu > 0$ and $E_+ = \emptyset$ or
(b) $\mu < 0$ and $E_- = \emptyset$ holds.
2) $\mu = 0$, $E_+ = E_- = \emptyset$.
Then the maximal logarithmic utility of the $\mathcal{G}'$-investor is finite for any function g.

Theorem 18 *Suppose that $c = 0$ and $\lambda < \infty$. Furthermore, suppose that one of the following two conditions is satisfied.*
1) Either $\mu \geq 0$ and $E_+ \neq \emptyset$ or $\mu \leq 0$ and $E_- \neq \emptyset$.
2) Assume that $\mu \neq 0$ and $\nu^{n*}(\{0\}) = 0$ for all $n \geq 2$.
Then the maximal logarithmic utility of the $\mathcal{G}'$-investor is infinite for any function g.

Remark 19 ***A.*** *Some of the results introduced in this subsection and in Corollary 6 can be summarized in table form. In this table, we only consider some of the pure jump cases with* $\alpha < 1$, $c = 0$, *supp*$(\nu_+) \neq \emptyset$ *and supp*$(\nu_-) \neq \emptyset$.

λ	μ	*support condition*	*insider*	*small investor*
$\lambda < \infty$	≥ 0	$E_+ \neq \emptyset$	$= \infty$	$< \infty$
$0 \notin supp(\nu)$	> 0	$E_+ = \emptyset$, $\exists n \geq 2, \nu^{n*}(\{0\}) > 0$	$< \infty$	$< \infty$
$\lambda < \infty$	≤ 0	$E_- \neq \emptyset$	$= \infty$	$< \infty$
$0 \notin supp(\nu)$	< 0	$E_- = \emptyset$, $\exists n \geq 2, \nu^{n*}(\{0\}) > 0$	$< \infty$	$< \infty$
$\lambda < \infty$	$\neq 0$	$\forall n \geq 2\ \nu^{n*}(\{0\}) = 0$	$= \infty$	$< \infty$
$0 \notin supp(\nu)$	$= 0$	$E_+ = E_- = \emptyset$	$< \infty$	$< \infty$

Case $c = 0$, *supp*$(\nu_+) \neq \emptyset$ *and supp*$(\nu_-) \neq \emptyset$

B. *All the results of this section about finite utility can also be directly generalized to the case of power utility functions under enough moment conditions. To prove that the power utility is finite one uses Hölder's inequality to obtain the result.*

6.3 Proofs of Theorems in Subsection 6.2

For the proof of theorems 18 and 17, we start with a series of Lemmas that establish some estimates on the number of jumps of the Lévy process given its value.

Lemma 20 *Assume that* $\lambda < \infty$. *For* $p \in \{+, -\}$, *the following properties are satisfied*
(1) *if* $A \in E_p$ *and* B *is a Borel subset of* A, *then* $\nu^{n*}(B) = \nu_p^{n*}(B)$ *for all* $n \geq 1$,
(2) *if* $A, B \in E_p$, *then* $A \cup B \in E_p$,
(3) *if* $\{A_n\}$ *is an increasing sequence of sets in* E_p, *then* $A = \cup_{k=1}^{\infty} A_k \in E_p$,
(4) *if* $E_p \neq \emptyset$, *then there exists* $A \in E_p$ *such that no Borel subset of* $A^c \cap \mathcal{S}_p$ *is contained in* E_p.

Proof It is enough to prove the statements for $p = +$. Note that $\nu^{n*}(A) = \sum_{k=0}^{n} \binom{n}{k} \nu_+^{k*} * \nu_-^{(n-k)*}(A)$ for $A \in \mathcal{B}(\mathbb{R})$.

To prove (1), let $A \in E_+$. Then

$$0 = \nu_+^{k*} * \nu_-^{(n-k)*}(A) \geq \nu_+^{k*} * \nu_-^{(n-k)*}(B)$$

for $n \geq 1$ and $k = 0, \ldots, n-1$. That is, $\nu^{n*}(B) = \nu_+^{n*}(B)$ for all $n \geq 1$.

To prove (2), let $A, B \in E_+$, then $\nu^{n*}(A \cap B) = \nu_+^{n*}(A \cap B)$ for all $n \geq 1$ by (1) and

$$\begin{aligned} &\nu_+^{k*} * \nu_-^{(n-k)*}(A \cup B) \\ &= \nu_+^{k*} * \nu_-^{(n-k)*}(A) + \nu_+^{k*} * \nu_-^{(n-k)*}(B) - \nu_+^{k*} * \nu_-^{(n-k)*}(A \cap B) = 0 \end{aligned}$$

for $n \geq 1$ and $k = 0, \ldots, n-1$. Furthermore, by definition of E_+, there exist $n_1, n_2 \geq 1$ such that $\nu^{n_1*}(A), \nu^{n_2*}(B) > 0$. Then, $\nu^{n_1*}(A \cup B), \nu^{n_2*}(A \cup B) > 0$. Hence $A \cup B \in E_p$.

Proof of (3) Since $\nu^{n*}(A_k) = \nu_+^{n*}(A_k)$ for all $n \geq 1$, $\nu^{n*}(A) = \nu_+^{n*}(A)$ for all $n \geq 1$. For A_k, there is $n \geq 1$ such that $\nu^{n*}(A_k) > 0$. Then $\nu^{n*}(A) \geq \nu^{n*}(A_k) > 0$ and $A \in E_+$.

