A Malliavin Calculus method to study densities of additive functionals of SDE's with irregular drifts*

Dedicated to the memory of Prof. Paul Malliavin

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Abstract

We present a general method which allows to use Malliavin Calculus for additive functionals of stochastic equations with irregular drift. This method uses the Girsanov theorem combined with Itô-Taylor expansion in order to obtain regularity properties for this density. We apply the methodology to the case of the Lebesgue integral of a diffusion with bounded and measurable drift.

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1 Introduction

Paul Malliavin in the eighties developed a stochastic method to study the regularity of probability laws arising on the Wiener space. In particular, using this stochastic method he obtained the existence and smoothness of densities of diffusions under the hypoelliptic condition. This celebrated theory has been applied successfully to a variety of stochastic equations providing useful properties of densities of random variables associated to these equations. It is based on the integration by parts formula which requires "stochastic" smoothness of the random variable in question.

For this reason, one usually requires smoothness of the coefficients of the stochastic equation. On the other hand, many results from partial differential equations provide the same results for the density of uniformly elliptic diffusions under weaker conditions on (some of) the coefficients, such as Hölder continuity or even bounded and measurable. Still, many stochastic equations (e.g. stochastic partial differential equations) do not have a partial differential equation counterpart and therefore the current results on existence and smoothness of probability laws are limited to the case of smooth coefficients.

Here we consider a d_1-dimensional diffusion

$$X_t^x = x + \int_0^t b(X_s^x)ds + \int_0^t \sigma(X_s^x) \circ dW_s, \tag{1}$$

where $\sigma : \mathbb{R}^{d_1} \to \mathbb{R}^{d_1 \times m}$, $b : \mathbb{R}^{d_1} \to \mathbb{R}^{d_1}$, $x \in \mathbb{R}^{d_1}$. We assume that σ is smooth and uniformly elliptic while b is measurable and bounded. Here W denotes a m-dimensional Brownian motion defined on the probability space $(\Omega, \mathcal{F}, P)$ and the above stochastic integral is the Fisk-Stratonovich integral. Under these conditions, we obtain the existence and smoothness of the density of Y_t where $Y_t = \int_0^t \psi(X_s^x)ds$ where $\psi : \mathbb{R}^{d_1} \to \mathbb{R}^{d_2}$ is a smooth function with bounded derivatives. Note that this random vector is non-elliptic in the sense that there is no Wiener component in the equation for Y and that the system (X, Y) is of dimension $d = d_1 + d_2$.

Now, we introduce our main condition. Define the vector fields

$$V_0 = \sum_{i=1}^{d_2} \psi_i \frac{\partial}{\partial x_{i+d_1}} - \frac{1}{2} \sum_{i,j=1}^{d_1} \sum_{l=1}^{m} \sigma_{jl} \partial_j \sigma_{il} \frac{\partial}{\partial x_i},$$

$$V_i = \sum_{j=1}^{d_1} \sigma_{ji} \frac{\partial}{\partial x_j}, \ 1 \leq i \leq m.$$

Then, define

$$A_0 = \{V_1, ..., V_m\}$$

$$A_n = \{[V_i, Z]; i = 0, ..., m, Z \in A_{n-1}\}, n \geq 1.$$

We assume that there exists $k \in \mathbb{N}$ and $c_0 > 0$ such that that for all $\xi \in \mathbb{R}^d$ and $x \in \mathbb{R}^{d_1}$

$$\sum_{V \in \cup_{l=1}^{k} A_l} \langle V(x), \xi \rangle^2 \geq c_0 \|\xi\|^2 . \tag{2}$$

For example, in the one dimensional case ($d_1 = d_2 = m = 1$), if we assume that $|\psi'(x)| \geq c_0$ and σ is uniformly elliptic then the above hypothesis (2) is satisfied.

Our goal is then to prove

Theorem 1. *Consider the random vector $Y_t = \int_0^t \psi(X_s^x) ds$ where X satisfies (1). Assume that the coefficients σ and ψ are smooth with bounded derivatives, b is a bounded and measurable function, σ is uniformly elliptic and that the system (X, Y) satisfies (2). Then the vector Y_t has a smooth density.*

The regularity of the joint density of time averages of diffusions is a problem that appears in various applied fields ranging from average quantities in economics, finance[1] and filtering problems[2] between others. One possible example is to consider one dimensional diffusions with drifts that have a discontinuity of the first kind (for some examples, see e.g. [27] and the references therein. Also see the references on existence and uniqueness of solutions for SDE's with irregular drift).

From the mathematical point of view, another related attempt to deal with irregularities in the coefficients of the diffusion using Malliavin Calculus can be found in [26]. Note that if we are only considering the density of the random vector X_t^x then the problem could be studied by using techniques from partial differential equations as can be seen in [15] or [24] but only limited regularity is obtained. The goal of the present

[1] Notably the Asian option considers as its main random variable the integral of the stock price

[2] The signal process is an example that belongs to an extended version of the above theorem to be considered in Theorem 7.

research is to try to establish an extension of the Malliavin Calculus technique to deal with situations where the drift is irregular.

Our method of proof can be succinctly explained as follows. First, we use Girsanov's theorem in order to remove the irregular drift from the stochastic equation. This is why we require uniform ellipticity. This reduces our study to the case where the stochastic equation is regular driven by a new Brownian motion, say B. Still, one has to deal with the non-smooth change of measure as this will appear in the expression for the characteristic function.

The change of measure is then expanded using Itô-Taylor expansion up to a high order. We prove that the residue is small. The problem is to be able to carry out the integration by parts formula without requiring derivatives of the multiple integrals as they contain the non-smooth drift..

Next, we divide the integration region of the multiple stochastic integral so that a fixed interval is not included in the region of integration. We call this interval, $[\alpha_1, \alpha_2]$. Using this interval we take advantage that the noise B within this interval regularizes the irregular drift by use of conditional expectations.

