Lower bounds for densities of uniformly elliptic random variables on Wiener space*

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Abstract

In this article, we generalize the lower bound estimates for uniformly elliptic diffusion processes obtained by Kusuoka and Stroock. We define the concept of uniform elliptic random variable on Wiener space and show that with this definition one can prove a lower bound estimate of Gaussian type for its density. We apply our results to the case of the stochastic heat equation under the hypothesis of unifom ellipticity of the diffusion coefficient.

1 Introduction

Professors S. Kusuoka and D. Stroock developed in a series of three long articles the set up for a variety of results about densities of diffusions that became one of the inspiring cornerstones on the topic of applications of Malliavin Calculus for random variables on Wiener space and in particular to solutions of various stochastic differential equations. Now we can use this technique not only to investigate the existence and smoothness of densities but also its positivity, the support of its law and large deviations principle between other properties. In Part III of their series of articles (see [13]), Kusuoka and Stroock proved that the density of a uniformly hypoelliptic diffusion whose drift is a smooth combination of its diffusion coefficients has a lower bound of Gaussian type. Their results were the first known detailed global extensions of analytical results obtained in [18] and [4]. In particular, they found particularly refined expressions that related these lower bounds with the large deviations principle for diffusions.

In this article, we intend to extend their results to various other uniformly elliptic situations that can not be directly deduced from their article as they specifically use the structure of a diffusion with a particular condition on the drift. This condition which is the result of the use of the Girsanov theorem essentially means that the drift has to be a smooth multiple of the diffusion coefficient. This restriction is not binding in one dimension, given that one is assuming the uniformly elliptic condition, but it is highly restrictive in higher dimensions. On the other hand, we partially give up on the idea of finding a very explicit expression for the exponent of the Gaussian density and instead use an Euclidean norm which is equivalent to the distance appearing in the large deviation principle under uniformly elliptic conditions.

The general idea of Kusuoka-Stroock's result is to expand the diffusion using the Itô-Taylor expansion, then consider the main term in this expansion. Their results rely heavily on a Lie algebra structure of multiple stochastic integrals generated by the Wiener process and therefore the use of the Girsanov theorem becomes natural.

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In this article we will neither use the Girsanov theorem nor the Lie algebra structure of the stochastic integrals involved. Thinking about several applications in stochastic processes at the same time it becomes obvious that one can not expect such a nice structure of the multiple integrals which will in general combine stochastic and Lebesgue integrals. On the other hand, one can expect that such lower bounds for the density of a big class of elliptic stochastic equations should be satisfied.

Instead of dealing with the problem on a case by case basis we will provide a general theory and a definition of uniformly elliptic random variable on Wiener space which most probably can be applied to a wide variety of situations. This definition implies, in particular, that the random variable is non-degenerate in the Malliavin Calculus sense. With this general definition in hand we will show that such random variables have densities with Gaussian type lower bounds. As an example, we apply this result to the stochastic heat equation. Various other cases possibly follow by applying the general theorem given in this article (Theorem 5). Other possible applications of our results are in the cases of the solutions of the stochastic Volterra equation, stochastic partial differential equations, and the functional delay equation. We treat one of these examples and leave the rest for future publications.

In our general theory, we admit that the variance of the random variable X_t in question could be of any order. In particular one could think of examples where the variance is of order t^α for $\alpha > 0$. In the diffusion case $\alpha = 1$. In a later section, we study the case of the stochastic heat equation where $\alpha = 1/2$. The reason for the decrease in the order is due to the degeneracy of the Green kernel around time 0. One could also develop other examples with time dependent coefficients with various values of α. For example, for the case of biparametric diffusions one has $\alpha = 2$.

One can certainly create particular situations were α takes any other value but those cases require to develop ad-hoc theorems of existence, uniqueness, smoothness, etc. For this reason we have preferred to use the stochastic heat equation as an example as most of its smoothness properties related to Malliavin Calculus are well known.

The main result of existence of lower bounds for densities of random variables in Wiener space will depend on two conditions. The main one being that for all sequence of partitions of the time interval of the subjacent Wiener process there exists a sequence of successive approximations to the random variable indexed in the partition (this is condition **(H1)** in Theorem 5). Next we require that these approximations have a series decompositions of the Itô-Taylor type. When this series expansion is truncated at some order we require a series of conditions encapsulated in **(H2a)**-**(H2d)**. These are mostly regularity properties except for the property describing the heart of the concept of uniform ellipticity, **(H2c)**.

This property essentially requires that each difference of two adjacent random variables in the approximation sequence can be decomposed as a non-trivial Gaussian term (this will be made explicit in condition **(H2c)**) and another term which is of smaller order than the Gaussian term (condition **(H2d)**).

The proof of the main theorem is based on ideas laid by Kusuoka and Stroock. Nevertheless several new problems appear due to the generality of the statement. The first is related to the fact that diffusions are Markov processes while in our general set-up such property can not be expected. In fact, Kusuoka and Stroock's approach is based on the Chapman-Kolmogorov formula. We deal with this problem using conditional expectations for the approximation sequence and at first hope that one can estimate these quantities uniformly for $\omega \in \Omega$ (see hypothesis **1** in Theorem 1 and hypothesis A3 in Theorem 3). At first we assume that there is a lower bound estimate for these conditional expectations in small time. Then in Theorem 5, we prove that our hypotheses (**H1**)-**(H2)** imply the existence of this lower bound. Obtaining this lower bound is done through the truncated approximation series expansion mentioned previously. Therefore the introduction of the truncated series expansion becomes natural as we want to control (uniformly) the Gaussian behavior of the approximation sequence.

Possible applications of these lower estimates for densities can be found in capacity theory (see [2]), statistical estimation theory (see [5] and [6]) and quantile estimation (see [19]).

Section 2 is composed of some notions of Malliavin Calculus used throughout the text. Section 3 contains the main definition of uniformly elliptic random variable and the proof of the main Theorem under the hypothesis of the local estimate for the conditional density in small time for the approximation process. In Section 4 we treat the case of the stochastic heat equation. In the Appendix we give some accessory results on the stability of the Malliavin covariance matrix of the approximations as well as some needed estimates for the study of the bounds for the density of the stochastic heat equation.

$C_b^\infty(\mathbb{R}^d)$ denotes the space of real bounded functions on $\mathbb{R}^d$ such that they are infinitely differentiable with bounded derivatives. $C_p^\infty(\mathbb{R}^d)$ stands for a similar space but the functions and their derivatives have polynomial growth instead. C, c, m and M denote constants in general that may change from one line to another unless stated otherwise. $\|\cdot\|$ without any subindices denotes the usual Euclidean norm in $\mathbb{R}^l$. The dimension l should be clear from the context.

2 Preliminaries

Let W be a k-dimensional Wiener process indexed in $[0,T]\times A$ with $A\subseteq\mathbb{R}^m$. Our base space will be a sample space $(\Omega,\mathcal{F},P)$ where the Wiener process will be defined (for details see [16], Section 1.1 and [17]). The associated filtration will be defined as $\{\mathcal{F}_t; 0\le t\le T\}$, where $\mathcal{F}_t$ is the $\sigma-$field generated by the random variables $\{W(s,x),\,(s,x)\in[0,t]\times A\}$ with $A\subseteq\mathbb{R}^m$. On the sample space $(\Omega,\mathcal{F},P)$ one can define a derivative operator D, associated domains $(\mathbb{D}^{n,p},\|\cdot\|_{n,p})$ where n denotes the order of differentiation and p denotes the $L^p(\Omega)$ space where the derivatives lie. We say that F is smooth if $F\in\mathbb{D}^\infty=\cap_{n\in\mathbf{N},p>1}\mathbb{D}^{n,p}$. For a $q-$dimensional random variable $F\in\mathbb{D}^{1,2}$, we denote by ψ_F the Malliavin covariance matrix associated with F. That is, $\psi_F^{i,j}=<DF^i,DF^j>_{L^2[0,T]\times A}$. One says that the random variable is non-degenerate if $F\in\mathbb{D}^\infty$ and the matrix ψ_F is invertible a.s. and $(\det\psi_F)^{-1}\in\cap_{p\ge1}L^p(\Omega)$. In such a case expressions of the type $E(\delta_y(F))$, where δ_y denotes the Dirac delta function, have a well defined meaning through the integration by parts formula. Obviously in such cases one can also define $E(\delta_y(F))$ as the limit of $E(\phi_r(F-y))$ as $r\to0$ with $\phi_r(x)=(2\pi r^d)^{-1/2}\exp(-\frac{\|x\|^2}{2r})$.

The integration by parts formula of Malliavin Calculus can be briefly described as follows. Suppose that F is a non-degenerate random variable and $G\in\mathbb{D}^\infty$. Then for any function $g\in C_p^\infty(\mathbb{R}^q)$ and a finite sequence of multi-indexes $\beta\in\cup_{l\ge1}\{1,...,q\}^l$, we have that there exists a random variable $H^\beta(F,G)$ so that

$$E(g^\beta(F)G)=E(g(F)H^\beta(F,G))\text{ with}$$
$$\left\|H^\beta(F,G)\right\|_{n,p}\le C(n,p,\beta)\left\|\det(\psi_F)^{-1}\right\|_{p'}^{a'}\|F\|_{d,b}^a\|G\|_{d',b'} \tag{1}$$

for some constants $C(n,p,\beta)$, a, b, d, p', a', b', d' and $\beta\in\cup_{l\ge1}\{1,...,q\}^l$. Here g^β denotes the high order derivative of order $l(\beta)$ and whose partial derivatives are taken according the index vector β. This inequality can be obtained following the calculations in Lemma 12 of [15]. In some cases we will consider the above norms and definitions on a conditional form. That is, we will use partial Malliavin Calculus. We will denote this by adding a further time sub-index in the norms. For example, if one completes the space of smooth functionals with the norm

$$\begin{aligned}\|F\|_{2,s}&=(E(\|F\|^2/\mathcal{F}_s))^{1/2}\\ \|F\|_{1,2,s}^2&=\|F\|_{2,s}^2+E(\int_s^T\|D_uF\|^2du/\mathcal{F}_s),\end{aligned}$$

we obtain the space $\mathbb{D}_s^{1,2}$. To simplify the notation we will sometimes denote $E_s(\cdot)=E(\cdot/\mathcal{F}_s)$ and P_s the respective conditional probability. Analogously we will write H_s^β and $\psi_F(s)$ when considering integration by parts formula and the Malliavin covariance matrix conditioned on $\mathcal{F}_s$. That is, $\psi_F^{i,j}(s)=<DF^i,DF^j>_{L^2[s,T]\times A}$. Also we say that $F\in\overline{\mathbb{D}}_s^{1,2}$ when $F\in\mathbb{D}_s^{1,2}$ and $\|F\|_{1,2,s}\in\cap_{p\ge1}L^p(\Omega)$. Similarly, we say that F is $s-$conditionally non-degenerate if $F\in\overline{\mathbb{D}}_s^\infty=\cap_{n\in\mathbf{N},p>1}\overline{\mathbb{D}}_s^{n,p}$ and $(\det\psi_F(s))^{-1}\in\cap_{p>1}L_s^p(\Omega)$. In such a case, as before, expressions like $E(\delta_y(F)/\mathcal{F}_s)$ have a well defined meaning through the partial integration by parts formula or via an approximation of the delta function.

We will also have to deal with similar situations for sequences F_i that are $\mathcal{F}_{t_i}-$measurable random variables, $i = 1, ..., N$ for a partition $0 = t_0 < t_1 < ... < t_N$. In this case we say that $\{F_i; i = 1, ..., N\} \subseteq \overline{\mathbb{D}}^{\infty}$ uniformly if $F_i \in \overline{\mathbb{D}}^{\infty}_{t_{i-1}}$ for all $i = 1, ..., N$ and for any $l > 1$ one has for each $n, p \in \mathbb{N}$

$$\sup_N \sup_{i=1,..,N} E \|F_i\|^l_{n,p,t_{i-1}} < \infty.$$

In what follows we will sometimes expand our basic sample space to include further increments of another independent Wiener process, $\overline{W}$ (usually these increments are denoted by $Z_i = \overline{W}(i+1) - \overline{W}(i) \sim N(0,1)$) independent of W in such a case we denote the expanded filtration by $\overline{\mathcal{F}}_t = \mathcal{F}_t \vee \sigma(\{\overline{W}(s); s \le i+1, t_i \le t\})$. We do this without further mentioning and suppose that all norms and expectations are considered in the extended space. Sometimes we will write $F \in \mathcal{F}_t$ which stands for F is a $\mathcal{F}_t$ -measurable random variable.

3 General theory

We start first with a result that will be useful only for very particular cases of random variables that are approximately Gaussian locally. Nevertheless this result shows clearly the natural idea that in order to obtain lower bounds for densities one has to generalize local estimates to global ones. In order to carry out this idea one usually uses some type of Markov property related to the random variable in question. Here we do this but without requiring explicitly this Markov property. In general all constants appearing in the rest of the article will be independent of T, ω, the variables y_1,...,y_N, or the chosen partition (see the next main set-up) unless explicitly stated otherwise. As the frame throughout the section is the same we will describe it here.

Main set-up: Let $F \in \mathcal{F}_t$.

a. Suppose that there exists $\epsilon > 0$ such that for any sequence of partitions $\pi_N = \{0 = t_0 < t_1 < ... < t_N = t\}$ whose norm is smaller than ϵ and $|\pi_N| = \max\{|t_{i+1} - t_i|; i = 0, ..., N-1\} \to 0$ as $N \to \infty$ there exists a sequence $F_i \in L^2(\Omega; \mathbb{R}^q)$, $i = 1, ..., N$ such that $F_N = F$. F_i is a $\mathcal{F}_{t_i}$-measurable random variable and is a t_{i-1}-conditionally non-degenerate random variable. F_i, $i = 0, .., N$ is an approximating sequence that will allow the application of the parallel of the Chapman-Kolmogorov formula.

b. Suppose that there exists a function $g : [0, T] \times A \to \mathbb{R}_{>0}$ and a positive constant C such that $\|g\|_{L^2([0,T]\times A)} \le C$. This function will measure the local variance of the r.v. F.

c. Define $\Delta_{i-1}(g) = \int_{t_{i-1}}^{t_i} \int_A \|g(t,x)\|^2 \, dxdt$. This quantity measures the local variance as explained in a. The general idea of the proof is to use a Chapman-Kolmogorov-like formula although the sequence F_i, $i = 0, ..., N$ is not necessarily Markovian. Instead of transition probabilities we will have conditional probabilities. Then we localize each conditional probability where we will obtain a Gaussian type lower bound. The localization will be done in the set A_i which is defined as $A_i = \{y \in \mathbb{R}^q; \|y - F_{i-1}\| \le c\Delta_{i-1}(g)^{1/2}\}$. This finishes the main set up and we are now ready to state the first theorem.

Theorem 1 *Under the main setup:*

1. *Suppose that there exists positive constants M, c and η_0 such that for $0 < \Delta_{i-1}(g) < \eta_0$ and $y_i \in A_i$*

$$E(\delta_{y_i}(F_i)/\mathcal{F}_{t_{i-1}}) \ge \frac{1}{M\Delta_{i-1}(g)^{q/2}}$$

for all $i = 1, ..., N$ and almost all $\omega \in \Omega$.

