Symplectic Structures and Current Algebras on Mapping Spaces from QP Manifolds

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§1. Introduction

$(N(=T^*M),\omega)$: a symplectic manifold.

ω: a symplectic structrue, $\{-,-\}_{P.B.}$: a Poisson bracket

Schouten-Nijenhuis brackets and Poisson bivector fields

$\wedge^\bullet TN$: the exterior algebra.

An odd Poisson bracket (the Schouten-Nijenhuis bracket) $[-,-]_S$ is defined on $\wedge^\bullet TN$ as an extension of a Lie bracket for vector fields.

Assume that there exists $P \in \wedge^2 TN$ such that

$$[P,P]_S = 0.$$

Then $[[-,P]_S,-]_S$ define a Poisson bracket $\{-,-\}_{P.B.}$ on N. P is called a Poisson bivector field.

Theorem 1. N を多様体とする。このとき、3 つ組 $(\wedge^\bullet TN, [-,-]_S, P)$ が $[P,P]_S = 0$ を満たすとき、N 上の *Poisson* 構造が $\{-,-\}_{P.B.} = [[-,P]_S,-]_S$ と構成される。

Our Purposes

前記は物理的には「力学」の場合に対応する。我々はこれを「場の理論」のセッティングに拡張することを考える。

1，２つの多様体Σ，Xの写像空間 $\mathrm{Map}(\Sigma, X)$ 上のシンプレクティック(または Poisson)構造への拡張。

2, 1が可能なX上に要求される代数、幾何構造の解析。

3, カレント代数の理論への応用とカレント代数の拡張。

§2. QP Manifolds

Definition 1. *A* **graded manifold** $\mathcal{M}$ *over a base manifold* M *is a sheaf of* $\boldsymbol{Z}$*-graded commutative algebras* $C^{\cdot}(M)$ *over a smooth manifold* M *locally isomorphic to an algebra of the form* $C^\infty(U) \times S^{\cdot}(V)$ *where* $U \subset M$ *is an open set,* V *is a graded vector space, and* $S^{\cdot}(V)$ *is the free graded-commutative algebra on* V*.*

Grading is called *degree*. If degree is nonnegative, a graded manifold is called a **N-manifold**.

Example 1. $T^*[1]N$. $x^{\tilde{I}}$ *is a local coordinate on* N *and* $\xi_{\tilde{I}}$ *is a local coordinate of degree* 1 *of the fiber of a graded manifold* $T^*[1]N$*.*

$\wedge^{\bullet} TN \cong C^\infty(T^*[1]N)$ *by idintfying* $\frac{\partial}{\partial x^{\tilde{I}}} \cong \xi_{\tilde{I}}$.

Definition 2. *A N-manifold equipped with a graded symplectic structure* Ω *of degree* n *is called a graded symplectic manifold or* **P-manifold** *of degree* n, $(\mathcal{M}, \Omega)$. Ω *is also called a* **P-structure**. *The graded Poisson bracket on* $C^\infty(\mathcal{M})$ *is defined from* Ω *as* $\{f, g\} = (-1)^{|f|+1} i_{X_f} i_{X_g} \Omega$, *where a Hamiltonian vector field* X_f *is defined by the equation* $\{f, g\} = X_f g$, *for* $f, g \in C^\infty(\mathcal{M})$.

Example 2. *A graded manifold* $T^*[1]N$ *has a graded symplectic structure*

$$\Omega = \ \delta x^{\tilde{I}} \wedge \delta \xi_{\tilde{I}},$$

of degree 1. $(T^*[1]N, \Omega)$ *is a P-manifold of degree* 1.

Definition 3. *Let* Q *be a differential of degree* $+1$ *with* $Q^2 = 0$ *on* $\mathcal{M}$. Q *is called a* **Q-structure**. *A triple* $(\mathcal{M}, \Omega, Q)$ *is called a* **QP-manifold** *of degree* n *and its structure is called a* **QP structure**, *if* Ω *and* Q *are compatible, that is,* $\mathcal{L}_Q \Omega = 0$.

A. Schwarz '93

A Hamiltonian of Q, $\Theta \in C^\infty(\mathcal{M})$ of degree $n+1$, satisfies $Q = \{\Theta, -\}$ and

$$Q^2 = 0 \Longleftrightarrow \{\Theta, \Theta\} = 0.$$

Example 3. $(T^*[1]N, \Omega)$: *A P-manifold of degree* 1.

