# Existence and regularity of the density for solutions to stochastic differential equations with boundary conditions *


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**Abstract**

We study the existence and regularity of the probability law associated with the solution of stochastic differential equations with boundary conditions. We focus our attention on two cases. In the one dimensional case we consider the general sde with a linear boundary condition. In the multidimensional case we consider a general boundary condition with constant diffusion coefficient.



**Keywords**: stochastic differential equations with boundary conditions, density, Malliavin calculus.

**AMS 1990 SCI**: 60H07, 60H10, 60H99 .



*Supported by a grant of the DGICYT No. PB 930052


†This article was partially written while the author was visiting Kyoto University, Department of Mathematics, under a JSPS fellowship

# 1 Introduction

Stochastic differential equations (sde) with boundary conditions have been studied by Ocone-Pardoux [8], Nualart- Pardoux [10], Donati-Martin [2], Garnier [5], among others. These are equations of the type

$$dX_t \;=\; \sigma(X_t) \circ dW_t \;+\; b(X_t)\, dt, \qquad 0 \le t \le 1, \tag{1.1}$$

with a boundary condition of the form

$$h(X_0, X_1) \;=\; \overline{h}. \tag{1.2}$$

Most of the studies on these equations concern existence and uniqueness of solutions and the Markov field property. In [8] the existence of density for the law of the solution is also analyzed.

These equations are anticipative in nature due to the boundary condition. Their solutions are evaluations of flows at random variables. That is, the solution to (1.1) and (1.2), when it exists, can be written as $X_t \;=\; \Phi_t(X_0)$, where $\{\Phi_t(x)\;\; t \in [0,1]\}$ is the stochastic flow associated with (1.1) and $X_0$ is the unique solution to (1.2).

A general type of sde with a given anticipating initial condition was first studied by Ocone-Pardoux [9]. They proved existence and uniqueness for solutions by means of an Itô-Ventzell type formula. Using the Malliavin Calculus, Masuda in [6] studied the existence of the density for the law of the solution. Furthermore, Caballero, Fernández and Nualart [1] studied the regularity of the laws. A further improvement of such results has been recen tly obtained by Rovira and Sanz-Solé [12]. Here we are interested in similar problems for the generic equation (1.1) with the restriction given by (1.2).

The results obtained in the above mentioned articles seemed to be good for the extensive class they study. Nevertheless, in the case of stochastic differential equations with boundary conditions, these results are far from being optimal. It is the belief of the authors that some middle point has to be found where the exchange of generality of the equation type and restrictions on the data is the best.

In this article we study two examples of sde's with boundary conditions. The first one is a one dimensional Stratonovich equation with linear boundary condition. The existence and uniqueness for this equation have been studied in [2]. The conditions we find for existence and regularity of densities are similar to the Hörmander's ones, except that we evaluate the Lie brackets at the initial random variable $X_0$, which could be non-adapted (see also [12]). Secondly, we consider the multidimensional sde with general boundary condition and constant diffusion coefficient studied in [10]. In this case, under ellipticity and some restriction on the boundary condition we obtain existence and regularity of the density for the law of the solution.

All the results that can be obtained applying (when applicable) theorems in [6], [1] and [12] require stronger hypothesis than the ones obtained here. More detailed comments on this matter will be made throughout the text (see e.g. the Remarks 2.2, 2.3 and the beginning of section 3.2).

## 2 The density for the solution to a one dimensional stochastic differential equation with linear boundary condition.

In this section we study the existence and regularity of the density for the probability law of the solution to a stochastic Stratonovich differential equation with a linear boundary condition. Let $\sigma$ and $b$ be real functions, $F_0$, $F_1$, $h_0 \in I\!R$. Consider the equation

$$
\begin{aligned}
dX_t \;&=\; \sigma(X_t) \,\circ\, dW_t \;+\; b(X_t)\, dt, \qquad 0 \le t \le 1, \\
F_0 X_0 \;+\; F_1 X_1 \;&=\; h_0.
\end{aligned}
\tag{2.3}
$$

Assume that $\sigma$ and $b_1 = b + \frac{1}{2}\sigma\sigma'$ are $C^4$ functions with bounded derivatives and $F_0\ F_1 > 0$. In [2] it is proved that under these assumptions, there exists a unique solution to (2.3) belonging to the space $I\!L_C^{1,loc}$. More exactly, the solution is given by $X_t := \Phi_t(X_0)$, where $\Phi_t(x),\ x \in I\!R$, is the stochastic flow

$$\Phi_t(x) \;=\; x \;+\; \int_0^t \sigma(\Phi_s(x)) \circ\, dW_s \;+\; \int_0^t b(\Phi_s(x))\, ds,\ \ t \in [0,1],$$

and $X_0$ is the unique measurable random variable satisfying $F_0 x + F_1 \Phi_1(x) = h_0$.

We will first prove that $X_0 \in \mathbb{D}^\infty$, assuming regularity of the coefficients. This property implies $X_t = \Phi_t(X_0) \in \mathbb{D}^\infty$, for any $t \in [0,1]$ (see [9]). The following lemma will be used repeatedly.