To prove (4), define the probability measure

$$P'(C) = \sum_{n=0}^{\infty} \frac{\nu^{n*}(C)}{n!} e^{-\lambda}$$

for $C \in \mathcal{B}(\mathbb{R})$ and set $a = \sup_{C \in E_+} P'(C)$. As $E_+ \neq \phi$ then $a > 0$. Choose $B_1, B_2, \cdots \in E_+$ so that $\lim_{n\to\infty} P'(B_n) = a$. Define $A_n = \cup_{k=1}^{n} B_k$ and $A = \cup_{n=1}^{\infty} A_n$. Then $A_n \in E_+$ by (2) and $A \in E_+$ by (3). By the inequality $P'(B_n) \leq P'(A_n) \leq P'(A) \leq a$, we have $P'(A) = a$. Suppose that there exists $B \subset A^c \cap \mathcal{S}_+$ such that $B \in E_+$. Then, $A \cup B \in E_+$ by (2). We have $P'(A \cup B) = P'(A) + P'(B) > P'(A) = a$. This contradicts the definition of a. The proof is finished. □

Lemma 21 *Assume that* $\lambda < \infty$ *and* $c = 0$. *For* $A \in E_+$, $s \in [0, T)$,

$$P(N_s((0,\infty)) > 0, X(s-) \in A) > 0, \tag{22}$$

$$P(N_s((-\infty,0)) > 0, X(s-) \in A) = 0. \tag{23}$$

Furthermore, if $a_+^* > 0$, *then*

$$E\left[N_s([a_+^*,\infty))/X(s-) = x\right] \leq \frac{x}{a_+^*} + 1,$$

$$E\left[N_s((-\infty,0))/X(s-) = x\right] = 0$$

for $P^{X(s-)}$*-a.a.* $x \in A$, $s \in [0,T)$. *Similar conclusions hold for* E_-. *In particular, for* $A \in E_-$ *and* $a_-^* < 0$ *we have*

$$E\left[N_s((-\infty,a_-^*])/X(s-) = x\right] \leq \frac{|x|}{a_-^*} + 1,$$

$$E\left[N_s((0,\infty))/X(s-) = x\right] = 0$$

for $P^{X(s-)}$*-a.a.* $x \in A$, $s \in [0,T)$.

Proof We fix $s \in [0,T)$ throughout the proof. We have that

$$P(N_s((0,\infty)) = n, X(s-) \in A) = e^{-\lambda(T-s+g(T-s))}(T-s+g(T-s))^n \frac{\nu^{n*}(A)}{n!} \tag{24}$$

for $n \geq 1$. Since $\nu^{n*}(A) > 0$ for some $n \geq 1$,

$$P(N_s((0,\infty)) > 0, X(s-) \in A) > 0.$$

Similarly, as $A \in E_+$, we also have that

$$P(N_s((-\infty,0)) > 0, X(s-) \in A) = 0.$$

Let $x \in A$ and let $B_\epsilon(x) = (x-\epsilon, x+\epsilon)$ for $0 < \epsilon < x$. We have by (24) and Lemma 20 (1),

$$\begin{aligned}
&E\left[N_s([a_+^*,\infty)) : X(s-) \in A \cap B_\epsilon(x)\right] \\
&\leq \lceil \frac{x+\epsilon}{a_+^*} \rceil e^{-\lambda(T-s+g(T-s))} \sum_{n=1}^{\lceil \frac{x+\epsilon}{a_+^*} \rceil} (T-s+g(T-s))^n \frac{\nu^{n*}(A \cap B_\epsilon(x))}{n!} \\
&= \lceil \frac{x+\epsilon}{a_+^*} \rceil P(X(s-) \in A \cap B_\epsilon(x)).
\end{aligned}$$

Here $\lceil y \rceil$ denotes the minimum integer which exceeds y. Since, by Lemma 20, the above inequality holds for any $\epsilon > 0$ then

$$E\left[N_s([a_+^*,\infty))/X(s-) = x\right] \leq \frac{x}{a_+^*} + 1$$

$P^{X(s-)}$-a.s. for $x \in A$. □

Proof of Theorem 18 First, we prove the theorem under 1) with $\mu \geq 0$ and $E_+ \neq \emptyset$. Let $A \in E_+$. In this case, we have, by (22) and (23) that

$$P(N_s((0,\infty)) > 0, X(s-) \in A) > 0$$

and

$$P(N_s((-\infty,0)) > 0, X(s-) \in A) = 0$$

for all $s \in [0,T)$. For $0 \leq s < t \leq T$, set $\tau^1 = s$ and $\tau^2 = t$. Then,

$$E\Big[\int_{\tau^1}^{\tau^2} \mathbb{I}(X(u-) \in A) \frac{N_u((0,\infty))}{T-u+g(T-u)} du\Big] > 0,$$
$$E\Big[\int_{\tau^1}^{\tau^2} \mathbb{I}(X(u-) \in A) \frac{N_u((-\infty,0))}{T-u+g(T-u)} du\Big] = 0.$$

Therefore the conclusion follows from Proposition 3 (1) with $\Upsilon_s^B = \mathbb{I}(X(s-) \in A)$. Note that $\bar{\pi}\Upsilon_s^B \mathbb{I}(\tau^1 < s \leq \tau^2)$ with $\bar{\pi} > 0$ is admissible due to the inequality $1 \leq 1 + (e^z - 1)\bar{\pi} \leq z\bar{\pi}$. The proof under 1) with $\mu \leq 0$ and $E_- \neq \emptyset$ is similar.

Now we suppose that $\nu^{n*}(\{0\}) = 0$ for all $n \geq 2$. That is, there is no combination of jumps that generate the value 0 for X. Therefore

$$P(X(s-)=0) = P\left(N_s((0,\infty))=0, N_s((-\infty,0))=0, X(s-)=0\right) = \exp\left(-\lambda(T-s+g(T-s))\right).$$

Define for $c > 0$ the following portfolio $\pi_s = c\mathbb{I}(X(s-)=0)sgn(\mu)$. Then $\pi \in \mathcal{A}$ and therefore from (5), $u(t,\pi) = c\,|\mu| \int_0^t P(X(s-)=0)ds > 0$ then taking $c \to \infty$ we obtain that the maximal logarithmic utility of the $\mathcal{G}'$-investor is infinite for any $t > 0$. □

In order to prepare for the proof of Theorem 17, we need some preliminary lemmas. The first one is a generalization of Lemma 21 under the stronger assumption $a^* > 0$.