Let us require a degree 2 *function* $\Theta \in C^\infty(T^*[1]N)$ *such that* $\{\Theta, \Theta\} = 0$. *A general solution of* Θ *is*

$$\Theta = \frac{1}{2} f^{\tilde{I}\tilde{J}}(x) \xi_{\tilde{I}} \xi_{\tilde{J}},$$

where $f^{\tilde{I}\tilde{J}}(x)$ *is skewsymmetric and* $\frac{\partial f^{\tilde{I}\tilde{J}}(x)}{\partial x^{\tilde{L}}} f^{\tilde{L}\tilde{K}}(x) + (\tilde{I}\tilde{J}\tilde{K}$ *cyclic*$) = 0$. *This is equivalent to a Poisson bivector field under* $\wedge^\bullet TN \cong C^\infty(T^*[1]N)$.

Theorem 2. *A Poisson manifold* $(N, \{-,-\}_{P.B.})$ *is a QP manifold of degree* 1, $(\mathcal{M} = T^*[1]N, \Omega, Q)$.

Example 4. *Set* $N = T^*M$ *and* $\mathcal{M} = T^*[1](T^*M)$.

• local coordinate expression

$\boldsymbol{x}^{\tilde{I}} = (x^I, p_I)$ has degree 0 and $\boldsymbol{\xi}_{\tilde{I}} = (\xi_I, \eta^I)$ has degree 1.

$$\Omega = \delta \boldsymbol{x}^{\tilde{I}} \wedge \delta \boldsymbol{\xi}_{\tilde{I}} = \delta x^I \wedge \delta \xi_I + \delta p_I \wedge \delta \eta^I.$$

$$\Theta = \eta^I \xi_I.$$

The Poisson brackets on the canonical conjugates are derived as

$$\{\{x^I, \Theta\}, x^J\} = 0, \quad \{\{x^I, \Theta, \}, p_J\} = \delta^I{}_J, \quad \{\{p_I, \Theta, \}, p_J\} = 0.$$

$\{\{-, \Theta\}, -\}$ with $\{\Theta, \Theta\} = 0$ is a called a derived bracket. Kosmann-Schwarzbach '03

• Connection and Curvature

Introducing a connection (gauge potential) $A_I(x)$ on M, the canonical momentum p_I is shifted as $p_I \to p_I + A_I$. The Poisson brackets of canonical conjugates are twisted by a closed 2-form (curvature) $H_{IJ} = \partial_I A_J - \partial_J A_I$ on the phase space as follows:

$$\{x^I, x^J\}_{P.B.} = \{\{x^I, \Theta\}, x^J\} = 0, \quad \{x^I, p_J\}_{P.B.} = \{\{x^I, \Theta, \}, p_J\} = \delta^I{}_J,$$
$$\{p_I, p_J\}_{P.B.} = \{\{p_I, \Theta, \}, p_J\} = -H_{IJ}.$$

where a Q-structure is

$$\Theta = \eta^I \xi_I + \frac{1}{2} H_{IJ}(x) \eta^I \eta^J.$$

§3. Generalizations to Field Theories

Symplectic Structure on Loop Spaces

S^1 with a local coordinate σ. M: a manifold.

$\mathrm{Map}(S^1, T^*M) \cong T^*LM$

A symplectic structure

$$\boldsymbol{\omega} = \int_{S^1} d\sigma \ \mathrm{ev}^*\omega$$

where ω is a symplectic structure on T^*M. $\mathrm{ev} : S^1 \times (T^*M)^{S^1} \longrightarrow T^*M$ is an evaluation map $\mathrm{ev} : (\sigma, \Phi) \longmapsto \Phi(\sigma)$, where $\sigma \in S^1$ and $\Phi \in (T^*M)^{S^1}$.

$x^I(\sigma) : S^1 \longrightarrow M$, $p_I(\sigma) \in \Gamma(S^1 \otimes x^*(T^*M))$: canonical conjugates with a symplectic structure

$$\omega = \int_{S^1} d\sigma \ \delta x^I \wedge \delta p_I.$$

Poisson bracket

$$\{x^I, x^J\}_{P.B.} = 0, \quad \{x^I, p_J\}_{P.B.} = \delta^I{}_J \delta(\sigma - \sigma'), \quad \{p_I, p_J\}_{P.B.} = 0.$$

The symplectic structrue can be twisted by a closed 3-form H as

$$\omega = \int_{S^1} d\sigma \ \delta x^I \wedge \delta p_I + \frac{1}{2} \int_{S^1} d\sigma \ H_{IJK} \partial_\sigma x^I \delta x^J \wedge \delta x^K.$$

$$\{x^I, x^J\}_{P.B.} = 0, \quad \{x^I, p_J\}_{P.B.} = \delta^I{}_J \delta(\sigma - \sigma'),$$
$$\{p_I, p_J\}_{P.B.} = -H_{IJK}(x) \partial_\sigma x^K \delta(\sigma - \sigma').$$

§4. A QP Manifold of Degree 2 and Loop Spaces

$T^*[2](T^*[1]M)$: a QP manifold of degree 2.