Then we perform a conditional integration by parts (ibp) in this interval with respect to B. If we perform a usual ibp we will not avoid considering derivatives of the drift.

The final idea consists in taking expectations with respect to the Wiener process, B, in the interval $[\frac{\alpha_1+\alpha_2}{2}, \alpha_2]$ in order to regularize the drift appearing in the multiple integrals, and then applying the ibp in the interval $[\alpha_1, \frac{\alpha_1+\alpha_2}{2}]$ (where no drift appears) as many times as necessary, in order to obtain the smoothness of the density. Here is where the additivity of Y plays an important role.

The method just described is rather general and can be applied to many situations. In fact, we will consider a slight generalization of the problem proposed here (see also the conclusions section at the end of the article).

This article has the following sections. In section 2, we give the main set up of the problem. In subsection 2.1, we give notation from Malliavin Calculus that will be used in this article. Then in Section 3 we give our general set-up for the problem and the generalization of Theorem 1 which we prove in Section 4. We close with the proof of Theorem 1 and an Appendix where some preliminary estimates are proven.

Throughout the article, we assume that all stochastic differential equations have unique strong solutions. In recent years, various applications of the system described here have appeared (in particular the 3D-Navier

Stokes equation between others) and has led to the study of existence and uniqueness of stochastic differential equations with irregular drifts in weak and pathwise form (the interested reader may look at [2], [25], [5], [3], [4], [6], [10], [11], [16], [17], [22], [18], [28] between others). In order to avoid this discussion, we decided to use a set-up where the Girsanov theorem has already been applied (see Section 3).

For a multi-index $\alpha = (\alpha_1, ..., \alpha_k) \in \mathbb{N}^k$, we denote its length by $|\alpha| = k$. Without further mention we use sometimes the convention of summation over double indices (Einstein convention). When the exact values of the constants are not important we may repeat the same symbol for constants that may change from one line to the next. These constants will be denoted by an over bar symbol. For a matrix A, A^T denotes the transpose of A.

Vectors will be always considered as column vectors except for vectors denoting time which are considered as line vectors. For a vector of time variables $s = (s_1, ..., s_n)$, we use the following notations: $ds = ds_n...ds_1$, $s^j = (s_1, ..., s_j)$, $\bar{s}^{n-j+1} = (s_j, ..., s_n)$. Similarly we denote by $\pi^i(s)$ and $\bar{\pi}^i(s)$ the projection operator for the first i components and the last i components of s respectively. Sometimes, we may use s to denote a real time (instead of a vector one) integration variable. The difference should be clear from the context. Partial derivatives will be denoted by $\partial_i f(x)$, $x \in \mathbb{R}^n$ where this means the i-th variable appearing in the function f. For a multi-index $\alpha = (\alpha_1, ..., \alpha_k) \in \{1, ..., d\}^k$ we denote $(i\theta)^{-\alpha} = i^{-k} \prod_{i=1}^k \theta_{\alpha_i}^{-1}$, $|\theta|^{-\alpha} = \prod_{i=1}^k |\theta_{\alpha_i}^{-1}|$ and $\partial_\alpha = \prod_{i=1}^k \partial_{\alpha_i}$. The inner product between two vectors u and v is denoted by $u \cdot v$. Similar notation is used for products of vectors and matrices or between matrices. Stochastic processes may be denoted interchangeably by $X(t)$ or X_t. As usual, constants may change from one line to the next even if the same notation is used. Finally, j will denote the imaginary unit and although there may be a slight abuse of notation (occasionally j may denote an index in a summation) the context will determine clearly the intended meaning.

2 Preliminaries

Let $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t \geq 0})$ be a filtered standard $m-$dimensional Wiener space with filtration given by $\mathcal{F}_t$ (see [8] Chapter 1, Section 2.25). We will use the following classical result which characterizes the existence and smoothness of densities

Lemma 2. *Let F be a random vector and denote by $h_F(\theta) = E[\exp(j\theta \cdot F)]$ its characteristic function. If* *for $k \in \mathbb{N}$, we have that*

$$\int \|\theta\|^k |h_F(\theta)| \, d\theta < \infty$$

then F has a density and it belongs to the class C_b^k.

Note that in order to obtain the smoothness of the density of F it is enough to prove that for any $k \in \mathbb{N}$ there exists a positive constant C_k such that for all $\|\theta\| \geq 1$, we have that $\|\theta\|^k |h_F(\theta)| \leq C_k$.

2.1 Malliavin Calculus

We give a brief introduction to Malliavin Calculus. For details, we refer the reader to [7], [19], [20] or [21]. From now on we fix $T > 0$. For any measurable function $h \in L^2([0,T]; \mathbb{R}^m) =: \mathcal{H}_{0,T}$ we denote its stochastic integral by $W(h) = \int_0^T h(s) \cdot dW_s$ and by $\mathcal{S}$ the class of smooth functionals

$$\mathcal{S} = \{F : F = f(W(h_1), ..., W(h_n)); h_1, ..., h_n \in \mathcal{H}_{0,T}; f \in C_p^\infty(\mathbb{R}^n; \mathbb{R}); n \in \mathbb{N}\}.$$

Here C_p^∞ denotes the class of C^∞-functions with at most polynomial growth at infinity. For $F \in \mathcal{S}$ we define the Malliavin derivative as

$$D_t^j F = \sum_{i=1}^n \partial_i f(W(h_1), ..., W(h_n)) h_i^j(t).$$

For $k \in \mathbf{Z}_+$ and $p \geq 1$ we define the following semi-norms

$$||F||_{k,p} = \left(E[|F|^p] + \sum_{i=1}^k E\left[||D^i F||^p_{\mathcal{H}_{0,T}^{\otimes i}} \right] \right)^{\frac{1}{p}}$$

$$||D^i F||_{\mathcal{H}_{0,T}^{\otimes i}} = \left(\int_0^T ... \int_0^T |D^i_{s_1,...,s_i} F|^2 ds_1 ... ds_i \right)^{\frac{1}{2}}$$