Lemma 22 *Assume that $c = 0$, $a^* > 0$.*
(1) If for some $p \in \{+,-\}$ there exist a Borel set $A \subset \mathcal{S}_p$ such that $P^{X(s-)}(A) > 0$ and no Borel proper subset of A belongs to E_p, then

$$P(N_s([a^*,\infty)) > 0, N_s((-\infty,-a^*]) > 0 / X(s-) = x) > 0 \tag{25}$$

for $P^{X(s-)}$-a.a. $x \in A$.
(2) If there is $n \geq 2$ such that $\nu^{n*}(\{0\}) > 0$ then (25) holds for $x = 0$.

Proof (1) For any Borel set $B \subset A$ satisfying $P^{X(s-)}(B) > 0$, there is $n \geq 2$ and $1 \leq k \leq n-1$ such that $\nu_+^{k*} * \nu_-^{(n-k)*}(B) > 0$. Then

$$P(N_s([a^*,\infty)) > 0, N_s((-\infty,-a^*]) > 0, X(s-) \in B) > 0. \tag{26}$$

Hence

$$P(N_s([a^*,\infty)) > 0, N_s((-\infty,-a^*]) > 0 / X(s-) = y) > 0 \tag{27}$$

for $P^{X(s-)}$-a.a. $y \in A$. In fact, assume that there exists a Borel set $C \subset \mathcal{S}_p$ such that $P^{X(s-)}(C) > 0$ and

$$P(N_s([a^*,\infty)) > 0, N_s((-\infty,-a^*]) > 0 / X(s-) = z) = 0$$

for $z \in C$. Then,

$$P(N_s[a^*,\infty)) > 0, N_s((-\infty,-a^*]) > 0, X(s-) \in C) = 0.$$

By the definition of $\mathcal{S}_p$, $0 \notin C$. As $P^{X(s-)}(C) > 0$ then $C \in E_p$. This contradicts the assumption.
(2) If $\nu^{n*}(\{0\}) > 0$, then exists $1 \leq k \leq n-1$ such that $\nu_+^{k*} * \nu_-^{(n-k)*}(\{0\}) > 0$. Then we easily get (25). □

Lemma 23 (1) *Assume that $\mu > 0$. Let π be a solution of*

$$\mu + E\Big[\int_{(-\infty,-a^*]} \frac{e^z - 1}{1 + (e^z-1)\pi} F_s(dz) \Big/ X(s-) = x\Big] = 0.$$

If $\pi \leq 0$, then π satisfies

$$\pi \geq 1 - \frac{E(N_s((-\infty,-a^*])|X(s-)=x)}{\mu(T-s+g(T-s))}.$$

(2) *Assume that* $\mu < 0$. *Let* π *be a solution of*

$$\mu + E\Big[\int_{[a^*,\infty)} \frac{e^z - 1}{1 + (e^z - 1)\pi} F_s(dz) \Big/ X(s-) = x\Big] = 0.$$

If $\pi \geq 0$, *then* π *satisfies*

$$\pi \leq -\frac{E(N_s((-\infty, -a^*])|X(s-) = x)}{\mu(T - s + g(T - s))}.$$

Proof If $\pi < 0$ and $z < 0$, then the inequality $\frac{-1}{1-\pi} \leq \frac{e^z-1}{1+(e^z-1)\pi}$ holds, and if $\pi > 0$ and $z > 0$, then the inequality $\frac{1}{\pi} \geq \frac{e^z-1}{1+(e^z-1)\pi}$ holds. By these inequalities we easily get the conclusions. □

Proof of Theorem 17

First, note that due to the conditions of the Theorem, we have that $a^* > 0$. Next, define

$$f_s(\pi) = \mu\pi + E\Big[\int_{\mathbb{R}} \log(1 + (e^z - 1)\pi) F_s(dz) \Big/ X(s-) = x\Big].$$

We start proving the theorem under hypothesis 2). Since $E_+ = E_- = \emptyset$, (25) holds for all $x \in \mathcal{S}_+ \cup \mathcal{S}_-$. Hence by Lemma 22, we have

$$\begin{aligned} E\left[N_s([a^*, \infty))/X(s-) = x\right] > 0, \\ E\left[N_s(-\infty, -a^*])/X(s-) = x\right] > 0. \end{aligned} \tag{28}$$

Therefore for any $\pi \in \mathcal{A}$, we have that $-(e^{a^*} - 1)^{-1} \leq \pi_s \leq -(e^{-a^*} - 1)^{-1}$ for $X(s-) = x \in \mathcal{S}_+ \cup \mathcal{S}_-$. Then, in this case any admissible portfolio is uniformly bounded (the bounds depend only on a^*) and hence

$$\sup_{\pi \in \mathcal{A}} \int_0^T E\left[f_s(\pi_s) : X(s-) \in \mathcal{S}_+ \cup \mathcal{S}_-\right] ds < \infty$$

follows from the proof of Proposition 4 (note that as $c = 0$ hypothesis **HIII** is not needed).

In order to finish the proof under hypothesis 2) we need to study the case when $X(s-) = 0$. This case is subdivided into two cases.

In the first case, assume that there exists $n \geq 2$ such that $\nu^{n*}(\{0\}) > 0$, then (25) holds for $x = 0$ then portfolios are again bounded and the same previous argument applies.

In the second case, assume that $\nu^{n*}(\{0\}) = 0$ for all $n \geq 1$, then

$$\begin{aligned} E\left[N_s([a^*, \infty))/X(s-) = 0\right] = 0, \\ E\left[N_s(-\infty, -a^*])/X(s-) = 0\right] = 0. \end{aligned}$$

Since $\mu = 0$, then $\int_0^T E\left[f_s(\pi_s) : X(s-) = 0\right] ds = 0$. In conclusion, in any of the above two complementary cases, we have that

$$\sup_{\pi \in \mathcal{A}} \int_0^T E\left[f_s(\pi_s) : X(s-) = 0\right] ds < \infty.$$

Hence the maximal logarithmic utility is finite under hypothesis 2).

Next, we prove the Theorem under hypothesis $1) - (a)$. Note that by Lemma 20(4), there exists $A \in E_-$ such that $B \notin E_-$ for all Borel sets $B \subset A^c \cap \mathcal{S}_-$. Here again, we separate the proof in two complementary cases. $E_- = \emptyset$ is the first case.