**Lemma 2.1** *Let $\{F_t(x),\ t \in [0,1]^m\}$, $x \in \mathbb{R}^k$, be a family of stochastic processes such that there exists $\alpha \geq 1$ with*

$$\sup_{t\in[0,1]} E(\sup_{|x|\leq 1} |F_t(\cdot, nx)|^p) \leq C n^{\alpha p}, \tag{2.4}$$

*for any $p \geq 1$, where $C$ is a constant depending on $p$ and $\alpha$. Let $Y : \Omega \to \mathbb{R}^k$ be a random vector belonging to $\cap_{p\geq 1} L^p(\Omega)$. Then*

$$\sup_{t\in[0,1]} E(|F_t(Y)|^p) < \infty, \text{ for any } p \geq 1.$$

*Proof*: Fix $p \geq 1$ and let $\frac{1}{r} + \frac{1}{s} = 1$. Hölder's inequality yield

$$\begin{aligned}
E(|F_t(Y)|^p) &\leq \sum_{n=1}^{\infty} E(\sup_{|x|\leq 1} |F_t(\cdot, nx)|^p I[n-1 \leq |Y| \leq n]) \\
&\leq \sum_{n=1}^{\infty} (E(\sup_{|x|\leq 1} |F_t(\cdot, nx)|^{pr})^{\frac{1}{r}} P[n-1 \leq |Y| \leq n]^{\frac{1}{s}} \\
&\leq C \sum_{n=1}^{\infty} n^{\alpha p} P[n-1 \leq |Y| \leq n]^{\frac{1}{s}} \\
&\leq C (\sum_{n=1}^{\infty} n^{-r})^{\frac{1}{r}} (\sum_{n=1}^{\infty} n^{(\alpha p+1)s} P[n-1 \leq |Y| \leq n])^{\frac{1}{s}} \\
&\leq C(1 + E|Y|^{(\alpha p+1)s})^{\frac{1}{s}} < \infty.
\end{aligned}$$

□

In the sequel we will assume

$(h_1)$ $\sigma$ and $b_1$ are $C^\infty$ functions with bounded derivatives of any order.

**Remark 2.1** *Using Sobolev's embedding theorem and standard estimates for stochastic flows it is easy to show that, under $(h_1)$, $\Phi_t(x)$, $\frac{d^k}{dx^k}\Phi_t(x)$, $\{D_r^k \Phi_t(x),\ r \in [0,1]^k\}$, for any $t \in [0,1]$, $k = 0, 1...$, satisfy condition (2.4) with $\alpha = 1$.*

**Proposition 2.1** *If $(h_1)$ holds and $F_0F_1 > 0$, the random variable $X_0$ belongs to $\mathbb{D}^\infty$.*

*Proof*: Clearly,

$$X_0 = \frac{h_0 - F_1\Phi_1(0)}{F_0 + F_1\Phi_1'(\xi_0)I[X_0 \neq 0]},$$

where $\xi_0$ is a random point lying between 0 and $X_0$, when $X_0 \neq 0$. Consequently, $X_0 \in \cap_{p\geq 1}L^p(\Omega)$. Moreover, Lemma 4.1 in [2], establishes that $X_0 \in \cap_{p\geq 2}\mathbb{D}^{1,p}_{loc}$ with $\Omega_n = \{\omega;\ |X_0| \leq n\}$, $n \geq 1$, as a localizing sequence. On $\Omega_n$,

$$D_rX_0 = -\frac{F_1(D_r\Phi_1)(X_0)}{F_0 + F_1\Phi_1'(X_0)}. \tag{2.5}$$

Since $D$ is a local operator, (2.5) defines uniquely the Malliavin derivative of $X_0$. We next prove that $E(\int_0^1 |D_rX_0|^2dr)^{\frac{p}{2}} < \infty$, for any $p \geq 2$. Indeed,

$$\begin{aligned}
E\left(\int_0^1 |D_rX_0|^2dr\right)^{\frac{p}{2}} &= \sum_{n=1}^\infty E\left(\left(\int_0^1 |D_rX_0|^2dr\right)^{\frac{p}{2}} I[n-1 \leq |X_0| < n]\right)\\
&= \sum_{n=1}^\infty E\left((\int_0^1 |\frac{F_1(D_r\Phi_1)(X_0)}{F_0 + F_1\Phi_1'(X_0)}|^2dr)^{\frac{p}{2}} I[n-1 \leq |X_0| < n]\right)\\
&\leq C\sum_{n=1}^\infty E\left((\int_0^1 |(D_r\Phi_1)(X_0)|^2dr)^{\frac{p}{2}} I[n-1 \leq |X_0| < n]\right)\\
&\leq C\sup_r E(|(D_r\Phi_1)(X_0)|^p).
\end{aligned}$$

Lemma 2.1 applied to the process $\{D_r\Phi_1(x),\ r \in [0,1]\}$ allows to conclude the proof.

The derivatives of higher order are defined on each subset $\Omega_n \subset \Omega$, $n \geq 1$ by differentiating the right hand-side of (2.5) and then on $\Omega$ using the local property of $D$. The fact that $D^kX_0 \in L^p(\Omega, L^2([0,1]^k))$, $k \geq 2$, $p \geq 2$ is proved by similar arguments as the ones developed for $k = 1$. □

**Remark 2.2** *Assume there exists $t > 0$ such that $\sigma(X_t) \neq 0$; then, using Proposition 2.1, one can prove existence of density for the probability law of $X_t$ for all $t \in [0,1]$. Furthermore, using the same method as in the proof for diffusions one also obtains that, if either $\sigma(X_0) \neq 0$ or there exists $n$ such that $(\frac{d^n}{dx^n}\sigma)(X_0)b(X_0) \neq 0$,* *the law of $X_t$ has a density for $t \in [0,1]$. In particular, the conditions in [6] require, for example, $X_0$ to be a functional of multiple Wiener-Stratonovich integrals of a fix degree.*

Before establishing the main result of this section we introduce sets of functions defined by

$$\begin{aligned} \Sigma_0 &= \{\sigma\} \\ \Sigma_j &= \left\{[\sigma, V],\ [b, V] + \frac{1}{2}[\sigma, [\sigma, V]]\,;\ V \in \Sigma_{j-1}\right\},\ j \geq 1, \end{aligned}$$

where $[\cdot,\cdot]$ denotes the Lie bracket.