Under these restrictions we have that there exists a constant $M' > 0$ which depends on all other constants (M, c, C, η_0, T, ϵ) such that

$$p_F(y) \ge \frac{\exp\left(-M' \frac{\|y - F_0\|^2}{\|g\|^2_{L^2([0,t]\times A)}}\right)}{M' \|g\|^{q/2}_{L^2([0,t]\times A)}}.$$

Proof. First, we assume without loss of generality that $\|g\|^2_{L^2([0,T]\times A)} \leq M$. Then for any N, there exists a partition $\pi = \{t_i;\, i = 0, ..., N\}$ with $0 = t_0 < ... < t_N = t$ defined by the equalities

$$\Delta_{i-1}(g) = \frac{\|g\|^2_{L^2([0,t]\times A)}}{N} \tag{2}$$

Now we show that there exists e_0 such that for any $e \leq e_0$ and N the smallest integer such that

$$N \geq e^{-1}\left(\frac{\|y - F_0\|^2}{\|g\|^2_{L^2([0,t]\times A)}} + 1\right)$$

we have that $|\pi_N| < \epsilon$. In fact, suppose by contradiction that for each e_0 there exists $t_i \equiv t_i(e_0)$ and $t_{i+1} \equiv t_{i+1}(e_0)$ such that $t_{i+1} - t_i \geq \epsilon$ then choosing a converging subsequence we have that if $(t_i, t_{i+1}) \to (a, b)$ for $b - a \geq \epsilon$. This implies $g(s, x) = 0$ for all $(s, x) \in (a, b) \times A$ which leads to a contradiction.

Let $\eta < \eta_0$ and assume that $e \leq \frac{\eta}{M} \wedge \frac{1}{2} \wedge \frac{c^2}{4}$. Without loss of generality we suppose that M is big enough so that $e \leq \frac{\eta}{M} \leq e_0$. We also have that

$$\Delta_{i-1}(g) \leq \frac{\eta \|g\|^2_{L^2([0,t]\times A)}}{M} \left(\frac{\|y - F_0\|^2}{\|g\|^2_{L^2([0,t]\times A)}} + 1\right)^{-1} \leq \eta < \eta_0.$$

Now choose $N - 1$ points $x_1, x_2, ..., x_{N-1}$ with $x_N = y$ and $x_0 = F_0$ so that $\|x_i - x_{i-1}\| = \|y - F_0\|/N$, $i = 1, ..., N$. Now, suppose that $y_i \in B(x_{i+1}, \frac{1}{4}c\Delta_{i-1}(g)^{1/2})$ for $i = 1, ..., N-1$, then

$$\begin{aligned}
\|y_{i-1} - y_i\| &\leq \|y_{i-1} - x_i\| + \|x_i - x_{i+1}\| + \|x_{i+1} - y_i\| \\
&\leq \frac{1}{4}c\Delta_{i-1}(g)^{1/2} + \frac{\|y - F_0\|}{N} + \frac{1}{4}c\Delta_{i-1}(g)^{1/2} \\
&\leq \frac{c\|g\|_{L^2([0,t]\times A)}}{2\sqrt{N}} + e^{1/2}\frac{\|g\|_{L^2([0,t]\times A)}}{\sqrt{N}} \\
&\leq c\Delta_{i-1}(g)^{1/2},
\end{aligned}$$

for $i = 2, ..., N-1$. In the following calculation, we obtain the lower bound estimate for the density. In the calculation to follow, we use expressions like $E(\delta_y(F)\delta_{y_{N-1}}(F_{N-1})...\delta_{y_1}(F_1))$ in the sense of Watanabe (see Chapter V.9 in [7], for another way to carry out these calculations, see the proof of Theorem 2). This and other terms of the same type have mathematical meaning through the partial integration by parts formula. Throughout we let $y_N = y$ and $y_0 = x_0$. By Fubini's theorem and the positivity of $E(\delta_y(F)\delta_{y_{N-1}}(F_{N-1})...\delta_{y_1}(F_1))$ for any $(y, y_1, ..., y_{N-1}) \in \mathbb{R}^{qN}$ we have that

$$\begin{aligned}
E(\delta_y(F)) &= \int_{\mathbb{R}^q} ... \int_{\mathbb{R}^q} E(\delta_y(F)\delta_{y_{N-1}}(F_{N-1})...\delta_{y_1}(F_1))dy_1...dy_{N-1} \\
&\geq \int_{B_N} ... \int_{B_2} E(\delta_y(F)\delta_{y_{N-1}}(F_{N-1})...\delta_{y_1}(F_1))dy_1...dy_{N-1}.
\end{aligned}$$

Here $B_i = B(x_i, \frac{1}{4}c\Delta_i(g)^{1/2})$. Next we use hypothesis **1** and the positivity of the Dirac delta function to obtain that

$$E(\delta_y(F)) \geq \int_{B_N}... \int_{B_2} \frac{1}{M\Delta_{N-1}(g)^{q/2}} \times E(\delta_{y_{N-1}}(F_{N-1})...\delta_{y_1}(F_1))dy_1...dy_{N-1}.$$

Then by induction it follows that iterating the above formula one has

$$p_F(y) \geq \int_{B_N}... \int_{B_2} \prod_{i=1}^{N} \frac{1}{M\Delta_{i-1}(g)^{q/2}} dy_1...dy_{N-1}.$$

Now we bound this term by below as follows

$$p_F(y) \geq \frac{1}{\Delta_{N-1}(g)^{q/2}} \left(\frac{1}{M}\right)^N \prod_{i=1}^{N-1} \frac{|B_{i+1}|}{\Delta_{i-1}(g)^{q/2}}.$$

Next, we use that $|B(x,r)| = C(q)r^q$ and (2). Then we have that the above lower bound can be rewritten as

$$\begin{aligned} p_F(y) &\geq \frac{C(q)^{N-1} N^{q/2} c^{q(N-1)}}{\|g\|^q_{L^2([0,t]\times A)} 4^{q(N-1)}} \exp\left(-N\log(M)\right) \\ &\geq \frac{C_q N^{q/2} \exp(-NC^*)}{\|g\|^q_{L^2([0,t]\times A)}}. \end{aligned}$$

Here $C_q = \frac{4^q}{C(q)c^q}$ and $C^* = \log(M) - \log(C(q)c^q/4^q) > 0$ (otherwise one may take a bigger constant M in hypothesis **1** and the previous sequence of inequalities follow as well). Finally we use that $e^{-1} \leq N \leq e^{-1}(\frac{\|y-F_0\|^2}{\|g\|^2_{L^2([0,t]\times A)}} + 1) + 1$ to obtain that

$$\begin{aligned} p_F(y) &\geq \frac{C_q e^{-q/2} \exp(-(1+e^{-1})C^*) \exp(-e^{-1}C^* \frac{\|y-F_0\|^2}{\|g\|^2_{L^2([0,t]\times A)}})}{\|g\|^q_{L^2([0,t]\times A)}} \\ &\geq \frac{\exp(-M' \frac{\|y-F_0\|^2}{\|g\|^2_{L^2([0,t]\times A)}})}{M' \, \|g\|^q_{L^2([0,t]\times A)}}, \end{aligned}$$

for some $M' \equiv M'(q, c, M, e) > 1$ where $e = e(\eta, M, c, C, \epsilon)$. ■

A common misconception about this theorem is that one may take limits in the above proof and therefore it becomes a consequence of a large deviation type result. This is not the case, as N in the proof has to be precisely taken within certain bounds determined by g, e, x_0 and y. Therefore the value of N is fixed in the above proof. Nevertheless, as we know exactly the value for N needed one could assume that the hypotheses of the theorem are satisfied for a partition having the properties as needed. Obviously then the partition will have to satisfy somewhat cumbersome conditions depending on the constants of the problem.

The restriction $\|g\|_{L^2([0,T]\times A)} \leq C$ actually says that the previous lower bound is only satisfied in bounded intervals as far as constants are concerned. For t, $T \to \infty$, one has to carry out a separate study. Also it is clear from the proof that the constant M' depends on all the other constants appearing in hypothesis **1** such as c, η_0 and M. Nevertheless g can still depend on other parameters but as long as C does not depend on them then M' will also be independent of these parameters. This will be the case in the next section when we treat the stochastic heat equation.

The above result can not easily be applied in most examples because the hypothesis **1** is as difficult to obtain as the claimed result itself. In fact hypothesis **1** is a uniformly (in $\omega \in \Omega$) localized (in A_i) lower bound of a conditional density of a random variable (F_i) of the same nature as the conclusion of the theorem.

Now we will try to establish an intermediary Theorem that can be applied in most examples. Here hypothesis **1** is replaced with an approximate local estimate of the conditional density of F_i given $\mathcal{F}_{t_{i-1}}$

Theorem 2 *Under the main set-up suppose that:*
I. There exist positive constants c, M, $\alpha > 1$, η_0 and random variables $C_i \in \overline{\mathcal{F}}_{t_i}$, $i = 0, ..., N-1$ satisfying that $\sup_{i=0,...,N} E\,|C_i| \leq M$ and such that for $0 < \Delta_{i-1}(g) < \eta_0$ and $y_i \in A_i$

$$E(\delta_{y_i}(F_i)/\mathcal{F}_{t_{i-1}}) \geq \frac{1}{M\Delta_{i-1}(g)^{q/2}} - C_{i-1}(\omega)\Delta_{i-1}(g)^\alpha$$

for almost all $\omega \in \Omega$ *and* $i = 1, ..., N$.
Then there exists a constant $M' > 0$ *that depends on all other constants such that*

$$p_F(y) \geq \frac{\exp\left(-M' \frac{\|y-F_0\|^2}{\|g\|^2_{L^2([0,t]\times A)}}\right)}{M' \|g\|^{q/2}_{L^2([0,t]\times A)}}.$$

Proof. As in the previous proof first we choose e_0 and let $\eta \leq \eta_0 \wedge 1$ and N be the smallest integer such that

$$N \geq e^{-1}(\frac{\|y - F_0\|^2}{\|g\|^2_{L^2([0,t]\times A)}} + 1)$$

for $e \in (\frac{c_0}{2}, c_0)$ with $c_0 = \frac{\eta}{M} \wedge \frac{1}{2} \wedge \frac{c^2}{4} \wedge e_0$. As before we have that there exists a partition π_N such that

$$\Delta_{i-1}(g) = \frac{\|g\|^2_{L^2([0,t]\times A)}}{N} \leq \eta.$$

In comparison with the previous proof, for reasons of clarity in the arguments, we prefer to follow an approximative argument for delta functions. Then we have by Fatou's lemma that for any r_1,...,$r_{N-1} > 0$

$$\begin{aligned} E(\delta_y(F)) &\geq \liminf_{r\to 0} \int_{B_N}...\int_{B_2} E(\phi_r(F-y)\phi_{r_{N-1}}(F_{N-1}-y_{N-1})...\phi_{r_1}(F_1-y_1))dy_1...dy_{N-1} \\ &\geq \int_{B_N}...\int_{B_2} E\left[\left(\frac{1}{M\Delta_{N-1}(g)^{q/2}} - C_{N-1}\Delta_{N-1}(g)^{\alpha}\right)...\phi_{r_1}(F_1-y_1)\right]dy_1...dy_{N-1} \\ &= \int_{B_N}...\int_{B_2} E\left(\frac{1}{M\Delta_{N-1}(g)^{q/2}}\phi_{r_{N-1}}(F_{N-1}-y_{N-1})...\phi_{r_1}(F_1-y_1)\right)dy_1...dy_{N-1} \\ &\quad -\Delta_{N-1}(g)^{\alpha} E\left|C_{N-1}\right|. \end{aligned}$$

Here we have used that

$$\int_{B_N}...\int_{B_2} E\left(\left|C_{N-1}\right|\phi_{r_{N-1}}(F_{N-1}-y_{N-1})...\phi_{r_1}(F_1-y_1)\right)dy_1...dy_{N-1} \leq E\left|C_{N-1}\right|,$$

which follows from Fubini's theorem. Next we take the limit when $r_{N-1} \to 0$ and repeat the same arguments. By induction and the arguments as in the proof of Theorem 1, we obtain that there exists a positive constant M depending on all the constants such that

$$p_F(y) \geq \frac{\exp\left(-M\frac{\|y-F_0\|^2}{\|g\|^2_{L^2([0,t]\times A)}}\right)}{M\|g\|^{q/2}_{L^2([0,t]\times A)}} - \sum_{i=1}^{N}\Delta_{i-1}(g)^{\alpha}E\left|C_{i-1}\right|\int_{B_N}...\int_{B_{i+1}}\prod_{j=i+1}^{N}\frac{1}{M\Delta_{j-1}(g)^{q/2}}dy_i...dy_{N-1}.$$

Here we define the previous integral as 1 when $i = N$. We bound the last integral as follows

$$\begin{aligned} \int_{B_N}...\int_{B_{i+1}}\prod_{j=i+1}^{N}\frac{1}{M\Delta_{j-1}(g)^{q/2}}dy_i...dy_{N-1} &\leq \prod_{j=i+1}^{N}\frac{|B_j|}{M\Delta_{j-1}(g)^{q/2}} \\ &\leq \left(\frac{C(q)c^q}{4^q M}\right)^{N-i}. \end{aligned}$$

From here one obtains that

$$p_F(y) \geq \frac{\exp\left(-M\frac{\|y-F_0\|^2}{\|g\|^2_{L^2([0,t]\times A)}}\right)}{M\,\|g\|^{q/2}_{L^2([0,t]\times A)}} - \sum_{i=1}^{N}\Delta_{i-1}(g)^{\alpha}E\,|C_{i-1}|\left(\frac{C(q)c^q}{4^qM}\right)^{N-i} \tag{3}$$

$$\geq \frac{\exp\left(-M\frac{\|y-F_0\|^2}{\|g\|^2_{L^2([0,t]\times A)}}\right)}{M\,\|g\|^{q/2}_{L^2([0,t]\times A)}} - C(M)\eta^{\alpha}N$$

where $C(M)$ is a positive constant and we have assumed without loss of generality that $M > \frac{C(q)c^q}{4^q}$. Now take η such that

$$\eta < \left(\frac{1}{C'}\frac{\exp\left(-M\frac{\|y-F_0\|^2}{\|g\|^2_{L^2([0,t]\times A)}}\right)}{2M\,\|g\|_{L^2([0,t]\times A)}{}^{q/2}}\right)^{1/(\alpha-1)} \wedge \frac{M}{2} \wedge \frac{c^2M}{4} \wedge Me_0 \wedge \eta_0 \wedge 1$$

with $C' = C(M)\left(2M(\frac{\|y-F_0\|^2}{\|g\|^2_{L^2([0,t]\times A)}}+1)+1\right)$, then $e \geq \frac{\eta}{2M} = \frac{c_0}{2}$ and therefore

$$\eta^{\alpha}N \leq \eta^{\alpha}\left(\frac{2M}{\eta}(\frac{\|y-F_0\|^2}{\|g\|^2_{L^2([0,t]\times A)}}+1)+1\right).$$

Putting these estimates in (3) we have the result with $M' = 2M$. ■

In this theorem, one uses the full sequence of partitions. In fact, note that as y becomes bigger η becomes smaller and therefore one refines the partition as e becomes smaller and therefore N becomes bigger. In the following theorem we establish that if a nice approximating sequence to F satisfying certain assumptions that ensure an efficient approximation with Malliavin Calculus then the above hypothesis I is satisfied. Essentially we require that for each i, there is a nicely behaved approximation $\overline{F}_i^l$ to F_i. This approximation sequence has to be as close as desired in the sense of the conditional norms defined in the preliminaries. The degree of closeness is measured through the parameter l. The higher the value of l, the better the approximation. γ will be a parameter that measures the quality of the approximation (for more on this, see Section 4). Usually this approximation will be obtained trough a truncation of the Itô-Taylor series expansion of $F_i - F_{i-1}$. For this reason we refer to $\overline{F}_i^l$ as the truncated approximation sequence.