As $E_- = \emptyset$, therefore $A = \emptyset$. Again by Lemma 22, we have that (28) is satisfied for $P^{X(s-)}$-a.a. $x \in \mathcal{S}_+ \cup \mathcal{S}_- \cup \{0\}$ as we are assuming that there exists $n \geq 2$ such that $\nu^{n*}(\{0\}) > 0$. Therefore $-(e^{a^*} - 1)^{-1} \leq \pi_s \leq -(e^{-a^*} - 1)^{-1}$ for $X(s-) \in \mathcal{S}_+ \cup \mathcal{S}_- \cup \{0\}$ for any $\pi \in \mathcal{A}$. Then, as in the proof under 2), any admissible portfolio is uniformly bounded and hence

$$\sup_{\pi \in \mathcal{A}} \int_0^T E\left[f_s(\pi_s) : X(s-) \in \mathcal{S}_+ \cup \mathcal{S}_- \cup \{0\}\right] ds < \infty$$

follows from Proposition 4.

Now, we consider the second case, $E_- \neq \emptyset$. Therefore, $A \neq \emptyset$. Then, by Lemma 21,

$$E\left[N_s((-\infty,-a^*])/X(s-)=x\right] \in \left(0, \frac{|x|}{a^*}+1\right], \qquad (29)$$
$$E[N([a^*,\infty))/X(s-)=x]=0$$

for $P^{X(s-)}$-a.s. $x \in A$. Therefore we can rewrite $f_s(\pi)$ as

$$f_s(\pi) = \mu\pi + E\Big[\int_{(-\infty,-a^*]} \log(1+(e^z-1)\pi)F_s(dz)\Big/X(s-)=x\Big]$$

for $P^{X(s-)}$-a.s. $x \in A$. We now study some properties of $f_s(\pi)$. From (4) and as $E\left[\int |z|F_s(dz)\right] = \int |z|\nu(dz) < \infty$, we have that

$$\lim_{\pi\to-\infty} \pi^{-1}E\Big[\int_{(-\infty,-a^*]} \log(1+(e^z-1)\pi)F_s(dz)\Big/X(s-)=x\Big] = 0.$$

Therefore as $\mu > 0$ then $\lim_{\pi\to-\infty} f_s(\pi) = -\infty$.

We prove now that the derivative of $f_s(\pi)$ exists. In fact, for $z < 0$ and $\pi < (e^{-a*}-1)^{-1}$ we have for sufficiently small h that

$$\left|\frac{\log(1+(e^z-1)(\pi+h)) - \log(1+(e^z-1)\pi)}{h}\right| \leq \frac{1}{|h|}\left|\int_\pi^{\pi+h} \frac{e^z-1}{1+(e^z-1)u}du\right|$$
$$\leq \max\{\big(1+(e^{-a*}-1)(\pi+h)\big)^{-1}, \big(1+(e^{-a*}-1)\pi\big)^{-1}\}.$$

Therefore the differentiability of $f_s(\pi)$ follows from the dominated convergence theorem.

Now, the optimality equation $f_s'(\pi) = 0$ becomes

$$\mu + E\left[\int_{(-\infty,-a^*]} \frac{e^z-1}{1+(e^z-1)\pi_s^*(x)}F_s(dz)\Bigg/ X(s-)=x\right] = 0. \qquad (30)$$

Therefore the optimal value that maximizes f_s exists. This value denoted by π_s^* is either $(1-e^{-a^*})^{-1}$ or is a solution of $f_s'(\pi) = 0$. We study the solution of the equation (30).

We remark that $f_s(\pi) \leq \mu\pi$ for $0 \leq \pi \leq (1-e^{-a^*})^{-1}$. Next, if $\pi_s^*(x) < 0$, then by Lemma 23

$$0 \wedge \left(1 - \frac{E\left[N_s((-\infty,-a^*])/X(s-)=x\right]}{\mu(T-s+g(T-s))}\right) \leq \pi_s^*(x) \leq 0$$

for $P^{X(s-)}$-a.s. $x \in A$ and consequently, using (29), we obtain $\pi_s^*(x) \geq 0 \wedge (1-h(x,s))$ for $X(s-)=x$, where $h(x,s) = \frac{|x|+a^*}{a^*\mu(T-s+g(T-s))}$. Then, for $z < 0$ and $X(s-)=x$, we have

$$1 \leq 1+(e^z-1)\pi_s^*(x) \leq (1-\pi_s^*(x)) \leq 1 \vee h(x,s).$$

We have using that $\pi^* \leq (1-e^{-a^*})^{-1}$ and (29),

$$\int_0^T E\left[f_s(\pi_s^*(X(s-))); X(s-)\in A\right]ds$$
$$\leq \int_0^T \int_A E\left[\int_{(-\infty,-a^*]} \log(1+(e^z-1)\pi_s^*(x))F_s(dz)\Bigg/ X(s-)=x\right] P(X(s)\in dx)ds$$
$$+\mu(1-e^{-a^*})^{-1}T$$
$$\leq \int_0^T \Big[\int_A |\log(1\vee h(x,s))||\mu|h(x,s)P(X(s-)\in dx)\Big]ds + \mu(1-e^{-a^*})^{-1}T. \qquad (31)$$