As usual, $\|F\|_{0,p} \equiv \|F\|_p$. If we complete these semi-norms appropriately, one defines the space $\mathbf{D^{k,p}}$. In such a case for $F \in \mathbf{D^{k,p}}$, we can define the k-th order derivative as $D_{s_1,...,s_i} F = D_{s_1} ... D_{s_i} F$. As with multiple derivatives we use the symbol $D_s^\alpha F$ for a multiple derivative with respect to the noises indexed in α at the times indicated by the time vector s. Furthermore we define the space of smooth random variables $\mathbf{D}^\infty = \cap_{k,p} \mathbf{D^{k,p}}$. Similarly, for a Hilbert space V and a V-valued random variable one can define $\mathbf{D^{k,p}}(V)$ and $\mathbf{D}^\infty(V)$. In particular, for a $\mathbb{R}$-valued random process $u_s, s \leq T$, we define the following semi-norm

$$||u||_{k,p} = \left(E[||u||^p_{\mathcal{H}_{0,T}}] + \sum_{i=1}^k E\left[||D^i u||^p_{\mathcal{H}_{0,T}^{\otimes i+1}} \right] \right)^{\frac{1}{p}}.$$

We define the Skorokhod integral δ as the dual operator of the closable operator D. When δ is restricted to $\mathcal{F}_t$-adapted L^2 stochastic processes $\{u_s, s \leq T\}$ then the Skorokhod integral coincides with the Itô integral $\delta(u)$. For this reason we sometimes write $\delta(u) = \int_0^T u_t \cdot dW_t$ or $\delta(u) = \int_0^T u_t \cdot \delta W_t$ if we want to be more explicit. Furthermore if u is an element in the domain of δ (this is the case if e.g. $u \in \mathbf{D}^{1,2}(L^2[0,T])$) and $F \in \mathbf{D}^{1,2}$ and it satisfies $E\left[F^2 \int_0^T u_t^2 dt\right] < \infty$ then

$$\delta(Fu_t) = F\delta(u) - \int_0^T (D_t F) \cdot u_t dt$$

is satisfied provided the right hand side is square integrable and furthermore we have the following duality formula

$$E[F\delta(u)] = E\left[\int_0^T D_t F \cdot u_t dt\right]. \tag{3}$$

Definition 3. *For a random vector $F = (F^1, ..., F^d) \in \left(\mathbf{D}^{1,2}\right)^d$, we define the $d \times d$ Malliavin covariance matrix M_F*

$$M_F^{ij} = < DF^i, DF^j >_{\mathcal{H}_{0,T}} . \tag{4}$$

If we have that for any $p \geq 1$,

$$E[(\det M_F)^{-p}] < +\infty$$

then we say that the random vector F is non-degenerate.

In order to simplify notation we sometimes consider the $d \times m$ matrix $DF = (D^j F^i)_{ij}$ for $F = (F^1, ..., F^d)$. We also have the following integration by parts formula.

Proposition 4. *Let $F \in (\mathbf{D}^\infty)^d$ be nondegenerate and let $G \in \mathbf{D}^\infty$, $\phi \in C_p^\infty$. Then for any multi-index $\alpha \in \{1, ..., d\}^{|\alpha|}$, we have*

$$E[\partial_\alpha \phi(F) G] = E[\phi(F) H_\alpha(F, G)]$$

for a random variable $H_\alpha(F, G) \in \mathbf{D}^\infty$ which has the following explicit expression

$$H_{(\alpha_1,..,\alpha_k)}(F, G) = H_{(\alpha_k)}(F, H_{(\alpha_1,...,\alpha_{k-1})}(F, G))$$

$$H_{(\alpha_1)}(F, G) = \delta(G \left(M_F^{-1}\right)^{\alpha_1 j} DF^j).$$

Furthermore there exists positive integers β, γ, q, n_1 and n_2 such that

$$\|H_\alpha(F, G)\|_p \leq C \left\|(\det M_F)^{-1}\right\|_\beta^{n_1} \|DF\|_{|\alpha|,\gamma}^{n_2} \|G\|_{|\alpha|,q} . \tag{5}$$

We will also use conditional Malliavin Calculus in various part of the article. For this we introduce the notation

$$E_t[G] = E[G|\mathcal{F}_t]$$

$$||F||_{t,t+h,k,p} = \left(E_t[|F|^p] + \sum_{i=1}^{k} E_t\left[||D^i F||^p_{\mathcal{H}^{\otimes i}_{t,t+h}} \right] \right)^{\frac{1}{p}}, \quad F \in \mathbf{D}^{k,p}, k \geq 1, p \geq 2, \ h > 0.$$

We also use the following notation for the Skorohod integral $\delta_{t,t+h}(u) = \int_t^{t+h} u_s \cdot dW_s$. We remark as before that if u is in the domain of δ and $E\left[F^2 \int_0^T u_t^2 dt\right] < \infty$ and $F \in \mathbf{D}^{1,2}$ then we have the duality formula

$$E_t\left[F\delta_{t,t+h}(u)\right] = E_t\left[\int_t^{t+h} D_s F \cdot u_s ds\right] \tag{6}$$

if both terms inside the expectations are conditionally square integrable. In many situations we will have to change probability spaces in order to carry the Malliavin Calculus in the standard Wiener space, given that the quantity to be computed is determined only by the law of W. We will do this without any further mention.

Note that usually the space $\mathcal{H}$ used in the definition of the Malliavin covariance matrix (4) is $\mathcal{H} = L^2[0,T]$. Some other times we may use $\mathcal{H} = L^2[t,t+h]$. In such a case we may use the notation $M_F^{[t,t+h]}$, $H_\alpha^{[t,t+h]}$ etc.