A graded symplectic form of degree 2:

$$\Omega = \delta x^I \wedge \delta \xi_I + \delta p_I \wedge \delta \eta^I$$

(x^I, p_I): local coordinates on $T^*[1]M$
(ξ_I, η^I): local coordinates on the fiber of $T^*[2](T^*[1]M)$

$\Theta \in C^\infty(T^*[2](T^*[1]M))$ of degree 3 is defined by $\{\Theta, \Theta\} = 0$. Solution

$$\Theta = \eta^I \xi_I + \frac{1}{3!} H_{IJK}(x) \eta^I \eta^J \eta^K,$$

where H is a closed 3-form.

Theorem 3. *A QP structure on $T^*[2](T^*[1]M)$ of degree 2 defines the* ***Courant algebroid*** *on $TM \oplus T^*M$.* *Roytenberg '99*

- Couran-Dorfman bracket

$(u, \alpha) \in \Gamma(TM \oplus T^*M)$

$[(u, \alpha), (u', \alpha')]_{CD} = ([u, u'], L_u \alpha' - L_{u'} \alpha + d(i_{u'} \alpha) + H(u, u', \cdot\ ))$: **Courant-Dorfman bracket** on $TM \oplus T^*M$.

$\langle (u, \alpha), (u', \alpha') \rangle = i_{u'} \alpha + i_u \alpha'$: symmetric scalar product on $TM \oplus T^*M$,

$j : TM \oplus T^*M \longrightarrow T^*[2](T^*[1]M)$ is a natural embedding map.

$j : (x^I, \frac{\partial}{\partial x^I}, dx^I, 0) \longmapsto (x^I, p_I, \eta^I, \xi_I)$.

A Courant-Dorfman bracket on $TM \oplus T^*M$ is constructed by the derived bracket:

$$[(u, \alpha), (u', \alpha')]_{CD} = j^{-1}\{\{j(u, \alpha), \Theta\}, j(u', \alpha')\}$$

A symmetric scalar product on $TM \oplus T^*M$ is constructed by

$$\langle (u, \alpha), (u', \alpha') \rangle = j^{-1}\{j(u, \alpha), j(u', \alpha')\}.$$

• Symplectic Structure on Loop Space from QP Structure

Theorem 4. $\mathcal{M} = \mathrm{Map}(T[1]S^1, T^*[2]T^*[1]M)$ *has a QP structure of degree* 1. *This induces a symplectic structure on* $\mathrm{Map}(S^1, T^*M)$.

Proof. Let $\mu = d\sigma d\theta$ be a Berezin measure on $T[1]S^1$. Then

$$\boldsymbol{\Omega} = \int_{T[1]S^1} \mu \ \mathrm{ev}^* \Omega, \qquad \boldsymbol{\Theta} = \int_{T[1]S^1} \mu \ \mathrm{ev}^* \Theta,$$

consist of a QP structure of degree 1, where $\mathrm{ev} : T[1]S^1 \times (T^*[2]T^*[1]M)^{T[1]S^1} \longrightarrow T^*[2]T^*[1]M$.

The natural projection $\mathrm{Map}(T[1]S^1, T^*[2]T^*[1]M) \longrightarrow \mathrm{Map}(S^1, T^*M)$ induces a symplectic structure. $\Box$

• local coordinate expressions on $\mathrm{Map}(T[1]S^1, T^*[2]T^*[1]M)$

Let (σ, θ) be a local coordinate on $T[1]S^1$.