Since

$$P(X(s-)\in dx)=\sum_{n=1}^{\infty}e^{-\lambda(T-s+g(T-s))}\frac{(T-s+g(T-s))^n}{n!}\nu^{n*}(dx)$$

for $x\neq 0$ and $X(s)$ has a second moment, the right-hand side of (31) is finite. Furthermore for $X(s-)\in(A^c\cap\mathcal{S}_-)\cup\mathcal{S}_+$, we apply Lemma 20(4) and an argument similar to the proof under hypothesis (2) to give that for any admissible portfolio $-(e^{a^*}-1)^{-1}\leq\pi_s\leq-(e^{-a^*}-1)^{-1}$ for $X(s-)\in(A^c\cap\mathcal{S}_-)\cup\mathcal{S}_+$. This gives that the logarithmic utility for any admissible portfolio on the set $\{(s,\omega);X(s-)\in(A^c\cap\mathcal{S}_-)\cup\mathcal{S}_+\}$ is uniformly bounded for all admissible portfolios. A similar argument (also used in the proof under hypothesis 2)) applies for the case $X(s-)=0$. Consequently the maximal logarithmic utility

$$\begin{aligned}u(t,\pi)&=\sup_{\pi\in\mathcal{A}}E\left[\int_0^T f_s(\pi_s)ds\right]\\&\leq\sup_{\pi\in\mathcal{A}}\int_0^T E\left[f_s(\pi_s);X(s-)\in A\right]ds+\sup_{\pi\in\mathcal{A}}\int_0^T E\left[f_s(\pi_s);X(s-)\in(A^c\cap\mathcal{S}_-)\cup\mathcal{S}_+\right]ds\\&\quad+\sup_{\pi\in\mathcal{A}}\int_0^T E\left[f_s(\pi_s);X(s-)=0\right]ds\end{aligned}$$

is uniformly bounded for $\pi\in\mathcal{A}$ and therefore the proof finishes.

Next we prove the theorem under $1)-(b)$. As the proof has many common points with the previous case 1)-(a), we briefly give the main points in the proof. We treat the case that $E_+\neq\emptyset$. Let $A\in E_+$ with $B\notin E_+$ for all $B\subset A^c\cap\mathcal{S}_+$. For $X(s-)\in\mathcal{S}_-\cup(A^c\cap\mathcal{S}_+)\cup\{0\}$, the argument is the same as in $1)-(a)$ and hence

$$\sup_{\pi\in\mathcal{A}}\int_0^T E\left[f_s(\pi_s):X(s-)\in\mathcal{S}_-\cup(A^c\cap\mathcal{S}_+)\cup\{0\}\right]ds<\infty.$$

Note that $0\notin A$. Using Lemmas 21 and 23, we have $\pi_s^*\leq h(s,x)$ for $P^{X(s-)}$-a.s. $x\in A$. Since $\log(1+(e^z-1)x)\leq z+(\log x)\vee 0$,

$$\log(1+(e^z-1)\pi_s^*)\leq z+\Big(\log(h(s,x))\Big)\vee 0$$

for $z>0$, and as in the case $1)-(a)$ we have

$$\begin{aligned}&\int_0^T\int_A E\left[\int_{[a^*,\infty)}\log(1+(e^z-1)\pi_s^*)F_s(dz)/X(s-)=x\right]P(X(s-)\in dx)ds\\&\leq\int_0^T\Big[\int_{[a^*,\infty)}\{x+\log(h(x,s))\}|\mu|h(x,s)P(X(s-)\in dx)\Big]ds<\infty.\end{aligned}$$

The rest of the proof is similar to the case 1.a). Hence the maximal logarithmic utility is finite. □

The proof of Theorems 11 use the notation in Subsection 6.2. For this reason, we give it here.

Proof of Theorem 11

Without loss of generality, we assume that there exists $a>0$ such that $\nu_{-a}(\cdot)=\nu(\cdot\cap(-\infty,-a])\neq 0$. Suppose that $P^{L(s-)}(B)=P(L(s-)\in B)>0$ for a Borel set B. Then, $P^{L(s-)}(B-y)>0$ for all y by the absolute continuity and $\nu_{-a}*P^{L(s-)}(B)=\int P^{L(s-)}(B-y)\nu_{-a}(dy)>0$. Let Q_s be the distribution of $L(s-)-\int_{(-\infty,-a]}xN_s(dx)$. We have

$$\begin{aligned}E\left(N_s((-\infty,-a]):L(s-)\in B\right)&=e^{-\lambda_{-a}(T-s+g(T-s))}\sum_{n=1}^{\infty}\frac{\nu_{-a}^{n*}*Q_s(B)}{(n-1)!}(T-s+g(T-s))^n\\&=(T-s+g(T-s))\nu_{-a}*P^{L(s-)}(B)\\&>0.\end{aligned}$$

Then

$$E\left(N_s((-\infty,-a])/L(s-)=x\right)>0$$

for $P^{L(s-)}$-a.a. x. In the same way,

$$E(N_s([a,\infty))/L(s-)=x)>0.$$

Therefore any admissible portfolio is in the interval $(-(e^a-1)^{-1},-(e^{-a}-1)^{-1})$. Therefore the conclusion follows from the proof of Proposition 4. This finishes the proof. □

Note that the previous proof applies as long as Q_s has a strictly positive density in $\mathbb{R}$. Other conditions besides $c\neq 0$ for this property to be satisfied can be found in [25], Chapter 5.

Acknowledgements: This research was partially supported with grants of the Spanish and Japanese governments. The authors would also like to thank Prof. T. Jeulin for his comments and interest on this work.

7 Appendices

7.1 Appendix A: Power utility functions

Now, we briefly discuss the case of utility functions satisfying the inequality $U(x)\leq C_0x^\alpha+C_1$ for $x>0$ and some positive constants C_0, C_1 with $0<\alpha<1$. This class of utility functions is related to the class of power utility. In fact, we have that

$$u(t,\pi)\leq C_0E[V_t^\alpha]+C_1=C_0E[\exp(\alpha R_t)]+C_1.$$

As before we define the class of admissible portfolios for power utility functions

Definition 24 *We say that π is an admissible portfolio ($\pi\in\mathcal{A}_\alpha$) if π is a $\mathcal{G}'$-predictable real valued process such that there exists a unique solution to the wealth equation (3) which satisfies that for all $t\in[0,T)$, $V_t^\pi>0$. Furthermore the following quantities are finite a.s. for all $t\in[0,T)$*

$$B_1^\pi(t):=\int_0^t|\pi_s|^2\,ds<\infty \text{ a.s.},$$

$$B_2^\pi(t):=\int_0^t\int_{\mathbb{R}}|(1+(e^x-1)\pi_s)^\alpha-1|F_s(dx)ds<\infty \text{ a.s.},$$

$$B_3^\pi(t):=\int_0^t\int_{|x|\leq 1}\{(1+(e^x-1)\pi_s)^\alpha-1\}^2F_s(dx)ds<\infty \text{ a.s.}$$

and

$$B_4^\pi(t):=\int_0^t\int_{|x|>1}|(1+(e^x-1)\pi_{s-})^\alpha-1)|N(dx,ds)<\infty \text{ a.s.}$$

Then we have the following Proposition.