Also we require that the behaviour of the Malliavin covariance matrix has to be as the one of the random variable being approximated. We also assume that the approximating sequence satisfies hypothesis **1** in Theorem 1 (*A3* below). Then we obtain that the hypothesis I in Theorem 2 is satisfied.

Theorem 3 *Under the main set-up suppose that for each $l \in \mathbb{N}$ and for each partition π_N, the sequence $\{F_i; , i = 1, ..., N\} \subseteq \overline{\mathbb{D}}^{\infty}$ uniformly and furthermore assume that there exists a sequence of truncated approximations $\{\overline{F}_i \equiv \overline{F}_i^l;\ i = 1, ..., N\}$ such that $\overline{F}_i \in \overline{\mathcal{F}}_{t_i} \cap\ \mathbb{D}^{\infty}_{t_{i-1}}$ and the following hypothesis are satisfied*

A1. $\left\|F_i - \overline{F}_i\right\|_{n,p,t_{i-1}} \leq C(n,p)\Delta_{i-1}(g)^{(l+1)\gamma}$ *for positive constants $C(n,p)$ and γ.*

A2. Define $\overline{F}_i(\rho) = \rho F_i + (1-\rho)\overline{F}_i$, $\rho \in [0,1]$. We assume that there exists a constant $C(p)$ such that

$$\sup_{\rho\in[0,1]}\left\|\det\psi^{-1}_{\overline{F}_i(\rho)}(t_{i-1})\right\|_{p,t_{i-1}} \leq C(p)\Delta_{i-1}(g)^{-q}.$$

A3. There exists positive constants M, η_0, c such that for $y_i \in A_i$ and $\Delta_{i-1}(g) < \eta_0$ one has that

$$E(\delta_{y_i}(\overline{F}_i)/\mathcal{F}_{t_{i-1}}) \geq \frac{1}{M\Delta_{i-1}(g)^{q/2}},$$

for almost all $\omega \in \Omega$. Then the hypothesis I *of Theorem 2 is satisfied with $\alpha = (l+1)\gamma - qc_1$ where c_1 is a positive constant. Therefore one has there exists a constant M' that depends on all other constants such that*

$$p_F(y) \geq \frac{\exp\left(-M' \frac{\|y-F_0\|^2}{\|g\|^2_{L^2([0,t]\times A)}}\right)}{M' \|g\|_{L^2([0,t]\times A)}{}^{q/2}}.$$

Example 4 *In this example we show that the condition A2 on the Malliavin covariance matrix of $\overline{F}_i$ is needed even if $\overline{F}_i$ is close to F_i according to condition 1. Let π_N be a uniform partition of size h and let $F_n = W(t_n)$ and $\overline{F}_n = W(t_{n-1}) + \int_{t_{n-1}}^{t_n} \psi_K(W(s))dW(s)$. $K \in \mathbb{R}$ is a constant to be fixed later and $\psi \in C_b^\infty(\mathbb{R}^k, [0,1])$ such that $\psi_K(x) = 1$ if $|x_i| \leq K$ for all $i = 1, ..., k$ and $\psi(x) = 0$ if $|x_i| \geq K+1$ for some $i = 1, ..., k$. In order for condition A1 to be satisfied one needs to choose $K \equiv K(l,h)$ and ψ_K such that*

$$\begin{aligned} E\int_{t_{n-1}}^{t_n} (1-\psi_K(W(s)))^2 ds &\leq C_K \int_{t_{n-1}}^{t_n} P(\max_{i=1,\dots,d} |W_i(s)| > K) ds \\ &\leq C_K \int_{t_{n-1}}^{t_n} 2\left(1 - \Phi\left(\frac{K}{\sqrt{s}}\right)\right)^k ds \\ &\leq C(t_n - t_{n-1})^{(l+1)/2}. \end{aligned}$$

We claim that this condition is not enough to have that the Malliavin covariance matrix of $\overline{F}_n$ is non-degenerate. In fact, on

$$\{\omega \in \Omega; \min_{s\in[t_{n-1},t_n]} \max_{i=1,\dots,k} |W_i(s)| > K\}$$

we have that the partial Malliavin covariance matrix of $\overline{F}_n$ conditioned on $\mathcal{F}_{t_{n-1}}$ is zero. It would be interesting to prove that for any approximation F_n there exists a sequence $\overline{F}_n$ with the required characteristics. We have not been able to prove this.
If one instead requires conditions that ensure A3 is satisfied then condition A2 can be simplified as shown in Proposition 12 in the Appendix.

Proof. It is enough to prove hypothesis I in Theorem 2. Consider for $y \in A_i$ and $\Delta_{i-1}(g) < \eta_0$

$$E_{t_{i-1}}(\delta_y(F_i)) \geq \frac{1}{M\Delta_{i-1}(g)^{q/2}} + E_{t_{i-1}}(\delta_y(F_i) - \delta_y(\overline{F}_i)).$$

Therefore is enough to prove that for almost all $\omega \in \Omega$, there exists $\overline{\mathcal{F}}_{t_{i-1}}$-measurable random variables C_{i-1} with the required characteristics such that

$$\left|E_{t_{i-1}}(\delta_y(F_i) - \delta_y(\overline{F}_i))\right| \leq C_{i-1}\Delta_{i-1}(g)^{(l+1)\gamma - qc_2}.$$

Now we estimate the error terms

$$\begin{aligned} &\left|E_{t_{i-1}}\left(\delta_{y_i}(F_i) - \delta_{y_i}(\overline{F}_i)\right)\right| \\ \leq\ & \sum_{\beta=1}^q \int_0^1 \left|E_{t_{i-1}}\left(\delta_{y_i}^{(\beta)}(\alpha F_i + (1-\alpha)\overline{F}_i)\left(F_i^\beta - \overline{F}_i^\beta\right)\right)\right| d\alpha \\ \leq\ & C\int_0^1 \sum_{\beta=1}^q \left\|H_{t_{i-1}}^{\gamma(\beta)}\left(\alpha F_i + (1-\alpha)\overline{F}_i, F_i^\beta - \overline{F}_i^\beta\right)\right\|_{1,t_{i-1}} d\alpha \\ \leq\ & C\int_0^1 \sum_{\beta=1}^q \left(\left\|F_i^\beta - \overline{F}_i^\beta\right\|_{n_3,p_3,t_{i-1}} \left\|\alpha F_i + (1-\alpha)\overline{F}_i\right\|_{n_2,p_2,t_{i-1}}^{c_2}\right. \\ &\left. \times \left\|\det\left(\psi_{\alpha F_i + (1-\alpha)\overline{F}_i}(t_{i-1})\right)^{-1}\right\|_{p_1,t_{i-1}}^{c_1}\right) d\alpha. \end{aligned}$$

Here $\gamma(\beta) = (1, ..., q, \beta)$ and the constants above are independent of i. Using our hypothesis A1 and A2, we have

$$\left|E_{t_{i-1}}\left(\delta_{y_i}(F_i) - \delta_{y_i}(\overline{F}_i)\right)\right|(\omega) \quad \leq \quad C_{i-1}(\omega)\Delta_{i-1}(g)^{(l+1)\gamma - qc_1}.$$

Here $C_{i-1} = \left\|\alpha F_i + (1-\alpha)\overline{F}_i\right\|_{n_2,p_2,t_{i-1}}^{c_2} \in \cap_{p\geq 1} L^p(\Omega)$ uniformly in $i = 1, ..., N$ and N. In fact, as $\{F_i; , i = 1, ..., N\} \subseteq \overline{\mathbb{D}}^{+\infty}$ uniformly and under hypothesis A1, we have for any $a > 1$,

$$\begin{aligned}\sup_N \sup_{i=1,..,N} E\left\|\overline{F}_i\right\|_{n,p,t_{i-1}}^a &\leq \sup_N \sup_{i=1,..,N}\left(E\left\|\overline{F}_i - F_i\right\|_{n,p,t_{i-1}}^a + E\left\|F_i\right\|_{n,p,t_{i-1}}^a\right) \\ &\leq C(a,n,p).\end{aligned}$$

Therefore taking l big enough and by Theorem 2 we obtain the conclusion. ■

In the previous theorem γ is a constant that may change depending on the characteristics of how the underlying noise appears in the structure of F and the quality of the truncated approximation sequence $\overline{F}_i$. In the following theorem we give conditions so that a sequence that approximates F_i as in the previous theorem can be constructed. In this setting we try to give conditions for the sequence as close as possible to the general set-up of stochastic equations and requiring the least amount of conditions so that the lower bound for the density of the approximative random variable can be obtained. In particular, in this set-up the condition of uniform ellipticity becomes clear. In the next theorem we use the notation $I_j^i(h) = \int_{t_{i-1}}^{t_i}\int h(s,x)dW^j(s,x)$ for $j = 1, ..., k$ and $h : \Omega \to L^2([t_{i-1}, t_i] \times A; \mathbb{R}^q)$ a $\mathcal{F}_{t_{i-1}}-$measurable smooth random processes. Also we remind the reader that the random variables Z_i are standard normal r.v.'s as defined in the Preliminaries and that therefore all norms considered from now are in an extended sample space.

Theorem 5 *Under the main set-up: Suppose that for each F_i and each $l \in \mathbb{N}$ there exists a (truncated) sequence $\overline{F}_i \equiv \overline{F}_i^l$ such that*

$$\overline{F}_i = \Delta_{i-1}(g)^{(l+1)\gamma} Z_i + F_{i-1} + \sum_{j=1}^{k} I_j^i(h_j) + G_i^l.$$

Here G_i^l are $\mathcal{F}_{t_i} \cap \overline{\mathbb{D}}_{t_{i-1}}^{\infty}$ random variables and $h_j \equiv h_j|_{[t_{i-1},t_i]} : \Omega \to L^2([t_{i-1}, t_i] \times A; \mathbb{R}^q)$ is a collection of $\mathcal{F}_{t_{i-1}}-$measurable smooth random processes which satisfies for almost all $\omega \in \Omega$:
(H1) *There exists a constant $C(n,p,T)$ such that*

$$\|F_i\|_{n,p} + \sup_{\omega\in\Omega}\|h_j\|_{L^2([t_{i-1},t_i]\times A)}(\omega) \leq C(n,p,T)$$

for any $j = 1, ..., k$, $i = 0, ..., N$ and n, $p \in \mathbb{N}$.
Furthermore the following four conditions are satisfied for the approximation sequence $\overline{F}_i$ and any $i = 1, ..., N$ and almost all $\omega \in \Omega$
(H2a) *There exists a constant $\gamma > 0$, such that for any n, p, $l \in \mathbb{N}$, $\left\|F_i - \overline{F}_i\right\|_{n,p,t_{i-1}} \leq C(n,p,T)\Delta_{i-1}(g)^{(l+1)\gamma}$.*
(H2b) *There exists a constant $C(p,T) > 0$ such that for any $p > 1$*

$$\left\|\det\psi_{F_i}^{-1}(t_{i-1})\right\|_{p,t_{i-1}} \leq C(p,T)\Delta_{i-1}(g)^{-q}.$$

(H2c) *Define*

$$A = \Delta_{i-1}(g)^{-1}\sum_{j=1}^{k}\begin{pmatrix} \int_{t_{i-1}}^{t_i}\left\langle h_j^1(s), h_j^1(s)\right\rangle_{L^2(A)} ds & ... & \int_{t_{i-1}}^{t_i}\left\langle h_j^1(s), h_j^q(s)\right\rangle_{L^2(A)} ds \\ : & . & : \\ \int_{t_{i-1}}^{t_i}\left\langle h_j^q(s), h_j^1(s)\right\rangle_{L^2(A)} ds & ... & \int_{t_{i-1}}^{t_i}\left\langle h_j^q(s), h_j^q(s)\right\rangle_{L^2(A)} ds\end{pmatrix}.$$

We assume that there exists strictly positive constants $C_1(T)$ and $C_2(T)$, such that for all $\xi \in \mathbb{R}^q$,

$$C_1(T)\xi'\xi \geq \xi' A\xi \geq C_2(T)\xi'\xi.$$

(H2d) *There exist constants $\varepsilon > 0$ and $C(n,p,l,T)$ such that*

$$\left\|G_i^l\right\|_{n,p,t_{i-1}} \leq C(n,p,l,T)\Delta_{i-1}(g)^{\frac{1}{2}+\varepsilon}.$$

Under the above conditions one has that there exists a constant $M > 0$ that depend on all other constants such that

$$p_F(y) \geq \frac{\exp\left(-M\frac{\|y-F_0\|^2}{\|g\|^2_{L^2([0,t]\times A)}}\right)}{M\left\|g\right\|_{L^2([0,t]\times A)}{}^{q/2}}.$$

Proof. The proof consists in showing that the conditions in Theorem 3 are satisfied. First, $\{F_i;, i = 1, ..., N\} \subseteq \overline{\mathbb{D}}^{+\infty}$ uniformly, due to hypothesis **(H1)** and the fact that $E\left\|F_i\right\|^k_{n,p,t_{i-1}} \leq C\left\|F_i\right\|^k_{n,pk}$. Next $\overline{F}_i \in \mathcal{F}_{t_i} \cap \overline{\mathbb{D}}^{+\infty}_{t_{i-1}}$ because of the definition of the truncated approximation sequence and **(H1)** (see Proposition 11 in the Appendix). Note that here the Wiener space has been expanded in order to include also the random variables Z_i. Condition A1 and **(H2a)** are the same. Verifying condition A2 is quite technical and it is done in Proposition 12 in the Appendix. Here, we verify condition A3.