$\boldsymbol{x}^I(\sigma,\theta) = x^I + \theta x^{(1)I}$: a smooth map from $T[1]S^1$ to M.
$\boldsymbol{p}_I(\sigma,\theta) = p_I^{(0)} + \theta p_I \in \Gamma(T^*[1]S^1 \otimes \boldsymbol{x}^*(T^*[1]M))$ of degree 1

'canonical conjugates'
$\boldsymbol{\xi}_I(\sigma,\theta) = \xi_I^{(0)} + \theta \xi_I^{(1)} \in \Gamma(T^*[1]S^1 \otimes \boldsymbol{x}^*(T^*[2]M))$ of degree 2
$\boldsymbol{\eta}^I(\sigma,\theta) = \eta^{(0)I} + \theta \eta^{(1)I} \in \Gamma(T^*[1]S^1 \otimes \boldsymbol{x}^*(T[1]M))$ of degree 1

The graded symplectic structure $\boldsymbol{\Omega}$ of degree 1 on $\mathcal{M}$ is induced from Ω on $T^*[2]T^*[1]M$:

$$\boldsymbol{\Omega} = \int_{T[1]S^1} d\sigma d\theta \ \mathrm{ev}^*\Omega = \int_{T[1]S^1} d\sigma d\theta \ (\delta \boldsymbol{x}^I \wedge \delta \boldsymbol{\xi}_I + \delta \boldsymbol{p}_I \wedge \delta \boldsymbol{\eta}^I),$$

A Q-structure super functional $\boldsymbol{\Theta}$ of degree 2 on $\mathcal{M}$ is constructed from Θ on $T^*[2]T^*[1]M$:

$$\boldsymbol{\Theta} = \int_{T[1]S^1} d\sigma d\theta \ \mathrm{ev}^*\Theta = \int_{T[1]S^1} d\sigma d\theta \left(\boldsymbol{\eta}^I \boldsymbol{\xi}_I + \frac{1}{3!} H_{IJK}(\boldsymbol{x}) \boldsymbol{\eta}^I \boldsymbol{\eta}^J \boldsymbol{\eta}^K \right).$$

The original Poisson brackets on the canonical conjugates are

$$\begin{aligned}
\{x^I, x^J\}_{P.B.} &= j^*\{\{\boldsymbol{x}^I(\sigma,\theta), \boldsymbol{\Theta}\}, \boldsymbol{x}^J(\sigma',\theta')\} = 0, \\
\{x^I, p_J\}_{P.B.} &= j^*\{\{\boldsymbol{x}^I(\sigma,\theta), \boldsymbol{\Theta}\}, \boldsymbol{p}_J(\sigma',\theta')\} = \delta^I{}_J \delta(\sigma - \sigma'), \\
\{p_I, p_J\}_{P.B.} &= j^*\{\{\boldsymbol{p}_I(\sigma,\theta), \boldsymbol{\Theta}\}, \boldsymbol{p}_J(\sigma',\theta')\} = -H_{IJK} \partial_\sigma x^K \delta(\sigma - \sigma')
\end{aligned}$$

§5. Generalizations of Current Algebras in two dimensions

Alekseev, Strobl '04

A Current Algebra is an algebraic structure on $\Gamma\mathrm{Map}(\Sigma, X)$ constructed from a Poisson bracket.

Example 5. *(Affine Lie (Kac-Moody) algebra in the WZW model) on* $\mathrm{Map}(S^1, \hat{G})$*: loop group*

$$\{J^A(\sigma), J^B(\sigma')\}_{P.B.} = f^{AB}{}_C J^C(\sigma)\delta(\sigma - \sigma') + \frac{k}{2\pi}\delta^{AB}\partial_\sigma\delta(\sigma - \sigma').$$

$$\begin{aligned} J^A(\sigma)T^A = J_L{}^A(\sigma)T^A &= \frac{k}{4\pi} g^{-1}\partial_\sigma g \\ &= \frac{k}{4\pi}\left(\left(e^A{}_J(x) - e^{AI}(x)B_{IJ}(x)\right)\partial_\sigma x^J + e^{AI}(x)p_I\right)T^A, \end{aligned}$$

where $g \in \mathrm{Map}(S^1, G)$, $T^A \in \mathfrak{g}$. ${e_A}^I(x)$ *is defined as* $T^A {e_A}^I(x) = g^{-1}\frac{\partial g}{\partial x^I}$, *satisfying* ${e^A}_I(x) {e_A}^J(x) = \delta^{IJ}$. $B_{IJ}(x) \in \Gamma(x^* \wedge^2 T^* \mathfrak{g})$ *s.t.* $\partial_I B_{JK} + (IJK \text{ cyclic}) = -\mathrm{tr}\left(g^{-1}\frac{\partial g}{\partial x^I}\left[g^{-1}\frac{\partial g}{\partial x^I}, g^{-1}\frac{\partial g}{\partial x^I}\right]\right)$.

