Higher Structures and Current Algebras

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Higher Structures in String Theory and M-Theory
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§1. Introduction

We discuss **current algebras** with **higher algebroid structures.**

Alekseev,Strobl '05, Bonelli,Zabzine '05, Ekstrand,Zabzine '09, NI,Koizumi '11, Hekmati,Mathai '12, etc.

We explain **Supergeometric technique** is to understand, compute and clarify current algebras.

Purposes and Applications

Physics

Topological and nonperturbative aspects of string and M-theory

Find new symmetries and new physics

Math

Analysis and geometric applications of higher-algebroids and higher-groupoids

Plan of Talk

Alekseev-Strobl type current algebras

Super symplectic geometry

Generalizations to higher dimensions

Quantization problems

§2. Generalized Current Algebras on Loop Space

Alekseev, Strobl '04

$X_2 = S^1 \times \boldsymbol{R}$ with a local coordinate σ on S^1.

$x^i(\sigma)$: $S^1 \to M$, $p_i(\sigma)$: canonical momentum.
$(\dim[x] = 0$ and $\dim[p] = 1$, $\dim[\partial_\sigma] = 1)$.

$$\{x^i, x^j\}_{PB} = 0, \quad \{x^i, p_j\}_{PB} = \delta^i{}_j \delta(\sigma - \sigma'), \quad \{p_i, p_j\}_{PB} = 0.$$

The Poisson bracket can be twisted by a closed 3-form H as

$$\{x^i(\sigma), x^j(\sigma')\}_{PB} = 0, \quad \{x^i(\sigma), p_j(\sigma')\}_{PB} = \delta^i{}_j \delta(\sigma - \sigma'),$$
$$\{p_i(\sigma), p_j(\sigma')\}_{PB} = -H_{ijk}(x)\partial_\sigma x^k \delta(\sigma - \sigma').$$

A generalization of currents on a target space $TM \oplus T^*M$:

$$J_{0(f)}(\sigma) = f(x(\sigma)), \quad J_{1(X+\alpha)}(\sigma) = \alpha_i(x(\sigma))\partial_\sigma x^i(\sigma) + X^i(x(\sigma))p_i(\sigma),$$

where $f(x(\sigma))$ is a function, $X + \alpha = X^i(x)\partial_i + \alpha_i(x)dx^i \in \Gamma(TM \oplus T^*M)$. $\dim[J_{0(f)}] = 0$ and $\dim[J_{1(X+\alpha)}] = 1$.

$$\begin{aligned}
\{J_{0(f)}(\sigma), J'_{0(g)}(\sigma')\}_{PB} &= 0, \\
\{J_{1(X+\alpha)}(\sigma), J'_{0(g)}(\sigma')\}_{PB} &= -\rho(X+\alpha)(\sigma)J'_{0(g)}(x(\sigma))\delta(\sigma-\sigma'), \\
\{J_{1(X+\alpha)}(\sigma), J'_{1(Y+\beta)}(\sigma')\}_{PB} &= -J_{1([X+\alpha,Y+\beta]_D)}(\sigma)\delta(\sigma-\sigma') \\
&\quad +\langle X+\alpha, Y+\beta\rangle(\sigma')\partial_\sigma\delta(\sigma-\sigma'),
\end{aligned}$$

where $X, Y \in \Gamma(TM)$, $\alpha, \beta \in \Gamma(T^*M)$,

$$\langle X+\alpha, Y+\beta \rangle = \iota_X \beta + \iota_Y \alpha = X^i \beta_i + Y^i \alpha_i,$$
$$\rho(X+\alpha) = X = X^i(x) \frac{\partial}{\partial x^i},$$
$$[X+\alpha, Y+\beta]_D = [X, Y] + L_X \beta - i_Y d\alpha + i_X i_Y H.$$

The 'algebra' of these operations is the **standard Courant algebroid**. Liu,Weinstein,Xu '97

•Anomaly cancellation condition

$$\langle X+\alpha, Y+\beta \rangle = 0.$$

Dirac Structure

This condition is satisfied on the **Dirac structure** of $TM \oplus T^*M$.

Definition 1. *If a subbundle L of the Courant algebroid $E \oplus E^*$ satisfies*

$$\langle e_1\,, e_2 \rangle = 0 \ (\textit{isotropic}), \quad [e_1, e_2]_D \in \Gamma(L) \ (\textit{close}),$$

for $e_1, e_2 \in \Gamma(L)$, and $\mathrm{rank}(L) = \frac{1}{2}\mathrm{rank}(E)$, L is called the ***Dirac structure****.*

Our Purposes

- Generalizations to higher dimensions.
- Find fundamental structures behind current algebras.

§3. Supergeometry

The Courant algebroid has the supergeometric construction corresponding to the BRST-BFV formalism.

A **graded manifold** $\mathcal{M} = (M, \mathcal{O}_M)$ on a smooth manifold M is a ringed space which structure sheaf $\mathcal{O}_M$ is $\boldsymbol{Z}$–graded commutative algebras over M, locally isomorphic to $C^\infty(U) \otimes S^{\cdot}(V)$, where U is a local chart on M, V is a graded vector space and $S^{\cdot}(V)$ is a free graded commutative ring on V.

Grading is called **degree**. We denote $\mathcal{O}_M = C^\infty(\mathcal{M})$.

If degrees are nonnegative, a graded manifold is called a **N-manifold**.