Proposition 25 *Let $\pi\in\mathcal{A}_{\alpha'}$ for $1>\alpha'>\alpha$ be an admissible portfolio such that there exists a constant $M>1$ with $|\pi(s)|\leq M$ for almost all $(s,\omega)\in[0,T]\times\Omega$. Furthermore, assume that*

$$E\left[\exp\left\{\left\{cM\alpha\left(1-\frac{\alpha}{\alpha'}\right)^{-1}\int_0^T|\beta(s)|ds\;+\alpha\left(1+\frac{e}{2}\right)\left(\frac{\alpha'}{\alpha}-1\right)^{-1}\int_0^T\int_{|x|\leq 1}|x|F_s(dx)ds\right.\right.\right.$$
$$\left.\left.\left.+\left(\frac{\alpha'}{\alpha}-1\right)^{-1}(2M)^{\alpha'}\int_0^T\int_{|x|>1}\exp(\alpha'x)\vee 1F_s(dx)ds\right\}\right\}\right]<\infty. \tag{32}$$

Then $u(t,\pi)<\infty$ for all $t\leq T$.

Proof In order to prove that the utility is finite we perform a change of measure. For this, we introduce the processes

$$\overline{X}_t = c\alpha p \int_0^t \pi_s dB_s + \int_0^t \int_{\mathbb{R}} \left((1 + (e^x - 1)\,\pi_{s-})^{\alpha p} - 1 \right) M(dx, ds), \tag{33}$$

$$\begin{aligned} \overline{Y}_t =& \int_0^t \left(\alpha q (b + c\beta(s))\pi_s - \frac{c^2}{2}\pi_s^2 \alpha q (1 - \alpha p) \right) ds \\ &+ \frac{q}{p} \int_0^t \int_{\mathbb{R}} \left((1 + (e^x - 1)\,\pi_s)^{\alpha p} - 1 \right) F_s(dx) ds \\ &- \int_0^t \int_{|x|\leq 1} \alpha q x \pi_s \nu(dx) ds \end{aligned}$$

for $p^{-1} + q^{-1} = 1$ with $p = \frac{\alpha'}{\alpha} > 1$ and $q = \left(1 - \frac{\alpha}{\alpha'}\right)^{-1}$. As in the proof of Proposition 4, we introduce

$$\tau_n = \inf\{t; \max\{B_i(t); i = 1, ..., 4\} \geq n\}.$$

Then using Hölder's inequality we have that

$$E[\exp(\alpha R_{t\wedge\tau_n})] \leq \left(E\left[\mathcal{E}(\overline{X})_{t\wedge\tau_n} \right] \right)^{1/p} \left(E\left[\exp(\overline{Y}_{t\wedge\tau_n}) \right] \right)^{1/q}. \tag{34}$$

Here $\mathcal{E}(\overline{X})$ stands for the Doléans-Dade exponential of the process $\overline{X}$. Note that since $V^\pi > 0$ a.s., the integrand of the second term of the right hand side of (33) is greater than -1 a.s. Hence the Doleans-Dade exponential $\mathcal{E}(\overline{X})$ is a positive local martingale and therefore its expectation is bounded by 1 in the first expectation on the right side of (34). For the second expectation, one uses the hypothesis (32) together with the following inequalities

$$(1 + (e^x - 1)y)^\alpha - 1 - \alpha x y \leq \frac{\alpha e M |x|^2}{2} \tag{35}$$

for $\alpha \in (0, 1)$, $|x| \leq 1$, $1 + (e^x - 1)y > 0$ and $|y| \leq M$

$$(1 + (e^x - 1)y)^\alpha - 1 \leq (2M)^\alpha \left(e^{\alpha x} \vee 1 \right) \tag{36}$$

for $\alpha > 0$, $|x| > 1$, $1 + (e^x - 1)y > 0$ and $|y| \leq M$, $M \geq 1$.

To prove (35) one considers the function $f(y) = (1 + (e^x - 1)y)^\alpha$ then using the Taylor expansion of order 2 together with the inequality $|e^x - 1 - x| \leq \frac{ex^2}{2}$ for $|x| \leq 1$ and $f''(y) \leq 0$ we have

$$\begin{aligned} (1 + (e^x - 1)y)^\alpha - 1 - \alpha x y &= \alpha(e^x - 1 - x)y + \int_0^y \int_0^z f''(w) dw dz \\ &\leq \frac{\alpha e M |x|^2}{2}. \end{aligned}$$

To prove (36), we divide the analysis in cases. First, in the case $x > 1$, we have that

$$(1 + (e^x - 1)y)^\alpha \leq (1 - y + e^x M)^\alpha \leq (2Me^x)^\alpha.$$

Also for $x < -1$ and $-M < y < 0$ we have that $(1 + (e^x - 1)y)^\alpha \leq (2M)^\alpha$. The other case has trivial bounds that are always smaller than $(2M)^\alpha \left(e^{\alpha x} \vee 1 \right)$. □

This proposition is the starting point to obtain similar results as the ones we have obtained for logarithmic utility in this article. For example, using this result as a base one can easily extend Corollaries 5 and 6.

7.2 Appendix B: Lévy driving processes with only positive jumps without Wiener part

In this case ($c = 0$, $\text{supp}(\nu_-) = \emptyset$), according to the result in Corollary 6, the small investor has finite maximal logarithmic utility if and only if $\mu < 0$. In contrast with this result, the insider's maximal logarithmic utility can be infinite even if $\mu < 0$. The following result treats a pure jump case with $g(T-s) = (T-s)^\alpha$, $0 < \alpha \le 1$. The integrability assumption for ν near 0 is slightly stronger than $\int_{|x|\le 1} |x|\nu(dx) < \infty$. In this subsection we use the notation introduced in section 6.2.