**Theorem 2.1** *Let $F_0 F_1 > 0$ and assume $(h_1)$. Suppose that there exist $j_0 \geq 0$ and $V \in \Sigma_{j_0}$ with $(V(X_0))^{-1} \in \cap_{p\geq 1} L^p(\Omega)$. Then, for any $t \in [0,1]$ the law of $X_t$ has a $C^\infty$ density.*

*Proof*: In view of Proposition 2.1 it suffices to check

$$Q^{-1} := \left(\int_0^1 (D_r X_t)^2 dr\right)^{-1} \in \cap_{p\geq 1} L^p(\Omega),\ t \in [0,1]. \tag{2.6}$$

Set $Y_t(x) = \frac{d}{dx}\Phi_t(x)$, $Z_t(x) = (\frac{d}{dx}\Phi_t(x))^{-1}$, for $t \in [0,1]$, $x \in I\!R$. Then

$$D_r X_t = \sigma\left(\Phi_r(X_0)\right) Z_r(X_0) \left\{ Y_t(X_0) \left( \frac{F_0}{F_0 + F_1 Y_1(X_0)} 1_{[0,t]}(r) - \frac{F_1 Y_1(X_0)}{F_0 + F_1 Y_1(X_0)} 1_{(t,1]}(r) \right) \right\}.$$

Fix $t \in (0,1]$. Clearly, $Q^{-1} \leq \left(\int_0^t (D_r X_t)^2 dr\right)^{-1}$. Moreover, it is well-known that

$$\left( Y_t(X_0) \frac{F_0}{F_0 + F_1 Y_1(X_0)} \right)^{-1} \in \cap_{p\geq 1} L^p(\Omega).$$

Thus the proof of (2.6) reduces to that of

$$(\int_0^t (\sigma(\Phi_r(X_0)) Z_r(X_0))^2 dr)^{-1} \in \cap_{p\geq 1} L^p(\Omega),$$

or, equivalently, for any $p \geq 1$, there exists $\epsilon_0 = \epsilon_0(p)$ such that, for every $\epsilon \in (0, \epsilon_0(p)]$,

$$P\left[\int_0^t \left(\sigma(\Phi_r(X_0)) Z_r(X_0)\right)^2 dr < \epsilon\right] \leq \epsilon^p. \tag{2.7}$$

Although Theorem 3.2 in [12] cannot be applied to this situation, its method of proof works here. For the sake of completeness we give some details.

Let $n(j) = 3^{-2j}$, $j \geq 1$,

$$\begin{aligned} E_0 &= \left\{ \int_0^t (\sigma(\Phi_r(X_0)) Z_r(X_0))^2 \, dr < \epsilon \right\}, \\ E_j &= \left\{ \sum_{V \in \Sigma_j} \int_0^t (V(\Phi_r(X_0)) Z_r(X_0))^2 \, dr < \epsilon^{n(j)} \right\}, \end{aligned}$$

$j = 1, ..., j_0$, $F = \cap_{j=0}^{j_0} E_j$. Clearly, $P(E_0) \leq \sum_{j=1}^{j_0} P(E_{j-1} \cap E_j^c) + P(F)$. Therefore (2.7) will follow from the following estimates:

$$P(F) \leq \epsilon^p, \quad P(E_{j-1} \cap E_j^c) \leq \epsilon^p, \ 1 \leq j \leq j_0, \tag{2.8}$$

for any $p \geq 1$ and $\epsilon \leq \epsilon_0(p)$. Let $|\Sigma_j|$ denote the cardinal of $\Sigma_j$, $j = 1, ..., j_0$. Then,

$$\begin{aligned} P(E_{j-1} \cap E_j^c) \leq \sum_{V \in \Sigma_{j-1}} P\Big( & \int_0^t (V(\Phi_r(X_0)) Z_r(X_0))^2 \, dr < \epsilon^{n(j-1)}, \int_0^t ((Z_r(X_0)[\sigma, V](\Phi_r(X_0)))^2 \\ & + (Z_r(X_0)([b, V] + \frac{1}{2}[\sigma, [\sigma, V]])(\Phi_r(X_0)))^2) dr > \epsilon^{n(j)} |\Sigma_{j-1}|^{-1} \Big). \end{aligned}$$

Itô's formula yields

$$V(\Phi_r(x)) \, Z_r(x) = V(x) + \int_0^r (\alpha_s(x) dW_s + a_s(x) ds),$$

with $\alpha_s(x) = Z_s(x)[\sigma, V](\Phi_s(x))$ and $a_s(x) = Z_s(x)([b, V] + \frac{1}{2}[\sigma, [\sigma, V]])(\Phi_s(x))$. Then, the estimate (2.8) follows from lemma 3.1 in [12] (a Norris type lemma) applied to the process

$$\overline{Y}_t(x) = V(X_0) + \int_0^t (\alpha_s(x) dW_s + a_s(x) ds), \ t \in [0, 1], \ x \in I\!R.$$

Let

$$\begin{aligned} S_1 &= \inf\{s > 0; \sup_{0 \leq r \leq s} |Z_r(X_0) - 1| \geq \frac{1}{2}\}, \\ S_2^\epsilon &= \inf\{s > 0; \sup_{0 \leq r \leq s} |\Phi_r(X_0) - X_0| \geq \epsilon^{\frac{\beta}{3}}\}, \ \beta, \epsilon > 0, \\ H_\epsilon &= \{|X_0| < \epsilon^{-w}\}, \ w \in (0, \frac{\beta}{18}). \end{aligned}$$