First, we renormalize the expression for the density. That is, for $F_{i-1} = z$

$$\begin{aligned} E_{t_{i-1}}(\delta_y(\overline{F}_i)) &= \frac{1}{\Delta_{i-1}(g)^{q/2}}\, E_{t_{i-1}}\Big[\delta_{(y-z)/\Delta_{i-1}(g)^{1/2}}\Big(\Delta_{i-1}(g)^{(l+1)\gamma-1/2}Z_i \\ &\quad + \sum_{j=1}^{k}\Delta_{i-1}(g)^{-1/2}I_j^i(h_j) + \Delta_{i-1}(g)^{-1/2}G_i^l\Big)\Big]. \end{aligned}$$

Next we consider the Taylor expansion of the delta function around the non-degenerate random variable $\Delta_{i-1}(g)^{-1/2}\sum_{j=1}^{k} I_j^i(h_j)$. To simplify the notation we will define

$$(X,Y) \equiv (X_i,Y_i) = \left(\Delta_{i-1}(g)^{-1/2}\sum_{j=1}^{k} I_j^i(h_j), \Delta_{i-1}(g)^{(l+1)\gamma-1/2}Z_i + \Delta_{i-1}(g)^{-1/2}G_i^l\right).$$

With this new notation we have by the mean value theorem

$$\begin{aligned} E_{t_{i-1}}(\delta_y(\overline{F}_i)) &= \frac{1}{\Delta_{i-1}(g)^{q/2}}E_{t_{i-1}}\left[\delta_{(y-z)/\Delta_{i-1}(g)^{1/2}}(X+Y)\right] \qquad (4) \\ &= \frac{1}{\Delta_{i-1}(g)^{q/2}}\left\{E_{t_{i-1}}\left[\delta_{(y-z)/\Delta_{i-1}(g)^{1/2}}\left(X\right)\right]+\int_0^1\sum_{\beta=1}^{q}E_{t_{i-1}}\left[\delta^{(\beta)}_{(y-z)/\Delta_{i-1}(g)^{1/2}}(X+\rho Y)Y^{\beta}\right]d\rho\right\}. \end{aligned}$$

Now we apply the integration by parts formula to each of the last q terms in the sum above. As the treatment is similar for every term we will consider one of these terms:

$$\begin{aligned} &\frac{1}{\Delta_{i-1}(g)^{q/2}}E_{t_{i-1}}\left[\delta^{(\beta)}_{(y-z)/\Delta_{i-1}(g)^{1/2}}\left(X+\rho Y\right)Y^{\beta}\right] \qquad (5) \\ &= \frac{1}{\Delta_{i-1}(g)^{q/2}}E_{t_{i-1}}\left[1\left(X+\rho Y \geq \frac{(y-z)}{\Delta_{i-1}(g)^{1/2}}\right)H^{\gamma(\beta)}(X+\rho Y, Y^{\beta})\right]. \end{aligned}$$

Here $\gamma(\beta) = (1, ..., q, \beta)$. Now we prove that all these terms in the sum above are bounded below by an expression of the order $\Delta_{i-1}(g)^{\epsilon}$. That is, by (1), the expression (5) is bounded above by

$$\left\|H^{\gamma(\beta)}(X+\rho Y, Y^{\beta})\right\|_{1,t_{i-1}} \leq C\Psi_{i-1}\left(X,Y\right)$$

where C is a universal constant which does not depend on ρ. Ψ is a random function defined for two smooth non-degenerate random variables X, Y as

$$\begin{aligned} \Psi_{i-1}(X,Y) &= \left\|X+\rho Y\right\|^{a_1}_{d_1,b_1,t_{i-1}}\left\|\det(\psi_{X+\rho Y}(t_{i-1}))^{-1}\right\|^{a_2}_{b_2,t_{i-1}}\left\|Y^{\beta}\right\|_{d_3,b_3,t_{i-1}} \\ &\leq C(d_1,b_1)^{a_1}C(b_2)^{a_2}C(d_3,b_3)\Delta_{i-1}(g)^{\epsilon}, \end{aligned}$$

where the last inequality is valid for l big enough. In fact, the middle term measures the $L^p(\Omega)$-norms of determinants of the Malliavin covariance matrix. This term is bounded by Proposition 12 in the Appendix (Note that $X+\rho Y=G_i^{'}(\rho)$). The first term, $\|X+\rho Y\|_{d_1,b_1,t_{i-1}}^{a_1}$, and the third, $\left\|Y^\beta\right\|_{d_3,b_3,t_{i-1}}$, are bounded due to **(H1)** and **(H2d)**. Therefore we can conclude that

$$\int_0^1 \sum_{\beta=1}^q E_{t_{i-1}}\left[\delta^{(\beta)}_{(y-z)/\Delta_{i-1}(g)^{1/2}}\left(X+\rho Y\right)Y^\beta\right]d\rho \le C\Delta_{i-1}(g)^\epsilon.$$

Now we have that the first term of (4), $\Delta_{i-1}(g)^{-1/2}\sum_{j=1}^k I_j^i(h_j)$ is Gaussian and due to **(H2c)**, its $\mathcal{F}_{t_{i-1}}$-conditional covariance matrix, A, is invertible. Therefore the exact conditional density in this case is clearly

$$E_{t_{i-1}}\left(\delta_{(y-z)/\Delta_{i-1}(g)^{1/2}}(X)\right)=\frac{1}{(2\pi)^{q/2}\det(A)^{1/2}}\exp(-\frac{(y-z)'A^{-1}(y-z)}{2\Delta_{i-1}(g)}).$$

Next, due to hypothesis (**H2c**) we have that

$$E_{t_{i-1}}\left(\delta_{(y-z)/\Delta_{i-1}(g)^{1/2}}\left(X\right)\right)\ge \frac{1}{(2\pi)^{q/2}C_1^{q/2}}\exp(-\frac{\|y-z\|^2}{C_2\Delta_{i-1}(g)}).$$

Therefore we have that

$$E_{t_{i-1}}(\delta_y(\overline{F}_i))\ge\frac{1}{\Delta_{i-1}(g)^{q/2}}\left(\frac{1}{(2\pi)^{q/2}C_1^{q/2}}\exp(-\frac{\|y-z\|^2}{C_2\Delta_{i-1}(g)})-C\Delta_{i-1}(g)^\epsilon\right).$$

Next, we have that if $\|y-z\|^2\le c\Delta_{i-1}(g)$ for a constant $c>0$ then

$$E_{t_{i-1}}(\delta_y(\overline{F}_i))\ge\frac{1}{\Delta_{i-1}(g)^{q/2}}\left(\frac{1}{(2\pi)^{q/2}C_1^{q/2}}\exp(-C_2^{-1}c)-C\Delta_{i-1}(g)^\epsilon\right).$$

Finally we choose the constants M and η_0 in hypothesis A3 as follows. Let M be a positive constant so that

$$\frac{1}{(2\pi)^{q/2}C_1^{q/2}}\exp(-C_2^{-1}c)>\frac{1}{M},$$

next we define

$$\eta_0:=\left(\frac{1}{(2\pi)^{q/2}C_1^{q/2}C}\exp(-C_2^{-1}c)-\frac{1}{MC}\right)^{1/\epsilon}.$$

With these definitions one obtains that if $\Delta_{i-1}(g)\le\eta_0$ and $\|y-z\|^2\le c\Delta_{i-1}(g)$ then

$$E_{t_{i-1}}(\delta_y(\overline{F}_i))\ge\frac{1}{M\Delta_{i-1}(g)^{q/2}}.$$

Therefore the estimate in A3 is proven. ■

When the conditions of the previous theorem are met we will say that the random variable F is a **uniformly elliptic random variable**. Note that in this theorem, $\overline{F}_i$ is measurable with respect to the expanded filtration $\overline{\mathcal{F}}_{t_i}$as we are adding the variables Z_i to its definition. Also we remark that the random variables $\overline{F}_i$, considered in this Theorem will not necessarily be non-degenerate unless one adds the independent random variable $\Delta_{i-1}(g)^{(l+1)\gamma}Z_i$. To see this is enough to consider the case $l=2$ with G_i^l a double stochastic integral.

Example 6 *For example, suppose that* $\overline{F}_i=F_{i-1}+\int_{t_{i-1}}^{t_i}f(W_{t_{i-1}})dW_s+\int_{t_{i-1}}^{t_i}\int_{t_{i-1}}^{s}g(W_{t_{i-1}})dW_udW_s$ *with* $h(s)=f(W_{t_{i-1}})$ *and* $G_i=\int_{t_{i-1}}^{t_i}\int_{t_{i-1}}^{s}g(W_{t_{i-1}})dW_udW_s$*. In this case one has that* $\psi_{\overline{F}_i}(t_{i-1})=(t_i-t_{i-1})\left(f(W_{t_{i-1}})+g(W_{t_{i-1}})\left(W_{t_i}-W_{t_{i-1}}\right)\right)^2$*. Then if* $f(x)\in[C_1,C_2]$ *for two positive constants* C_1 *and* C_2 *then the random variable* F_i *satisfies* ***(H2c)****. Furthermore if* $g(x)\neq 0$ *and bounded then one has that* $E_{t_{i-1}}\psi_{\overline{F}_i}(t_{i-1})^{-p}=+\infty$ *for all* $p\ge 1$*. Obviously the same example can be used for* F_i*.*

Related with these comments we also emphasize that the above proof can not be used to obtain a lower bound on the local density of F_i conditioned on $\mathcal{F}_{t_{i-1}}$. The main reason being that once the main first order stochastic integrals are taken out of F_i , we are not able to prove the stability of the Malliavin covariance matrix of $X + \rho Y$ in the proof (with $Z_i \equiv 0$). Probably, as the previous example shows, this stability is not satisfied in general.

Nevertheless, one can refine the above proof to obtain that for $\|y-z\|^2 \leq c\Delta_{i-1}(g)$ then

$$E_{t_{i-1}}(\delta_y(\overline{F}_i)) \geq \frac{\exp\left(-\frac{M\|y-z\|^2}{\Delta_{i-1}(g)}\right)}{M\Delta_{i-1}(g)^{q/2}}.$$

4 The stochastic heat equation

In most situations like the diffusion case one expects the function g to be constant and therefore the variance to be of the order t^α for $\alpha = 1$. In the example to be treated in this section we consider a case where $\alpha = 1/2$ and the local variance function g is not constant. Other examples using stochastic differential equations can be constructed if the coefficients were allowed to be degenerate as functions of time. For example, $\sigma(t,x) = t^{-\alpha} f(x)$ for a smooth function f and $0 < \alpha < 1/2$. In cases of this type one will need to develop an ad-hoc succession of existence, uniqueness and smoothness results as the coefficient degenerates at $t = 0$.

Instead of taking this long and tedious road, we have chosen to show an example where most of the needed properties are known but still the model is quite general. This is the case of the stochastic heat equation. In fact, most of the smoothness and estimates for the Malliavin variance will follow from results in [1]. Our main result in this section, Theorem 10, is the characterization of a specific Gaussian type lower bound for the density of the solution of the stochastic heat equation.

We believe this is the first study of the kind. The Varadhan estimates for the stochastic heat equation were obtained in [9] using a general theorem taken from [16]. It is clear that these two results are deeply related. Nevertheless the arguments to obtain Varadhan's estimate can not be extended to obtain inequalities for any time $t > 0$. In contrast, one can obtain estimates for small time from our results here but the specification of the distance function is not as accurate as in Varadhan's estimate.

Without loss of generality we will assume throughout the text that $t < 1$. The hypotheses stated in this section are valid in all that follows.

Now we introduce the stochastic heat equation with Neumann conditions. Let us start by considering $u(t,x)$ to be the weak solution of the stochastic parabolic equation with Neummann type conditions of the form

$$\begin{aligned}
\frac{\partial u}{\partial t}(t,x) &= \frac{\partial^2 u}{\partial x^2}(t,x) + b(u(t,x)) + \sigma(u(t,x))\frac{\partial^2 W}{\partial t \partial x}(x,t) \\
u(0,x) &= u_0(x),\ \frac{\partial u}{\partial x}(t,0) = \frac{\partial u}{\partial x}(t,1) = 0,\ t \in [0,1].
\end{aligned}$$

Here $b, \sigma : \mathbb{R} \to \mathbb{R}$ are bounded smooth functions with bounded derivatives with $|\sigma(x)| \geq c_0 > 0$ for all $x \in \mathbb{R}$, $u_0 : [0,1] \to \mathbb{R}$ with $u_0 \in C([0,1])$. $\{W(t,x); (t,x) \in [0,1]^2\}$ is a Wiener sheet. We are interested in obtaining lower bounds for the density of $F = u(t,x)$, therefore in reference to the notation in the previous section we have $m = q = 1$ and $A = [0,1]$. It is well known (see e.g. Nualart (1998), Section 2.4) that the solution to the above equation exists, is unique, smooth and non-degenerate (for details, see [1]). The solution can be expressed as

$$u(t,x) = G(u_0)(t,x) + \int_0^t \int_0^1 G_{t-s}(x,y)b(u(s,y))dyds + \int_0^t \int_0^1 G_{t-s}(x,y)\sigma(u(s,y))W(dy,ds).$$

The above stochastic integral is the one defined by Walsh (for details see e.g. Nualart's section 2.4). $G_t(x,y)$ is the Green kernel associated to the heat equation with Neumann boundary conditions. That is,

$$\begin{aligned} G_t(x,y) &= \frac{1}{\sqrt{4\pi t}}\sum_{n=-\infty}^{+\infty}\left\{\exp\left(-\frac{(y-x-2n)^2}{4t}\right)+\exp\left(-\frac{(y+x-2n)^2}{4t}\right)\right\} \\ G(u_0)(t,x) &= \int_0^1 G_t(x,y)u_0(y)dy. \end{aligned}$$

Recall that as noted by [1], Remark 2.1 the above kernel satisfies the same properties as in the Dirichlet boundary condition (A.1), (A.3) and (A.5) established in the Appendix of their paper. The same results are valid for both cases. In particular we recall that there exists a positive universal constant c_1 such that

$$c_1\phi_{4(t-s)}(x-y) \geq G_{t-s}(x,y) \geq \phi_{t-s}(x-y). \tag{6}$$

Here, as before, $\phi_r(x)$ denotes the density of a normal random variable with variance r. Note that $G_{t-s}(x,y)$ is degenerate at $t=s$. Therefore the local variance g is not constant in this case. In particular, consider the trivial case $b\equiv 0$ and $\sigma\equiv 1$. In this case $u(t,x)$ is a Gaussian random variable with mean $G(u_0)(t,x)$ and variance $\int_0^t\int_0^1 G_{t-s}(x,y)^2dyds$. The density can then be written explicitly and the behavior of the local variance g is clear: it is constant away from $s=t$ but it degenerates at a rate $(t-s)^{-1/2}$. In fact, we set for the rest of the section $g(s,y)=\phi_{t-s}(x-y)$.

The goal of the next subsections is to prove that under strong ellipticity conditions one has a Gaussian type lower bounds for the density of $u(t,x)$. In this case the main technical problem lies in the fact that there is not an Ito's formula that adapts well to form an Itô-Taylor expansion which could lead to the definition of G_i^l. This means that the verifications of some of the hypothesis (in particular **(H2a)** and **(H2d)**) may become slightly more complicated.

Here, some of the calculations related to the behavior of the Malliavin variance are related to existing ones in the literature. Still we have to do them as we have to keep exact track of all the time dependent constants. These are long calculations which we briefly sketch for the sake of completeness. In general we refer the reader to [1].

4.1 The lower bound

Now we start the description of all the ingredients towards proving that $F=u(t,x)$ is a uniformly elliptic random variable. First for any partition $0=t_0<...<t_N=t$, define

$$F_i = G(u_0)(t,x)+\int_0^{t_i}\int_0^1 G_{t-s}(x,y)b(u(s,y))dyds+\int_0^{t_i}\int_0^1 G_{t-s}(x,y)\sigma(u(s,y))W(dy,ds).$$

It is clear from this definition that $F_i\in\mathcal{F}_{t_i}=\sigma\{W(s,x),\,(s,x)\in[0,t_i]\times[0,1]\}$ for $i=0,...,N$. We will prove the needed properties in the main set up and in **(H1)-(H2)** through a sequence of lemmas. In the proofs to follow, we frequently use the following estimates.

$$\begin{aligned} C_1t^{1/4} &\leq \|g\|_{L^2([0,t]\times[0,1])}\leq C_2t^{1/4} \\ C_1(\sqrt{t-s_1}-\sqrt{t-s_2})^{1/2} &\leq \|g\|_{L^2([s_1,s_2]\times[0,1])} \\ \|g\|_{L^2([s_1,s_2]\times[0,1])} &\leq C_2(\sqrt{t-s_1}-\sqrt{t-s_2})^{1/2} \\ (s_2-s_1)^{1/2} &\leq C_2(\sqrt{t-s_1}-\sqrt{t-s_2})^{1/2}, \end{aligned} \tag{7}$$

for any $s_1\leq s_2\leq t$, and some positive constants C_1, C_2 independent of t. We will use them without further mentioning. In many of the subsequent lemmas we will use the following notation for high order stochastic derivatives. For a vector $v=(y_1,s_1,...,y_n,s_n)\in[0,1]^{2n}$, define $D_v^n\equiv D_{(s_1,y_1)}...D_{(s_n,y_n)}$, $dv=dy_nds_n...dy_1ds_1$ and $v-=(y_1,s_1,....,y_{n-1},s_{n-1})$.