Theorem 26 *Assume that $c = 0$, $\mu < 0$, $\text{supp}(\nu_-) = \emptyset$, $\text{supp}(\nu_+) = \mathbb{R}_+$ and $g(T-s) = (T-s)^\alpha$. Furthermore, assume that either*
1) $\int_0^1 x^\beta \nu(dx) < \infty$ for some $0 < \beta < 1$ if $\alpha = 1$ or
2) $\int_0^1 x^\beta \nu(dx) < \infty$ for $\beta = \alpha$ if $\alpha < 1$.
Then the maximal logarithmic utility of the $\mathcal{G}'$-investor is infinite.

Proof Since $\text{supp}(\nu_-) = \emptyset$, then $E[N_s((y,\infty))/X(s-) = y] = 0$ for $y > 0$. Hence as $\text{supp}(\nu_+) = \mathbb{R}_+$, then for any admissible process we have that $\pi_s \ge \frac{-1}{e^y - 1}$ if $X(s-) = y > 0$.

Define $\pi_s(y) = \frac{-1}{2(e^y-1)}(\frac{T-s}{T})^\gamma$ for $X(s-) = y > 0$, where $\gamma = \frac{\alpha}{\beta} - 1 \ge 0$. Next, we note that as

$$\left|\frac{1}{2}(e^x - 1)\pi_s(X(s-))\right| \le \frac{1}{2},\ F_s(dx)\ a.s.$$

then $\pi_\cdot(X(\cdot-)) \in \mathcal{A}$.
Now, we estimate each of the terms that will appear later in the utility function. First, using that $0 < e^y - 1 \le 2y$ for $y \in [0,1]$ we have the following estimate

$$\begin{aligned}\int_0^\infty \mu\pi_s(y)P(X(s-) \in dy) &\ge \frac{|\mu|}{2}\int_0^1 \frac{1}{e^y - 1}(\frac{T-s}{T})^\gamma P(X(s-) \in dy) \\ &\ge \frac{|\mu|}{4}(\frac{T-s}{T})^\gamma \int_0^1 \frac{1}{y} P(X(s-) \in dy).\end{aligned}$$

To estimate the above integral, we use

$$\begin{aligned}&E[(X(s-))^{-1}\mathbb{I}(X(s-) \le 1)] \\ &= E\Big[\Big(\int_0^1 xN_s(dx)\Big)^{-1} - \Big(\int_0^1 xN_s(dx)\Big)^{-1}\mathbb{I}(\int_0^1 xN_s(dx) > 1)\Big] \\ &\ge \Big(\int_0^\infty E\Big[\exp\{-\theta\int_0^1 xN_s(dx)\}\Big]d\theta - 1\Big) \\ &= \Big(\int_0^\infty \exp\Big\{(T-s+g(T-s))\int_0^1 (e^{-\theta x} - 1)\nu(dx)\Big\}d\theta - 1\Big) \\ &\ge \Big(\beta^{-1}\Gamma(\beta^{-1})\Big\{(T-s+g(T-s))\int_0^1 x^\beta\nu(dx)\Big\}^{-1/\beta} - 1\Big).\end{aligned}$$

Here, we have used the following inequality for $\theta > 0$

$$\begin{aligned}-\int_0^1 \frac{1-e^{-\theta x}}{x^\beta}x^\beta\nu(dx) &\ge -\int_0^1 (\frac{1-e^{-\theta x}}{x})^\beta x^\beta \nu(dx) \\ &\ge -\theta^\beta \int_0^1 x^\beta\nu(dx)\end{aligned}$$

and $\int_0^\infty e^{-\theta^\beta}d\theta = \beta^{-1}\Gamma(\beta^{-1})$. Hence, in any of the two cases 1) or 2), we have

$$\int_0^T\int_0^\infty \mu\pi_s(y)P(X(s-) \in dy)ds = \infty.$$

On the other hand, we have $\frac{e^x-1}{x} \leq \frac{e^y-1}{y}$ for $0 < x < y$, which in turn implies

$$\left|\log\left(1 - \frac{(e^x-1)}{2(e^y-1)}(\frac{T-s}{T})^\gamma\right)\right| \leq \frac{x}{y}(\frac{T-s}{T})^\gamma$$

for $0 < x < y$. Therefore

$$\begin{aligned}
&\left|\int_0^{+\infty} E\left[\int_0^y \log(1+(e^x-1)\pi_s)F_s(dx)/X(s-)=y\right] P(X(s-)\in dy)\right| \\
&\leq \int_0^{+\infty} E\left[\int_0^y \frac{1}{y}(\frac{T-s}{T})^\gamma x F_s(dx)/X(s-)=y\right] P(X(s-)\in dy) \\
&= \int_0^{+\infty} \frac{1}{y}(\frac{T-s}{T})^\gamma E\left[\frac{X(s-)}{T-s+g(T-s)}/X(s-)=y\right] P(X(s-)\in dy) \\
&= \int_0^{+\infty} (\frac{T-s}{T})^\gamma \frac{P(X(s-)\in dy)}{T-s+g(T-s)} = \frac{((T-s)/T)^\gamma}{T-s+g(T-s)}.
\end{aligned}$$

Then

$$\begin{aligned}
&\int_0^T \int_0^\infty \left|E\Big[\int_0^y \log(1+(e^x-1)\pi_s)F_s(dx)|X(s-)=y\Big]\right| P(X(s-)\in dy)ds \\
&\leq \int_0^T \frac{((T-s)/T)^\gamma}{T-s+g(T-s)} ds < \infty.
\end{aligned}$$

Hence putting these two estimates together, we obtain that

$$u(T,\pi) = \int_0^T E\left[\mu\pi_s\right] ds + \int_0^T E\left[E\left[\int_0^\infty \log(1+(e^x-1)\pi_s)F_s(dx)/\mathcal{G}_s\right]\right] ds.$$

where the first term is infinite and the second is finite. Therefore the maximal logarithmic utility of the $\mathcal{G}'$-investor is infinite. □

If $c = 0$, $\mu \geq 0$, $\text{supp}(\nu_+) \neq \emptyset$ and $\text{supp}(\nu_-) = \emptyset$, then by Proposition 3(1), $\mathcal{G}$-maximal logarithmic utility is infinite.