Since $X_0 \in \cap_{p\geq 1} L^p(\Omega)$, $P(H_\epsilon^c) \leq C\epsilon^{wp}$, for any $p \geq 1$. Furthermore,

$$P([S_2^\epsilon \leq \epsilon^\beta] \cap H_\epsilon) + P([S_1 \leq \epsilon^\beta] \cap H_\epsilon) \leq C\epsilon^{(\frac{\beta}{6}-3w)q},$$

for any $q > 1$ (see the estimate (2.15) in [12]). Fix $\beta, w > 0$ with $\beta > 18w$. It suffices to check

$$P\left(F \cap [S_2^\epsilon > \epsilon^\beta] \cap [S_1 > \epsilon^\beta]\right) \leq \epsilon^p, \tag{2.9}$$

for any $p \geq 1$ and $\epsilon$ sufficiently small. The left hand-side of (2.9) is bounded by $p_1(\epsilon) + p_2(\epsilon)$, where

$$\begin{aligned} p_1(\epsilon) &= P\left(\sum_{j=0}^{j_0} \sum_{V\in\Sigma_j} \int_0^{\epsilon^\beta} (V(\Phi_r(X_0)) - V(X_0))^2\, dr > 4(j_0+1)\epsilon^{n(j_0)},\ S_2^\epsilon > \epsilon^\beta\right), \\ p_2(\epsilon) &= P\left(\sum_{j=0}^{j_0} \sum_{V\in\Sigma_j} \int_0^{\epsilon^\beta} V^2(X_0) dr \leq 16(j_0+1)\epsilon^{n(j_0)}\right). \end{aligned}$$

Using the mean value theorem and Chebyshev's inequality we obtain

$$p_1(\epsilon) \leq C\epsilon^{(\frac{5}{3}\beta - n(j_0))q},\ q \geq 1.$$

Let $\beta < n(j_0)$. The assumptions yield, for any $p \geq 1$ and $\epsilon$ sufficiently small, that $p_2(\epsilon) \leq \epsilon^p$. This concludes the proof of (2.7) and therefore the proof of the theorem for $t \in (0,1]$.

For $t = 0$ the proof is almost the same. Indeed,

$$\int_0^1 |D_r X_0|^2 dr = \left(\frac{F_1 Y_1(X_0)}{F_0 + F_1 Y_1(X_0)}\right)^2 \int_0^1 (\sigma(\Phi_r(X_0)) Z_r(X_0))^2\, dr$$

and consequently, (2.6) follows from

$$\left(\int_0^1 (\sigma(\Phi_r(X_0)) Z_r(X_0))^2 dr\right)^{-1} \in \cap_{p\geq 1} L^p(\Omega),$$

which has been proven in the first part of this proof (that is, (2.7) for $t = 1$). $\square$

**Example 1.** Consider the linear case $\sigma(x) = Ax$, $b(x) = Bx$, $AB \neq 0$. Suppose that $h_0 \neq 0$. In this case, $X_0^{-1} \in \cap_{p\geq 1} L^p(\Omega)$ and consequently the hypothesis of Theorem 2.1 are satisfied. Indeed, it is well known that

$$\Phi_t(x) = x\exp(Bt + AW_t),\ t \in [0,1].$$

Consequently,

$$X_0 = \frac{h_0}{F_0 + F_1\exp(B + AW_1)}.$$

**Example 2.** Let $\sigma(x) = Ax$, $b(x) = B$ with $h_0F_1B < 0$. Easy computations yield

$$\Phi_t(x) = \exp(AW_t)(x + \int_0^t \exp(-AW_s)Bds).$$

Furthermore,

$$X_0 = \frac{h_0 - F_1B\exp(AW_1)\int_0^1 \exp(-AW_s)ds}{F_0 + F_1\exp(AW_1)}.$$

One then obtains that

$$|h_0 - F_1B\exp(AW_1)\int_0^1 \exp(-AW_s)ds| \geq h_0.$$

Thus, $X_0^{-1} \in \cap_{p\geq 1} L^p(\Omega)$.

**Example 3.** Let $\sigma(x) = cos(x)$, $b(x) = sin(x)$. In this example,

$$[b,\sigma](x) = \sigma'(x)b(x) - b'(x)\sigma(x) = -sin^2(x) - cos^2(x) = -1.$$

Therefore $([b,\sigma](X_0))^{-1} \in \cap_{p\geq 1} L^p(\Omega)$ and the assumptions of Theorem 2.1 hold with $j_0 = 1$.

**Remark 2.3** *It is not difficult to write an extension of Theorem 2.1 changing the condition* $(V(X_0))^{-1} \in \cap_{p\geq 1} L^p(\Omega)$ *into the following one: there exists* $t \in [0,1]$ *such that* $(V(X_t))^{-1} \in \cap_{p\geq 1} L^p(\Omega)$.

