A large trader-insider model*

Arturo Kohatsu-Higa and Agnès Sulem
INRIA-Rocquencourt
Domaine de Voluceau, Rocquencourt, B.P. 105,
F–78153 Le Chesnay Cedex, France

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Abstract

We give some remarks on the anticipating approach to insider modelling introduced by the authors recently. In particular, we define forward integrals by using limits of Riemmann sums. This definition is well adapted to financial applications.

As an application, we consider a portfolio maximization problem of a large trader with insider information. We show that the forward integral is a natural tool to handle such problems and we compute the optimal portfolios for an insider and a small trader.

1 Introduction

In this article, we would like to explain the anticipating approach to insider information. The section on the forward integrals properties relies on Chapter 3 of Nualart (1995). Nevertheless, as we have not found a standard reference for this material in the form of the forward integral we will do it here in detail. For this we need to introduce the basic tools of differentiation on the Wiener space.

Consider the interval $[0,T]$ and a complete probability space $(\Omega,\mathcal{F},P)$ on which a standard one dimensional Brownian motion W is defined; $\{\mathcal{F}_t\}_{t\in[0,T]}$ denotes the filtration generated by W, augmented with the $P-$null sets and made right continuous. Since all the results in the paper rely heavily on Malliavin calculus, we introduce some of its terminology briefly.

We denote by $\mathbb{C}_b^\infty(\mathbb{R}^n)$ the set of $\mathbb{C}^\infty$ bounded functions f from $\mathbb{R}^n$ to $\mathbb{R}$, with bounded derivatives of all orders. If S is the class of real random variables F that can be represented as $f(W_{t_1},\dots,W_{t_n})$ for some $n\in\mathbf{N}$, $t_1,\dots,t_n\in[0,T]$ and $f\in\mathbb{C}_b^\infty(\mathbb{R}^n)$, we can complete this space under the Sobolev norm $\|\cdot\|_{1,p}$ given by

$$\| F \|_{1,p}^p = E(|F|^p) + E\left((\int_0^T |D_sF|^2 ds)^{\frac{p}{2}} \right),$$

where D is defined as $D_sF = \displaystyle\sum_{i=1}^n \frac{\partial f}{\partial x_i}(W_{t_1},\dots,W_{t_n})1_{[0,t_i]}(s)$, obtaining a Banach space, usually indicated with $\mathbf{D}^{1,p}$. Analogously, we can construct the space $\mathbf{D}^{k,p}$ by completing S under the Sobolev norm

$$\| F \|_{k,p}^p = E(|F|^p) + \sum_{i=1}^k E\Big[(\int_0^T \dots \int_0^T |D^i_{s_i\dots s_1}F|^2 ds_1\dots ds_i)^{\frac{p}{2}} \Big],$$

where $D^i_{s_i\dots s_1}F = D_{s_i}\dots D_{s_1}F$. Finally, we denote $\mathbf{D}^\infty = \displaystyle\bigcap_{p\geq 1}\bigcap_{k\geq 1}\mathbf{D}^{k,p}$.

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We denote the adjoint of the closable unbounded operator

$$D : \mathbf{D}^{1,2} \subseteq L^2(\Omega) \longrightarrow L^2([0,T] \times \Omega)$$

by δ_0^T. This operator is called the Skorohod integral. The domain of δ_0^T is the set of all processes u in $L^2([0,T] \times \Omega)$ such that

$$\left| E\left(\int_0^T D_t F u_t dt \right) \right| \le C \parallel F \parallel_2 \qquad \forall\, F \in S,$$

for some constant C possibly depending on u.

If $u \in Dom(\delta_0^T)$, then $\delta_0^T(u)$ is the square integrable random variable determined by the duality relation

$$E(\delta_0^T(u) F) = E(\int_0^T D_t F u_t dt) \qquad \forall\, F \in \mathbf{D}^{1,2}.$$

Note that the above construction can be carried through for any fixed time interval $[s,S]$, in the space $L^2([s,S] \times \Omega)$. We will also use the notation

$$\delta_0^T(u) = \int_0^T u(s) \delta W(s).$$

For a stochastic process ϕ, we say that $\phi \in \mathbf{L}^{1,2}$ if the following norm is finite:

$$\|\phi\|_{1,2}^2 = E\left[\int_0^T \|\phi(s)\|^2 ds \right] + E\left[\int_0^T \int_0^T \|D_u \phi(s)\|^2 ds du \right].$$

2 The forward integral

Consider an insider, that is an agent that has sensible information about the future values of a stock, who may also have an influence on the evolution of the stock price. This is called a large trader-insider.

In general one would like to study models of the type

$$S(t) = S(0) + \int_0^t \mu(s, \pi(s)) S(s) ds + \int_0^t \sigma(s, \pi(s)) S(s) d^- W(s).$$

Here π represents the insider's strategy which is adapted to a filtration $\mathcal{G}$, which may be bigger (or just different) than the filtration generated by the Wiener process W with natural filtration $\mathcal{F}$. Therefore S is also adapted to $\mathcal{G}$ and the above stochastic integral will be an anticipating integral commonly known as the forward integral of Russo-Vallois.

Next, we define the forward integral. For this, first define for any partition $0 = t_0 < ... < t_n = T$ such that $\max\{t_{i+1} - t_i; i = 0, ..., n-1\} \to 0$ as $n \to \infty$

$$\eta(s) := \max\{t_i; t_i \le s\}.$$

Then we can define the forward integral as follows:

Definition 1 *Let $\phi : [0,T] \times \Omega \to$ be a measurable continuous process. The* forward integral *of ϕ with respect to $W(.)$ is defined by*

$$\int_0^T \phi(t) d^- W(t) = \lim_{n \to +\infty} \sum_{i=0}^{n-1} \phi(t_i)(W(t_{i+1}) - W(t_i)), \tag{1}$$

if the limit exists in probability and is independent of the partition sequence taken.

This definition does not coincide exactly with the original definition of Russo-Vallois, unless we put some additional assumptions.

Note that the above definition is local. That is, let ϕ be forward integrable such that for a measurable set $A \subset \Omega$ we have that $\phi 1_A = 0$, Then $\int_0^T \phi(t) 1_A d^- W(t) = 0$. In that sense, as in Nualart, (1995), page 45 we will use the local defintion of all the spaces to appear below.

First let us start proving that the expectation of this integral is not zero and therefore the usual rules of calculus do not apply. In particular, usual martingale properties are not true, see the interesting articles of Tudor and Pecatti-Theieullen-Tudor.