In the next result we give an extension of various classical theorems that can be found in [20], [9], [12], [13] and [14]. The proof is quite similar so we leave it for the reader. We will apply the following results in various situations.

3 Problem set-up and main result

The problem that we will treat is the integration by parts formula for the following expression

$$h(\theta) = E\left[\exp(j\theta \cdot Y_t^{0,z})\rho(t)\right]$$

$$\rho(t) = \exp\left(\int_0^t \bar{b}(X_s^{0,x}) \cdot dW_s - \frac{1}{2}\int_0^t \bar{b}^T\bar{b}(X_s^{0,x})ds\right)$$

where W is a m-dimensional Wiener process and

$$X_t^{s,x} = x + \int_s^t b(X_u^{s,x})du + \int_s^t \sigma(X_u^{s,x}) \circ dW_u, \tag{7}$$

$$Y_t^{s,z} = y + \int_s^t \psi(X_u^{s,x})du + \int_s^t \varphi(X_u^{s,x}) \circ dW_u \tag{8}$$

for $z = (x, y)$. The following hypotheses will be used in our main theorem:

Hypothesis (A1): $\bar{b} : \mathbb{R}^{d_1} \to \mathbb{R}^m$ is a bounded and measurable function.

Hypothesis (A2): All the coefficients $b : \mathbb{R}^{d_1} \to \mathbb{R}^{d_1}$, $\sigma : \mathbb{R}^{d_1} \to \mathbb{R}^{d_1 \times m}$, $\varphi : \mathbb{R}^{d_1} \to \mathbb{R}^{d_2 \times m}$ and $\psi : \mathbb{R}^{d_1} \to \mathbb{R}^{d_2}$ are smooth with bounded derivatives.

Hypothesis (A3):$\sigma^T \sigma(x)$ is an invertible matrix for all $x \in \mathbb{R}^m$.

Remark 5. *1. The process $Y^{s,(x,0)}$ will appear frequently in our calculations so we may simplify the notation to $Y^{s,x} \equiv Y^{s,(x,0)}$. Similarly, we will also use $Y_t = Y_t^{0,x}$ and $X_t \equiv X_t^{0,x}$. Note that due to the fact that the initial value of Y is zero the flow properties are slightly modified. In fact, the classical flow property is written as*

$$\exp(j\theta \cdot Y_t^{0,z}) = \exp(j\theta \cdot Y_s^{0,z}) \exp(j\theta \cdot Y_t^{s,(X_s^{0,x}, Y_s^{0,z})}). \tag{9}$$

2. We also remark that under hypothesis (A2), we have that there exists a pathwise unique solution for (X, Y) and it has smooth flows satisfying the Markov property. We define its associated semigroup by

$$P_t f(z) = E[f(Z_t^{0,z})]$$

for $Z_t^{s,z} := (X_t^{s,x}, Y_t^{s,z})$. Note that the following basic estimate is satisfied

$$|P_t f(z)| \leq C \|f\|_\infty .$$

3. Furthermore under hypothesis (A3) we remark that

$$(\sigma^T \sigma)^{-1} \sigma(X_t)^T \left(\tilde{b}(X_t)dt - dX_t \right) = dW_t$$

and therefore

$$\sigma(X_s, s \leq t) = \sigma(W_s, s \leq t)$$

is satisfied. Here

$$\tilde{b}_i = b_i = \frac{1}{2} \sum_{j=1}^{d_1} \sum_{l=1}^{m} \partial_j \sigma_{il} \sigma_{jl}.$$

Therefore we also have that X satisfies the Markov property(see e.g. [23] Section V.6 Theorem 32). For example, we have

$$E\left[f\left(\int_a^b g_1(X_s)dW_s + \int_a^b g_2(X_s)ds\right)\middle|\mathcal{F}_a\right] = E\left[f\left(\int_a^b g_1(X_s)dW_s + \int_a^b g_2(X_s)ds\right)\middle|X_a\right].$$

This property can also be generalized to multiple stochastic integrals.

We define the vector fields associated to Z

$$V_0 = \sum_{i=1}^{d_1} b_i \frac{\partial}{\partial x_i} + \sum_{i=1}^{d_2} \psi_i \frac{\partial}{\partial x_{i+d_1}} - \frac{1}{2}\sum_{i,j=1}^{d_1}\sum_{l=1}^{m} \sigma_{jl}\partial_j\sigma_{il}\frac{\partial}{\partial x_i} - \frac{1}{2}\sum_{i=1}^{d_2}\sum_{j=1}^{d_1}\sum_{l=1}^{m}\sigma_{jl}\partial_j\varphi_{il}\frac{\partial}{\partial x_{i+d_1}}$$
$$V_i = \sum_{j=1}^{d_1}\sigma_{ji}\frac{\partial}{\partial x_j} + \sum_{j=1}^{d_2}\varphi_{ji}\frac{\partial}{\partial x_{j+d_1}}, \ 1 \le i \le m.$$

We assume that there exists $k \in \mathbb{N}$ and $c_0 > 0$ satisfying that for all $\xi \in \mathbb{R}^d$ and $x \in \mathbb{R}^{d_1}$

$$\sum_{V \in \cup_{l=1}^k A_l} \langle V(x), \xi\rangle^2 \ge c_0 \|\xi\|^2. \tag{10}$$

This condition is a uniform version of the Hörmander condition. Furthermore we also have the following classical result.