Definition 2. *A following triple $(\mathcal{M}, \omega, Q)$ is called a QP-manifold (a differential graded symplectic manifold) (Symplectic NQ-manifold) of degree n.* *Schwarz '92*

• $\mathcal{M}$: N-manifold (nonnegatively graded manifold)

• ω: P-structure
A graded symplectic form of degree n on $\mathcal{M}$.

• Q: Q-structure (a homological vector field)
A graded vector field of degree $+1$ such that $Q^2 = 0$, is a symplectic vector field, $L_Q \omega = 0$.

Note:

If degree $n \neq 0$, there exists a Hamiltonian function (a homological function) $\Theta \in C^\infty(\mathcal{M})$ of degree $n+1$ such that $\iota_Q \omega = -\delta\Theta$.

$Q^2 = 0$ is equal to the classical master equation, $\{\Theta, \Theta\} = 0$.

The QP-manifold triple can be replaced to $(\mathcal{M}, \omega, \Theta)$.

Derived bracket construction (supergeometric construction) of the (standard) Courant algebroid

Roytenberg '99

We take $n = 2$.

An N-manifold is $\mathcal{M} = T^*[2]T[1]M = (M, \mathcal{O}_M)$.

A local coordinate of $T[1]M$ is (x^i, q^i) of degree $(0, 1)$ and the conjugate coordinate is (ξ_i, p_i) of degree $(2, 1)$.

- P-structure is a canonical graded symplectic form,

$$\omega = \delta x^i \wedge \delta \xi_i + \delta q^i \wedge \delta p_i.$$

$\{-,-\}$ of degree -2

- Q-structure   the Hamiltonian function of degree 3,

$$\Theta = \xi_i q^i + \frac{1}{3!} H_{ijk}(x) q^i q^j q^k.$$

Impose the classical master equation,

$$\{\Theta, \Theta\} = 0 \Longleftrightarrow dH = 0.$$

This derives that H is a closed 3-form.

Functions on graded manifold $\mathcal{O}_M = C^\infty(\mathcal{M})$

We decompose $C^\infty(\mathcal{M}) = \sum_{i\geq 0} C_i(\mathcal{M})$, where $C_i(\mathcal{M})$ is functions of degree i.

degree 0: $f(x) \in C_0(\mathcal{M}) \simeq C^\infty(M)$,

degree 1: $\alpha_i(x)q^i + X^i(x)p_i \in C_1(\mathcal{M}) \leftrightarrow \alpha + X = \alpha_i(x)dx^i + X^i(x)\partial_i \in \Gamma(TM \oplus T^*M)$. i.e. $C_1(\mathcal{M}) \simeq \Gamma(TM \oplus T^*M)$,

($C_2(\mathcal{M}) \simeq \Gamma(\wedge^2(TM \oplus T^*M) \oplus T^*M)$, etc.)

Note : $C_0(\mathcal{M}) \oplus C_1(\mathcal{M})$ make a closed subalgebra by the **derived bracket** $\{\{-, \Theta\}, -\}$. (Count degree!)

Operations

The 'operations' on E is defined by **graded Poisson brackets** and **derived brackets**.

For $f, g \in C_0(\mathcal{M})$, $e, e_1, e_2 \in C_1(\mathcal{M})$, Poisson brackets

$C_0 \times C_0, \quad 0 = \{f, g\}$

$C_1 \times C_0, \quad 0 = \{e, f\}$

$C_1 \times C_1, \quad \langle e_1 , e_2 \rangle = \{e_1, e_2\}$ (inner product)

Derived brackets

$C_0 \times C_0 \to 0, \quad 0 = \{\{f, \Theta\}, g\}$

$C_1 \times C_0 \to C_0, \quad \rho(e)f = -\{\{e, \Theta\}, f\}$ (anchor map)

$C_1 \times C_1 \to C_1, \quad [e_1, e_2]_D = -\{\{e_1, \Theta\}, e_2\}$ (Dorfman bracket)

§4. QP Structure of 2D Current Algebras NI, Xu '13

Construction of Poisson brackets from supergeometric data

First we derive the following Poisson bracket from supergeometry.

$$\{x^i, x^j\}_{PB} = 0, \quad \{x^i, p_j\}_{PB} = \delta^i{}_j \delta(\sigma - \sigma'),$$
$$\{p_i, p_j\}_{PB} = -H_{ijk}(x)\partial_\sigma x^k \delta(\sigma - \sigma').$$

Recall supergeometric data of the standard Courant algebroid.

1, $\mathcal{M} = T^*[2]T[1]M$

2, $\omega = \delta x^i \wedge \delta\xi_i + \delta q^i \wedge \delta p_i$

3, $\Theta = \xi_i q^i + \frac{1}{3!} H_{ijk}(x) q^i q^j q^k$.

• **Observation** in the target space computation.

For $(x^i, p_i) \in T^*[1]M$, the graded target space, the derived bracket is

$$\{\{x^i, \Theta\}, x^j\} = 0,$$
$$\{\{x^i, \Theta\}, p_j\} = \delta^i{}_j,$$
$$\{\{p_i, \Theta\}, p_j\} = -H_{ijk}(x) q^k$$

Canonical Transformation and Lagrangian Submanifold

Definition 3. *Let* $\alpha \in C^\infty(\mathcal{M})$ *be a function of degree* n. *A canonical transformation* e^{δ_α} *is defined by* $f' = e^{\delta_\alpha} f = f + \{f, \alpha\} + \frac{1}{2}\{\{f, \alpha\}, \alpha\} + \cdots$. e^{δ_α} *is also called twisting.*

If $\{\Theta, \Theta\} = 0$, $\{e^{\delta_\alpha}\Theta, e^{\delta_\alpha}\Theta\} = e^{\delta_\alpha}\{\Theta, \Theta\} = 0$ for any twisting.