Definition 2 *Let $\phi : [0,T] \times \Omega \to \mathbb{R}$ be a measurable process such that $\phi(t) \in \mathbb{L}^{1,2}$. We say that $\phi \in \mathbb{L}_+^{1,2}$ if the following stability property is satisfied: for any sequence of partitions $0 = t_0 < ... < t_n = T$ such that its norm tends to zero as $n \to \infty$, there exists the trace process $D_{s+}\phi \in L^2([0,T] \times \Omega)$ such that*

$$\|\phi(\eta(\cdot)) - \phi(\cdot)\|_{1,2} + E\left[\int_0^T |D_s\phi(\eta(s)) - D_{s+}\phi|^2 \, ds\right] \to 0$$

In such a case we say that $\phi \in \mathbb{L}_+^{1,2}$ and we define

$$\|\phi\|_{1,2,+}^2 := \|\phi\|_{1,2}^2 + E\left[\int_0^T |D_{s+}\phi|^2 \, ds\right].$$

This norm will serve to control the variance of the forward integral as it is shown in the next Theorem.

Theorem 3 *Suppose that $\phi \in \mathbb{L}_+^{1,2}$. Then the forward integral of ϕ exists, the limit in the definition 1 being satisfied in $L^1(\Omega)$ and furthermore*

$$\int_0^T \phi(t) d^- W(t) = \int_0^T \phi(t) \delta W(t) + \int_0^T D_{t+}\phi dt,$$

where δ denotes the Skorohod integral. Furthermore,

$$E\left[\left(\int_0^T \phi(t) d^- W(t)\right)^2\right] \le 2 \|\phi\|_{1,2,+}^2 .$$

Proof. In order to prove that the integral exists we use the following formula (see formula (1.12) in page 130 in Nualart (1995a))

$$\phi(t_i)(W(t_{i+1}) - W(t_i)) = \int_{t_i}^{t_{i+1}} \phi(t_i) \delta W(s) + \int_{t_i}^{\tau_{i+1}} D_s\phi(t_i) ds.$$

Then the existence of the forward integral follows from Definition 2. Furthermore we have that each element in this expression belongs to $L^2(\Omega)$ and therefore we have that

$$\begin{aligned} \sum_{i=0}^{n-1} E\left[\phi(t_i)(W(t_{i+1}) - W(t_i))\right] &= E\left[\int_0^T D_s\phi(\eta(s)) ds\right] \\ &\to E\left[\int_0^T D_{s+}\phi ds\right]. \end{aligned}$$

The last estimate is obtained similarly. We have

$$\sum_{i=0}^{n-1} \phi(t_i)(W(t_{i+1}) - W(t_i)) = \int_0^T \phi(\eta(s)) \delta W(s) + \int_0^T D_s\phi(\eta(s)) ds.$$

Therefore,

$$E\left[\left(\sum_{i=0}^{n-1}\phi(t_i)(W(t_{i+1})-W(t_i))\right)^2\right]\leq 2\left\{E\left[\left(\int_0^T\phi(\eta(s))\delta W(s)\right)^2\right]+E\left[\left(\int_0^T D_s\phi(\eta(s))ds\right)^2\right]\right\}.$$

Then the Riemmann sum sequence is bounded in $L^2(\Omega)$ and therefore converges in $L^2(\Omega)$ as it converges in $L^1(\Omega)$. Then taking limits in the above inequality we obtained the desired result. ■

Next we prove that the integral process is a continuous process.

Theorem 4 *Suppose that* $\phi\in\mathbb{L}_+^{1,2}$, *such that* $E\left[\int_0^T\left(\int_0^T|D_s\phi(u)|^2\,ds\right)^{p/2}du\right]<\infty$ *for some* $p>2$ *then the process* $\{\int_0^t\phi(s)d^-W(s);t\in[0,T]\}$ *has a continuous version.*

Proof. Use Proposition 5.1.1 in Nualart (1995a). ■

Now we give the formula for the quadratic variation.

Theorem 5 *Given any sequence of partitions of the interval* $[0,t]$, $\pi_n:0=t_0<...<t_n=t$ *such that* $\max\{t_{i+1}-t_i;i=0,...,n-1\}\to 0$ *as* $n\to\infty$, *we have that*

$$\sum_{i=0}^{n-1}\left(\int_{t_i}^{t_{i+1}}\phi(s)d^-W(s)\right)^2\to\int_0^t|\phi(s)|^2\,ds\ a.s.$$

for $\phi\in\mathbb{L}_+^{1,2}$.

Proof. First suppose the simple case that there exists a fixed partition $0=s_0<...<s_m=t$ such that

$$\phi(s)=\sum_{i=0}^{m-1}F_i1(s_i<s\leq s_{i+1}),$$

where $F_i\in\mathbb{D}^{1,2}$. In such a case we obviously have that ϕ is forward integrable and furthermore

$$\int_0^t\phi(s)d^-W(s)=\sum_{i=0}^{m-1}F_i(W(s_{i+1})-W(s_i)).$$

We then also have that for the sequence of partitions $\pi_n'=\{t_i;i=0,...,n\}\cup\{s_j;j=0,...,m\}$ then

$$\sum_{t_i\in\pi_n}\left(\int_{t_i}^{t_{i+1}}\phi(s)d^-W(s)\right)^2-\sum_{s_j\in\pi_n'}\left(\int_{s_j}^{s_{j+1}}\phi(s)d^-W(s)\right)^2\to 0,$$

as $n\to\infty$ because the partition $\{s_j;j=0,...,m\}$ is fixed and the forward integrals are L^2- continuous in the time variable. Therefore without loss of generality we will suppose that $\{s_j;j=0,...,m\}\subset\pi_n$. Then we have that

$$\sum_{i=0}^{n-1}\left(\int_{t_i}^{t_{i+1}}\phi(s)d^-W(s)\right)^2=\sum_{j=0}^{m-1}F_j^2\sum_{s_j<t_i<s_{j+1}}(W(t_{i+1})-W(t_i))^2.$$

As the partition $\{s_j;j=0,...,m\}$ is fixed we have that

$$\sum_{s_j<t_i<s_{j+1}}(W(t_{i+1})-W(t_i))^2\to s_{j+1}-s_j$$

as $n \to \infty$. Therefore

$$\sum_{i=0}^{n-1} \left(\int_{t_i}^{t_{i+1}} \phi(s) d^- W(s) \right)^2 \to \sum_{j=0}^{m-1} F_j^2 \left(s_{j+1} - s_j \right).$$

Finally the result follows from the following density argument:

$$\begin{aligned}
&E \left[\left| \sum_{i=0}^{n-1} \left(\int_{t_i}^{t_{i+1}} \phi(s) d^- W(s) \right)^2 - \sum_{i=0}^{n-1} \left(\int_{t_i}^{t_{i+1}} \psi(s) d^- W(s) \right)^2 \right| \right] \\
&\leq \left\{ E \left[\sum_{i=0}^{n-1} \left(\int_{t_i}^{t_{i+1}} \left(\phi - \psi \right)(s) d^- W(s) \right)^2 \right] \right\}^{1/2} \times \left\{ E \left[\sum_{i=0}^{n-1} \left(\int_{t_i}^{t_{i+1}} \left(\phi + \psi \right)(s) d^- W(s) \right)^2 \right] \right\}^{1/2} \\
&\leq 2 \left\| \phi - \psi \right\|_{1,2,+} \left\| \phi + \psi \right\|_{1,2,+}
\end{aligned}$$

■

Now we give the Itô formula that is necessary for our calculations. Before we need a preliminary Lemma.