*In this case the smoothness of the density is only guaranted for* $t \in (0,1]$. *In particular this extended version of Theorem 2.1 contains as a particular case the results that could also be obtained by applying Proposition 3.2 in [1].*

## 3 The density of a multidimensional stochastic differential equations with general boundary conditions.

In this section we will consider the stochastic differential equation

$$dX_t \;+\; f(X_t)\; dt \;=\; B\; dW_t, \qquad t \in [0,1], \tag{3.1}$$

with boundary condition

$$h(X_0, X_1) \;=\; \overline{h}. \tag{3.2}$$

We assume that $\{X_t\ ,\ t\in[0,1]\}$ is $I\!R^d$- valued, $\{W_t\ ,\ t\in[0,1]\}$ is a $I\!R^k$- valued Brownian motion with $k\geq d$, $f: I\!R^d \to I\!R^d$ takes the form

$$f(x) \;=\; Ax \;+\; B\widetilde{f}(x),$$

where $A$ is a $d\times d$ matrix, $\widetilde{f}: I\!R^d \to I\!R^k$ is measurable and locally bounded and $B$ is a $d\times k$ matrix. Finally, $h: I\!R^{2d}\to I\!R^d$ and $\overline{h}\in I\!R^d$. In [10] a theorem on existence and uniqueness of solution for this kind of equations has been established. More explicitely, let $\mathcal{C}_0\left([0,1];I\!R^k\right)$ be the set of continuous, $I\!R^d$- valued functions vanishing at 0; set $F=\left\{\int_0^1\ e^{At}B\ d\varphi(t)\ ;\ \varphi\in\mathcal{C}\left([0,1];I\!R^k\right)\right\}$, where the integrals are defined using integration by parts. Assume

**(H1)** For any $z\in F$ the equation $h\left(y, e^{-A}(y+z)\right) \;=\; \overline{h}$ has a unique solution $y=g(z)$.

In order to find the solution of (3.1), (3.2) we consider the linear equation

$$dY_t \;+\; AY_t\; dt \;=\; B\; dW_t, \qquad t\in[0,1],$$

with boundary condition (3.2). This equation has a unique solution given by

$$Y_t \;=\; e^{-At}\left[g\left(\int_0^1\ e^{As}B\ dW_s\right) \;+\; \int_0^t\ e^{As}B\ dW_s\right].$$

Let $\Sigma=\left\{\xi\in\mathcal{C}\left([0,1];I\!R^k\right)\ ,\ \xi_t-\xi_0+\int_0^t\ A\xi_s\ ds\in \mathrm{Im}B, 0\leq t\leq 1, h(\xi_0,\xi_1)=\overline{h}\right\}$. There exists a bijection $\psi:\mathcal{C}_0\left([0,1];I\!R^k\right)\to\Sigma$ such that $Y_t=(\psi(W))_t$. Finally we define the mapping $T : \mathcal{C}_0\left([0,1]; I\!R^k\right) \to \mathcal{C}_0\left([0,1]; I\!R^k\right)$ by

$$T(\theta) \;=\; \theta \;+\; \int_0^{\cdot} \widetilde{f}((\psi(\theta))_s)\; ds.$$

**Theorem 3.1** *(Nualart and Pardoux, [10]) Assume $T$ is a bijection and* (H1). *Then equation (3.1) with boundary condition (3.2) possess a unique solution given by*

$$X \;=\; \psi\left(T^{-1}(W)\right). \tag{3.3}$$

In this section we will give conditions that assure the existence and smoothness of densities for solutions of (3.3). From now on we assume (H1).

### 3.1 Existence of densities

Here we will consider two different approaches. The first one will consist in using an appropiate Girsanov theorem to reduce the analysis to the linearized equation. In the second one we will make a direct computation of the stochastic derivative of the solution (3.3) and then we will find conditions to apply the Bouleau and Hirsch theorem for existence of densities. A third approach using Masuda's results requires strong smoothness properties that we do not impose here. Furthermore our calculations in the proof of Theorem 3.3 will help to develop the theorem for smoothness of the density that is not covered in Masuda's article. Let $\xi_t = \int_0^t e^{As} B dW_s$. Define $\{\Phi_t; 0 \le t \le 1\}$ as the solution of the ordinary differential equation

$$\begin{aligned} \frac{d\Phi_t}{dt} &= -f'(Y_t)\Phi_t \\ \Phi_0 &= I \end{aligned}$$

**Theorem 3.2** *Assume that $g, f \in C^1(I\!R^d, I\!R^d)$, $T$ is bijective,* $\det BB^T \neq 0$ *and*

$$\det(I - e^A \Phi_1 g'(\xi_1) + g'(\xi_1)) \neq 0, \quad a.s., \tag{3.4}$$

*then we have*

*(i) if $P\{\omega;\; g'(\xi_1)$ or $g'(\xi_1) + I$ is invertible $\} = 1$, the law of $X_t$ is absolutely continuous with respect to the Lebesgue measure for $t \in (0,1)$.*

*(ii) if $P\{\omega;\ g'(\xi_1)$ is invertible $\} = 1$, the law of $X_0$ is absolutely continuous with respect to the Lebesgue measure.*

*(iii) if $P\{\omega;\ g'(\xi_1) + I$ is invertible $\} = 1$, the law of $X_1$ is absolutely continuous with respect to the Lebesgue measure.*

*Proof*: By the extended Girsanov theorem, under condition (3.4), $P \circ T$ is absolutely continuous with respect to the Wiener measure $P$ (see Proposition 3.2 in [10]). Therefore, since $X = \psi\left(T^{-1}(W)\right)$, we only need to analyze the absolute continuity of the law of the solution to the linear equation, $Y_t$, with respect to the Lebesgue measure. Note that under the present hypothesis $Y_t \in \cap_{p>1} \mathbb{D}^{1,p}_{loc}$ for all $t$ and furthermore

$$D_r Y_t = e^{-At}\{g'(\xi_1) + I 1_{[0,t]}(r)\} e^{Ar} B.$$

Notice that $r \to D_r Y_t$ is *a.s.* continuous. The Malliavin covariance matrix of $Y_t$ is given by

$$\begin{aligned} Q_t &= e^{-At}\{g'(\xi_1) + I\} \int_0^t e^{Ar} B B^T e^{A^T r} dr \{(g'(\xi_1))^T + I\} e^{-A^T t} \\ &\quad + e^{-At} g'(\xi_1) \int_t^1 e^{Ar} B B^T e^{A^T r} dr (g'(\xi_1))^T e^{-A^T t}. \end{aligned} \tag{3.5}$$

Assume $\det Q_t = 0$ in a set $\Omega_0 \subset \Omega$ of positive probability. Then, necessarily $\det BB^T = 0$ on $\Omega_0$, violating the assumptions. □

The fact that $\det BB^T \neq 0$ is a natural condition because this makes the noise in equation (3.1) 'fill' $\mathbb{R}^d$. In fact this is an ellipticity condition on the difussion coefficient.