Lemma 7 *Suppose that* σ, $b \in C_b^\infty(\mathbb{R})$, $\|\sigma(x)\| \geq c_0 > 0$ *for all* $x \in \mathbb{R}$. *Then* $F_i \in \mathbb{D}^{n,p}$ *and* $\|F_i\|_{n,p} \leq C_{F_i}(n,p)$ *for a positive constant* $C_{F_i}(n,p)$ *and any* $n \in \mathbb{N}$ *and* $p > 1$. *Furthermore, there exists a constant* $C(p) > 0$ *such that for any* $p > 1$ *and* $i = 1, ..., N$

$$\left\|\psi_{F_i}^{-1}(t_{i-1})\right\|_{p,t_{i-1}} \leq C(p)\Delta_{i-1}(g)^{-1}.$$

Proof. In [1], it is shown that $u(t,x) \in \mathbb{D}^{+\infty}$ and $\sup_{t,x} \|u(t,x)\|_{n,p} \leq C_u(n,p)$ (one can also use Lemma 14 in the Appendix to reprove this result). We use Lemma 14 to prove that $\|F_i\|_{n,p} \leq C(n,p)$. In fact, applying Lemma 14 for F_i and $p > 6$, we have (for $a = 0$, $b = t_i$, $X = F_i$, $I_i = 1$, $f(x) = b(x)$, $g(x) = \sigma(x)$, $u'(s,y) = u(s,y)$, $X_0 = G(u_0)(t,x), c_u(l, p(l,j), 0, s) = C_u(l, p(l,j))$)

$$\|F_i\|_{n,p} \leq |G(u_0)(t,x)| + C(n,p)C_u(n,p')^n \left\{t_i + \left(t^{3/2-q} - (t-t_i)^{3/2-q}\right)^{(p-2)/(2p)} t_i^{1/p}\right\}.$$

Now we prove the estimate on the Malliavin variance of F_i conditioned to $\mathcal{F}_{t_{i-1}}$. In order to obtain this estimate one has to follow carefully the same steps as in [1]. Here we only sketch the main points, referring the reader to [1] for details

$$\psi_{F_i}(t_{i-1}) \geq c_0^2 \int_{t_{i-1}}^{t_i} \int_0^1 S_{s,y}^{i-1}(t_i,x)^2 dyds,$$

where $S_{s,y}^{i-1}(t_1,x_1)$ is defined for s, $t_1 \in [t_{i-1}, t_i]$, x_1, $y \in [0,1]$

$$\begin{aligned}
S_{s,y}^{i-1}(t_1,x_1) &= G_{t-s}(x_1,y) + Q_{s,y}^{i-1}(t_1,x_1) \\
Q_{s,y}^{i-1}(t_1,x_1) &= \int_s^{t_1}\int_0^1 G_{t-s_1}(x_1,y_1)b'(u(s_1,y_1))S_{s,y}(s_1,y_1)dy_1ds_1 \\
&\quad + \int_s^{t_1}\int_0^1 G_{t-s_1}(x_1,y_1)\sigma'(u(s_1,y_1))S_{s,y}(s_1,y_1)W(dy_1,ds_1) \\
S_{s,y}(s_1,y_1) &= \frac{D_{s,y}u(s_1,y_1)}{\sigma(u(s,y))}.
\end{aligned}$$

Then we will estimate the probability $p_{t_{i-1}} = P_{t_{i-1}}\left(\int_{t_{i-1}}^{t_i}\int_0^1 S_{s,y}^{i-1}(t_i,x)^2 dyds \leq \frac{c}{2}\|g\|^2_{L^2([t_i-\varepsilon,t_i]\times[0,1])}\right)$ for $0 < \varepsilon \leq t_i - t_{i-1}$ and c is a positive constant such that $c\|g\|^2_{L^2([t_i-\varepsilon,t_i]\times[0,1])} < \frac{2}{3}\int_{t_i-\varepsilon}^{t_i}\int_0^1 G_{t-s}(x,y)^2 dyds$. For this, note that

$$\begin{aligned}
p_{t_{i-1}} &\leq P_{t_{i-1}}\left(\frac{2}{3}\int_{t_i-\varepsilon}^{t_i}\int_0^1 G_{t-s}(x,y)^2dyds - 2\int_{t_i-\varepsilon}^{t_i}\int_0^1 Q_{s,y}^{i-1}(t_i,x)^2dyds \leq \frac{c}{2}\|g\|^2_{L^2([t_i-\varepsilon,t_i]\times[0,1])}\right) \\
&\leq E_{t_{i-1}}\left(\int_{t_i-\varepsilon}^{t_i}\int_0^1 Q_{s,y}^{i-1}(t_i,x)^2dyds\right)^p\left(\frac{c}{2}\|g\|^2_{L^2([t_i-\varepsilon,t_i]\times[0,1])}\right)^{-p}.
\end{aligned}$$

Now we only need to estimate the above conditional expectation. In order to shorten the length of the equations we assume without loss of generality that $b' = 0$. The general case is similar. The needed estimate is obtained using Burkholder's inequality for martingales in Hilbert spaces (see [14], E.2, p. 212) then one has that using (6) and Lemma 13

$$\begin{aligned}
& E\left(\int_{t_i-\varepsilon}^{t_i}\int_0^1 Q_{s,y}^{i-1}(t_i,x)^2dyds\right)^p \\
\leq\ & CE_{t_{i-1}}\left[\int_{t_i-\varepsilon}^{t_i}\int_0^1 G_{t-s_1}^2(x,y_1)\int_{t_i-\varepsilon}^{s_1}\int_0^1 S_{s,y}(s_1,y_1)^2dydsdy_1ds_1\right]^p \\
\leq\ & C\left(\int_{t_i-\varepsilon}^{t_i}\int_0^1 G_{t-s_1}^{2q}(x,y_1)dy_1ds_1\right)^{p/q}\int_{t_i-\varepsilon}^{t_i}\int_0^1 E_{t_{i-1}}\left(\int_{t_i-\varepsilon}^{s_1}\int_0^1 S_{s,y}(s_1,y_1)^2dyds\right)^p dy_1ds_1 \\
\leq\ & C\left((t-t_i+\varepsilon)^{3/2-q} - (t-t_i)^{3/2-q}\right)^{p/q}\varepsilon^{(p+2)/2}\exp\left(C\varepsilon^{(p-1)/2}\right).
\end{aligned}$$

Here $p^{-1}+q^{-1}=1$ and C is a constant independent of t. Then we obtain, using (7) that for $p>3$

$$\begin{aligned} & P_{t_{i-1}}\left(\psi_{F_i}(t_{i-1}) \leq \frac{c\left(C_1 c_0\right)^2}{2}\left(\sqrt{t-t_i+\varepsilon}-\sqrt{t-t_i}\right)\right) \leq p_{t_{i-1}} \\ \leq & \quad f(\varepsilon)=C\varepsilon^{(p+2)/2}\left((t-t_i+\varepsilon)^{3/2-q}-(t-t_i)^{3/2-q}\right)^{p-1}\left(\sqrt{t-t_i+\varepsilon}-\sqrt{t-t_i}\right)^{-p} \end{aligned}$$

for two constants c and C independent of t and $\varepsilon \leq t_i - t_{i-1} \leq 1$.

Now we choose $\varepsilon \equiv \varepsilon(y)=\left(\frac{2}{\overline{c}y^{1/k}}+\sqrt{t-t_i}\right)^2-(t-t_i)$ with $\overline{c}=c\left(c_0 C_2\right)^2$ and $y \geq 2^k\left(cc_0^2\Delta_{i-1}(g)\right)^{-k}$. Under these conditions we have that $\varepsilon \leq t_i - t_{i-1}$ and

$$\int_{2^k\left(cc_0^2\Delta_{i-1}(g)\right)^{-k}}^{+\infty} P_{t_{i-1}}(\psi_{F_i}(t_{i-1}) \leq \frac{1}{y^{1/k}})dy \leq \int_{2^k\left(cc_0^2\Delta_{i-1}(g)\right)^{-k}}^{+\infty} f(\varepsilon)dy.$$

Therefore

$$\begin{aligned} E_{t_{i-1}}(\psi_{F_i}^{-k}) &= \int_0^{+\infty} P_{t_{i-1}}(\psi_{F_i} \leq \frac{1}{y^{1/k}})dy \\ &\leq 2^k\left(cc_0^2\Delta_{i-1}(g)\right)^{-k}+\int_{2^k\left(cc_0^2\Delta_{i-1}(g)\right)^{-k}}^{+\infty} f(\varepsilon)dy. \end{aligned}$$

First, suppose that $t_i=t$ then we have

$$\begin{aligned} E_{t_{i-1}}(\psi_{F_i}^{-k}) &\leq 2^k\left(cc_0^2\Delta_{i-1}(g)\right)^{-k}+C(p)\int_{2^k\left(cc_0^2\Delta_{i-1}(g)\right)^{-k}}^{+\infty}\frac{1}{y^{(p-1)/k}}dy \\ &\leq C(p,k)\left(\Delta_{i-1}(g)^{-k}+\Delta_{i-1}(g)^{p-1-k}\right). \end{aligned}$$

Therefore in this case we have that $\left\|\psi_{F_i}^{-1}\right\|_{k,t_{i-1}} \leq C(p,k)\Delta_{i-1}(g)^{-1}$ for $p>1$. In the case $t_i<t$ we perform the change of variables $w=y\left(\frac{\overline{c}\sqrt{t-t_i}}{2}\right)^k$, one has that the above is bounded by

$$\begin{aligned} & E_{t_{i-1}}(\psi_{F_i}^{-k}) \leq C(k)\Delta_{i-1}(g)^{-k} \\ & +C(t-t_i)^{(p-k-1)/2}\int_{C_2^{2k}(t-t_i)^{k/2}\Delta_{i-1}(g)^{-k}}^{+\infty} w^{p/k}\left[\left(1+\frac{1}{w^{1/k}}\right)^2-1\right]^{(p+2)/2}\left[\left(1+\frac{1}{w^{1/k}}\right)^{3-2q}-1\right]^{p-1}dw \end{aligned}$$

We also have that for any positive constant $\overline{C}_0$ there exists a positive constant $\overline{C}_1$ such that for any $w \geq \overline{C}_0$

$$\left(1+\frac{1}{w^{1/k}}\right)^{3-2q}-1 \leq \frac{\overline{C}_1}{w^{1/k}}.$$

Therefore taking $\overline{C}_0$ small enough so that $C_1^{2k}(t-t_i)^{k/2}\Delta_{i-1}(g)^{-k} \geq \overline{C}_0$, we have that

$$\begin{aligned} E_{t_{i-1}}(\psi_{F_i}^{-k}) &\leq C\Delta_{i-1}(g)^{-k}+C(t-t_i)^{(p-k-1)/2}\int_{C_2^{2k}(t-t_i)^{k/2}\Delta_{i-1}(g)^{-k}}^{+\infty}\frac{1}{w^{p/(2k)}}\,dw \\ &= C\Delta_{i-1}(g)^{-k}+C(k,p)(t-t_i)^{(p-2)/4}\Delta_{i-1}(g)^{\frac{p}{2}-k}. \end{aligned}$$

Therefore $\left\|\psi_{F_i}^{-1}\right\|_{k,t_{i-1}} \leq C(k,p)\Delta_{i-1}(g)^{-1}$ for $p>2k\vee 2$. ∎

With this lemma we have proven that the hypotheses in the main set-up and **(H2b)** in Theorem 5 are satisfied. Now we proceed with the definition of $\overline{F}_i$. In order to do this one needs to obtain some kind of $\mathcal{F}_{t_{i-1}}$-conditional high order Itô-Taylor formula for the difference F_i-F_{i-1} then consider the truncated series approximation and prove all the properties established in Theorem 5. Note that

$$F_i-F_{i-1}=\int_{t_{i-1}}^{t_i}\int_0^1 G_{t-s}(x,y)b(u(s,y))dyds+\int_{t_{i-1}}^{t_i}\int_0^1 G_{t-s}(x,y)\sigma(u(s,y))W(dy,ds).$$

The next objective is to find a Taylor expansion for the terms on the right above. To do this one uses a Taylor expansion of b and σ around another point u_{i-1} to be defined as

$$\begin{aligned} u_{i-1}(s_1,y_1) &= G(u_0)(s_1,y_1) + \int_0^{t_{i-1}} \int_0^1 G_{s_1-s_2}(y_1,y_2) b(u(s_2,y_2)) dy_2 ds_2 \\ &\quad + \int_0^{t_{i-1}} \int_0^1 G_{s_1-s_2}(y_1,y_2)\sigma(u(s_2,y_2)) W(dy_2,ds_2). \end{aligned}$$

Note that $u_{i-1} \in \mathcal{F}_{t_{i-1}}$ and is a smooth process. Our first result gives an estimate of the distance between u and u_{i-1}.

Lemma 8 *Suppose that σ, $b \in C_b^\infty(\mathbb{R})$. For $s \in [t_{i-1}, t_i]$, we have that $\|u(s,y) - u_{i-1}(s,y)\|_{n,p,t_{i-1}} \leq C(s-t_{i-1})^{1/8}$.*

Proof. In fact, we have that

$$u(s,y) - u_{i-1}(s,y) = \int_{t_{i-1}}^s \int_0^1 G_{s-s_1}(y,y_1) b(u(s_1,y_1)) dy_1 ds_1 + \int_{t_{i-1}}^s \int_0^1 G_{s-s_1}(y,y_1)\sigma(u(s_1,y_1)) W(dy_1,ds_1).$$

First we remark that applying Lemma 14 we obtain that $\sup_{t,x} \|u(t,x)\|_{n,p,t_{i-1}} \leq C(n,p)$. The argument is done through induction on n. As this proof is similar to the one that follows we leave it for the reader. In order to prove the estimate on $\|u(s,y) - u_{i-1}(s,y)\|_{n,p,t_{i-1}}$ apply Lemma 14 with $X = u(s,y) - u_{i-1}(s,y)$, $X_0 = 0$, $I_i(s,y) \equiv 1$, $\gamma = 0$, $p^* = p$, $f = b$, $g = \sigma$, $u' = u$, $a = t_{i-1}$, $b = s$, $t = s$ which gives for $n = 0$

$$\|u(s,y) - u_{i-1}(s,y)\|_{p,t_{i-1}} \leq C(n,p)\left\{(s-t_{i-1}) + \left((s-t_{i-1})^{3/2-q}\right)^{(p-2)/(2p)} (s-t_{i-1})^{1/p}\right\}.$$

The result follows as $(3/2-q)(p-2)/(2p) + 1/p = 1/4 - 1/(2p) \leq 1/8$ if $p \geq 4$. Next suppose that $\|u(s,y) - u_{i-1}(s,y)\|_{j,p,t_{i-1}} \leq C(s-t_{i-1})^{1/8}$ for $j \leq n-1$ then we have that $A(u(s,y))_{n,p,t_{i-1}} = A(u(s,y) - u_{i-1}(s,y))_{n,p,t_{i-1}}$ as $u_{i-1}(s,y)$ is a $\mathcal{F}_{t_{i-1}}$-measurable random variable (A is defined just before Lemma 14 in the Appendix). Therefore using again Lemma 14, we have that for $p > 6$

$$\begin{aligned} A(u(s,y) - u_{i-1}(s,y))_{n,p,t_{i-1}} &\leq C(n,p)\Bigg\{(s-t_{i-1}) + (s-t_{i-1})^{1/8} + \int_{t_{i-1}}^s A(u(r,y) - u_{i-1}(r,y))_{n,p,t_{i-1}} dr \\ &\quad + (s-t_{i-1})^{1/4-3/(2p)} \left(\int_{t_{i-1}}^s A(u(r,y) - u_{i-1}(r,y))^p_{n,p,t_{i-1}} dr\right)^{1/p}\Bigg\}. \end{aligned}$$

The result follows from Gronwall's lemma applied to $A(u(s,y) - u_{i-1}(s,y))^p_{n,p,t_{i-1}}$ and the inductive hypothesis. ∎

In order to proceed in the definition of the truncated approximations we will first study all the terms that appear in the Taylor expansion of $u(s_1,y_1) - u_{i-1}(s_1,y_1)$ in terms of stochastic and Lebesgue integrals depending only on u_{i-1}.