Lemma 6 *Suppose that $\phi \in \mathbb{L}_+^{1,2} \cap \mathbb{L}^{2,4}$ with $D_{s+}\phi \in \mathbb{L}^{1,2}$ and b is a stochastic process with $b \in \mathbb{L}^{1,2}$. Define the process*

$$X(t) = x + \int_0^t b(s) ds + \int_0^t \phi(s) d^- W(s).$$

Then $f(\cdot, X)\phi \in \mathbb{L}_+^{1,2}$ for any $f \in C_b^{1,2}([0,T] \times \mathbb{R})$.

Proof. First, note that $\int_0^t b(s)ds \in \mathbb{L}_+^{1,2}$. In fact, $D_u \int_0^t b(s) ds = \int_0^t D_u b(s) ds$. Furthermore, one clearly has that

$$E \left[\int_0^T \left| D_u \int_0^{\eta(u)} b(s) ds - \int_0^u D_u b(s) ds \right|^2 du \right] \to 0.$$

The other properties being clear the assertion $\int_0^t b(s)ds \in \mathbb{L}_+^{1,2}$ follows. Next, consider

$$\begin{aligned}
D_u \left(\int_0^t \phi(s) d^- W(s) \right) &= D_u \left(\int_0^t \phi(s) \delta W(s) + \int_0^t D_{s+} \phi ds \right) \\
&= \phi(u) 1(u \leq \eta(t)) + \int_0^t D_u \phi(s) \delta W(s) + \int_0^t D_u D_{s+} \phi ds.
\end{aligned}$$

Therefore we have that

$$D_{u+} \left(\int_0^{\cdot} \phi(s) d^- W(s) \right) = \int_0^u D_u \phi(s) \delta W(s) + \int_0^u D_u D_{s+} \phi ds.$$

Finally by the chain rule and product rule, we have that

$$D_s \left(f(t, X(t)) \phi(t) \right) = \frac{\partial f}{\partial x}(t, X(t)) D_s X(t) \phi(t) + f(t, X(t)) D_s \phi(t)$$

and

$$D_{s+} \left(f(\cdot, X) \phi \right) = \frac{\partial f}{\partial x}(\cdot, X) D_{s+} X \phi + f(s, X(s)) D_{s+} \phi.$$

■

Theorem 7 *Suppose that $\phi \in \mathbb{L}_{+}^{1,2} \cap \mathbb{L}^{2,4}$ with $D_{s+}\phi \in \mathbb{L}^{1,2}$ and b is a stochastic process with $b \in \mathbb{L}^{1,2}$ then for any $f \in C_b^{1,2}([0,T] \times \mathbb{R})$ we have that*

$$f(t, X(t)) = f(0,x) + \int_0^t \frac{\partial f}{\partial t}(s, X(s)) + \frac{\partial f}{\partial x}(s, X(s))b(s) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(s, X(s))\phi(s)^2 ds$$
$$+ \int_0^t \frac{\partial f}{\partial x}(s, X(s))\phi(s)d^- W(s),$$

for

$$X(t) = x + \int_0^t b(s)ds + \int_0^t \phi(s)d^- W(s).$$

Proof. In order to prove that the integral exists we find first a smooth approximation of the process ϕ of the type

$$\phi^n(s) = \sum_{i=0}^{n-1} F_i^n 1(s_i < s \leq s_{i+1})$$

where $F_i^n = \phi(s_i) \in \mathbb{D}^{1,2}$ and $0 = s_0 < ... < s_n = T$ is a fixed partition and

$$\|\phi^n - \phi\|_{1,2,+} \to 0$$

as $n \to \infty$. Note that in this case one has that

$$\int_0^t \phi^n(s)d^- W(s) = \sum_{i=0}^{n-1} F_i^n (W(s_{i+1}) - W(s_i)).$$

Now define $\eta_1(s) = \inf\{s_i; s_i > s\}$ and $\eta_2(s) = \sup\{s_i; s_i \leq s\}$. We define similarly the approximation process

$$X^n(t) = x + \int_0^t \frac{1}{\eta_1(s) - \eta_2(s)} \int_{\eta_2(s)}^{\eta_1(s)} b(u)duds + \int_0^t \phi^n(s)d^- W(s).$$

Consider any partition $0 = t_0 < ... < t_m = t$ such that it contains all the points s_j, $j = 0, ..., n$. Using the Taylor expansion we have

$$f(t, X^n(t)) = f(0,x) + \sum_{i=0}^{m-1} (\partial_x f(t_i, X^n(t_i))(X^n(t_{i+1}) - X^n(t_i)) + \partial_t f(t_i, X^n(t_i))(t_{i+1} - t_i))$$
$$+ \frac{1}{2}\sum_{i=0}^{m-1} \partial_{xx} f(t_i, \overline{X}^n(t_i))(X^n(t_{i+1}) - X^n(t_i))^2.$$

Here $\overline{X}^n(t_i)$ denotes a value between $X^n(t_i)$ and $X^n(t_{i+1})$. Obviously, $f(t, X^n(t))$ converges a.s. to $f(t, X(t))$ as $n \to \infty$. The last term above, as in the previous Theorem 5 converges to

$$\frac{1}{2}\int_0^t \partial_{xx} f(s, X(s))\phi(s)^2 ds.$$

In fact, one can easily reduce the problem to the calculation of the limit of

$$\sum_{i=0}^{m-1} \partial_{xx} f(t_i, \overline{X}^n(t_i)) \left(\int_{t_i}^{t_{i+1}} \phi^n(s)d^- W(s)\right)^2.$$
$$= \sum_{i=0}^{m-1} \left(\partial_{xx} f(t_i, \overline{X}^n(t_i)) - \partial_{xx} f(\eta_2(t_i), X^n(\eta_2(t_i)))\right) \left(\int_{t_i}^{t_{i+1}} \phi^n(s)d^- W(s)\right)^2$$
$$+ \sum_{i=0}^{n-1} \partial_{xx} f(s_j, X^n(s_j))\phi^n(s_j) \sum_{s_j \leq t_i < s_{j+1}} (W(t_{i+1}) - W(t_i))^2.$$

The first term converges to zero as $n \to \infty$ and the second converges first as $m \to \infty$ to

$$\int_0^t \partial_{xx} f(\eta_2(s), X^n(\eta_2(s)))\phi^n(\eta_2(s))ds,$$

and to $\int_0^t \partial_{xx} f(s, X(s))\phi(s)^2 ds$ as $n \to \infty$. The other terms converge clearly to

$$\int_0^t \left(\frac{\partial f}{\partial t}(s, X(s)) + \frac{\partial f}{\partial x}(s, X(s))b(s) \right) ds.$$