Let $\mathcal{L}$ be a Lagrangian submanifold of $\mathcal{M}$ and $pr : \mathcal{M} \to \mathcal{L}$ be a natural projection.

Theorem 1. *Let* Θ *be a homological function such that* $\{\Theta, \Theta\} =$

0. *Then*

$$\{f,g\}_{\mathcal{L}} \equiv pr_*\{\{pr^*f,\Theta\},pr^*g\}$$

is a graded Poisson bracket on $\mathcal{L}$*, where* $f,g \in C^\infty(\mathcal{L})$*.*

Superfields

Next, we consider the mapping space.

Take the supermanifold $\mathcal{X} = T[1]S^1$ with local coordinates (σ, θ).

Local coordinates on $\mathrm{Map}(T[1]S^1, T^*[2]T[1]M)$ are superfields,
$\boldsymbol{x}^i(\sigma, \theta) : T[1]S^1 \to M$ of degree 0
$\boldsymbol{q}^i(\sigma, \theta) \in \Gamma(T^*[1]S^1 \otimes \boldsymbol{x}^*(T_x[1]M))$ of degree 1

and canonical conjugates

$\boldsymbol{\xi}_i(\sigma, \theta) \in \Gamma(T^*[1]S^1 \otimes \boldsymbol{x}^*(T_x^*[2]M))$ of degree 2,
$\boldsymbol{p}_i(\sigma, \theta) \in \Gamma(T^*[1]S^1 \otimes \boldsymbol{x}^*(T_q^*[2]T_x[1]M))$ of degree 1.

AKSZ Construction Alexandrov, Kontsevich, Schwartz, Zaboronsky '97

The AKSZ construction is, by definition, a procedure to construct a QP-manifold structure on a mapping space of two graded manifolds, $\mathrm{Map}(\mathcal{X}, \mathcal{M})$.

$(\mathcal{X}, D, \mu)$: $\mathcal{X} = T[1]X$, where X is an $n-1$ dimensional manifold. D is a differential on $\mathcal{X}$. μ is a D-invariant nondegenerate Berezin measure.

$(\mathcal{M}, \omega, Q)$: A QP-manifold of degree n

An *evaluation map* $\mathrm{ev} : \mathcal{X} \times \mathcal{M}^{\mathcal{X}} \longrightarrow \mathcal{M}$ is defined as $\mathrm{ev} : (z, \Phi) \longmapsto \Phi(z)$, where $z \in \mathcal{X}$ and $\Phi \in \mathcal{M}^{\mathcal{X}}$.

A *chain map* $\mu_* : \Omega^\bullet(\mathcal{X} \times \mathcal{M}^{\mathcal{X}}) \longrightarrow \Omega^\bullet(\mathcal{M}^{\mathcal{X}})$ is defined as an integration on $\mathcal{X}$, $\mu_* F = \int_{\mathcal{X}} \mu\, F$ where $F \in \Omega^\bullet(\mathcal{X} \times \mathcal{M}^{\mathcal{X}})$.

$\mu_* \mathrm{ev}^* : \Omega^\bullet(\mathcal{M}) \longrightarrow \Omega^\bullet(\mathcal{M}^{\mathcal{X}})$ is called a *transgression map*.

• P-structure (BV antibracket)

We define the graded symplectic form on $\mathrm{Map}(\mathcal{X}, \mathcal{M})$ as

$$\boldsymbol{\omega} = \mu_* \mathrm{ev}^* \omega = \int_{\mathcal{X}} d\sigma d\theta (\delta \boldsymbol{x}^i \wedge \delta \boldsymbol{\xi}_i + \delta \boldsymbol{q}^i \wedge \delta \boldsymbol{p}_i),$$

- Q-structure (BV action)

$$
\begin{aligned}
\boldsymbol{Q} &= \{S, -\} = \{S_0, -\} + \{S_1, -\} (= \hat{D} + \hat{Q}), \\
S &= S_0 + S_1 = \iota_{\hat{D}} \mu_* \mathrm{ev}^* \vartheta + \mu_* \mathrm{ev}^* \Theta \\
&= \int_{\mathcal{X}} d\sigma d\theta \left(-\boldsymbol{\xi}_i \boldsymbol{d}\boldsymbol{x}^i + \boldsymbol{p}_i \boldsymbol{d}\boldsymbol{q}^i \right) + \left(\boldsymbol{\xi}_i \boldsymbol{x}^i + \frac{1}{3!} H_{ijk}(\boldsymbol{x}) \boldsymbol{q}^i \boldsymbol{q}^j \boldsymbol{q}^k \right),
\end{aligned}
$$

where $\omega = -\delta\vartheta$.

S_0 is a Hamiltonian for D and S_1 is a Hamiltonian for Q. S satisfies the *classical master equation*, $\{S, S\} = 0$.