We say that a process $J_1(s,t,y)$ for $s \leq t \leq 1$, $y \in [0,1]$ is of order 1 (in the interval $[t_i, t_{i-1}]$) if it can be written as

$$J_1(s,t,y) = \int_{t_{i-1}}^s \int_0^1 G_{t-s_1}(y,y_1)\sigma(u_{i-1}(s_1,y_1)) W(dy_1,ds_1).$$

In the particular case $s = t$ we define $I_1(s,y) = J_1(s,s,y)$ and we say that I_1 is a diagonal process of order 1. We define by induction a process of high order as: A process J_k is a process of order k if either:

1. $$J_k(s,t,y) = \int_{t_{i-1}}^s \int_0^1 G_{t-s_1}(y,y_1)\sigma^{(l)}(u_{i-1}(s_1,y_1)) \prod_{j=1}^l I_{m_j}(s_1,y_1) W(dy_1,ds_1) \quad (8)$$

where $l \leq k-1$ and $I_{m_1}, ..., I_{m_l}$ are diagonal processes of order m_1,...,m_l respectively with $m_1 + ... + m_l = k-1$.

2.
$$J_k(s,t,y) = \int_{t_{i-1}}^{s} \int_0^1 G_{t-s_1}(y,y_1) b^{(l)}(u_{i-1}(s_1,y_1)) \prod_{j=1}^{l} I_{m_j}(s_1,y_1) dy_1 ds_1 \qquad (9)$$

is a process of order k where $l \leq k-2$ and $I_{m_1}, ..., I_{m_l}$ are processes of order m_1,...,m_l respectively with $m_1 + ... + m_l = k-2$. As before we define $I_k(s,y) = J_k(s,s,y)$ and we say that I_k is a diagonal process of order k.

We expand the above set of processes by assuming that the process

$$J_2(s,t,y) = \int_{t_{i-1}}^{s} \int_0^1 G_{t-s_1}(y,y_1) b(u_{i-1}(s_1,y_1)) dy_1 ds_1$$

is a process of order 2. Obviously the set of processes of order k is finite and we index them using a finite set $\mathcal{A}_k$. If $\alpha \in \mathcal{A}_k$ then J_k^{α} denotes the corresponding process of order k indexed by α.

Next we define the set of residue processes. We say that a process R_1 is a residue process of order 1 if it is defined as either

$$R_1(s,t,y) = \int_{t_{i-1}}^{s} \int_0^1 G_{t-s_1}(y,y_1) \left(\int_0^1 \sigma'(u^m(\lambda,s_1,y_1)) d\lambda \right) (u(s_1,y_1) - u_{i-1}(s_1,y_1)) W(dy_1, ds_1)$$

where $u^m(\lambda,s_1,y_1) = \lambda u(s_1,y_1) + (1-\lambda) u_{i-1}(s_1,y_1)$ or

$$R_1(s,t,y) = \int_{t_{i-1}}^{s} \int_0^1 G_{t-s_1}(y,y_1) b(u(s_1,y_1)) dy_1 ds_1.$$

Similarly, as before, we define the diagonal residue process of order 1 as $R_1(s,y) = R_1(s,s,y)$.

The following process R_2 is a residue process of order two

$$R_2(s,t,y) = \int_{t_{i-1}}^{s} \int_0^1 G_{t-s_1}(y,y_1) \left(\int_0^1 b'(u^m(\lambda,s_1,y_1)) d\lambda \right) (u(s_1,y_1) - u_{i-1}(s_1,y_1)) dy_1 ds_1.$$

By induction one says that a stochastic process is a residue process of order k if it can be expressed as either

$$1. R_k(s,t,y) = \frac{1}{k-1!} \int_{t_{i-1}}^{s} \int_0^1 G_{t-s_1}(y,y_1) \left(\int_0^1 (1-\lambda)^{k-1} \sigma^{(k)}(u^m(\lambda,s_1,y_1)) d\lambda \right) (u(s_1,y_1) - u_{i-1}(s_1,y_1))^k W(dy_1, ds_1) \qquad (10)$$

or

$$R_k(s,t,y) = \int_{t_{i-1}}^{s} \int_0^1 G_{t-s_1}(y,y_1) \sigma^{(l)}(u_{i-1}(s_1,y_1)) \prod_{j=1}^{l} \overline{R}_{m_j}(s_1,y_1) W(dy_1, ds_1),$$

where $\overline{R}_{m_j}$ is either a diagonal residue process of order m_j or a diagonal process of order m_j and at least one of the $\overline{R}_{m_j}$, $j = 1, ..., l$ is a residue process. As before $l \leq k-1$ with $m_1 + ... + m_l = k-1$.

$$2. R_k(s,t,y) = \frac{1}{k-2!} \int_{t_{i-1}}^{s} \int_0^1 G_{t-s_1}(y,y_1) \left(\int_0^1 (1-\lambda)^{k-2} b^{(k-1)}(u^m(\lambda,s_1,y_1)) d\lambda \right) (u(s_1,y_1) - u_{i-1}(s_1,y_1))^{k-1} dy_1 ds_1$$

or

$$R_k(s,t,y) = \int_{t_{i-1}}^{s} \int_0^1 G_{t-s_1}(y,y_1) b^{(l)}(u_{i-1}(s_1,y_1)) \prod_{j=1}^{l} \overline{R}_{m_j}(s_1,y_1) dy_1 ds_1,$$

where $\overline{R}_{m_j}$ is either a diagonal residue process of order m_j or a diagonal process of order m_j and at least one of the $\overline{R}_{m_j}$, $j = 1, ..., l$ is a diagonal residue process. Here $l \leq k-2$ with $m_1 + ... + m_l = k-2$.

We denote the index set for the residues of order k as $\mathcal{B}_k$. The next lemma gives the Taylor expansion for F_i conditioned on $\mathcal{F}_{t_{i-1}}$ and studies the order of each term in the $\mathbb{D}^{n,p}_{t_{i-1}}$−norms

Lemma 9 *Suppose that σ, $b \in C_b^\infty(\mathbb{R})$. For $r \geq 1$, one has the following expansion of the approximation sequence $\{F_i; i = 1, ..., N\}$*

$$F_i - F_{i-1} = \sum_{k=1}^{r} \sum_{\alpha \in \mathcal{A}_k} C_1(\alpha, k) J_k^\alpha(t_i, t, x) + \sum_{k=r}^{r+1} \sum_{\alpha \in \mathcal{B}_k} C_2(\alpha, k) R_k^\alpha(t_i, t, x),$$

for some appropriate constants $C_j(\alpha, k)$ for $j = 1, 2$. Furthermore the following estimates are satisfied for any $(s, y) \in [t_{i-1}, t_i] \times [0, 1]$

$$\|J_k^\alpha(t_i, t, y)\|_{n,p,t_{i-1}} + \left\|R_{k-1}^\alpha(t_i, t, y)\right\|_{n,p,t_{i-1}} \leq C(n, p, k)(t_i - t_{i-1})^{k/16}. \tag{11}$$

The above norm estimate is obviously non-optimal but we prefer to do this as the proof becomes easier to follow.

Proof. The proof is done by induction. As the proof is long and tedious we only give the main steps here. For $k = 1$ is not difficult to see that by the mean value theorem we have

$$\begin{aligned} F_i - F_{i-1} &= \int_{t_{i-1}}^{t_i} \int_0^1 G_{t-s}(x, y) \sigma(u_{i-1}(s, y)) W(dy, ds) + \int_{t_{i-1}}^{t_i} \int_0^1 G_{t-s}(x, y) b(u(s, y)) dy ds + \\ & \int_{t_{i-1}}^{t_i} \int_0^1 G_{t-s}(x, y) \left(\int_0^1 \sigma'(u^m(\lambda, s_1, y_1)) d\lambda \right) (u(s, y) - u_{i-1}(s, y)) \, W(dy, ds). \end{aligned}$$

The first term is the only process of order 1 and the next two terms are residues of order 1. The above formula can be extended with the same steps for $k = 2$. In doing so one also checks that the residue of order 1 can be written as the sum of processes of order 2 and residues of order 2 and 3. By inductive hypothesis suppose that the above formula is true for r and that any residue of order $k < r$ can be expressed as sums of processes of order $k + 1$ and residue processes of order $k + 1$ and $k + 2$. Then we consider every residue term of order r and develop it as follows: If the residue process of order r is of the type (10) then one rewrites it as

$$\begin{aligned} & \frac{1}{r!} \int_{t_{i-1}}^{s} \int_0^1 G_{t-s_1}(y, y_1) \sigma^{(r)}(u_{i-1}(s_1, y_1))(u(s_1, y_1) - u_{i-1}(s_1, y_1))^r W(ds_1, dy_1) + \\ & \frac{1}{r!} \int_{t_{i-1}}^{s} \left(\int_0^1 G_{t-s_1}(y, y_1) \int_0^1 (1 - \lambda)^r \sigma^{(r)}(u^m(\lambda, s_1, y_1)) d\lambda \right) (u(s_1, y_1) - u_{i-1}(s_1, y_1))^{r+1} W(ds_1, dy_1). \end{aligned}$$

The last term is a residue process of order $r + 1$. The first term is decomposed, using the first order decomposition

$$u(s_1, y_1) - u_{i-1}(s_1, y_1) = I_1(s_1, y_1) + \sum_{\alpha \in \mathcal{B}_1} R_1^\alpha(s_1, y_1)$$

so that the first term is decomposed in sums of terms of order $r + 1$ and further residue processes of order $r + 1$ as follows

$$\begin{aligned} & \frac{1}{r!} \int_{t_{i-1}}^{s} \int_0^1 G_{t-s_1}(y, y_1) \sigma^{(r)}(u_{i-1}(s_1, y_1)) I_1(s_1, y_1)^r W(ds_1, dy_1) + \\ & \frac{1}{r!} \sum_{j=0}^{r-1} \sum_{\alpha_l \in \mathcal{B}_1} \int_{t_{i-1}}^{s} \int_0^1 G_{t-s_1}(y, y_1) \sigma^{(r)}(u_{i-1}(s_1, y_1)) I_1(s_1, y_1)^j \prod_{l=1}^{r-j} R_1^{\alpha_l}(s_1, y_1) W(ds_1, dy_1). \end{aligned}$$

Next suppose that one has a residue process of the type

$$R_r(s, t, y) = \int_{t_{i-1}}^{s} \int_0^1 G_{t-s_1}(y, y_1) \sigma^{(l)}(u_{i-1}(s_1, y_1)) \prod_{j=1}^{l} \overline{R}_{m_j}(s_1, y_1) W(ds_1, dy_1).$$

Here, by the induction hypotheses, for each residue process $\overline{R}_{m_j}$ one has that it can be rewritten as sums of terms of order m_j plus residues of order $m_j + 1$ and $m_j + 2$ therefore generating processes of order $r + 1$ and residues of order $r + 1$ or $r + 2$.

Similar operations have to be done when the drift coefficients b appears instead of σ and the Lebesgue integral replaces the stochastic one.

Now we prove the norm estimates by double induction on n and k. For $n=0$ and $k=1$ or $k=2$ the estimates can be obtained from straightforward estimates of the integrals. Suppose that the estimates are true for $k-1$, $n=0$ and that J_k is of type (8) then we have using Lemma 14 with $X_0=0$, $f(x)=0$, $g(x)=\sigma^{(l)}(u_{i-1}(s_1,y_1))$, $a=t_{i-1}$, $b=s$, $\alpha_j=m_j$, $\alpha=k-1$, $i_0=l$, $\gamma=1/16$, $u'\equiv 1$ that

$$\|J_k(s,t,y)\|_{p,t_{i-1}} \leq C(p)\Big\{(t_{i-1}-s)^{(k+15)/16}+\Big((t-t_{i-1})^{3/2-q}-(t-s)^{3/2-q}\Big)^{(p-2)/(2p)}(s-t_{i-1})^{(k-1)/16+1/p}\Big\},$$

for $s\in[t_{i-1},t_i]$. As $\big((t-t_{i-1})^{3/2-q}-(t-s)^{3/2-q}\big)^{(p-2)/(2p)} \leq C(s-t_{i-1})^{1/4-3/(2p)}$ the estimate follows for $p\geq 8$. Similarly one proceeds in the case that J_k is of the type (9). Next we consider the estimates for the $\mathbb{D}^{n,p}_{t_{i-1}}$-norms. This estimate is also obtained by induction on the order of differentiation and on the order of the process being considered. For this, suppose that we have that the estimate (11) is satisfied for $j\leq n$ and $k\leq r-1$ we will prove that the same estimate is satisfied for $k=r$. Suppose then that we have a process of order r, J_r, of the type (8) then one estimates $A(J_r(s,t,y))_{n,p,t_{i-1}}$, using Lemma 14 with the same choices as before we then obtain that

$$A(J_r(s,t,y))_{n,p,t_{i-1}} \leq C(n,p)(s-t_{i-1})^{r/16}.$$

For the residues of order k the proof is also similar. In fact, suppose we have a residue R_k of the type (10) then one has as before with $X_0=0$, $f(x)=0$, $g(x)=\sigma^{(k)}(\lambda x+u_{i-1}(s,y))$, $a=t_{i-1}$, $b=s$, $I_i(s,y)=(u-u_{i-1})(s,y)$, $\alpha_j=1$, $\alpha=k$, $i_0=k$, $\gamma=1/8$, $u'\equiv u-u_{i-1}$ that for $s\in[t_{i-1},t_i]$

$$\begin{aligned}\|R_k(s,t,y)\|_{n,p,t_{i-1}} &\leq C(n,p)\Big\{(s-t_{i-1})^{(k+8)/8}+\Big((t-t_{i-1})^{3/2-q}-(t-s)^{3/2-q}\Big)^{(p-2)/(2p)}(s-t_{i-1})^{k/8+1/p}\Big\}\\ &\leq C(n,p)(t_{i-1}-s)^{(k+1)/16}.\end{aligned}$$

Here we have used Lemma 8 and Lemma 14. ■

With this result one defines the approximation of order r, $\overline{F}^r_i$ as

$$\overline{F}_i \equiv \overline{F}^r_i = \Delta_{i-1}(g)^{\frac{r+1}{8}}Z_i + F_{i-1} + \sum_{k=1}^{r}\sum_{\alpha\in\mathcal{A}_k} C_1(\alpha,k)J^\alpha_k(t_i,t,x).$$

Theorem 10 *Assume that the coefficients b and $\sigma\in C^\infty_b(\mathbb{R})$. Furthermore suppose that $\|\sigma(x)\|\geq c_0>0$ for all $x\in\mathbb{R}$. Then $u(t,x)$ has a smooth density for $0<t<1$ and $x\in[0,1]$ denoted by $p(t,x,\cdot)$ furthermore it satisfies*

$$p(t,x,y) \geq \frac{\exp(-M\frac{\|G(u_0)(t,x)-y\|^2}{t^{1/2}})}{Mt^{1/4}}$$

for a constant $M\in[1,+\infty)$.