So we only have to consider the last term which is $\sum_{i=0}^{m-1} \partial_x f(t_i, X^n(t_i)) \int_{t_i}^{t_{i+1}} \phi^n(s)d^- W(s)$. First, as $m \to \infty$ this term converges a.s. as all the other terms converge. Therefore this limit is the forward integral $\int_0^t \partial_x f(s, X^n(s))\phi^n(s)d^- W(s)$. The forward integral $\int_0^t \partial_x f(s, X(s))\phi(s)d^- W(s)$ exists due to Lemma 6. The rest of the argument follows by a subsequence that converges at a fast speed. That is, consider a uniform partition of the interval $[0, T]$, say $s_i = Ti/n$, then consider the sequence ϕ^n such that $\sup_{t \le T} |X(t) - X^n(t)| \le n^{-\epsilon}$ for $\epsilon > 1/2$. We will then have that for the same sequence $t_i = Ti/m$

$$\begin{aligned} &\int_0^t \left(\partial_x f(s, X(s))\phi(s) - \partial_x f(s, X^n(s))\phi^n(s)\right) d^- W(s) \\ &= \lim_{m\to\infty} \sum_{i=0}^{m-1} \left(\partial_x f(t_i, X(t_i))\phi(t_i) - \partial_x f(t_i, X^n(t_i))\phi^n(t_i)\right) (W(t_{i+1}) - W(t_i)). \end{aligned}$$

Now we consider the subsequence for which $n = m$ to obtain that the above limit converges to zero. Then the result follows. ■

Remark 8 *1.The previous proof also gives a sense to the integral* $\int_0^t \partial_x f(s, X(s))d^- X(s)$.
2. In fact the original definition of the forward integral by Russo-Vallois is somewhat different to the one given here. In general, their definition is far more general. Nevertheless, once one wants that this integral becomes the limit of Riemman sums then one is forced to the above framework. Still, we remark that the above conditions can be somewhat relaxed but the general idea remains.
3. For example, the above proof is also satisfied in local form. That is, the result is also satisfied if $\phi \in \mathbb{L}^{1,2}_{+,loc} \cap \mathbb{L}^{2,4}_{loc}$ *with* $D_{s+}\phi \in \mathbb{L}^{1,2}_{loc}$ *and* b *is a stochastic process with* $b \in \mathbb{L}^{1,2}_{loc}$ *and* $f \in C^{1,2}([0, T] \times \mathbb{R})$*. For the definition of these spaces see Nualart [25].*
4. The fact that the above Itô formula demands an extra condition ($D_{s+}\phi \in \mathbb{L}^{1,2}$) *in comparison with its counterpart in Skorohod integral form is well documented in the literature. In particular, in the case of the Stratonovich-Skorohod integral. Nevertheless as our restriction comes from the finantial interpretation of the models to be used we accept them as natural.*

3 A first toy example

Rather than following the general theory exposed in Kohatsu-Sulem, we will expose the examples in order to illustrate the theory. In this section, we consider a first toy model where the dynamics of the prices are given by

$$dS(t) = S(t)(\mu + bW(T))dt + \sigma S(t)d^- W(t) \tag{2}$$

where μ and b are real numbers, $\sigma > 0$. We suppose moreover that $\rho(t) = \rho =$ constant. The interpretation of this model when $b \ge 0$ is that the insider introduces a higher appreciation rate in the stock price if $W(T) > 0$. Given the linearity of the equation of S this indicates that the higher the final stock price the bigger the value of the drift of the equation driving S. Some cases of negative values for b can also be studied but the practical interpretation of such a study is dubious.

Furthermore we remark that usually in this model we assume that the trades of the insider are not revealed to the public. This is also an interesting modelling issue which is studied in detail by

Kyle and Back. They assume that the cumulative trades of the insider plus a Wiener process in the insider's filtration are public information. The Wiener process is interpreted as the effect of the so-called noise traders.

This interpretation can also be applied in any of the cases studied with the enlargement of filtration approach and as we will see it can also be applied here.

The difference here is that we will introduce large trader-insider models with finite utility where there can also be small traders that act rationally.

In order to compare with the theory given in our previous article, we decide to first give an approach which is easier to introduce at this stage but that later will not be possible to apply. This is the set-up of enlargement of filtration. For this, consider the filtration $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(W(T))$. In this filtration it is well known that W is a semimartingale and its semimartingale decomposition is given by

$$W(t) = \hat{W}(t) + \int_0^t \frac{W(T) - W(s)}{T - s} ds,$$

where $\hat{W}$ is a Wiener process in $\mathcal{G}$. Therefore in this case, as the forward integral becomes a semimartingale integral we have that the model for S is

$$dS(t) = S(t)\left(\mu + bW(T) + \sigma\frac{W(T) - W(t)}{T - t}\right) dt + \sigma S(t) d\widehat{W}(t).$$

$$S(t) = S(0)\exp\left(\left(\mu - \frac{\sigma^2}{2}\right) t + bW(T)t + \sigma W(t)\right).$$

Therefore the optimization of the logarithmic utility for this model is done through classical methods. Briefly, one has that the wealth process associated with this price process is given by

$$V(t) = V(0) + \int_0^t \frac{\pi(s)V(s)}{S(s)} dS(s) + \int_0^t (1 - \pi(s))V(s) r e^{rs} ds.$$

Then the discounted wealth, $\hat{V}(t) = e^{-rt}V(t)$ can be written as

$$\hat{V}(t) = V(0) + \int_0^t \left(\mu - r + bW(T) + \sigma\frac{W(T) - W(s)}{T - s}\right)\pi(s)\hat{V}(s) ds + \int_0^t \sigma\pi(s)\hat{V}(s) d\widehat{W}(s).$$

The solution to the above equation is

$$\hat{V}(t) = V(0)\exp\left(\int_0^t \left(\mu - r + bW(T) + \sigma\frac{W(T) - W(s)}{T - s}\right)\pi(s) - \frac{\sigma^2}{2}\pi(s)^2 ds + \int_0^t \sigma\pi(s) d\widehat{W}(s)\right).$$

Therefore if we consider the optimization of the logarithmic utility we have the following problem

$$\max_{\pi \in \mathcal{A}_{\mathcal{H}}} J(\pi)$$

where

$$J(\pi) \equiv J_{\mathcal{H}}(t, \pi) = E\left[\int_0^t \left(\mu - r + bW(T) + \sigma\frac{W(T) - W(s)}{T - s}\right)\pi(s) - \frac{\sigma^2}{2}\pi(s)^2 ds + \int_0^t \sigma\pi(s) d\widehat{W}(s)\right],$$

and for any filtration $\mathcal{H}$ satisfying the usual conditions we define

$$\mathcal{A}_{\mathcal{H}}(t) = \left\{\pi \text{ is } \mathcal{H} \text{ adapted; } \int_0^t |\pi(s)|^2 ds < \infty\right\}.$$