Theorem 6. *Assume Hypothesis (A2) and (10). Then $Z_t^{s,z} := (X_t^{s,x}, Y_t^{s,z}) \in \mathbf{D}^\infty$ and in fact $\|Z_t^{s,z}\|_{s,t,k,p} \le C(1+\|z\|)^\mu$ for any $s < t$ and $k,\ p \in \mathbb{N}$ and some positive constants C and μ. The Malliavin covariance matrix has all inverse moments. That is, there exists positive constants C and μ such that*

$$E\left[\left(\det M_{Z_t^{s,z}}^{[a,t]}\right)^{-p}\right] \le C(t-a)^{-\mu}(1+\|z\|)^\mu$$

for all $a \in [s,t)$ and there exists positive constants $C \equiv C(\alpha)$ and $\mu \equiv \mu(\alpha)$ such that for any multi-index α

$$\left\|H_\alpha^{[a,t]}(Z_t^{s,z},1)\right\|_p \le C(1+\|z\|)^\mu(t-a)^{-\mu}. \tag{11}$$

Therefore the semigroup associated to $Z_t^{s,z}$ satisfies

$$|\partial_\alpha P_t f(z)| \le C(1+\|z\|)^\mu t^{-\mu}\|f\|_\infty$$

and in particular, the characteristic function of $Z_t^{s,z}$ satisfies

$$\left|E\left[e^{j\theta\cdot Z_t^{s,z}}\right]\right| \le C(1+\|z\|)^\mu(t-s)^{-\mu}\|\theta\|^{-k}$$

for any $\theta \in \mathbb{R}^{d+1} - \{0\}$, any $k \in \mathbb{N}$ and some constants C and μ. Therefore $Z_t^{s,z}$ has a smooth density $p_{t-s}(z,\cdot)$.

We will prove the following result which is more general than Theorem 1.

Theorem 7. *Assume hypotheses (A1), (A2), (A3) and that the system (7)-(8) is uniformly hypoelliptic in the sense of (10). Then there exists constants C_k such that for all $k \in \mathbb{N}$, we have for all $\theta \in \mathbb{R}^{d_2}$*

$$\|\theta\|^k |h(\theta)| = \|\theta\|^k \left| E\left[\exp(j\theta \cdot Y_t^{0,x})\rho(t)\right]\right| \leq C_k$$

The conclusions on the densities of Y will follow from Lemma 2.

4 Proof of Theorem 7

The goal is to prove that $\|\theta\|^k |h(\theta)|$ is a bounded function of θ for any $k > 0$. We start with a Lemma which expands $h(\theta)$. The proof uses the Itô-Taylor expansion.

Lemma 8. *For any $\theta \in \mathbb{R}$ and $t \geq 0$, then*

$$h(\theta) = E[\exp(j\theta \cdot Y_t)] + E[I_N(t,\theta) + R_N(t,\theta)] \tag{12}$$

where

$$I_N(t,\theta) = \exp(j\theta \cdot Y_t) \sum_{n=1}^{N} \tilde{I}_n(t), \tag{13}$$

$$\tilde{I}_n(t) = \int_0^t \int_0^{s_1} ... \int_0^{s_{n-1}} \bar{b}(X_{s_n}) \cdot dW_{s_n} ... \bar{b}(X_{s_1}) \cdot dW_{s_1}$$

$$R_N(t,\theta) = \xi_t(0,x) \int_0^t \int_0^{s_1} ... \int_0^{s_N} \rho_{s_{N+1}} \bar{b}(X_{s_{N+1}}) \cdot dW_{s_{N+1}} ... \bar{b}(X_{s_1}) \cdot dW_{s_1}.$$

Furthermore

$$E[|R_N(t,\theta)|] \leq \exp\left(2^{-1} t \left\|\bar{b}\right\|_\infty^2\right) \frac{\left(C(d)t \left\|\bar{b}\right\|_\infty\right)^{(N+1)/2}}{\sqrt{(N+1)!}}. \tag{14}$$

Here $C(d)$ is a universal constant that only depends on the dimension d.

Proof: First we just need to apply the Itô -Taylor expansion of ρ as follows

$$\rho_t = \sum_{n=1}^{N} \tilde{I}_n(t) + \int_0^t \int_0^{s_1} ... \int_0^{s_N} \rho_{s_{N+1}} \bar{b}(X_{s_{N+1}}) \cdot dW_{s_{N+1}} ... \bar{b}(X_{s_1}) \cdot dW_{s_1}.$$

In order to obtain the estimate on $R_N(t,\theta)$ we just perform an $L^2(\Omega)$ estimate as follows

$$\begin{aligned} E[|R_N(t,\theta)|] &\leq E\left[\left|\int_0^t \int_0^{s_1} ... \int_0^{s_N} \rho_{s_{N+1}} \bar{b}(X_{s_{N+1}}) \cdot dW_{s_{N+1}} ... \bar{b}(X_{s_1}) \cdot dW_{s_1}\right|^2\right]^{1/2} \\ &\leq \exp\left(2^{-1} t \left\|\bar{b}\right\|_\infty^2\right) \frac{\left(C(d)t \left\|\bar{b}\right\|_\infty\right)^{(N+1)/2}}{\sqrt{(N+1)!}}. \end{aligned}$$

Note that here we have used that

$$\begin{aligned}
&E\left[\rho_t^2\right] \\
&= E\left[\exp\left(2\int_0^t \bar{b}(X_s)\cdot dW_s - 2\int_0^t \bar{b}^T\bar{b}(X_s)ds\right)\exp\left(\int_0^t \bar{b}^T\bar{b}(X_s)ds\right)\right] \\
&\leq \exp\left(t\left\|\bar{b}\right\|_\infty^2\right).
\end{aligned}$$

□

Considering the result in Lemma 8, we have to find upper bounds for each of the terms in (12). We will start with each of the terms in the sum (13). The important idea is to separate the domain of integration in each integral $\tilde{I}_n$ in a clever way so that we can apply the integration by parts formula. In order to do this, we define for each $n \in \mathbb{N}$, $i = 1, ..., n$ the following points: $a_i = \frac{t}{2^i}$ and $b_i = \frac{3t}{2^{i+1}}$. and the following sets

$$\begin{aligned}
A^n &= \{(s_1, ..., s_n) \in [0,t]^n; s_n \leq ... \leq s_1 \leq t\} \\
A_i^n &= \{(s_1, ..., s_n) \in [0,t]^n; s_1 > a_1, ..., s_{i-1} > a_{i-1}, s_i \leq a_i\} \cap A^n \\
A_n^* &= \{(s_1, ..., s_n) \in [0,t]^n; s_1 > a_1, ..., s_{n-1} > a_{n-1}, s_n > a_n\} \cap A^n.
\end{aligned}$$