The transgression induces the derived bracket on superfields.

$$\begin{aligned}
\{\{\boldsymbol{x}^i(\sigma,\theta), S\}, \boldsymbol{x}^j(\sigma',\theta')\} &= 0, \\
\{\{\boldsymbol{x}^i(\sigma,\theta), S\}, \boldsymbol{p}_j(\sigma',\theta')\} &= -\delta^i{}_j \delta(\sigma-\sigma')\delta(\theta-\theta'), \\
\{\{\boldsymbol{p}_i(\sigma,\theta), S\}, \boldsymbol{p}_j(\sigma',\theta')\} &= H_{ijk}(\boldsymbol{x})\boldsymbol{q}^k(\sigma,\theta)\delta(\sigma-\sigma')\delta(\theta-\theta').
\end{aligned}$$

Note

- $(\mathrm{Map}(T[1]X, \mathcal{M}), \boldsymbol{\omega}, \boldsymbol{Q} = \{S, -\})$ is of degree 1. Therefore, $\{-,-\}$ is of degree -1.

Twisting

The derived bracket $pr_*\{\{-,\Theta\},-\}$ induce the graded Poisson bracket on the Lagrangian submanifold $\mathcal{L}$ spanned by (x^i, p_i) with the symplectic form,

$$\omega_{\mathcal{L}} = \delta x^i \wedge \delta p_i,$$

The Liouville 1-form on the Lagrangian submanifold $\widehat{\mathcal{L}}_0 = \mathrm{Map}(\mathcal{X}, \mathcal{L})$ is

$$\alpha_0 = \iota_{\hat{D}} \mu_* \mathrm{ev}^* \vartheta_{\mathcal{L}} = -\int_{\mathcal{X}} \mu\, \boldsymbol{p}_i \boldsymbol{d}\boldsymbol{x}^i.$$

Twisting by α_0 gives rise to the transformation $\boldsymbol{q}^k \to \boldsymbol{q}^k - \boldsymbol{dx}^k$.

If we reduce to the canonical Lagrangian submanifold $\widehat{\mathcal{L}}_0$ defined by $\boldsymbol{\xi}_i = \boldsymbol{q}^i = 0$, we obtain a normal Poisson bracket,

$$\{-,-\}_{PB} = pr_* e^{\delta_{\alpha_0}} \{\{-, S_1\}, -\}.$$

and

$$\begin{aligned}
\{\boldsymbol{x}^i(\sigma,\theta), \boldsymbol{x}^j(\sigma',\theta')\}_{PB} &= 0, \\
\{\boldsymbol{x}^i(\sigma,\theta), \boldsymbol{p}_j(\sigma',\theta')\}_{PB} &= -\delta^i{}_j \delta(\sigma-\sigma')\delta(\theta-\theta'), \\
\{\boldsymbol{p}_i(\sigma,\theta), \boldsymbol{p}_j(\sigma',\theta')\}_{PB} &= -H_{ijk}(\boldsymbol{x}(\sigma,\theta))\boldsymbol{dx}^k(\sigma,\theta)\delta(\sigma-\sigma')\delta(\theta-\theta').
\end{aligned}$$

Physical Fields

We expand the superfields to component fields by the local coordinate θ on $T[1]S^1$,

$\boldsymbol{\Phi}(\sigma, \theta) = \Phi^{(0)}(\sigma) + \theta\Phi^{(1)}(\sigma)$.

The degree zero component in the expansion is the physical field (and degree nonzero components are ghost fields).

Physical fields are $x^i(\sigma) = x^{(0)i}(\sigma)$ and $p_i(\sigma) = p_i^{(1)}(\sigma)$.

The Poisson brackets of the physical canonical quantities are degree zero components:

$$\begin{aligned}
\{x^i(\sigma), x^j(\sigma')\}_{PB} &= 0, \\
\{x^i(\sigma), p_j(\sigma')\}_{PB} &= \delta^i{}_j \delta(\sigma - \sigma'), \\
\{p_i(\sigma), p_j(\sigma')\}_{PB} &= -H_{ijk}(x)\partial_\sigma x^k(\sigma)\delta(\sigma - \sigma').
\end{aligned}$$

Construction of currents from supergeometric data

$C^\infty(\mathcal{M}) = \sum_{i \geq 0} C_i(\mathcal{M})$, where $C_i(\mathcal{M})$ is functions of degree i.

We take $C_0 \oplus C_1$ as space of currents.

Note : $C_0(\mathcal{M}) \oplus C_1(\mathcal{M})$ make a closed subalgebra by the **derived bracket** $\{\{-, \Theta\}, -\}$.

$j_{0(f)} = f(x) \in C_0(\mathcal{M}) \simeq C^\infty(M)$,

$j_{1(X+\alpha)} = \alpha_i(x) q^i + X^i(x) p_i \in C_1(\mathcal{M}) \leftrightarrow$
$\alpha + X = \alpha_i(x) dx^i + X^i(x) \partial_i \in \Gamma(TM \oplus T^*M)$. i.e. $C_1(\mathcal{M}) \simeq \Gamma(TM \oplus T^*M)$.

Currents are identified with twisted functions on the Lagrangian submanifold of we apply the transgression map to j_0 and j_1. Then, we twist them by α_0 and finally restrict the resulting functions to the Lagrangian submanifold defined by $\boldsymbol{\xi}_i = \boldsymbol{q}^i = 0$.