We then have the following theorem

Theorem 9 *Assume that $\mathcal{H}$ is any filtration included in $\mathcal{G}$. Then the optimal portfolio for the above problem is given by*

$$\hat{\pi}(s) = \frac{\mu - r}{\sigma^2} + E\left[\frac{b}{\sigma^2}W(T) + \sigma^{-1}\frac{W(T) - W(s)}{T - s} \Big/ \mathcal{H}_s\right]$$

and the optimal value is given by

$$\frac{(\mu - r)^2 t}{2\sigma^2} + \frac{1}{2\sigma^2}\int_0^t E\left[E\left[bW(T) + \sigma\frac{W(T) - W(s)}{T - s} \Big/ \mathcal{H}_s\right]^2\right] ds.$$

In particular,

$$\lim_{t \to T} J_{\mathcal{G}}(t, \hat{\pi}) = \infty, \tag{3}$$

while

$$\lim_{t \to T} J_{\mathcal{H}}(t, \hat{\pi}) < \infty$$

for $\mathcal{H}_t = \sigma(S(s); s \leq t)$. Furthermore the functions $J_{\mathcal{G}}(t, \hat{\pi})$ and $J_{\mathcal{H}}(t, \hat{\pi})$ are increasing in b.

A far more general theorem was given in Kohatsu-Sulem.

Proof. In order to obtain the result first note that given that $\pi \in \mathcal{A}_{\mathcal{H}}(t)$, we have that

$$E\left[\int_0^t \sigma\pi(s)d\widehat{W}(s)\right] = 0.$$

Next the function

$$f_s(\pi) = \left(\mu - r + E\left[bW(T) + \sigma\frac{W(T) - W(s)}{T - s} \Big/ \mathcal{H}_s\right]\right)\pi - \frac{\sigma^2}{2}\pi^2$$

is a strictly convex function adapted to the filtration $\mathcal{H}$. Therefore the maximal value is obtained for the value $\hat{\pi}$ given in the statement of the theorem. The limit wealth for the full insider is infinite because

$$E\left[E\left[bW(T) + \sigma\frac{W(T) - W(s)}{T - s} \Big/ \mathcal{G}_s\right]^2\right] = b^2T + +2b\sigma + \frac{\sigma^2}{T - s}.$$

The last result follows by noting that

$$\mathcal{H}_t = \sigma\left(bW(T)s + \sigma W(s); s \leq t\right).$$

By using a formula for conditional expectations of Gaussian random variables (see Kohatsu-Sulem) one obtains that

$$E[W(T) - W(t)|\mathcal{H}_t] = \frac{b(T - t)}{(b^2T + 2b\sigma)t + \sigma^2}(bW(T)t + \sigma W(t)),$$
$$E[W(T)|\mathcal{H}_t] = \frac{(bT + \sigma)}{(b^2T + 2b\sigma)t + \sigma^2}(bW(T)t + \sigma W(t)).$$

Therefore the result follows because

$$E\left[\left(\frac{E[W(T) - W(t)|\mathcal{H}_t]}{T - t}\right)^2\right] = \frac{b^2t}{(b^2T + 2b\sigma)t + \sigma^2}.$$

To finish one only needs to note that

$$J_{\mathcal{H}}(t, \hat{\pi}) = \frac{(\mu - r)^2 t}{2\sigma^2} + \frac{1}{2\sigma^2}\int_0^t E\left[E\left[bW(T) + \sigma\frac{W(T) - W(s)}{T - s} \Big/ \mathcal{H}_s\right]^2\right] ds$$
$$= \frac{(\mu - r)^2 t}{2\sigma^2} + \frac{b^2}{2\sigma^2}\int_0^t s\frac{((bT + \sigma) + \sigma(T - s))^2}{(b^2T + 2b\sigma)s + \sigma^2} ds.$$

Finally differentiating with respect to b it follows that $J_{\mathcal{H}}(t,\hat{\pi})$ is increasing. ■

There are various other interesting remarks that are made in Kohatsu-Sulem with respect to the interpretation of this result. This result says that in various situations the insider which acts as a large trader may have effects in the market and the small trader only uses a projection of this market in order to optimize its utility.

This projection does not transfer the information from the insider to the small investor. This example also reflects the fact that there is not only one insider but various insiders that may act depending on the filtration that one takes between $\mathcal{H}_t = \sigma(S(s); s \leq t)$ and $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(W(T))$. Finding examples where the calculations can be done explicitely will be an interesting subject of future research.

This toy example, which can be solved using the simple technique showed here was solved in Kohatsu-Sulem using a powerful technique consisting on optimization in an anticipating framework. We will show in the next section an example which can be considered as a non-trivial application which cannot be solved using the previous technique.

Before that we will discuss another issue related with (3). In fact with a small modification we can obtain that the optimal logarithmic utility of the insider is finite.

Theorem 10 *Consider the filtration $\mathcal{G}'_t = \mathcal{F}_t \vee \sigma\left(W(T) + W'((T-s)^\theta); s \leq t\right)$ where W' is another Wiener process independent of W and $\theta \in (0,1)$. Then we have that*

$$\lim_{t \to T} J_{\mathcal{G}'}(t,\hat{\pi}) < \infty.$$

Proof. First note that the previous Theorem 9. The proof and the result also follow for the filtration $\bar{\mathcal{G}}_t = \mathcal{F}_t \vee \sigma(W(T)) \vee \sigma(W'(s); s \leq T^\theta)$. Therefore we only need to compute

$$E\left(\frac{W(T)-W(s)}{T-s} \Big/ \mathcal{G}'_s\right) = \frac{W(T)-W(s)+W'((T-s)^\theta)}{T-s+(T-s)^\theta}.$$

From here it follows that the logarithmic utility is finite if $\theta < 1$ ■

To finish we prove a theorem that can be interpreted as the non-existence of arbitrage or the issue of non-conspicuous insider trader.

Theorem 11 *For any filtration $\mathcal{H}$ included in $\mathcal{G}$ such that S is $\mathcal{H}$-adapted, suppose that there exists an optimal portfolio in $\mathbb{L}^{1,2}_+$ which leads to a finite logarithmic utility. Then there exists an $\mathcal{H}$ Wiener process $W_{\mathcal{H}}$ such that*

$$\log(S(t)/S(0)) = \int_0^t \left(r + \sigma^2 \hat{\pi}(s)\right) ds + \sigma W_{\mathcal{H}}(t).$$

Proof. Just to avoid explicit notation let $\mu_s(\omega) = \mu + bW(T)$. If there exists an optimal portfolio $\hat{\pi}$ then it minimizes the logarithmic utility of this trader which is

$$E\left(\int_0^t (\mu_s - r)\pi(s) - \frac{\sigma^2}{2}\pi(s)^2 ds + \int_0^t \sigma\pi(s) dW(s)\right).$$

Applying variational calculus to the above expression we obtain that

$$E\left(\int_u^t (\mu_s - r) - \sigma^2\hat{\pi}(s)ds + \sigma(W(t) - W(u)) \Big/ \mathcal{H}_u\right) = 0.$$

Furthermore note that

$$\log(S(t)/S(0)) = \int_0^t \mu_s - \frac{\sigma^2}{2} ds + \sigma W(t).$$

Then

$$E\left(\log(S(t)/S(u)) / \mathcal{H}_u\right) = r(t-u) + \int_u^t \sigma^2 \hat{\pi}(s) ds.$$

Therefore by Lévy's characterization of the Wiener process we have the result. ■

Note that in the classical Merton model $\hat{\pi}(s) = \frac{\mu - r}{\sigma^2}$. Therefore the previous theorem states that the small trader will not find any anomaly in his trading of the stock even if this is influenced by an insider.