Proof. We have already defined F_i, G^l_i and g. Define $h(s,y)=G_{t-s}(x,y)\sigma(u_{i-1}(s,y))$. With these definitions and Lemma 7 we have that condition **(H1)** is satisfied. Next, one has that

$$\begin{aligned}F_i-\overline{F}^r_i &= F_i-F_{i-1}-\Delta_{i-1}(g)^{\frac{r+1}{8}}Z_i - \sum_{k=1}^{r}\sum_{\alpha\in\mathcal{A}_k} C_1(\alpha,k)J^\alpha_k(t_i,t,x)\\ &= -\Delta_{i-1}(g)^{\frac{r+1}{8}}Z_i - \sum_{k=r}^{r+1}\sum_{\alpha\in\mathcal{B}_k} C_2(\alpha,k)R^\alpha_k(t_i,t,x).\end{aligned}$$

Therefore using Lemma 9 one obtains that

$$\left\|F_i-\overline{F}^r_i\right\|_{k,p,t_{i-1}} \leq C(k,p)\Delta_{i-1}(g)^{(r+1)/8}$$

so that property **(H2a)** is satisfied with $\gamma = 1/8$. **(H2b)** follows from Lemma 7. Now to obtain **(H2c)** is just a matter of computing

$$\Delta_{i-1}(g)^{-1} \int_{t_{i-1}}^{t_i} \int_0^1 G_{t-s_1}^2(x, y_1)\sigma^2(u_{i-1}(s_1, y_1))dy_1 ds_1$$

this is bounded above and below due to the estimate (A.1) in [1] (or (6)). In order to verify **(H2d)**, one has that the result in Lemma 9 is insufficient and therefore we compute an exact estimate using the same induction method of Lemma 9 to estimate :

$$\|R_1(t_i, t, y)\|_{n,p,t_{i-1}} \le \left\| \int_{t_{i-1}}^{t_i} \int_0^1 G_{t-s}(x, y) \left(\int_0^1 \sigma'(u^m(\lambda, s_1, y_1))d\lambda \right) (u(s, y) - u_{i-1}(s, y))W(dy, ds) \right\|_{n,p,t_{i-1}} \quad (12)$$
$$+ \left\| \int_{t_{i-1}}^{t_i} \int_0^1 G_{t-s}(x, y)b(u(s, y))dyds \right\|_{n,p,t_{i-1}}.$$

First we have that

$$\begin{aligned} \left\| \int_{t_{i-1}}^{t_i} \int_0^1 G_{t-s}(x, y)b(u(s, y))dyds \right\|_{n,p,t_{i-1}} &\le \int_{t_{i-1}}^{t_i} \int_0^1 G_{t-s}(x, y) \|b(u(s, y))\|_{n,p,t_{i-1}} dyds \\ &\le C(t_i - t_{i-1}). \end{aligned}$$

The last inequality follows because $\|b(u(s, y))\|_{n,p,t_{i-1}} \le C(n, p)$. In fact, $A(b(u(s, y))_{0,p,t_{i-1}} \le C(p)$ and there exists $p' > 0$ such that

$$A(b(u(s, y)))_{n,p,t_{i-1}} = C(n, p) \sum_{j=1}^n A((u - u_{i-1})(s, y))_{n,p',t_{i-1}}^n.$$

For the first term in (12) we have as before that applying Lemma 14 with $X_0 = 0$, $f(x) = 0$, $g(x) = \sigma'(\lambda x + u_{i-1}(s, y))$, $a = t_{i-1}$, $b = t_i$, $I_i(s, y) = (u - u_{i-1})(s, y)$, $\alpha_j = 1$, $\alpha = 1$, $i_0 = 1$, $\gamma = 1/8$, $u' \equiv u - u_{i-1}$

$$\begin{aligned} &\left\| \int_{t_{i-1}}^{t_i} \int_0^1 G_{t-s}(x, y) \left(\int_0^1 \sigma'(u^m(\lambda, s_1, y_1))d\lambda \right) (u(s, y) - u_{i-1}(s, y)) W(dy, ds) \right\|_{n,p,t_{i-1}} \\ \le\ & C(n, p)(t_i - t_{i-1})^{3/8 - 1/(2p)} \le C(n, p)\Delta_{i-1}(g)^{3/4 - 1/p} \end{aligned}$$

for $s \in [t_{i-1}, t_i]$. Therefore $\varepsilon = 1/4 - 1/p > 0$ if $p > 4$. Then the result follows from Theorem 5. In particular note that although g depends on (t, x) as $\|g\|_{L^2([0,t]\times[0,1])} \le C$ where C is independent of (t, x) then the constant M appearing in the conclusion of the Theorem 5 is independent of (t, x). ■

A Appendix

In this section we give some accessory results used in Sections 3 and 4.1. In the first part we study of the behavior of Malliavin covariance matrix for truncated approximation sequences used in Section 3, Theorem 5. In the second part we give some estimates on norms of various random variables associated with the solution of the stochastic heat equation. These estimates were used throughout Section 4.1.

We start proving some differentiability properties of the approximating and truncated sequences. In the next two propositions we use the notation introduced in the proof of Theorem 5 $X_i = \Delta_{i-1}(g)^{-1/2} \sum_{j=1}^k I_j^i(h_j)$ and $Y_i = \Delta_{i-1}(g)^{(l+1)\gamma - 1/2} Z_i + \Delta_{i-1}(g)^{-1/2} G_i^l$.

Proposition 11 *Let F be a uniformly elliptic random variable with truncated approximating sequence $\overline{F}_i$, then F_i, $\overline{F}_i \in \overline{\mathbb{D}}^{\infty}_{t_{i-1}}$ uniformly. Furthermore, assume that $(l+1)\gamma - 1/2 > \varepsilon > 0$ then $G_i'(\rho) = X_i + \rho Y_i \in \overline{\mathbb{D}}^{n,p}_{t_{i-1}}$, uniformly for $\rho \in [0,1]$. Also there exist a positive constant $C(\alpha)$ such that*

$$E\left(\left|\det \psi_{G_i'(\rho)} - \det \psi_{X_i}\right|^{\alpha} / \mathcal{F}_{t_{i-1}}\right) \leq C(\alpha)\rho^{\alpha}\Delta_{i-1}(g)^{\varepsilon\alpha}.$$

Proof. $E\left(\left\|\overline{F}_i\right\|^a_{n,p,t_{i-1}}\right) + E\left(\left\|F_i\right\|^a_{n,p,t_{i-1}}\right)$ is uniformly bounded due to the condition **(H1)** and **(H2d)**. Also, due to **(H1)**, **(H2c)** and **(H2d)**, we have that

$$\begin{aligned} \|X_i\|_{n,p,t_{i-1}} &\leq C(n,p) \\ \Delta_{i-1}(g)^{-1/2}\left\|G_i^l\right\|_{n,p,t_{i-1}} &\leq C(n,p), \end{aligned}$$

for a constant $C(n,p)$. Furthermore $\left\|\Delta_{i-1}(g)^{(l+1)\gamma-1/2} Z_i\right\|_{n,p,t_{i-1}} \leq C$ for a constant C. Therefore one obtains that

$$\|G_i'(\rho)\|_{n,p,t_{i-1}} \leq C(n,p).$$

where $C(n,p)$ is a positive constant that does not depend on ρ. For the last inequality one has to use the definition of the determinant and estimate each difference. That is,

$$\det \psi_{G_i'(\rho)} - \det \psi_{X_i} = \sum_{\sigma \in S_q}\left(\prod_{j=1}^{q}\left(\psi_{G_i'(\rho)}\right)_{j\sigma(j)} - \prod_{j=1}^{q}\left(\psi_{X_i}\right)_{j\sigma(j)}\right)$$

where S_q denotes the set of all permutations of order q. The difference within the sum can be rewritten as

$$\sum_{p=1}^{q}\prod_{j=1}^{p-1}\left(\psi_{G_i'(\rho)}\right)_{j\sigma(j)}\left(\left(\psi_{G_i'(\rho)}\right)_{p\sigma(p)} - \left(\psi_{X_i}\right)_{p\sigma(p)}\right)\prod_{j=p+1}^{q}\left(\psi_{X_i}\right)_{j\sigma(j)}.$$

Each term $\left(\psi_{G_i'(\rho)}\right)_{j\sigma(j)}$, $\left(\psi_{X_i}\right)_{j\sigma(j)} \in \overline{\mathbb{D}}^{\infty}_{t_{i-1}}$ uniformly and the middle term can be bounded as follows

$$\left\|\left(\psi_{G_i'(\rho)}\right)_{p\sigma(p)} - \left(\psi_{X_i}\right)_{p\sigma(p)}\right\|_{\alpha,t_{i-1}} \leq \|G_i'(\rho) - X_i\|_{1,\alpha_1,t_{i-1}}\left(\|G_i'(\rho)\|_{1,\alpha_2,t_{i-1}} + \|X_i\|_{1,\alpha_3,t_{i-1}}\right).$$

Therefore the result follows from the previous estimates on the norms of X_i and G_i^l. ■

Now we show the stability of the Malliavin covariance matrices associated with any point in between the approximating sequence and the truncated approximating sequence as a consequence of the definition of uniformly elliptic random variables.

Proposition 12 *Assume that F is a uniformly elliptic random variable with approximation sequence $\overline{F}_i$. Then we have that for any l such that $(l+1)\gamma - 1/2 > \varepsilon > 0$, there exists a positive constant $C(p)$ such that*

$$\begin{aligned} \left\|\det \psi^{-1}_{\overline{F}_i}(t_{i-1})\right\|_{p,t_{i-1}} &\leq C(p)\Delta_{i-1}(g)^{-2(l+1)\gamma} \\ \sup_{\rho\in[0,1]}\left\|\det \psi^{-1}_{G_i'(\rho)}(t_{i-1})\right\|_{p,t_{i-1}} &\leq C(p) \\ \sup_{\rho\in[0,1]}\left\|\det \psi^{-1}_{\overline{F}_i(\rho)}(t_{i-1})\right\|_{p,t_{i-1}} &\leq C(p)\Delta_{i-1}(g)^{-q}. \end{aligned}$$

Proof. The first statement follows because one considers the Malliavin covariance matrix of $\overline{F}_i$ in the extended space. In fact, if one denotes by $\widetilde{D}$, the stochastic derivative with respect to the Wiener process that generates the increments Z_i one has that

$$\det \psi_{\overline{F}_i}(t_{i-1}) \geq \Delta_{i-1}(g)^{-2(l+1)\gamma}\det\left(\sum_{j=1}^{q}\int_i^{i+1}\widetilde{D}_s^{\,j} Z_i^{r_1}\widetilde{D}_s^{\,j} Z_i^{r_2} ds\right)_{q\times q} = q\Delta_{i-1}(g)^{-2(l+1)\gamma}.$$

From here the first estimate follows. For the rest of the proof in order to simplify the notation we will write $\psi_{F_i} \equiv \psi_{F_i}(t_{i-1})$ assuming that the time interval is understood. Define the set

$$B = \left\{ w \in \Omega;\ \left|\det \psi_{G_i'(\rho)} - \det \psi_{X_i}\right| < \frac{1}{4} \left|\det \psi_{X_i}\right| \right\}.$$

Note that $\psi_{X_i} = A$ defined in hypothesis **(H2c)**. Therefore there exists a deterministic constant $C(p)$ independent of t and ρ such that

$$\begin{aligned} E\left(\left(\det \psi^{-1}_{G_i'(\rho)}\right)^p 1_B / \mathcal{F}_{t_{i-1}}\right) &\leq \left(\frac{4}{3}\right)^p E\left(\left(\det \psi^{-1}_{X_i}\right)^p / \mathcal{F}_{t_{i-1}}\right) \\ &\leq C(p). \end{aligned}$$

On the other hand, repeating the same argument as for the estimate of $\psi_{\overline{F}_i}(t_{i-1})$ we have that $\psi_{G_i'(\rho)} \geq C\rho^2 \Delta_{i-1}(g)^{2((l+1)\gamma-1/2)}$. Therefore using the definition of uniformly elliptic r.v., we have for $\rho \in (0,1]$ and $\alpha > 0$

$$\begin{aligned} E\left(\left(\det \psi^{-1}_{G_i'(\rho)}\right)^p 1_{\overline{B}} / \mathcal{F}_{t_{i-1}}\right) &\leq E\left(\left(\inf_{\|v\|=1} v' \psi_{G_i'(\rho)} v\right)^{-qp} 1_{\overline{B}} / \mathcal{F}_{t_{i-1}}\right) \\ &\leq C \Delta_{i-1}(g)^{-2((l+1)\gamma-1/2)qp} \rho^{-2qp} P(\overline{B} / \mathcal{F}_{t_{i-1}}) \\ &\leq C \Delta_{i-1}(g)^{-2((l+1)\gamma-1/2)qp} 4^\alpha \rho^{-2pq} \\ &\quad \times E\left(\left|\det \psi_{G_i'(\rho)} - \det \psi_{X_i}\right|^\alpha \left|\det \psi_{X_i}\right|^{-\alpha} / \mathcal{F}_{t_{i-1}}\right) \\ &\leq C(\alpha) \Delta_{i-1}(g)^{-2((l+1)\gamma-1/2)qp} 4^\alpha \rho^{-2pq} \rho^\alpha \Delta_{i-1}(g)^{\varepsilon\alpha}. \end{aligned}$$

In the last inequality we have used the Proposition 11. Taking α big enough the result follows. The third estimate follows with a similar argument replacing X_i by F_i. ∎

We now start the second part of this section. We start proving an L^p upper estimate on the derivative of the solution of the stochastic heat equation.

Lemma 13 *Define* $S_{s,y}(s_1, y_1) = \frac{D_{s,y} u(s_1,y_1)}{\sigma(u(s,y))}$, *for* $s \leq s_1 \leq 1$, $y, y_1 \in [0,1]$. *Then there exists a positive constant* $C(p)$ *such that for* $p > 1$ *and* $\alpha \leq s_1$ *one has*

$$E_{t_{i-1}}\left[\int_\alpha^{s_1} \int_0^1 S_{s,y}(s_1,y_1)^2 dy ds\right]^p \leq C(p) \left(s_1 - \alpha\right)^{p/2} \exp\left(C(p) \left(s_1 - \alpha\right)^{(p-1)/2}\right).$$

Proof. We assume for simplicity that $b' \equiv 0$. Note that $S_{s,y}(s_1, y_1)$ is a solution of the equation

$$S_{s,y}(s_1, y_1) = G_{s_1-s}(y_1, y) + \int_s^{s_1} \int_0^1 G_{s_1-s_2}(y_1, y_2) \sigma'(u(s_2, y_2)) S_{s,y}(s_2, y_2) W(dy_2, ds_2).$$

Then we have that $E_{t_{i-1}}\left[\int_\alpha^{s_1} \int_0^1 S_{s,y}(s_1,y_1)^2 dy ds\right]^p$, for $\alpha \leq s_1$ can be bounded by

$$\begin{aligned} &C(p)\left(\left(\int_\alpha^{s_1} \int_0^1 G^2_{s_1-s}(y_1, y) dy ds\right)^p\right. \\ &\left. + E_{t_{i-1}}\left[\int_\alpha^{s_1} \int_0^1 \left(\int_s^{s_1} \int_0^1 G_{s_1-s_2}(y_1,y_2)\sigma'(u(s_2,y_2)) S_{s,y}(s_2,y_2) W(dy_2, ds_2)\right)^2 dy ds\right]^p\right). \end{aligned}$$

Applying the Burkholder's inequality for martingales in Hilbert spaces and Cauchy-Schwartz inequality we have for $q^{-1}+p^{-1}=1$ that the above is bounded by

$$\begin{aligned}
&E_{t_{i-1}}\left[\int_\alpha^{s_1}\int_0^1 S_{s,y}(s_1,y_1)^2dyds\right]^p\\
\leq\;& C(p)\,(s_1-\alpha)^{p/2}+C(p)E_{t_{i-1}}\left[\int_\alpha^{s_1}\int_0^1 G_{s_1-s_2}(y_1,y_2)^2\int_\alpha^{s_2}\int_0^1 S_{s,y}(s_2,y_2)^2dydsdy_2ds_2\right]^p\\
\leq\;& C(p)\,(s_1-\alpha)^{p/2}+C(p)\,(s_1-\alpha)^{(3/2-q)(p-1)}\times\\
&\int_\alpha^{s_1}\int_0^1 E_{t_{i-1}}\left(\int_\alpha^{s_2}\int_0^1 S_{s,y}(s_2,y_2)^2dyds\right)^p dy_2ds_2.
\end{aligned}$$

Then using Gronwall's inequality on $\sup_{y\in[0,1]}E_{t_{i-1}}\left[\int_\alpha^{s_1}\int_0^1 S_{s,y}(s_1,y_1)^2dyds\right]^p$ we have the result. ■

Now we give a result on norm estimates used in Section 4.1. This result applied to various situations that appear in that Section and uses ideas of the previous proof. This leads to an inequality that allows to estimate various $\mathbb{D}^{n,p}_{t_{i-1}}$-norms of random variables associated with the stochastic heat equation.