The corresponding currents are

$$\boldsymbol{J}_{(0)(f)}(\epsilon_{(1)}) = pr_* e^{\delta_{\alpha_0}} \mu_* \epsilon_{(1)} \mathrm{ev}^* j_{(0)(f)} = \int_{T[1]S^1} \mu \epsilon_{(1)} f(\boldsymbol{x}),$$

$$\boldsymbol{J}_{(1)(X+\alpha)}(\epsilon_{(0)}) = pr_* e^{\delta_{\alpha_0}} \mu_* \epsilon_{(0)} \mathrm{ev}^* j_{(1)(u,\alpha)}$$

$$= \int_{T[1]S^1} \mu \epsilon_{(0)} (-\alpha_i(\boldsymbol{x}) \boldsymbol{d}\boldsymbol{x}^i + X^i(\boldsymbol{x}) \boldsymbol{p}_i),$$

where $\epsilon_{(i)} = \epsilon_{(i)}(\sigma, \theta)$ is a test function of degree i on the super circle $\mathcal{X} = T[1]S^1$. The integrands of degree zero components of $\boldsymbol{J}_0$ and $\boldsymbol{J}_1$ are

$$
\begin{aligned}
&J_{(0)(f)}(\sigma) = f(x(\sigma)), \\
&J_{(1)(X+\alpha)}(\sigma) = \alpha_i(x)\partial_\sigma x^i(\sigma) + X^i(x)p_i(\sigma),
\end{aligned}
$$

which are the correct AS currents.

We compute the Poisson algebra of these supergeometric currents from the Poisson brackets of canonical quantities $(\boldsymbol{x}^i, \boldsymbol{p}_i)$,

$$\begin{aligned}
\{\boldsymbol{J}_{0(f)}(\epsilon), \boldsymbol{J}_{0(g)}(\epsilon')\}_{PB} &= 0, \\
\{\boldsymbol{J}_{1(X+\alpha)}(\epsilon), \boldsymbol{J}_{0(g)}(\epsilon')\}_{PB} &= \rho(X+\alpha)\boldsymbol{J}_{0(g)}(\epsilon\epsilon'), \\
\{\boldsymbol{J}_{1(X+\alpha)}(\epsilon), \boldsymbol{J}_{1(Y+\beta)}(\epsilon')\}_{PB} &= \boldsymbol{J}_{1([X+\alpha, Y+\beta]_H)}(\epsilon\epsilon') \\
&+ \int_{T[1]S^1} \mu\, \boldsymbol{d}\epsilon_{(0)}\epsilon'_{(0)}\langle X+\alpha\,, Y+\beta\rangle(\boldsymbol{x}),
\end{aligned}$$

where $\boldsymbol{J}'_{0(g)} = \int_{T[1]S^1} \mu\epsilon_{(1)} g(\boldsymbol{x})$, $\boldsymbol{J}'_{1(Y+\beta)} = \int_{T[1]S^1} \mu\epsilon_{(0)}(-\beta_i(\boldsymbol{x})\boldsymbol{d}\boldsymbol{x}^i + Y^i(\boldsymbol{x})\boldsymbol{p}_i)$. The AS current algebra is given by the physical components of the supergeometric currents, i.e., degree zero components,

$$
\begin{aligned}
\{J_{0(f)}(\sigma), J_{0(g)}(\sigma')\}_{PB} &= 0, \\
\{J_{1(X+\alpha)}(\sigma), J_{0(g)}(\sigma')\}_{PB} &= -\rho(X+\alpha)J_{0(g)}(x(\sigma))\delta(\sigma-\sigma'), \\
\{J_{1(X+\alpha)}(\sigma), J_{1(Y+\beta)}(\sigma')\}_{PB} &= -J_{1([X+\alpha,Y+\beta]_H)}(\sigma)\delta(\sigma-\sigma') \\
&+ \langle (X+\alpha),(Y+\beta)\rangle(\sigma')\partial_\sigma\delta(\sigma-\sigma').
\end{aligned}
$$

This coincides with the generalized current algebra described in Section 2.

Righthand side of the current algebra

The current terms are twist of the pullback of $\{\{j_a, \Theta\}, j_b\}$, where $a, b = 0, 1$.

The anomaly terms are the pullback of $\{J_a, J_b\}$.

Two terms are derived from the derived bracket on the mapping space,

$$\{\{\boldsymbol{J}_a, S_1 + S_0\}, \boldsymbol{J}_b\},$$

and twisting.

§5. BFV-AKSZ Formalism of Current Algebras

Data

$(\mathcal{X}, D, \mu)$: Here $\mathcal{X} = T[1]\Sigma$, where $\Sigma \times \boldsymbol{R}$ is an n dimensional manifold. D is a differential on $\mathcal{X}$. μ is a D-invariant nondegenerate Berezin measure.
$(\mathcal{M}, \omega, \Theta, \mathcal{L})$: A QP manifold of degree n and a Lagrangian submanifold.

Assume $\{-,-\}_{\mathcal{L}} = pr_*\{\{-,\Theta\},-\}$ is nondegenerate.

We consider the induced symplectic structure $\omega_{\mathcal{L}}$ defined from $\{-,-\}_{\mathcal{L}}$ on $\mathcal{L}$.

Twisting by canonical 1-form on Lagrangian submanifold

Let $\vartheta_{\mathcal{L}}$ be the canonical 1-form for $\omega_{\mathcal{L}}$ such that $\omega_{\mathcal{L}} = -\delta\vartheta_{\mathcal{L}}$.