This result also says that if we interpret $W_{\mathcal{H}}$ as the effect of $\mathcal{H}$ noise traders then the market maker will only see the information in the stock price itself.

4 Continuous stream of information

In this section, we consider for $\delta > T$ fixed

$$S(t) = S(0) + \int_0^t (\mu + bW(s+\delta)) S(s)ds + \int_0^t \sigma S(s) d^- W(s). \tag{4}$$

In this model, the insider has an effect on the drift of the diffusion through information that is δ units of time in the future. This continuous deformation of information may be used to model streams of information rather than one single piece of information. In this case, it is difficult to see what is the information held by the insider but his/her effect on the market is known. One first important remark is the following proposition.

Proposition 12 *W is not a semimartingale on the filtration $(\mathcal{F}_{t+\delta})_{t\in[0,T]}$.*

Proof. Consider the definition of semimartingale as given in Protter page 52. If W is a $(\mathcal{F}_{t+\delta})$-semimartingale, then for any partition whose norm tends to zero and always smaller than δ, consider the process

$$H(t) = \sum_{i=0}^{n-1} (W(t_{i+1}) - W(t_i)) 1_{(t_i, t_{i+1}]}(t).$$

This process is then $(\mathcal{F}_{t+\delta})$-adapted and converges uniformly to zero but its stochastic integral converges to the quadratic variation of W leading to a contradiction. ■

This shows that the insider filtration does not even correspond to $(\mathcal{F}_{t+\delta})_{t\in[0,T]}$. The definition for the insider's filtration in the particular case that $\delta \geq T$ is

$$\mathcal{G}_t = \mathcal{F}_t \vee \sigma(W(T)) \vee \sigma(W(s+\delta) - W(T); s \leq t).$$

Then the calculations can be carried out as in the previous section. Nevertheless, we need to be more precise here in the general case. We do this here.

In such a situation, we have to clearly use the anticipative set-up given in the first section. Therefore we have to find the solution for the equation of the prices.

Proposition 13

$$S(t) = S(0) \exp\left(\left(\mu - \frac{1}{2}\sigma^2 \right) t + b \int_\delta^{t+\delta} W(s)ds + \sigma W(t) \right).$$

is the unique solution of equation (4) *in the space* $\mathbb{L}^{1,2}_{+,loc}$.

The proof of this result follows directly from the Itô formula given in theorem 7. We are interested in computing the optimal policy of the small investor with iltration $\mathcal{H}_t = \sigma(S_s; s \leq t)$. From the previous proposition, we have that

$$\mathcal{H}_t = \sigma\left(Y(s); s \leq t\right),$$

where $Y(s) = b\int_\delta^{s+\delta} W(r)dr + \sigma W(s)$. Now we study the wealth process associated with this price process. The wealth process is defined as the solution of

$$V(t) = V(0) + \int_0^t \frac{\pi(s)V(s)}{S(s)} d^- S(s) + \int_0^t (1 - \pi(s)) V(s) r e^{rs} ds$$

where the interpretation of $d^- S(t)$ is as in Definition 1. Note that in order that this equation among others has a sensible financial interpretation we introduced in Section 2 the forward integral as a limit of Riemmann sums.

Then the discounted wealth, $\hat{V}(t) = e^{-rt}V(t)$ can be written as

$$\hat{V}(t) = V(0) + \int_0^t (\mu - r + bW(t+\delta))\pi(s)\hat{V}(s)ds + \int_0^t \sigma\pi(s)\hat{V}(s)d^- W(s).$$

As before the solution to the above equation is

$$\hat{V}(t) = V(0)\exp\left(\int_0^t (\mu - r + bW(t+\delta))\pi(s) - \frac{\sigma^2}{s}\pi(s)^2 ds + \int_0^t \sigma\pi(s)d^- W(s)\right).$$

We will later show that the optimal portfolios proposed satisfy the conditions stated in Section 3. With these assumptions, we have that the limit of the logarithmic wealth process can be written as

$$\begin{aligned} J(\pi) = J_{\mathcal{H}}(t,\pi) &:= E\log(\hat{V}(t)) - \log(V_0) \\ &= E\left[\int_0^t \left(\pi(s)(\mu - r + bW(s+\delta)) - \frac{1}{2}\sigma^2\pi(s)^2\right) ds + \sigma\int_0^t \pi(s)d^- W(s)\right]. \end{aligned}$$

The class of admissible portfolios is given by

$$\mathcal{A} = \{\pi \text{ is } \mathcal{H} \text{ adapted}; \pi \in \mathbb{L}^{1,2}_+\}.$$

Theorem 14 *Define the following portfolio*

$$\hat{\pi}(s) = \frac{\mu - r}{\sigma^2} + \sigma^{-2}E\left(bW(s+\delta)/\,\mathcal{H}_{\ s}\right) + \sigma^{-1}a(s)$$

where

$$a(s) = L^1(\Omega) - \lim_{h\to 0} E\left[\frac{W(s+h) - W(s)}{h}\Big/\,\mathcal{H}_{\ s}\right].$$

If $\hat{\pi} \in \mathbb{L}^{1,2}_+$ then $\hat{\pi}$ is the optimal portfolio for the above problem for any filtration $\mathcal{H}$ and the optimal value is given by

$$J(t,\pi^*) = \frac{\sigma^2}{2}E\left[\int_0^t \hat{\pi}(s)^2 ds\right]$$

A more general theorem was proved in Kohatsu-Sulem.

Proof. In order to obtain the result we have to prove first that the functional J is strictly convex. For this, let π_0 and $\pi_1 \in \mathcal{A}$. Then we have that for any $\alpha \in (0,1)$

$$J(\alpha\pi_0 + (1-\alpha)\pi_1) < \alpha J(\pi_0) + (1-\alpha)J(\pi_1).$$

This property clearly comes from the factor $-\frac{\sigma^2}{2}\pi(s)^2$ in the expression for J. Next, we find the first directional derivative of J.