- **$\hat{G}$ を $TM \oplus T^*M$ に拡張することを考える。**

A generalization of current algebras containing a large class of current algebras.

- affine Lie (Kac-Moody) algebra
- String theory with a 3-form flux H
- topological field theory (the Poisson sigma model)

$(\mathrm{Map}(S^1, T^*M), \omega)$: loop space with a symplectic structure

canonical conjugates: $x^I(\sigma) : S^1 \longrightarrow M$, $p_I(\sigma) \in \Gamma(T^*S^1 \otimes x^*(T^*M))$

For $f \in C^\infty(M)$ and $u + \alpha \in \Gamma(TM \oplus T^*M)$,

$$J_{0(f)}(\sigma) = x^* f(x) = f(x(\sigma)) \in C^\infty(\mathrm{Map}(S^1, M)),$$

$$J_{1(u,\alpha)}(\sigma) = x^*\alpha + \langle x^* u \,, p \rangle = \alpha_I(x(\sigma)) d_\sigma x^I(\sigma) + u^I(x(\sigma)) p_I(\sigma)$$
$$\in \Gamma(T^*S^1 \otimes x^*(TM \oplus T^*M)),$$

where $\langle - \,, - \rangle$ is a pairing of TM and T^*M.

$\mathcal{C}^1_{S^1}(TM \oplus T^*M) = \{J_{0(f)}, J_{1(u,\alpha)} | f \in C^\infty(M), u + \alpha \in \Gamma(TM \oplus T^*M)\}$ は、
ループ群上のカレント代数の自然な拡張。

$$
\begin{aligned}
\{J_{0(f)}(\sigma), J_{0(f')}(\sigma')\}_{P.B.} &= 0, \\
\{J_{1(u,\alpha)}(\sigma), J_{0(f')}(\sigma')\}_{P.B.} &= -u^I \frac{\partial f'}{\partial x^I}(x(\sigma))\delta(\sigma - \sigma'), \\
\{J_{1(u,\alpha)}(\sigma), J_{1(u',\alpha')}(\sigma')\}_{P.B.} &= -J_{1([(u,\alpha),(u',\alpha')]_{CD})}(\sigma)\delta(\sigma - \sigma') \\
&\quad + \langle (u,\alpha), (u',\alpha') \rangle (\sigma') \partial_\sigma \delta(\sigma - \sigma'),
\end{aligned}
$$

Theorem 5. L を $TM \oplus T^*M$ の部分バンドルとする。このとき、$\mathcal{C}^1_{S^1}(L) = \{J_{0(f)}, J_{1(u,\alpha)} | f \in C^\infty(M), u + \alpha \in \Gamma(L)\}$ が *Poisson* 括弧$\{-,-\}_{P.B.}$で閉じる $\Longleftrightarrow L \subset TM \oplus T^*M$ が *Dirac* 構造

Alekseev, Strobl '04

The Dirac structure is a maximally isotropic subbundle of $TM \oplus T^*M$, whose sections are closed under the Courant-Dorfman bracket.

Questions

• Natural interpretation of Alekseev and Strobl.

• Generalizations to higher dimensions (S^1 to Σ_{n-1}).

• Fundamental algebraic and geometric structures behind symplectic structures and current algebras.

Our Idea

QP Manifolds

§6. QP Structures of Current Algebras in Two Dimensions

$(\mathrm{Map}(S^1, T^*M), \omega)$ has been constructed from a QP manifold $(T^*[2]T^*[1]M, \Omega, \Theta)$. and $\mathrm{Map}(T[1]S^1, T^*[2]T^*[1]M)$.

Generalized currents are described by superfunctions: $j_* \mathcal{C}^1_{S^1}(TM \oplus T^*M) = \boldsymbol{\mathcal{C}}^1_{S^1}(TM \oplus T^*M) = \left\{ \boldsymbol{J}_{0(f)}, \boldsymbol{J}_{1(u,\alpha)} \right\}$, where

$$j_* J_{0(f)} = \boldsymbol{J}_{0(f)} = f(\boldsymbol{x}), \quad \text{and} \quad j_* J_{1(u,\alpha)} = \boldsymbol{J}_{1(u,\alpha)} = \alpha_I(\boldsymbol{x})\boldsymbol{\eta}^I + u^I(\boldsymbol{x})\boldsymbol{p}_I.$$

where j is an embedding map:

$$j : TM \oplus T^*M \longrightarrow T^*[2]T^*[1]M, \quad j : (x^I, \tfrac{\partial}{\partial x^I}, dx^I, 0) \longmapsto (x^I, p_I, \eta^I, \xi_I).$$