We define the function α_0 on the mapping space,

$$\alpha_0 = \iota_{\hat{D}}\mu_* \mathrm{ev}^* pr^* \vartheta_s.$$

Poisson bracket

$$\{-,-\}_{PB} = pr^* e^{\delta_{\alpha_0}} \{\{-, S\}, -\}.$$

is a graded Poisson bracket of degree 0, therefore, a normal Poisson bracket.

Current functions on target space $\mathcal{M}$

We consider a closed subalgebra of the structure sheaf $C^\infty(\mathcal{M})$ not only under the Poisson bracket $\{-,-\}$, but also under the derived bracket $\{\{-,\Theta\},-\}$:

$$C^{(n-1)}(\mathcal{M}) = \sum_{i=0}^{n-1} C_i(\mathcal{M}) = \{f \in C^\infty(\mathcal{M}) | |f| \leq n-1\}$$

Currents

The pullbacks of $j_i \in C^{(n-1)}(\mathcal{M})$ to $\mathrm{Map}(T[1]\Sigma_{n-1}, \mathcal{L})$, is a BFV current

$$\boldsymbol{J}_i(\epsilon) = pr_* e^{\delta_{\alpha_0}} \mu_* \epsilon \, \mathrm{ev}^* j_i,$$

where ϵ is a test function on $T[1]\Sigma_{n-1}$ of degree $n-1-i$.

$$\mathcal{C}\mathcal{A}^{(n-1)}(\mathcal{X}, \mathcal{M}) = \{\boldsymbol{J}_i\}.$$

Supergeometic (BFV-AKSZ) Current Algebras

The Poisson algebra on the space $\mathcal{CA}^{(n-1)}(\mathcal{X},\mathcal{M})$ is a current algebra.

Theorem 2. *NI, Xu '13, (NI, Strobl)*

For currents $\boldsymbol{J}_{j_1}$ and $\boldsymbol{J}_{j_2}$ associated to current functions j_1, j_2 $\in C^{(n-1)}(\mathcal{M})$ respectively, the commutation relation is given by

$$
\begin{aligned}
&\{\boldsymbol{J}_{j_1}(\epsilon_1), \boldsymbol{J}_{j_2}(\epsilon_2)\}_{PB} \\
&= -pr_* e^{\delta_{\alpha_0}}\{\{\mu_*\epsilon_1 \mathrm{ev}^* j_1, S_1 + S_0\}, \mu_*\epsilon_2 \mathrm{ev}^* j_2\} \\
&= -\boldsymbol{J}_{[j_1,j_2]_D}(\epsilon_1\epsilon_2) - pr_* e^{\delta_{\alpha_0}} \iota_{\hat{D}}\mu_*(d\epsilon_1)\epsilon_2 \mathrm{ev}^*\{j_1, j_2\}
\end{aligned}
$$

where ϵ_i are test functions for j_i and $[j_1, j_2]_D = \{\{j_1, \Theta\}, j_2\}$ is the derived bracket.

Due to the second term in the equation, it fails to be a Poisson algebra. The anomalous terms vanish if j_1 and j_2 commute, $\{j_1, j_2\} = 0$. Therefore we have

Corollary 1. *If anomaly terms vanish for current functions on a subspace of commutative functions of $C^{(n-1)}(\mathcal{M})$ under the graded Poisson bracket $\{-, -\}$.*

Physical Currents

For a superfield, introduce the second degree, the **form degree** $\deg f$, which is the order of θ^μ.

$\mathrm{gh}\ f = |f| - \deg f$ is called the **ghost number**.

We define a physical current $J_{cl} = \boldsymbol{J}|_{\mathrm{gh} f = 0}$.

Theorem 3. *The* ***ghost number zero components*** *of the supergeometric current algebra gives the physical current algebra:*

$$\{J_{j_1,cl}(\epsilon_1), J_{j_2,cl}(\epsilon_2)\}_{P.B}$$
$$= \left(-J_{[j_1,j_2]_D}(\epsilon_1\epsilon_2) - pr_* e^{\delta_{S_s}} \iota_{\hat{D}} \mu_*(d\epsilon_1)\epsilon_2 \mathrm{ev}^* \{j_1, j_2\}_b|_{\mathrm{gh}=0}\right),$$

§6. Results

• Many known current algebras are included as a special case, such as Lie algebras (gauge currents), Kac-Moody algebras, Alekseev-Strobl types (algebroids), topological membranes, L_∞-algebra, etc.,except for energy-moment tensors.

• We have constructed new current algebras with algebroid structures, which are generalizations of Alekseev-Strobl, Bonelli-Zabzine.

• Anomaly cancellation conditions are characterized in terms of supergeometry as a Lagrangian submanifold in the target space.

• Noether currents of AKSZ sigma models are nontrivial examples of this current algebra.

• Duality between standard and Poisson Courant algebroid and Poisson Courant algebroid is reinterpreted in terms of the current algebra. Bessho, Heller, NI, Watamura '16

§7. Membrane Current Algebras

As one example, we construct a new current algebra of Alekseev-Strobl type on a $2+1$ dimensional manifold $X_3 = \Sigma_2 \times \boldsymbol{R}$.

Target space

1, Consider a QP manifold of degree 3, $\mathcal{M} = T^*[3]\mathcal{L}$, where $\mathcal{L} = T^*[2]E[1]$.

QP Manifold of degree 3

Let us consider the classical phase space $(x^I(\sigma), q_\mu^A(\sigma))$, $\mathrm{Map}(\Sigma_2, T^*M) \oplus \mathrm{Map}(T\Sigma_2, T^*E)$. This is the phase space for the physical models in 3 dimensions.