What will be important in what follows is that on A_i^n, we have that as $s_i \leq a_i$ and $s_{i-1} > a_{i-1}$ then $s_{i-1} - s_i > a_{i-1} - a_i$. Then we also define the sets for $n \in \mathbb{N}$

$$\begin{aligned}
\Lambda_i^n &= \pi^{i-1}(A_i^n),\ i = 2, ..., n \\
\bar{\Lambda}_i^n &= \bar{\pi}^{n-i+1}(A_i^n),\ i = 1, ..., n.
\end{aligned}$$

Then we have the following important decompositions. The proof is straightforward

Lemma 9. *For $n \in \mathbb{N}$ and $i = 1, ..., n$ we have*

$$A^n \quad = \quad \sum_{i=1}^n A_i^n + A_n^*. \tag{15}$$

Furthermore $A_1^n = \bar{\Lambda}_1^n$ and for $n \in \mathbb{N}$ and $i = 2, ..., n$ we have

$$A_i^n = \Lambda_i^n \times \bar{\Lambda}_i^n. \tag{16}$$

From this lemma one obtains the following decomposition for the integrals in (13).

Lemma 10. *For $n \in \mathbb{N}$, we have*

$$\tilde{I}_n(t) \quad = \quad \bar{U}_1^n + \sum_{i=2}^{n} \bar{U}_i^n U_i^n + U_{n+1}^{n+1} \tag{17}$$

where U_i^n and $\bar{U}_i^n \in \mathcal{F}_{a_i}$ are given by

$$U_i^n = \int_{[0,t]^{i-1}} \mathbf{1}_{\Lambda_i^n}(s^{i-1})\bar{b}(X_{s_{i-1}}) \cdot dW_{s_{i-1}} ... \bar{b}(X_{s_1}) \cdot dW_{s_1}, \; i = 2, ..., n \tag{18}$$

$$\bar{U}_i^n = \int_{[0,t]^{n-i+1}} \mathbf{1}_{\bar{\Lambda}_i^n}(\bar{s}^{n-i+1})\bar{b}(X_{s_n}) \cdot dW_{s_n} ... \bar{b}(X_{s_i}) \cdot dW_{s_i}, \; i = 1, ..., n. \tag{19}$$

The following $L^2(\Omega)$ estimates are also satisfied: $\|\bar{U}_i^n\|_2 \leq \|\bar{b}\|_\infty^{n-i+1} \frac{a_i^{n-i+1}}{(n-i+1)!}$ and $\|U_i^n\|_{a_{i-1},t,0,2} \leq \|\bar{b}\|_\infty^{i-1}(t - a_{i-1})^{i-1}$.

Proof. The proof starts from the previous lemma as

$$\tilde{I}_n(t) \quad = \int_{[0,t]^n} \mathbf{1}_{A^n} \bar{b}(X_{s_n}) \cdot dW_{s_n} ... \bar{b}(X_{s_1}) \cdot dW_{s_1}.$$

Therefore by (15), we first decompose the above stochastic integral in $n+1$ stochastic integrals. The first and the last are already the integrals announced in (17). The last term in (17) follows because $\Lambda_{n+1}^{n+1} = A_n^*$.

For the other stochastic integrals one uses (16) and finally note that $\bar{U}_i^n \in \mathcal{F}_{a_i}$ and therefore the first inner $n - i + 1$ stochastic integrals can be taken outside of the external $i - 1$ integrals as their region of integration is always taken for values of time strictly greater than a_i. The $L^2(\Omega)$ estimates for U_i^n and $\bar{U}_i^n$ follows straightforwardly from the Itô-isometry. □

Proof of Theorem 7. Our goal is to prove that $\|\theta\|^k \left|E[e^{i\theta \cdot Y_t}]\right| \leq C_k$ for $\|\theta\|$ big enough. Throughout the proof various constants may depend on k but this is not important as k is fixed for the rest of the proof (although we may try to be explicit at some points).

First, we note that due to (12) we will only consider its middle term (the other terms which are simpler will be analyzed at the end of the proof):

$$E\left[I_N(t,\theta)\right] \quad = \quad \sum_{n=1}^{N} E\left[\exp(j\theta \cdot Y_t)\tilde{I}_n(t)\right]. \tag{20}$$

Furthermore considering (13) and (17) our goal can be decomposed as follows: Prove that for any $k \in \mathbb{N}$, any $\theta \in \mathbb{R}^{d_2}$ with $\|\theta\|$ big enough, there exists a constant $C_{n,k}$ so that for $i = 1, ..., n$,

$$\|\theta\|^k \left|E\left[\exp(j\theta \cdot Y_t)\bar{U}_1^n\right]\right| \quad \leq \quad C_{n,k} \tag{21}$$

$$\|\theta\|^k \left|E\left[\exp(j\theta \cdot Y_t)\bar{U}_i^n U_i^n\right]\right| \quad \leq \quad C_{n,k}, \; i = 2, ..., n \tag{22}$$

$$\|\theta\|^k \left|E\left[\exp(j\theta \cdot Y_t)U_{n+1}^{n+1}\right]\right| \quad \leq \quad C_{n,k}. \tag{23}$$