The derived brackets of current superfunctions describe the Poisson bracket of current algebras:

$$
\begin{aligned}
&\{\{\boldsymbol{J}_{0(f)}(\sigma,\theta),\boldsymbol{\Theta}\},\boldsymbol{J}_{0(f')}(\sigma',\theta')\}=0,\\
&\{\{\boldsymbol{J}_{1(u,\alpha)}(\sigma,\theta),\boldsymbol{\Theta}\},\boldsymbol{J}_{0(f')}(\sigma',\theta')\}=-u'^I\frac{\partial f}{\partial \boldsymbol{x}^I}\delta(\sigma-\sigma')\delta(\theta-\theta'),\\
&\{\{\boldsymbol{J}_{1(u,\alpha)}(\sigma,\theta),\boldsymbol{\Theta}\},\boldsymbol{J}_{1(u',\alpha')}(\sigma',\theta')\}=-\boldsymbol{J}_{1([(u,\alpha),(u',\alpha')]_{CD})}\delta(\sigma-\sigma')\delta(\theta-\theta').
\end{aligned}
$$

Anomaly terms are derived by the graded Poisson brackets of currents as

$$
\begin{aligned}
&\{\boldsymbol{J}_{0(f)}(\sigma,\theta),\boldsymbol{J}_{0(f')}(\sigma',\theta')\}=0,\quad \{\boldsymbol{J}_{0(f)}(\sigma,\theta),\boldsymbol{J}_{1(u',\alpha')}(\sigma',\theta')\}=0,\\
&\{\boldsymbol{J}_{1(u,\alpha)}(\sigma,\theta),\boldsymbol{J}_{1(u',\alpha')}(\sigma',\theta')\}=(\alpha_I u'^I+\alpha'_I u^I)\delta(\sigma-\sigma')\delta(\theta-\theta').
\end{aligned}
$$

Anomaly cancellation conditions are that current super functions are commutative under the graded Poisson bracket.

Proposition 1. [N.I, Koizumi '11] *(1). A current algebra* $\mathcal{C}^1_{S^1}(TM \oplus T^*M)$ *in two dimensions has a QP structures of degree* 2*. That is, they have a structure of a Courant algebroid.*

(2). Lが *Dirac* 構造$\Longleftrightarrow$ $\mathcal{C}^1_{S^1}(L)$が、*Poisson* 括弧で *Lie* 代数（カレント代数）となる。$\Longleftrightarrow$ $\boldsymbol{\mathcal{C}}^1_{S^1}(L)$ が *graded Poisson* 括弧で可換な部分代数をなす。

§6. QP Structures of Current Algebras in n Dimensions

Let Σ be a manifold in $n-1$ dimensions and a target space is N.

$\mathrm{Map}(\wedge^\bullet T\Sigma, N)$ with a symplectic structure ω.

An embedding map $j : N \longrightarrow \mathcal{N}$, where $(\mathcal{N}, \Omega, Q)$ is a QP manifold of degree n. Then $\mathrm{Map}(T[1]\Sigma, \mathcal{N})$ is a QP manifold of degree 1 by $\boldsymbol{\Omega} = \int_{T[1]\Sigma} \mu \; \mathrm{ev}^* \Omega$ and $\boldsymbol{\Theta} = \int_{T[1]\Sigma} \mu \; \mathrm{ev}^* \Theta$.

The pull back of j induces a symplectic structure ω on $\mathrm{Map}(\wedge^\bullet T\Sigma, N)$.

Example 6. *Let* $E = \bigoplus_{i=1}^{\lfloor \frac{n}{2} \rfloor} E_i$ *be a vector bundle on* M. $N = T^* E$. *Then* $\mathcal{N} = T^*[n]\left(T^*[n-1]\left(\bigoplus_{i=1}^{\lfloor \frac{n}{2} \rfloor} E_i[i]\right)\right)$ *becomes a QP-manifold of degree* n. $\mathrm{Map}(T[1]\Sigma_{n-1}, \mathcal{N})$ *has a QP structure of degree* 1. *This induces a canonical symplectic structure on* $\mathrm{Map}(\wedge^\bullet T\Sigma, T^* E)$.