Consider for π, $v \in \mathcal{A}$, then

$$D_\pi J(\pi) := \lim_{\epsilon\to 0}\frac{J(\pi+\epsilon v) - J(\pi)}{\epsilon} = E\left[\int_0^t (\mu - r + bW(s+\delta))v(s) - \sigma^2\pi(s)v(s)ds + \int_0^t \sigma v(s)d^- W(s)\right].$$

If we set the above equation equal to zero for all $v \in \mathcal{A}$ and in particular for $v = X1_{[s_0,t_0]}$ for $X \in \mathbb{D}^{1,2}$ we have by a density argument that

$$E\left[\int_{s_0}^{t_0} (\mu - r + bW(s+\delta))ds - \sigma^2\pi(s)ds + \sigma\left(W(t_0) - W(s_0)\right)\Big/\,\mathcal{H}_{s_0}\right] = 0.$$

Now note that $\hat{\pi}$ satisfies the above equation. In fact, replacing $\hat{\pi}$ in the above equation, we have

$$E\left[\int_{s_0}^{t_0} -\sigma \lim_{h\to 0} E\left[\frac{W(s+h)-W(s)}{h}\Big/ \mathcal{H}_{\ s}\right] ds + \sigma\left(W(t_0)-W(s_0)\right)\Big/ \mathcal{H}_{s_0}\right]$$
$$= -\sigma \lim_{h\to 0} E\left[\int_{s_0}^{t_0} \frac{W(s+h)-W(s)}{h} ds + \left(W(t_0)-W(s_0)\right)\Big/ \mathcal{H}_{s_0}\right] = 0,$$

by continuity of the paths of the Wiener process. Therefore $\hat{\pi}$ has to be optimal. In fact, for all $\beta \in \mathcal{A}$ and $\varepsilon \in (0,1)$, we have

$$\begin{aligned} J(\hat{\pi}+\varepsilon\beta) - J(\hat{\pi}) &= J((1-\varepsilon)\frac{\hat{\pi}}{1-\varepsilon} + \varepsilon\beta) - J(\hat{\pi}) \\ &\geq (1-\varepsilon)J(\frac{\hat{\pi}}{1-\varepsilon}) + \varepsilon J(\beta) - J(\hat{\pi}) \\ &= J(\frac{\hat{\pi}}{1-\varepsilon}) - J(\hat{\pi}) + \varepsilon(J(\beta) - J(\frac{\hat{\pi}}{1-\varepsilon})). \end{aligned}$$

Now, with $\dfrac{1}{1-\varepsilon} = 1+\eta$ we have

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon}(J(\frac{\hat{\pi}}{1-\varepsilon}) - J(\hat{\pi})) = \lim_{\eta\to 0} \frac{1+\eta}{\eta}(J(\hat{\pi}+\eta\hat{\pi}) - J(\hat{\pi})) = D_{\hat{\pi}}J(\hat{\pi}).$$

Then we get

$$D_\beta J(\hat{\pi}) = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}(J(\hat{\pi}+\varepsilon\beta) - J(\hat{\pi})) \geq D_{\hat{\pi}}J(\hat{\pi}) + J(\beta) - J(\hat{\pi}).$$

We conclude that

$$J(\beta) - J(\hat{\pi}) \leq D_\beta J(\hat{\pi}) - D_{\hat{\pi}}J(\hat{\pi}) \ \ ; \hat{\pi}, \beta \in \mathcal{A}.$$

In particular, using that $D_\beta J(\pi^*) = 0$, we get

$$J(\beta) - J(\pi^*) \leq 0,$$

which proves that π^* is optimal.

To find the optimal expression for the utility it is enough to note that

$$E\left[\int_0^t (\mu - r + bW(s+\delta))\hat{\pi}(s) - \sigma^2\pi(s)\hat{\pi}(s)ds + \int_0^t \sigma\hat{\pi}(s)d^- W(s)\right] = 0,$$

therefore the optimal utility is

$$E\left[\int_0^t \left(\hat{\pi}(s)(\mu - r + bW(s+\delta)) - \frac{1}{2}\sigma^2\hat{\pi}(s)^2\right) ds - \int_0^t (\mu - r + bW(s+\delta))\hat{\pi}(s) - \sigma^2\pi(s)\hat{\pi}(s)ds)\right].$$

From here the result follows. ■

A very useful property is that the optimal portfolios in a smaller filtration is just a projection.

Proposition 15 *Let $\mathcal{H}^1 \subset \mathcal{H}^2 \subset \mathcal{G}$ be two filtrations satifying the usual conditions such that there is an optimal portfolio $\hat{\pi}_2$ in $\mathcal{H}^2$ within a class of protfolios $\mathcal{A}_{\mathcal{H}^2}$. If $\mathcal{A}_{\mathcal{H}^1} \subset \mathcal{A}_{\mathcal{H}^2}$, then there is an optimal portfolio $\hat{\pi}_1$ in $\mathcal{H}^1$ which satisfies*

$$\hat{\pi}_1(s) = E\left[\hat{\pi}_2(s)/\,\mathcal{H}_s^1\right],$$
$$J_{\mathcal{H}^1}(t, \widehat{\pi}_1) \leq J_{\mathcal{H}^2}(t, \widehat{\pi}_2).$$

Therefore in order to prove the existence of the optimal portfolio it is essential to compute a or at least obtain its existence and some regularity properties. We do this, first in the case that $\delta \geq T$. This is done in the next proposition.

Proposition 16 *Suppose that $\delta \geq T$. The optimal logarithmic utility portfolio in the filtration $\mathcal{H} \subset \mathcal{G}$ is given by*

$$\hat{\pi}(s) = \frac{\mu - r}{\sigma^2} + \sigma^{-2} E\left[bW(s+\delta) + \sigma \frac{W(T) - W(s)}{T - s} \Big/ \mathcal{H}_s \right].$$

The optimal value is given by

$$\frac{(\mu - r)^2 t}{2\sigma^2} + \frac{1}{2\sigma^2} \int_0^t E\left[E\left[bW(s+\delta) + \sigma \frac{W(T) - W(s)}{T - s} \Big/ \mathcal{H}_s \right]^2 \right] ds.$$

In particular,

$$\lim_{t \to T} J_{\mathcal{G}}(t, \hat{\pi}) = \infty,$$

while

$$\lim_{t \to T} J_{\mathcal{H}}(t, \hat{\pi}) < \infty$$

for $\mathcal{H}_t = \sigma(S(s); s \leq t)$.Furthermore the functions $J_{\mathcal{G}}(t, \hat{\pi})$ and $J_{\mathcal{H}}(t, \hat{\pi})$ are increasing in b.

Proof. Define $Y(t) = b \int_\delta^{t+\delta} W(r)dr + \sigma W(t)$. Then for $\delta \geq T$

$$\lim_{s \downarrow t} E\left[\frac{W(s) - W(t)}{s - t} \Big/ \mathcal{H}_t \right] = bM \int_0^t g(t,u) dY(u).$$

$$E\left[W(t+\delta) / \mathcal{H}_t \right] = (b(t+\delta) + \sigma) M \int_0^t g(t,u) dY(u)$$

where $M \equiv M_t = \sigma^{-1} \left((b\delta + 2\sigma) \left(e^{\frac{2bt}{\sigma}} - 1 \right) + \sigma \left(e^{\frac{2bt}{\sigma}} + 1 \right) \right)^{-1}$ and $g(t,u) = e^{\frac{b}{\sigma}(2t-u)} + e^{\frac{b}{\sigma}u}$.