N ow we will deduce a similar result to the previous one but through a detailed calculation of the stochastic derivative in the non-linear case. The conditions are slightly different due to the nature of the analysis. Let

$$h(s,y) = -e^{As} B \widetilde{f}' \left(e^{-As}\left[g\left(\xi_1 + y\right) \ + \ \xi_s + u_s(y)\right]\right) e^{-As}.$$

We will also use a different way of expressing the solution to (3.1), (3.2). For this let $\theta = T^{-1}(W)$ and $\varphi_t = \displaystyle\int_0^t e^{Au} B \, d\theta_u$. Then,

$$\begin{aligned} X_t &= e^{-At} \left[ g \left( \int_0^1 e^{Au} B \, d\theta_u \right) + \int_0^t e^{Au} B \, d\theta_u \right] \\ &= e^{-At} \left[ g\left(\varphi_1\right) + \varphi_t \right]. \end{aligned}$$

As in [4], one easily obtains $\varphi_t = \xi_t + u_t$

$$u_t = - \int_0^t e^{As} B \widetilde{f} \left( e^{-As} \left[ g\left(\xi_1 + u_1\right) + \left(\xi_s + u_s\right) \right] \right) ds.$$

Fix $y \in I\!\!R$ and assume

$$u_t(y) = - \int_0^t e^{As} B \widetilde{f} \left( e^{-As} \left[ g\left(\xi_1 + y\right) + \xi_s + u_s(y) \right] \right) ds \tag{3.6}$$

has a unique solution. Moreover, suppose that the mapping $y \mapsto u_1(y)$ has a unique fixed point $Y$. Then, clearly

$$\varphi_t = \xi_t + u_t(Y).$$

Let $|M|$ denote the norm of a matrix $M$, that is, $|M| = \sup\limits_{|x|=1} |Mx|$. We need the following assumption:

**(H2)** $g, f \in C^1(I\!\!R^d, I\!\!R^d)$ and

$$K\left(L_{\widetilde{f}}, L_g\right) = L_g \left[ \exp\left[ \left( e^{2|A|} - 1 \right) \frac{|B|}{2|A|} L_{\widetilde{f}} \right] - 1 \right] < 1, \tag{3.7}$$

where $\widetilde{f}$ and $g$ are Lipschitz functions with Lipschitz constant $L_{\widetilde{f}}$ and $L_g$, respectively.

In the proof of Theorem 3.3 we need the statement provided by the next lemma.

**Lemma 3.1** *Assume* (H2). *Then $I - u_1'(y)$ is an invertible matrix for all $y \in I\!\!R^d$ a.s.*

*Proof*: It suffices to prove that the biggest eigenvalue of $u_1'(y)$ is smaller than one. Let this eigenvalue be $\lambda$ and $\epsilon$ the corresponding eigenvector. Then, by the definition of derivative, there exists a residue $\kappa(y, \epsilon)$ such that

$$u_1(y + \epsilon) - u_1(y) - \kappa(y, \epsilon) = \lambda \epsilon,$$

with $\lim_{|\epsilon|\to 0} \frac{|\kappa(y,\epsilon)|}{|\epsilon|} = 0$. Then, using the contraction property of $u_1$ (see Proposition 3.1 in [4]) and letting $\epsilon$ go to zero, we have $\lambda \le K\left(L_{\widetilde{f}}, L_g\right) < 1$. $\square$

**Lemma 3.2** *Assume* (H2). *Then* $u_t(y) \in \cap_{p>1} \mathbb{D}^{1,p}$, *for all* $y \in \mathbb{R}^d$ *and* $t \in [0,1]$, $Y \in \cap_{p>1} \mathbb{D}^{1,p}$ *and* $u_t(Y) \in \cap_{p>1} \mathbb{D}^{1,p}$ *for all* $t \in [0,1]$. *Furthermore,* $D_s(u_t(Y)) = (D_s u_t)(Y) + u_t'(Y) D_s Y$ *and* $D_s Y = (I - u_1'(Y))^{-1}(D_s u_t)(Y)$.

The proof of the above Lemma is done using standard methodology developed to prove stochastic differentiability for solutions of differential equations. In particular, it uses the methodology developed in Proposition 2.1, Lemma 2.1 and estimates in Section 3 of [4] as well as Lemma 3.1.

**Theorem 3.3** *Assume* (H2) *and* $\det BB^T \neq 0$. *Then*

*(i) if* $P\{\omega;\ g'(\xi_1)$ *or* $g'(\xi_1) + I$ *is invertible* $\} = 1$, *the law of* $X_t$ *is absolutely continuous with respect to the Lebesgue measure for* $t \in (0,1)$.