The idea of the previous proof can be extended in a variety of ways. For example, the proof of the following theorem is exactly the symmetric of the previous proof.

Theorem 27 *Assume that* $c = 0$, $\mu > 0$, $\text{supp}(\nu_+) = \emptyset$, $\text{supp}(\nu_-) = \mathbb{R}_-$ *and* $g(T-s) = (T-s)^\alpha$. *Furthermore, assume that either*
1) $\int_{-1}^0 |x|^\beta \nu(dx) < \infty$ *for some* $0 < \beta < 1$ *with* $\alpha = 1$ *or*
2) $\int_{-1}^0 |x|^\beta \nu(dx) < \infty$ *for* $\beta = \alpha$ *if* $\alpha < 1$.
Then the maximal logarithmic utility of the $\mathcal{G}'$*-investor is infinite.*

References

[1] Amendiger, J., Imkeller, P., Schweizer, M. : Additional logarithmic utility of an insider. Stochastic Proc. Appl. 75, 263-286 (1998).

[2] Applebaum, D. : Lévy Processes and Stochastic Calculus, Cambridge University Press (2004).

[3] Back, K. : Insider Trading in Continuous Time. Rev. Financial Stud. 5, 387-409 (1992).

[4] Cont, R. Tankov, P.: Financial Modelling With Jump Processes. Chapman & Hall/CRC Financial Mathematics Series, Boca Raton 2004

[5] Corcuera J. M., Imkeller P., Kohatsu-Higa A., Nualart D. : Additional utility of insiders with imperfect dynamical information. Finance Stoch. 8, 437-450 (2004).

[6] Corcuera J.M., Guerra J., Nualart D., Schoutens W. : Optimal investment in a Lévy market. Appl. Math. Optim. 53, 279-309, (2006).

[7] Delbaen, F., Schachermayer, W. : The Mathematics of Arbitrage. Springer Finance, 2006.

[8] Di Nunno G., Meyer-Brandis T., Oksendal B., Proske F. : Optimal portfolio for an insider in a market driven by Lévy processes. Quant. Finance, 6, 83-94, (2006).

[9] Framstad N.C., Oksendal B., Sulem A. : Optimal consumption and portfolio in a jump diffusion model. In : Shiryaev, A., Sulem, A. (eds.) Proceedings of the Workshop on Mathematical Finance, INRIA, Paris (1998).

[10] Goll T., Kallsen J. : Optimal Portfolios for Logarithmic Utility. Stochastic Proc. Appl. 89, 31-48 (2000).

[11] Goll T., Kallsen J. : A Complete Explicit Solution to the Log-Optimal Portfolio Problem. Ann. Appl. Probab. 13, 774-799 (2003).

[12] Grorud, A. : Asymmetric information in a financial market with jumps. Int. J. Theor. Appl. Finance 3, 641-659 (2000).

[13] Grorud, A., Pontier, M. : Insider Trading in a continuous Time Market Model. Int. J. Theor. Appl. Finance 1, 331-347 (1998).

[14] Ikeda, N., Watanabe, S., : Stochastic Differential Equations and Diffusion Processes. North-Holland/Kodansha, Amsterdam New York Oxford Tokyo (1989).

[15] Imkeller, P. : Enlargement of the Wiener filtration by an absolutely continuous random variable via Malliavin's calculus. Probab. Th. Rel . Fields 106, 105-135 (1996).

[16] Imkeller, P. : Enlargement of the Wiener filtration by a manifold valued random element via Malliavin's calculus. In Statistics and Control of Stochastic Processes. In : Kabanov, Y. M., Rozovskii, B.L. , Shiryaev, A.N.(eds.) The Lipster Festschrift, World Scientific, Singapore (1997).

[17] Imkeller, P. : Random times at which insiders can have free lunches. Stochastics Stochastics Rep. 74 465-487, (2002).

[18] Imkeller, P., Pontier, M., Weisz, F. : Free lunch and arbitrage possibilities in a financial market with an insider. Stochastic Proc. Appl. 92, 103-130 (2001).

[19] Jacod, J. : Grossissement initial, hypothèse (H'), et théorème de Girsanov. In : Jeulin, T., Yor, M. (eds.) Grossissements de filtrations: exemples et applications. Lect. Notes in Math. 1118, 15-35, Springer-Verlag, Berlin Heidelberg New York (1985).

[20] Jacod, J., Shiryaev, A.N.: Limit Theorems for Stochastic Processes. Second Edition. A series of Comprehensive Studies in Mathematics. Springer-Verlag, 2003.

[21] Karatzas, I., Pikovsky, I. : Anticipative portfolio optimization. Adv. Appl. Prob. 28, 1095–1122 (1996).

[22] Kohatsu-Higa A., Yamazato M. : Enlargement of filtrations with random times for processes with jumps. Stochastic Proc. Appl. 1136-1158 (2008).

[23] Kunita H. : Mathematical Finance for price processes with jumps. In : Akahori et al. (eds) Stochastic Processes and Applications to Mathematical Finance, Proceedings of the Ritsumeikan International Symposium, World Scientific, Singapore (2003).

[24] Protter, P. : Stochastic Integration and Differential Equations. A New Approach. Springer-Verlag, Berlin Heidelberg New York (2004).

[25] Sato K. : Lévy processes and infinite divisible distributions. Cambridge University Press, Cambridge (1999).

[26] Yor, M. : Grossissement de filtrations et absolute continuité de noyaux. In : Jeulin, T., Yor, M. (eds.) Grossissements de filtrations: exemples et applications, Lect. Notes in Math. 1118, 6-14, Springer-Verlag, Berlin Heidelberg New York (1985).