In fact, note that Y is a Gaussian process. Therefore $E\left[W(s) / \mathcal{H}_t \right] = \int_0^t h(s,t,u) dY(u)$ for a deterministic function h. To compute h we compute the covariances between $W(s)$ and the stochastic integral and $Y(v)$ for some $v \leq t \leq s \leq T$. First

$$E\left[W(s) Y(v) \right] = bsv + \sigma(s \wedge v).$$

Also

$$\begin{aligned} E\left[\int_0^t h(s,t,u) dY(u) Y(v) \right] &= b^2 \int_0^t \int_0^v h(s,t,\theta_1)(\theta_1 \wedge \theta_2 + \delta) d\theta_2 d\theta_1 \\ &\quad + 2b\sigma v \int_0^t h(s,t,\theta) d\theta + \sigma^2 \int_0^v h(s,t,\theta) d\theta. \end{aligned} \tag{5}$$

Therefore the above two expressions have to be equal. After differentiation of the equality with respect to $v \leq t$ three times, we obtain

$$-b^2 h(s,t,u) + \sigma^2 \frac{\partial^2 h}{\partial u^2}(s,t,u) = 0.$$

Solving this differential equation gives

$$h(s,t,u) = C_1(s,t) e^{-\frac{b}{\sigma}u} + C_2(s,t) e^{\frac{b}{\sigma}u}.$$

Next one verifies that for the following constants, the covariances coincide.

$$\begin{aligned} C_2(s,t) &= \sigma^{-1}(bs + \sigma) \left((b\delta + 2\sigma) \left(e^{\frac{2bt}{\sigma}} - 1 \right) + \sigma \left(e^{\frac{2bt}{\sigma}} + 1 \right) \right)^{-1} \\ C_1(s,t) &= e^{\frac{2bt}{\sigma}} C_2(s,t). \end{aligned}$$

Therefore, we have that

$$E\left(\frac{W(s)-W(t)}{s-t}\Big/\mathcal{H}_t\right)=\int_0^t \frac{h(s,t,u)-h(t,t,u)}{s-t}dY(u).$$

Then the result follows.

Next, using Theorem 14, we have that the possible optimal portfolio π^* is defined by

$$\pi^*(t)=\frac{\mu-r}{\sigma^2}+bM_t\left(\frac{b(t+\delta)+2\sigma}{\sigma^2}\right)\int_0^t g(t,u)dY(u).$$

satisfies that $\pi^*\in\mathbb{L}_+^{1,2}$. In fact, all the properties are obtained through the process Y. We do not give the details of this verification.

Then the optimal utility is finite as it is given by

$$J(t,\pi^*)=\log(V_0)+\frac{\sigma^2}{2}E\left[\int_0^t \pi^*(s)^2ds\right].$$

■

Remark 17 *When $s\le T$, we have that*

$$E\left[W(s)/\mathcal{H}_T\right]=\int_0^s h(s,T,u)dY(u)+\int_s^t \bar{h}(s,T,u)dY(u),$$

where

$$\begin{aligned}\bar{h}(s,t,u)&=\bar{C}_1(s,t)e^{-\frac{b}{\sigma}u}+\bar{C}_2(s,t)e^{\frac{b}{\sigma}u}\\ \bar{C}_2(s,t)&=\sigma^{-1}\left(1+\sigma h(s,T,s)\right)\left(e^{\frac{b(2T-s)}{\sigma}}+e^{\frac{2bs}{\sigma}}\right)^{-1}\\ \bar{C}_1(s,t)&=e^{\frac{2bt}{\sigma}}\bar{C}_2(s,t).\end{aligned}$$

This shows that even the information on all the prices of the interval $[0,T]$ does not reveal the information held by the insider to the small trader.

As before we can also show that the insider's utility is finite if we use the filtration $\mathcal{G}_t'=\mathcal{F}_t\vee\sigma\left(W(s+\delta)+W'((T-t)^\theta);s\le t\right)$ for $\theta<1$. Similarly we can also obtain a representation theorem such as Theorem 11. Instead we will take a look at the case $\delta<T$. We use a different shortcut through the anticipating Girsanov's theorem. For details and notation we refer to Chapter 4 in [25].

Theorem 18 *Consider the case $\delta<T$. Then there is no arbitrage for the filtration $\mathcal{H}_t=\sigma\left(S(s);s\le t\right)$ and the logarithmic utility for the optimal portfolio value for this investor is finite.*

Proof. We apply Theorem 4.1.2 in [25] in the interval $[0,T+\delta]$ with the transformation

$$T(\omega)=\omega+b1(\cdot\le T)\int_0^{\cdot}\omega(s+\delta)ds,$$

defined in $C[0,T+\delta]$. Then we have that if $T(\omega)=0$ then $\omega(t)=0$ for all $t\in[T,T+\delta]$. Therefore

$$T(\omega)=\omega+b1(\cdot\le T-\delta)\int_0^{\cdot}\omega(s+\delta)ds.$$

That is, by finite induction we have that T is an injection. To prove that it is surjective one follows a similar pattern.

Next we have that

$$\det{}_2\left(I+Du\right)>0$$

and that under the change of measure

$$\frac{dQ}{dP} = \det{}_2 (I + Du) \exp\left(- \int_0^T bW(s+\delta) dW(s) - \frac{b^2}{2} \int_0^T W(s+\delta)^2 ds \right)$$

then $\hat{W} = T(W)$ has the law of a Wiener process under Q. Therefore there exists an equivalent martingale measure for this problem.

In order to compute the optimal portfolio one uses the dual method. That is, denote $m = \sigma^{-2}(\mu - r)$ and define

$$\frac{dQ'}{dP} = \det{}_2 (I + Du) \exp\left(- \int_0^T (bW(s+\delta) - m) \, dW(s) - \frac{1}{2} \int_0^T (bW(s+\delta) + m)^2 \, ds \right).$$

Then the optimal portfolio value is

$$\hat{V}_T = V_0 \frac{dQ'}{dP}.$$

The optimal portfolio value is finite because $E\left[\log\left(\frac{dQ'}{dP}\right)\right] < \infty$. ∎

Off course an interesting problem is to compute explicitely the optimal portfolio for the case $\delta < T$. It seems that this calculation is heavy and we hope to find an easier way of computing the optimal portfolio and wealth for the small trader and the insider.

Although one may consider that the large trader effect is somewhat hidden in this paper through the process appearing in the drift. We remark that this may be considered as a first learning step towards more complex models. Some of these models were presented in Kohatsu-Sulem or Kohatsu.

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