The proof of the first and third statement follow by a simplification of the argument for the second inequality, so we just prove the second. In fact, using the flow property (see Remark 5.1 (9)), we have

$$
\begin{aligned}
E\left[\exp(j\theta\cdot Y_t)\bar{U}_i^n U_i^n\right] &= E\left[\exp(j\theta\cdot Y_{a_{i-1}})\bar{U}_i^n E\left[\exp(j\theta\cdot Y_t^{a_{i-1},X_{a_{i-1}}})U_i^n\Big|\,\mathcal{F}_{a_{i-1}}\right]\right] \\
&= E\left[\exp(j\theta\cdot Y_{b_i})\bar{U}_i^n E\left[\exp(j\theta\cdot Y_{a_{i-1}}^{b_i,X_{b_i}})F_i^n(X_{a_{i-1}})\Big|\,\mathcal{F}_{b_i}\right]\right].
\end{aligned}
$$

Here we have obviously defined

$$
F_i^n(x)\equiv F_i^n(x;\theta):=E\left[\exp(j\theta\cdot Y_t^{a_{i-1},X_{a_{i-1}}})U_i^n\Big|\,X_{a_{i-1}}=x\right]. \tag{24}
$$

The important issue in the above decomposition and the calculations to follow is that as $\bar{b}$ is not smooth then U_i^n and $\bar{U}_i^n$ are not smooth random variables in the Malliavin sense and therefore a direct application of the integration by parts formula in (22) or (24) using Proposition 4 would not work. Instead, we will use the partition of time intervals given in Lemma 9 so as to allow for the application of partial Malliavin Calculus or conditional integration by parts in the appropriately selected time interval $[a_i, b_i]$.

Now, let $\phi_i^n(z)\equiv\phi_i^n(z,\theta)=\exp(j\theta\cdot y)F_i^n(x)$ where $z=(x,y)$. Therefore using the definition of the semigroup P and the flow property as explained in Remark 5.1, we have that

$$
\begin{aligned}
&E\left[\exp(j\theta\cdot Y_t)\bar{U}_i^n U_i^n\right] = E\left[\exp(j\theta\cdot Y_{b_i})\bar{U}_i^n P_{a_{i-1}-b_i}\phi_i^n(X_{b_i},0)\right] \\
&= E\left[\exp(j\theta\cdot Y_{a_i})\bar{U}_i^n E\left[\exp(j\theta\cdot Y_{b_i}^{a_i,X_{a_i}})P_{a_{i-1}-b_i}\phi_i^n(X_{b_i},0)\Big|\,\mathcal{F}_{a_i}\right]\right].
\end{aligned} \tag{25}
$$

Now, we perform integration by parts in the inner conditional expectation with respect to the multi-index $\beta\in\{d_1+1,...,d_2\}^{|\beta|}$ on the interval $[a_i,b_i]$ and the function $g_i^n(z)=\exp(j\theta\cdot y)P_{a_{i-1}-b_i}\phi_i^n(x,0)$. This gives (here we abuse the notation, letting $Z_{b_i}^{a_i,X_{a_i}}\equiv Z_{b_i}^{a_i,(X_{a_i},0)}$)

$$
\begin{aligned}
&E\left[\exp(j\theta\cdot Y_{b_i}^{a_i,X_{a_i}})P_{a_{i-1}-b_i}\phi_i^n(X_{b_i},0)\Big|\,\mathcal{F}_{a_i}\right] \\
=\;& (j\theta)^{-\tilde{\beta}}E\left[\partial_\beta g_i^n(Z_{b_i}^{a_i,X_{a_i}})\Big|\,\mathcal{F}_{a_i}\right] \\
=\;& (j\theta)^{-\tilde{\beta}}E\left[g_i^n(Z_{b_i}^{a_i,X_{a_i}})H_\beta^{[a_i,b_i]}\left(Z_{b_i}^{a_i,X_{a_i}},1\right)\Big|\,\mathcal{F}_{a_i}\right] \\
=\;& (j\theta)^{-\tilde{\beta}}E\left[\exp(j\theta\cdot Y_{b_i}^{a_i,X_{a_i}})P_{a_{i-1}-b_i}\phi_i^n(X_{b_i},0)H_\beta^{[a_i,b_i]}\left(Z_{b_i}^{a_i,X_{a_i}},1\right)\Big|\,\mathcal{F}_{a_i}\right],
\end{aligned} \tag{26}
$$

where $\tilde{\beta}=(\beta_1-d_1,...,\beta_{|\beta|}-d_1)$.

We will now bound each of the terms in (26). Note that from Remark 5.2, (24) and Lemma 10, we have

$$
\begin{aligned}
\|P_{a_{i-1}-b_i}\phi_i^n(X_{b_i},0)\|_{a_i,t,0,2} &\le C\|\phi_i^n\|_\infty\le C\|F_i^n\|_\infty\le C\|U_i^n\|_{a_{i-1},t,1} \\
&\le C\|\bar{b}\|_\infty^{i-1}(t-a_{i-1})^{i-1}
\end{aligned} \tag{27}
$$

Furthermore, using Theorem 6 (in particular (11)), the following estimate is valid for some non-random constants p_i, $i = 1, ..., 4$, C and μ

$$\left\| H_\beta^{[a_i,b_i]}\left(Z_{b_i}^{a_i,X_{a_i}},1\right)\right\|_{a_i,b_i,0,2} \leq C\left(b_i-a_i\right)^{-\mu}\left(1+\|X_{a_i}\|\right)^{\mu}. \tag{28}$$

Putting (27) and (28) together in (26) and (25) successively with the obvious equality $|\exp(j\theta\cdot y)| = 1$, we obtain the following upper bound

$$\begin{aligned}&|E\left[\exp(j\theta\cdot Y_t)\bar{U}_i^n U_i^n\right]| \\ &\qquad \leq C\left(b_i-a_i\right)^{-\mu}\|\bar{b}\|_\infty^{i-1}(t-a_{i-1})^{i-1}|(j\theta)^{-\tilde{\beta}}|E\left[|\bar{U}_i^n|\left(1+\|X_{a_i}\|\right)^{\mu}\right].\end{aligned}$$