*(ii) if* $P\{\omega;\ g'(\xi_1)$ *is invertible* $\} = 1$, *the law of* $X_0$ *is absolutely continuous with respect to the Lebesgue measure.*

*(iii) if* $P\{\omega;\ g'(\xi_1) + I$ *is invertible* $\} = 1$, *the law of* $X_1$ *is absolutely continuous with respect to the Lebesgue measure.*

*Proof*: From (3.7) it follows that $u$ is a contraction and therefore $T$ is a bijection. Define $Z_t = e^{At} X_t$. It suffices to prove the theorem for $Z$. We have

$$D_r Z_t = g'(\varphi_1) D_r \varphi_1 + D_r \varphi_t,$$

$$D_r \varphi_t = e^{Ar} B 1_{[0,t]}(r) + (D_r u_t)(Y) + u_t'(Y)(I - u_1'(Y))^{-1}(D_r u_1)(Y).$$

From (3.6), one obtains the following equations:

$$u_t'(y) = \int_0^t h(s,y)\left(g'(\xi_1 + y) + u_s'(y)\right) ds,$$

$$D_r u_t(y) = \int_0^t h(s,y)\left(g'(\xi_1 + y) + 1_{[0,s]}(r)\right) e^{As} B + h(s,y) D_r u_s(y) ds.$$

The solution to these equations are given by (see Dunford and Schwartz [3], vol II, page 1282):

$$u'_t(Y) = \sum_{j=0}^{\infty} \Psi^j G(t), \qquad (D_r u_t)(Y) = \sum_{j=0}^{\infty} \Psi^j (H_r + G)(t) e^{Ar} B,$$

where

$$G(t) = \left( \int_0^t h(s,Y) ds \right) g'(\varphi_1), \qquad H_r(t) = \int_0^t h(s,Y) 1_{[0,s]}(r) ds.$$

And $\Psi$ is the operator defined by $(\Psi Z)(t) = \int_0^t h(s,Y)Z(s)ds$. The above solutions are unique and the series converge uniformly in $t$ for $r$ in compact sets, due to the condition $\sup_{s\in[0,1]} |h(s,Y)| < e^{2|A|}|B|L_{\tilde{f}}$, which is satisfied because of assumption (H2). Let $Y_t^r$ be the unique solution of the following random ordinary differential equation:

$$Y_t^r = \int_0^t h(s,Y) 1_{[0,s]}(r) ds + \int_0^t h(s,Y) Y_s^r ds.$$

Notice that $Y_t^r = 0$ for any $t \le r$. Furthermore, by uniqueness of solutions one has $u'_t(Y) = Y_t^0 g'(\varphi_1)$ and $(D_r u_t)(Y) = (Y_t^r + Y_t^0 g'(\varphi_1))e^{Ar}B$. Then, we can rewrite $D_r Z_t$ in the following form:

$$\begin{aligned} D_r Z_t \;\; = \;\; & \{g'(\varphi_1)(I + Y_1^0 g'(\varphi_1) + Y_1^r + Y_1^0 g'(\varphi_1)(I - u'_1(Y))^{-1}(Y_1^0 g'(\varphi_1) \\ & + Y_1^r)) + Y_t^0 g'(\varphi_1) + Y_t^0 g'(\varphi_1)(I - u'_1(Y))^{-1}(Y_1^0 g'(\varphi_1) + Y_1^r) \\ & + 1_{[0,t]}(r) I + Y_t^r\} e^{Ar} B. \end{aligned} \tag{3.8}$$

Finally, set $X_t^r = I + Y_t^r$, which satisfies the equation

$$\begin{aligned} X_t^r &= I + \int_r^t h(u,Y) X_u^r du, \quad \text{for } t \ge r, \\ X_t^r &= I \qquad \text{for } t < r. \end{aligned}$$

In terms of $X_t^r$, $D_r Z_t$ can be rewritten as:

$$\begin{aligned} D_r Z_t \qquad &= (1_{[0,t]}(r)(X_1^t)^{-1}(I + g'(\varphi_1))(I - Y_1^0 g'(\varphi_1))^{-1} X_1^r \\ &+ 1_{[t,1]}(r) X_t^0 g'(\varphi_1)(I - Y_1^0 g'(\varphi_1))^{-1} X_1^r) e^{Ar} B. \end{aligned}$$

Note that the matrix $X_1^t$ is invertible a.s. for $t \in [0,1]$ and it is the only solution to the equation

$$
\begin{aligned}
(X_t^r)^{-1} &= I - \int_r^t (X_u^r)^{-1} h(u,Y)du, \quad \text{for } t \geq r, \\
(X_t^r)^{-1} &= I \qquad \text{for } t < r.
\end{aligned} \tag{3.9}
$$

Thus, $X_t^r = X_t^0 (X_r^0)^{-1}$ for $t \geq r$.

Let $\Gamma_t$ be the Malliavin covariance matrix of $Z_t$. Assume that $\det \Gamma_t = 0$ on a set of positive probability. Then there exists $v \in I\!R^d$ with $|v| = 1$ and

$$
\begin{aligned}
v^T \Gamma_t v &= \int_0^t |v^T (X_1^t)^{-1} (I + g'(\varphi_1))(I - Y_1^0 g'(\varphi_1))^{-1} X_1^r e^{Ar} B|^2 dr \\
&\quad + \int_t^1 |v^T X_t^0 g'(\varphi_1)(I - Y_1^0 g'(\varphi_1))^{-1} X_1^r e^{Ar} B|^2 dr = 0.
\end{aligned} \tag{3.10}
$$

Then we conclude following the analysis carried out in the proof of Theorem 3.2. □

## 3.2 Regularity of the density

In this section we study the regularity of the density for the law of the solutions to (3.1) and (3.2). A first approach could consist in applying a multi-dimensional generalization of Proposition 3.2 in [1], which gives the awkward condition for regularity of the density

$$
E(|(I + g'(\varphi_1)(I - u_1'(Y))^{-1}(I + Y_1^0))B|^{-p}) < \infty,
$$

for all $p > 1$. Using formula (3.10), one can obtain a better result.