We introduce $T^*[3]\mathcal{M} = T^*[3]T^*[2]E[1]$, a graded symplectic manifold of degree 3.

Take local coordinates (x^I, q^A, p_I) of degree $(0,1,2)$ on $T^*[2]E[1]$, and conjugate Darboux coordinates (ξ_I, η^A, χ^I) of degree $(3,2,1)$ on $T^*[3]$.

A graded symplectic structure on $T^*[3]\mathcal{M}$ is

$$\omega_b = \delta x^I \wedge \delta\xi_I - k_{AB}\delta q^A \wedge \delta\eta^B + \delta p_I \wedge \delta\chi^I,$$

where k_{AB} is a fiber metric on E. We take the following Q-structure function

$$\Theta = \chi^I \xi_I + \frac{1}{2} k_{AB}\eta^A\eta^B + \frac{1}{4!} H_{IJKL}(x)\chi^I\chi^J\chi^K\chi^L.$$

$\{\Theta, \Theta\} = 0$ if H is a closed 4-form and $dk = 0$.

A Lagrangian submanifold $\mathcal{L} = T^*[2]E[1]$ spanned by (x^I, q^A, p_I).

Lie 3-Algebroid

NI, Uchino '10, Grützmann '10

Let $F = E \oplus TM$ and $F^* = E \oplus T^*M$.

$C_0(\mathcal{M}) = C^\infty(M)$.
$C_1(\mathcal{M}) = \Gamma(F^*)$ which is spanned by degree one basis q^A, χ^I.
$C_2(\mathcal{M}) = \Gamma(F \oplus \wedge^2 F^*)$ which is spanned by degree two basis $p_I, \eta^A, q^A q^B, q^A \chi^I, \chi^I \chi^J$, etc.

Let $f \in C_0(\mathcal{M})$, $t, t_1, t_2, \cdots \in C_1(\mathcal{M})$ and $s, s_1, s_2, \cdots \in C_2(\mathcal{M})$.

A graded Poisson bracket

induces a natural pairing $\Gamma(F^*) \times \Gamma(F \oplus \wedge^2 F^*) \to \boldsymbol{C}$, and a bilinear form on $\Gamma(F \oplus \wedge^2 F^*) \times \Gamma(F \oplus \wedge^2 F^*) \to \Gamma(F^*)$,

$$\langle t\,, s\rangle \;=\; \{t, s\}, \quad \langle s_1\,, s_2\rangle = \{s_1, s_2\}.$$

Derived brackets

$C_1 \times C_1 \to \boldsymbol{C}$, a bilinear symmetric form on ΓF^*,

$$(t_1, t_2) \;=\; \{\{t_1, \Theta\}, t_2\}.$$

$C_2 \times C_0 \to C_0$, an anchor map $\rho : F \oplus \wedge^2 F^* \to TM$ is

$$\rho(s) f(x) \;=\; -\{\{s, \Theta\}, f(x)\}.$$

$C_2 \times C_1 \to C_1$, a Lie type derivative $L : (F \oplus \wedge^2 F^*) \times F^* \to F^*$ is

$$L_s t \;\; = \;\; -\{\{s, \Theta\}, t\}.$$

$C_2 \times C_2 \to C_2$, a higher Dorfman bracket on $\Gamma F \oplus \wedge^2 F^*$,

$$[s_1, s_2]_3 \;\; = \;\; -\{\{s_1, \Theta\}, s_2\}.$$

Definition 4. *We impose* $\{\Theta, \Theta\} = 0$. *then* $(E, \langle - , - \rangle, (-, -), \rho, L, [-, -]_3)$ *is called a Lie* 3*-algebroid.*

Superfields

$(\boldsymbol{x}^I(\sigma,\theta), \boldsymbol{q}^A(\sigma,\theta), \boldsymbol{p}_I(\sigma,\theta))$ and $(\boldsymbol{\xi}_I(\sigma,\theta), \boldsymbol{\eta}^A(\sigma,\theta), \boldsymbol{\chi}^I(\sigma,\theta)$ are corresponding superfields.

Poisson brackets

Derived bracket

$$\{\{x^I, \Theta\}, p_J\} = \delta^I{}_J,$$
$$\{\{q^A, \Theta\}, q^B\} = k^{AB},$$
$$\{\{p_I, \Theta\}, p_J\} = -\frac{1}{2} H_{IJKL}(x) \chi^K \chi^L.$$

A canonical 1-form is $\alpha_0 = \int_{\mathcal{X}} \mu \, (\delta \boldsymbol{x}^I \wedge \delta \boldsymbol{p}_I + \frac{1}{2} k_{AB} \delta \boldsymbol{q}^A \wedge \delta \boldsymbol{q}^B)$.