Using Lemma 10 (bound of the $L^2(\Omega)$-norm of $\bar{U}_i^n$)and Theorem 6 (bound for the $L^2(\Omega)$-norm of $\|X_{a_i}\|$), we obtain that there exists constants C (depending on x and $|\beta|$) and μ (depending on $|\beta|$) such that

$$|E\left[\exp(j\theta\cdot Y_t)\bar{U}_i^n U_i^n\right]| \leq C\left(b_i-a_i\right)^{-\mu}|\theta|^{-\tilde{\beta}}\|\bar{b}\|_\infty^n\frac{a_i^{n-i+1}(t-a_{i-1})^{i-1}}{(n-i+1)!}. \tag{29}$$

This finishes the proof of (22). Similarly, one also obtains the proof of (21) and (23). That is,

$$\begin{aligned}&\left|E\left[\exp(j\theta\cdot Y_t)\bar{U}_1^n\right]\right| \\ =\;&\left|E\left[\exp(j\theta\cdot Y_{a_1})\bar{U}_1^n E\left[(j\theta)^{-\tilde{\beta}}\exp(j\theta\cdot Y_t^{a_1,X_{a_1}})H_\beta^{[a_1,t]}(Z_t^{a_1,X_{a_1}},1)\middle|\mathcal{F}_{a_1}\right]\right]\right| \\ \leq\;& C|\theta|^{-\tilde{\beta}}\|\bar{b}\|_\infty^n\frac{a_1^n}{n!}(t-a_1)^{-\mu}, \qquad (30)\\ &\left|E\left[\exp(j\theta\cdot Y_t)U_{n+1}^{n+1}\right]\right| \\ =\;&\left|E\left[\exp(j\theta\cdot Y_{a_n})F_{n+1}^{n+1}(X_{a_n})\right]\right| \leq C|\theta|^{-\tilde{\beta}}\|\bar{b}\|_\infty^n(t-a_n)^n a_n^{-\mu}. \qquad (31)\end{aligned}$$

Summarizing, we have that the first term in (12) satisfies by direction application of Theorem 6 that

$$|E[\exp(j\theta\cdot Y_t^{0,x})]| \leq Ct^{-\mu}|\theta|^{-\tilde{\beta}}.$$

Similarly, the second term which has been decomposed as in (20), had been studied in three separate terms (29), (30) and (31). Adding all of them, one obtains a bound of the type

$$|E\left[I_N(t,\theta)\right]| \leq C|\theta|^{-\tilde{\beta}}M^N,$$

for some positive constants C and M. And finally the third term in (12), had been bounded in (14). Therefore putting all these terms together we obtain the following estimate for some positive constant M,

$$|h(\theta)| \leq |\theta|^{-\tilde{\beta}}M^N + M^{(N+1)/2}\left(N+1\right)!^{-1/2}.$$

As $\lim_{N\to\infty} \frac{N!}{c^N} = \infty$ for any constant $c > 0$, we have that there exists N_0 such that for any $N \geq N_0$

$$M^{(N+1)/2} (N+1)!^{-1/2} \leq M^{-2N}.$$

Finally one takes $N = \log_M(|\theta|^{\tilde{\beta}})^{1/2}$ with θ satisfying $\log_M(|\theta|^{\tilde{\beta}})^{1/2} \geq N_0$. From here we obtain our final result:

$$|h(\theta)| \leq C(|\theta|^{-\tilde{\beta}})^{1/2},$$

for $|\theta|^{\tilde{\beta}} > 1$. Therefore the statement of Theorem 7 follows from the inequality $\|\theta\| \leq \max\{|\theta_i|;\ i = 1, .., d_2\}$

□

5 Proof of Theorem 1

Here we assume that σ is smooth and uniformly elliptic and therefore of full rank. Then given a bounded measurable $b : \mathbb{R}^d \to \mathbb{R}^{d_1}$ there exists $\bar{b} : \mathbb{R}^d \to \mathbb{R}^m$ measurable and bounded such that $b = \sigma\bar{b}$. We apply the setting in section 3 as follows : $b = 0$, $\phi = 0$ and let X be the unique pathwise solution of the equation

$$X_t = x + \int_0^t \sigma(X_s) \circ dW_s.$$

In this setting we apply the results of the previous section. Then if we define the change of measure

$$\left.\frac{dP}{dQ}\right|_{\mathcal{F}_t} = \exp\left(-\int_0^t \bar{b}(X_s) \cdot dW_s + \frac{1}{2}\int_0^t \bar{b}^T\bar{b}(X_s)ds\right).$$

Then we have that under Q, $B_t = W_t + \int_0^t \bar{b}(X_s)ds$ is a $\mathcal{F}$-Brownian motion. Therefore the result in Theorem 7 implies that the characteristic function of

$$Y_t = \int_0^t \psi(X_s)ds$$

decreases rapidly to zero as $\|\theta\| \to \infty$ where X is given by

$$X_t = x + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s) \circ dB_s. \tag{32}$$

Therefore by weak uniqueness of solutions, the law of X in (32) and the law of the solution of (1) are the same. From here we obtain the result. □

Final Remarks: We have presented a general and simple method to prove the smoothness of densities of random vectors generated by additive functionals of stochastic systems with measurable bounded drift.

From the arguments is clear that one may further weaken some of the conditions on b as long as the Girsanov change of measure can be performed. In that sense, b may depend on time, be somewhat random or have linear growth.

Also one can use the same method of proof to deal with the case

$$Y_t^1 = \int_0^t \psi(X_s^x) d\mu(s)$$

where μ has an accumulation point in its support. That is, there exists x_0 such that for any $\epsilon > 0$, we have $\mu\left[(x_0 - \epsilon, x_0 + \epsilon) - \{x_0\}\right] > 0$.

Other possibilities include the possibility that this method can be used to obtain upper bound for the densities. Lower bounds can also be possibly be obtained but refined techniques are needed (see e.g. [1] or [9]). Many other extensions can be considered, such as the case of stochastic partial differential equations.

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