**Theorem 3.4** *Assume that $g'(x)$ (resp. $I + g'(x)$) has an inverse for all $x \in I\!R^d$ and that its inverse has at most polynomial growth at infinity. Furthermore assume* (H2) *and* $\det BB^T \neq 0$.

*Then the law of the solution of (3.3) is infinitely differentiable for $t \in [0,1)$ (resp. $t \in (0,1]$).*

*Proof*: The proof follows the lines of the classical proof of regularity of densities for diffusions. See, for example, Theorem 2.3.3 in [11]. We will outline the case when the hypothesis are satisfied for $I + g'(x)$ and $t > 0$. Since $BB^T$ is positive definite and $\det BB^T \neq 0$, there exists a positive constant $C$ such that, for all $v \neq 0$, $v^T BB^T v \geq C|v|^2$.

Fix $t > 0$, as in the proof of Theorem 3.3 it is enough to consider $Z_t$ instead of $X_t$. Denote by $\Gamma_t$ the Malliavin covariance matrix of $Z_t$. It suffices to prove

$$\sup_{|v|=1} P\{v^T \Gamma_t v \leq \epsilon\} \leq \epsilon^p \tag{3.11}$$

for any $p \geq 2$ and for $\epsilon \leq \epsilon_0(p)$, for some $\epsilon_0(p) > 0$.

Fix a unitary $d$-dimensional vector $v$; define

$$F = \{\int_0^t |v^T (X_1^t)^{-1}(I + g'(\phi_1))(I - Y_1^0 g'(\phi_1))^{-1} X_1^r e^{Ar} B|^2 dr \leq \epsilon\},$$

$v_0^T = v^T (X_1^t)^{-1}(I + g'(\varphi_1))(I - Y_1^0 g'(\varphi_1))^{-1} X_1^0$ and

$S^v = \inf\{\sigma \geq 0; \sup_{r \leq \sigma} |v_0^T (X_r^0)^{-1} e^{Ar} BB^T e^{A^T r}((X_r^0)^{-1})^T v_0 - v_0^T BB^T v_0| \geq \frac{1}{2} v_0^T BB^T v_0\} \wedge t.$

Note that $v_0^T BB^T v_0 > 0$, *a.s.* Then $P\{v^T \Gamma_t v \leq \epsilon\}$ is bounded by

$$P(F \cap [S^v \geq \epsilon^\beta]) + P(S^v \leq \epsilon^\beta),$$

for $\beta < 1$. Carrying out a similar calculation as in step 1, page 121 in [11], the above probability can be bounded by

$$P(|v_0^T|^2 \leq C\epsilon^{1-\beta}) + P(S^v \leq \epsilon^\beta), \tag{3.12}$$

for some positive constant $C$.

For any invertible matrix $A$ and any unitary vector $v$, $|v^T A|^{-1} \leq |A^{-1}|$; therefore, for any $p \geq 1$,

$$\begin{aligned} &P(|v_0^T|^2 \leq C\epsilon^{1-\beta}) \leq C^p \epsilon^{(1-\beta)p} E|v_0|^{-2p} \\ &\leq C^p \epsilon^{(1-\beta)p} E\left(|((X_1^0)^{-1}(I - Y_1^0 g'(\varphi_1))(I + g'(\varphi_1))^{-1} X_1^t|^{2p}\right) \end{aligned}$$

Assumption (H2) yields $\sup_{s\in[0,1]}|h(s,Y)| \leq e^{2|A|}|B|L_{\tilde f}$, $\|g'\|_\infty \leq L_g$ and $Y \in \cap_{p\geq 1}L^p$ (see Lemma 3.2 in [4]). Hence, $Y_s^t$, $X_s^t$, $(X_s^t)^{-1}$ are in $\cap_{p\geq 1}L^p(\Omega)$ for $s,t\in[0,1]$. Therefore implies

$$P(|v_0^T|^2 \leq C\epsilon^{1-\beta}) \leq C\epsilon^{(1-\beta)p}$$

and consequently, the first term in (3.12) satisfies (3.11).

We now prove the same property for the second term in (3.12). Let $V_r = (X_r^0)^{-1}e^{Ar}BB^Te^{A^Tr}((X_r^0)^{-1})^T$, $r\in[0,1]$. Note that $\{V_r, r\in[0,1]\}$ is a process with a.s. continuous paths and $V_0 = BB^T$. In addition $E(\sup_{r\in[0,1]}|V_r'|^q) < \infty$, for all $q>1$, where $V'$ denotes the derivative of $V$. Consequently,

$$\begin{aligned} P(S^v \leq \epsilon^\beta) &\leq P(\sup_{r\leq\epsilon^\beta}|v_0^TBB^Tv_0|^{-1}|v_0^T(X_r^0)^{-1}e^{Ar}BB^Te^{A^Tr}((X_r^0)^{-1})^Tv_0 - v_0^TBB^Tv_0| \geq \frac{1}{2}) \\ &\leq 2^qE\left(\sup_{r\leq\epsilon^\beta}|v_0^TBB^Tv_0|^{-1}|v_0|^2|(X_r^0)^{-1}e^{Ar}BB^Te^{A^Tr}((X_r^0)^{-1})^T - BB^T|\right)^q \\ &\leq CE\left(\sup_{r\leq\epsilon^\beta}|V_r - BB^T|\right)^q \leq C\epsilon^{\beta q}. \end{aligned}$$

The proof is now complete. □

**Example 4.** Let $h(x,y) = F_0x+F_1y$ whith $F_0$, $F_1$ $d\times d$ matrices. Assume $F_0+F_1e^{-A}$ has an inverse and either $F_0$ or $F_1$ has an inverse. Then, if (H2) is satisfied, the hypotheses of Theorem 3.4 hold true.

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