Define for $a\leq b\leq t$

$$X=X_0(t,a,b)+\int_a^b\int_0^1 G_{t-s}(x,y)f(u'(s,y))\prod_{i=1}^{i_0}I_i(s,y)dyds+\int_a^b\int_0^1 G_{t-s}(x,y)g(u'(s,y))\prod_{i=1}^{i_0}I_i(s,y)W(dy,ds).$$

Here $X_0(t,a,b)$ is a $\mathcal{F}_a$ measurable random variable and define for a smooth random variable X and a smooth process $u'(s,y)$, $s,y\in[0,1]^2$,

$$\begin{aligned}
A(u')_{j,p,a}\;&:=\left[E_a\left(\int_a^b\int_0^1\cdots\int_a^b\int_0^1\left(D_v^ju'(s,y)\right)^2dvdsdy\right)^{p/2}\right]^{1/p}\\
A(X)_{j,p,a}\;&:=\left[E_a\left(\int_a^b\int_0^1\cdots\int_a^b\int_0^1\left(D_v^jX\right)^2dv\right)^{p/2}\right]^{1/p}.
\end{aligned}$$

Lemma 14 *Suppose that $f,\ g\in C_b^\infty(\mathbb{R})$, u' and I are smooth processes and that there exists constants $c_{u'}(j,p,a,s)>1$ which are increasing in p and $C(n,p)$ such that*

$$\begin{aligned}
A(u'(s,y))_{j,p,a}&\leq c_{u'}(j,p,a,s)\\
A\left(I_i(s,y)\right)_{j,p,a}&\leq C(n,p)(s-a)^{\gamma\alpha_i},
\end{aligned}$$

for any $p>0$, $a\leq s\leq t$, $y\in[0,1]$, $j=0,...,n$ and some $\gamma\geq 0$, $\alpha_1>0$,...,$\alpha_{i_0}>0$. Then $X\in\mathbb{D}^{n,\infty}$ and we have that, for $p>6$, $q=\frac{p}{p-2}$ and $\alpha_1+...+\alpha_{i_0}=\alpha$, there exists p' and p^* with $p^*=p$ if I_i is a constant for all i and the following inequality is satisfied for all $n\geq 0$ and $p>6$

$$\begin{aligned}
&A(X)_{n,p,a}\\
\leq\;& A(X_0(t,a,b))_{n,p,a}+C(n,p)\left\{\int_a^b\left(c^*_{u'}(n-1,p',a,s)^n+c_{u'}(n,p^*,a,s)\right)\times(s-a)^{\gamma\alpha}ds\right.\\
&\left.+\left((t-a)^{3/2-q}-(t-b)^{3/2-q}\right)^{(p-2)/(2p)}\left(\int_a^b\left(c^*_{u'}(n-1,p',a,s)^{pn}+c_{u'}(n,p^*,a,s)^p\right)\times(s-a)^{\gamma\alpha p}ds\right)^{1/p}\right\}.
\end{aligned}$$

We define $c_{u'}(-1,p,a,s)=0$ and $c_{u'}(0,p,a,s)$ as the constant such that $A\left(f(u')(s,y)\right)_{0,p,a}+A\left(g(u')(s,y)\right)_{0,p,a}\leq c_{u'}(0,p,a,s)$ and $c^*_{u'}(n-1,p',a,s)=\max_{j\leq n-1}c_{u'}(j,p',a,s)$.

Note that if we have $I_i \equiv$constant we can take $\gamma = 0$. Another particular case occurs if we suppose that all the constants $c_{u'}(j, p', a, s) < 1$ for $j = 1, .., n$. In such a case one has

$$\begin{aligned} & A(X)_{n,p,a} \\ \leq\ & A(X_0(t,a,b))_{n,p,a} + C(n,p)\left\{\int_a^b (c^*_{u'}(n-1,p',a,s) + c_{u'}(n,p^*,a,s)) \times (s-a)^{\gamma\alpha} ds \right. \\ & \left. + \left((t-a)^{3/2-q} - (t-b)^{3/2-q}\right)^{(p-2)/(2p)} \left(\int_a^b (c^*_{u'}(n-1,p',a,s)^p + c_{u'}(n,p^*,a,s)^p)(s-a)^{\gamma\alpha p} ds\right)^{1/p}\right\}. \end{aligned}$$

Proof. One does the estimation in various steps. First we have for $n \geq 1$

$$\begin{aligned} A(X)_{n,p,a} \leq & \int_a^b \int_0^1 G_{t-s}(x,y) A(f(u'(s,y)) \prod_{i=1}^{i_0} I_i(s,y))_{n,p,a} dy ds \qquad (13) \\ & + A\left(\int_a^b \int_0^1 G_{t-s}(x,y) g(u'(s,y)) \prod_{i=1}^{i_0} I_i(s,y) W(dy,ds)\right)_{n,p,a}. \end{aligned}$$

To estimate the first term we first have that

$$D_v^n f(u'(s,y)) = \sum_{\substack{k\in\Theta(n)\\ \sigma\in S_n}} f^{(k_0)}(u'(s,y)) \prod_{l=1}^{n}\prod_{j=1}^{k_l} D^l_{\sigma(K(l,j),K(l,j+1))} u'(s,y).$$

Here the summation is done for

$$k \in \Theta(n) = \{(k_0,...,k_n) \in \mathbb{N}^{n+1}; k_1 + 2k_2 + ... + nk_n = n, k_1 + ... + k_n = k_0\}.$$

S_n denotes the set of permutations of the index set $\{1,...,n\}$. We have used the following notation

$$\begin{aligned} \sigma(i_1,i_2) &= (y_{\sigma(i_1+1)}, s_{\sigma(i_1+1)}, ..., y_{\sigma(i_2)}, s_{\sigma(i_2)}) \\ K(l,j) &= k_1 + ... + (l-1)k_{l-1} + (j-1)l. \end{aligned}$$

Also one has that

$$D_v^n \prod_{j=1}^{i_0} I_{m_j}(s,y) = \sum_{\substack{\omega\in\Gamma(n)\\ \sigma\in S_n}} \prod_{j=1}^{i_0} D^{\omega_j}_{\sigma(\omega_{j-1},\omega_j)} I_{m_j}(s,y)$$

where

$$\Gamma(n) = \{(\omega_1,...,\omega_{i_0}) \in \{0,...,n\}^{i_0}; \omega_1 + ... + \omega_{i_0} = n\}.$$

Finally one has that

$$D_v^n \left(f(u'(s,y)) \prod_{i=1}^{i_0} I_i(s,y)\right) = \sum_{r=0}^n \sum_{\sigma\in S_n} D^r_{\sigma(0,r)} f(u'(s,y)) D^{n-r}_{\sigma(r,n-r)} \left(\prod_{i=1}^{i_0} I_i(s,y)\right).$$

Then using the Cauchy Schwartz inequality we have that for $n \geq 1$ (in the case that $r = 0$, $k_l = 0$ or $i_0 = 0$ we set the product equal to 1)

$$\begin{aligned} A\left(f(u'(s,y)) \prod_{i=1}^{i_0} I_i(s,y)\right)^p_{n,p,a} \leq\ & C(n,p) \sum_{r=0}^n \left(\sum_{\substack{k\in\Theta(r)\\ \sigma\in S_r}} \prod_{l=1}^{r}\prod_{j=1}^{k_l} A(u'(s,y))^p_{l,p(l,j),a}\right) \\ & \times \left(\sum_{\substack{\omega\in\Gamma(n-r)\\ \sigma\in S_{n-r}}} \prod_{i=1}^{i_0} A\left(I_i(s,y)\right)^p_{\omega_i,q(\omega_i),a}\right) \\ \leq\ & C(n,p)\left(c^*_{u'}(n-1,p',a,s)^n + c_{u'}(n,p^*,a,s)\right)^p \times (s-a)^{\gamma\alpha p}. \end{aligned}$$

Here $p' = \max_{\substack{l=1,\ldots,n \\ j=1,\ldots,l}} p(l,j)$ where $p(l,j) > 1$, $q(\omega_l) > 1$ are a set of positive real numbers such that $\sum_{l=1}^{r}\sum_{j=1}^{l} p(l,j)^{-1} + \sum_{j=1}^{i_0} q(\omega_j)^{-1} = p^{-1}$ for each set of indices. $p(n,1) = p$ if I_i is a constant for all i and therefore $p^* = p$ if I_i is a constant, otherwise $p^* = p'$. Here we have used the assumption that $c_{u'} > 1$. If one assumes that $c_{u'}(j,p',a,s) \le 1$ then note that the above bound becomes $C(n,p)\left(c^*_{u'}(n-1,p',a,s) + c_{u'}(n,p^*,a,s)\right)^p \times (s-a)^{\gamma\alpha p}$. The case $n = 0$ follows directly as

$$A\left(f(u'(s,y))\prod_{i=1}^{i_0} I_i(s,y)\right)^p_{0,p,a} \le c_{u'}(0,p',a,s)^p(s-a)^{\gamma\alpha p}.$$

The second term in (13) is a bit more involved but uses similar techniques. First note that for $s_1,\ldots,s_n \in [a,b]$ we have

$$D^n_v\left(\int_a^b\int_0^1 G_{t-s}(x,y)g(u'(s,y))\prod_{i=1}^{i_0} I_i(s,y)W(dy,ds)\right) = G_{t-s_n}(x,y_n)\sum_{j=1}^{n} D^{n-1}_{v(j)}\left(g(u'(s_j,y_j))\prod_{i=1}^{i_0} I_i(s_j,y_j)\right)$$
$$+\int_{a\vee s_1\vee\ldots\vee s_n}^b\int_0^1 G_{t-s}(x,y)D^n_v\left(g(u'(s,y))\prod_{i=1}^{i_0} I_i(s,y)\right)W(dy,ds).$$

Here $v(j)$ denotes the set v without the variables (s_j,y_j). Therefore we have

$$A\left(\int_a^b\int_0^1 G_{t-s}(x,y)g(u'(s,y))\prod_{i=1}^{i_0} I_i(s,y)W(dy,ds)\right)^p_{n,p,a} \tag{14}$$
$$\le\; C(n,p)\left\{A\left(G_{t-\cdot}(x,\cdot)g(u')\prod_{i=1}^{i_0} I_i\right)^p_{n-1,p,a} + \right.$$
$$\left. E_a\left(\int_a^b\int_0^1\ldots\int_a^b\int_0^1\left(\int_{a\vee s_1\vee\ldots\vee s_n}^b\int_0^1 G_{t-s}(x,y)D^n_v\left(g(u'(s,y))\prod_{i=1}^{i_0} I_i(s,y)\right)W(dy,ds)\right)^2 dv\right)^{p/2}\right\}.$$

To estimate the first term we integrate with respect to the variables (s_n,y_n) last and apply Cauchy Schwartz inequality to obtain that

$$A\left(G_{t-\cdot}(x,\cdot)g(u')\prod_{i=1}^{i_0} I_i\right)^p_{n-1,p,a}$$
$$\le\;\left(\int_a^b\int_0^1 G_{t-s_n}(x,y_n)^{2q}dy_n ds_n\right)^{p/2q}\times\int_a^b\int_0^1 A\left(g(u'(s_n,y_n))\prod_{i=1}^{i_0} I_i(s_n,y_n)\right)^p_{n-1,p,a} dy_n ds_n.$$

Here $q = \frac{p}{p-2}$, $p > 6$. Now we estimate the first integral on the right using (6) and the second using the same steps as in the first term of (13) to obtain that for $n \ge 1$

$$A\left(G_{t-\cdot}(x,\cdot)g(u')\prod_{i=1}^{i_0} I_i\right)^p_{n-1,p,a}$$
$$\le\; C(n-1,p)\left((t-a)^{3/2-q}-(t-b)^{3/2-q}\right)^{(p-2)/2}\int_a^b c^*_{u'}(n-1,p',a,s)^{(n-1)p}(s-a)^{\gamma\alpha p}ds.$$

To estimate the second term in (14) one uses the Burkholder inequality for martingales in Hilbert spaces and Fubini's theorem which gives that

$$
\begin{aligned}
& E_a\left(\int_a^b\int_0^1 ... \int_a^b\int_0^1\left(\int_{a\vee s_1\vee...\vee s_n}^b\int_0^1 G_{t-s}(x,y)D_v^n\left(g(u'(s,y))\prod_{i=1}^{i_0}I_i(s,y)\right)W(dy,ds)\right)^2 dv\right)^{p/2} \\
\leq\ & C(p)E_a\left(\int_a^b\int_0^1 G_{t-s}(x,y)^2\int_a^s\int_0^1 ...\int_a^s\int_0^1\left(D_v^n\left(g(u'(s,y))\prod_{i=1}^{i_0}I_i(s,y)\right)\right)^2 dvdyds\right)^{p/2} \\
\leq\ & C(p)\left(\int_a^b\int_0^1 G_{t-s}(x,y)^{2q}dyds\right)^{p/2q}\int_a^b\int_0^1 A\left(g(u'(s,y))\prod_{i=1}^{i_0}I_i(s,y)\right)^p_{n,p,a} dyds \\
\leq\ & C(n,p)\left((t-a)^{3/2-q}-(t-b)^{3/2-q}\right)^{(p-2)/2} \\
& \times\int_a^b\sum_{r=0}^n\left(\sum_{\substack{k\in\Theta(r)\\ \sigma\in S_r}}\prod_{l=1}^r\prod_{j=1}^{k_l}A(u'(s,y))^p_{l,p(l,j),a}\right)\times\left(\sum_{\substack{\omega\in\Gamma(n-r)\\ \sigma\in S_{n-r}}}\prod_{i=1}^{i_0}A\left(I_i(s,y)\right)^p_{\omega_i,q(\omega_i),a}\right)ds \\
\leq\ & C(n,p)\Big((t-a)^{3/2-q}-(t-b)^{3/2-q}\Big)^{(p-2)/2}\int_a^b(c^*_{u'}(n,p',a,s)^{(n-1)p}+c_{u'}(n,p^*,a,s)^p)(s-a)^{\gamma\alpha p}ds.
\end{aligned}
$$

As before the case $n = 0$ is treated separately using the same proof line as in Lemma 13 obtaining a similar bound. Putting all the above estimates together we have the result. ■

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