Current Algebras

$\widehat{N}$ is a manifold such that $\widehat{N} = \pi(\mathcal{N})$ is a sub graded manfiold $\mathcal{N}$ of degree $\leq n-1$. $N \subset \widehat{N}$. $\pi \circ j = \mathrm{id}$.

$\mathcal{C}_{\Sigma}^{n-1}(\widehat{N}) = \{J_{(v)} | v \in \Gamma\left(\widehat{N}\right)\}$: current algebra

A current $J_{(v)}$ is described by a super function $\boldsymbol{J}_{(v)}$ such that $\pi^* : J_{(v)} \longrightarrow \boldsymbol{J}_{(v)}$, where $\pi^* \mathcal{C}_{\Sigma}^{n-1}(\widehat{N}) = \boldsymbol{\mathcal{C}}_{\Sigma}^{n-1}(\widehat{N}) = \{\boldsymbol{J}_v \in \Gamma(T^*[1]\Sigma \otimes x^*(\mathcal{N}))| \, |\boldsymbol{J}_v| \leq n-1\}$,

Then the current algebra is obtained as

$$\{J_{(v)}(\sigma), J_{(v')}(\sigma')\}_{P.B.} = -J_{([v,v'])}\delta(\sigma - \sigma') + \langle J_{(v)}, J_{(v')}\rangle(\sigma') d_\sigma \delta(\sigma - \sigma'),$$

where

$$\boldsymbol{J}_{([v,v'])} = \{\{\boldsymbol{J}_{(v)}, \boldsymbol{\Theta}\}, \boldsymbol{J}_{(v')}\}$$

and $\langle J_{(v)}, J_{(v')} \rangle$ is obtained as

$$\pi_* \{ \boldsymbol{J}_{(v)}(\sigma), \boldsymbol{J}_{(v)}(\sigma') \} = \langle J_{(v)}, J_{(v)} \rangle (\sigma'),$$

under proper restriction to a submanifold. We have obtained the following theorem.

Theorem 6. *A symplectic structure on* $\mathrm{Map}(\wedge^\bullet T\Sigma, N)$ *is derived from a QP structure of degree* n*, where* Σ *is a manifold in* $n-1$ *dimensions.*

A current algebra in n *dimensions,* $\mathcal{C}^{n-1}_{\Sigma}(\widehat{N})$*, has a structure of a QP manifold of degree* n*. The anomalies cancel if and only if currents are commutative under the Q-structure.*

A commutative subalgebras of $\boldsymbol{\mathcal{C}}^{n-1}_{\Sigma}(\widehat{N})$ corresponds to as generalizations of the Dirac structure on $L \subset \widehat{N}$.

§7. QP Manifolds and Loday(Leibniz) Algebroids

Definition 4. *A vector bundle* $E = (E, \rho, [-,-])$ *is called* **an algebroid** *if there is a bracket product* $[e_1, e_2]$*, where* $e_1, e_2 \in \Gamma E$*, and a bundle map* $\rho : E \to TM$ *which is called an anchor map, satisfying the conditions below:*

$$\rho[e_1, e_2] = [\rho(e_1), \rho(e_2)],$$
$$[e_1, fe_2] = f[e_1, e_2] + \rho(e_1)(f)e_2,$$

where the bracket $[\rho(e_1), \rho(e_2)]$ *is the usual Lie bracket on* ΓTM*.*

A Loday algebroid is an algebroid version of a Loday(Leibniz) algebra.

Definition 5. *An algebroid* $E = (E, \rho, [-,-])$ *is called* **a Loday algebroid** *if* *there is a bracket product* $[e_1, e_2]$ *satisfying the Leibniz identity:*

$$[e_1, [e_2, e_3]] \;=\; [[e_1, e_2], e_3] + [e_2, [e_1, e_3]],$$

where $e_1, e_2, e_3 \in \Gamma E$*, A Loday algebroid is also called a Leibniz algebroid.*

Theorem 7. *Let* $n > 1$*. Functions of degree* $n-1$ *on a QP manifold can be identifies as sections of a vector bundle* E*. This subalgebra induces a Loday algebroid structure on* E*.* *'10 Kotov,Strobl*

Let $\boldsymbol{x}$ be an element of degree 0 and $\boldsymbol{e}^{(n-1)}$ be the element of degree $n-1$. If we define

$$[e_1, e_2] \;=\; \pi_*\{\{\boldsymbol{e}_1^{(n-1)}, \Theta\}, \boldsymbol{e}_2^{(n-1)}\}, \qquad \rho(e_1) f(x) = \pi_*\{\{\boldsymbol{e}^{(n-1)}, \Theta\}, f(\boldsymbol{x})\},$$

Theorem 8. *A current algebra in* n *dimensions has a Loday algebroid structure.*

Example 7. *Let* $\mathfrak{g}$ *be a* $\mathbf{Lie\ algebra}$ *with a Lie bracket* $[-,-]$. *Then* $T^*[n]\mathfrak{g}[1]$ *is a QP manifold of degree* n. *A natural P-structure* Ω *is induced from the canonical symplectic structure on* $T^*\mathfrak{g}$, *constructed from the canonical pairing of* $\mathfrak{g}$ *and* $\mathfrak{g}^*$, $\langle -,- \rangle$. *Define* $\Theta = \frac{1}{2}\langle p, [q,q]\rangle$, *where* $q \in \mathfrak{g}[1]$ *and* $p \in \mathfrak{g}^*[n]$. *Since* $\{\Theta,\Theta\} = 0$ *from a Lie algebra structure,* Θ *defines a Q-structure. If we take a structure constant* $f^A{}_{BC}$ *of Lie algebra,* $\Theta = \frac{1}{2} f^A{}_{BC} p_A q^B q^C$.

Example 8. *Let* $n = 1$. *Then* $\mathcal{M}$ *is canonically* $\mathcal{M} = T^*[1]M$ *and a Poisson bracket* $\{-,-\}$ *in the P-structure is a Schouten-Nijenhuis bracket. A Q-structure* Θ *has degree* 2 *and* $Q^2 = 0$ *is that* Θ *is a Poisson bivector field. Thus a QP manifold of degree* 1 *is a* $\mathbf{Poisson\ manifold}$ *on* M.

Example 9. *Let* $n = 2$. *A P-structure* Ω *is an even form of degree* 2. *A Q-structure* Θ *has degree* 3 *and* $Q^2 = 0$ *defines a Courant algebroid structure on a vector bundle* E. *A QP manifold of degree* 2 *is a* $\mathbf{Courant\ algebroid}$.

Example 10. *Let $n = 3$. One of examples of a N-manifold is $\mathcal{M} := T^*[3]E[1]$. A P-structure Ω is an odd form of degree 3. A Q-structure $Q^2 = 0$ defines a* ***Lie algebroid up to homotopy*** *(the splittable H-twisted Lie algebroid) on E. A general nonsplittable algebroid is the H-twisted Lie algebroid*

Example 11. *Let E be a vector bundle on M and $\mathcal{M} = T^*[n]E[1]$. If a QP structure is defined on $\mathcal{M} = T^*[n]E[1]$ and $n \geq 4$, E becomes a Lie algebroid and $\Gamma E \oplus \wedge^{n-1}E^*$ is a subalgebroid. A Lie algebroid is a Loday algebroid which bracket $[-,-]$ is skewsymmetric. A QP structure induces the* ***higher Courant-Dorfman bracket*** *on the subalgebroid $E \oplus \wedge^{n-1}E^*$ by the derived bracket $[-,-] = \{\{-,\Theta\},-\}$, which has the following form,*

$$[u + \alpha, v + \beta] = [u, v] + L_u\beta - i_v d\alpha + H(u, v),$$

where $u, v \in \Gamma E$, $\alpha, \beta \in \Gamma \wedge^{n-1} E^*$ and H is a closed $(n+1)$-form on E.

§8. Summary and Discussions

$\mathrm{Map}(\wedge^{\bullet} T\Sigma, N)$ に対して、

• Σ が 1 次元のとき、symplectic 構造を（次数2の）QP 構造で再構成した。背後の構造が Courant algebroid であることを示した。

(• 力学の一般化されたカレント代数は（次数1の）QP構造（Schouten-Nijenhuis 括弧と Poisson bevector）から構成できることはよく知られている。)

• Σ が $n-1$ 次元のとき、symplectic 構造を次数 n の QP 構造から導出した。

• 一般に $n-1$ 次元のカレント代数の代数構造は次数 n の QP 構造であることを示した。この代数構造は Loday algebroid の構造を持つ。

• consistency anomaly cancellation の条件が次数 n の QP 構造の上の可換条件と同値であることを示した。これは一般化された Dirac 構造と理解できることを示した。

- j ? moment map?
- Quantization? A generalization of deformation quantizations
- Relations to a topological field theory?

A current algebra in n dimensions $\longleftrightarrow$ a topological field theory in $n+1$ dimensions