These derive the Poisson brackets for canonical conjugates

$$
\begin{aligned}
&\{\boldsymbol{x}^I(\sigma,\theta), \boldsymbol{p}_J(\sigma',\theta')\}_{PB} = \delta^I{}_J \delta^2(\sigma-\sigma')\delta^2(\theta-\theta'), \\
&\{\boldsymbol{q}^A(\sigma,\theta), \boldsymbol{q}^B(\sigma',\theta')\}_{PB} = k^{AB} \delta^2(\sigma-\sigma')\delta^2(\theta-\theta'), \\
&\{\boldsymbol{p}_I(\sigma), \boldsymbol{p}_J(\sigma')\}_{PB} = -\frac{1}{2} H_{IJKL} \boldsymbol{dx}^K \boldsymbol{dx}^L \delta^2(\sigma-\sigma')\delta^2(\theta-\theta'),
\end{aligned}
$$

Currents

Let us consider the space of functions of degree equal to or less than 2. $C_2(T^*[3]T^*[2]E[1]) = \{f \in C^\infty(T^*[3]T^*[2]E[1]) || f| \leq 2\}$. General functions of $C_2(T^*[3]T^*[2]E[1])$ of degree 0, 1 and 2 are

$$
\begin{aligned}
J_{(0)(f)} &= f(x), \\
J_{(1)(a,u)} &= a_I(x)\chi^I + u_A(x)q^A, \\
J_{(2)(G,K,F,B,E)} &= G^I(x)p_I + K_A(x)\eta^A + \frac{1}{2}F_{AB}(x)q^A q^B \\
&\quad + \frac{1}{2}B_{IJ}(x)\chi^I\chi^J + E_{AI}(x)\chi^I q^A.
\end{aligned}
$$

Next we pullback and twist current functions with canonical 1-form, $\alpha_0 = \int_{T[1]\Sigma_2} \mu \ \left(-\boldsymbol{p}_I \boldsymbol{d}\boldsymbol{x}^I + \frac{1}{2} k_{AB} \boldsymbol{q}^A \boldsymbol{d}\boldsymbol{q}^B\right)$. Currents of degree 0, 1 and 2 are

$$
\begin{aligned}
\boldsymbol{J}_{(0)(f)} &= \int_{T[1]\Sigma_2} \mu \epsilon_{(2)} f(\boldsymbol{x}), \\
\boldsymbol{J}_{(1)(a,u)} &= \int_{T[1]\Sigma_2} \mu \epsilon_{(1)} (a_I(\boldsymbol{x}) \boldsymbol{d}\boldsymbol{x}^I + u_A(\boldsymbol{x}) \boldsymbol{q}^A), \\
\boldsymbol{J}_{(2)(G,K,F,B,E)}(\sigma, \theta) &= \int_{T[1]\Sigma_2} \mu \epsilon_{(0)} (G^I(\boldsymbol{x}) \boldsymbol{p}_I + K_A(\boldsymbol{x}) \boldsymbol{d}\boldsymbol{q}^A \\
&+ \frac{1}{2} F_{AB}(\boldsymbol{x}) \boldsymbol{q}^A \boldsymbol{q}^B + \frac{1}{2} B_{IJ}(\boldsymbol{x}) \boldsymbol{d}\boldsymbol{x}^I \boldsymbol{d}\boldsymbol{x}^J + E_{AI}(\boldsymbol{x}) \boldsymbol{d}\boldsymbol{x}^I \boldsymbol{q}^A).
\end{aligned}
$$

The current algebra is as follows:

$$\{\boldsymbol{J}_{(0)(f)}(\epsilon), \boldsymbol{J}_{(0)(f')}(\epsilon')\}_{P.B.} = 0,$$

$$\{\boldsymbol{J}_{(1)(u,a)}(\epsilon), \boldsymbol{J}_{(0)(f')}(\epsilon')\}_{P.B.} = 0,$$

$$\{\boldsymbol{J}_{(2)(G,K,F,H,E)}(\epsilon), \boldsymbol{J}_{(0)(f')}(\epsilon')\}_{P.B.} = -G^I \frac{\partial \boldsymbol{J}_{(0)(f')}}{\partial \boldsymbol{x}^I}(\epsilon\epsilon'),$$

$$\{\boldsymbol{J}_{(1)(u,a)}(\epsilon), \boldsymbol{J}_{(1)(u',a')}(\epsilon')\}_{P.B.} = -\int_{T[1]\Sigma_2} \mu\epsilon_{(1)}\epsilon'_{(1)}\mathrm{ev}^* k^{AB} u_A u'_B,$$

$$\{\boldsymbol{J}_{(2)(G,K,F,B,E)}(\epsilon), \boldsymbol{J}_{(1)(u',a')}(\epsilon')\}_{P.B.}$$
$$= -\boldsymbol{J}_{(1)(\bar{u},\bar{\alpha})}(\epsilon\epsilon') - \int_{T[1]\Sigma_2} \mu(\boldsymbol{d}\epsilon_{(0)})\epsilon'_{(1)}(G^I\alpha'_I - k^{AB}K_A u'_B),$$

$$\{\boldsymbol{J}_{(2)(G,K,F,B,E)}(\epsilon), \boldsymbol{J}_{(2)(G',K',F',B',E')}(\epsilon')\}_{P.B.} = -\boldsymbol{J}_{(2)(\bar{G},\bar{K},\bar{F},\bar{B},\bar{E})}(\epsilon\epsilon')$$
$$-\int_{T[1]\Sigma_2} \mu(\boldsymbol{d}\epsilon_{(0)})\epsilon'_{(0)}\left[(G^J B'_{JI} + G'^J B_{JI} + k^{AB}(K_A E'_{BI} + E_{AI}K'_B))\boldsymbol{d}\boldsymbol{x}^I\right.$$
$$\left. + (G^I E'_{AI} + G'^I E_{AI} + k^{BC}(K_B F'_{AC} + F_{AC}K'_B))\boldsymbol{q}^A\right].$$

Here

$$
\begin{aligned}
&\bar{\alpha} = (i_G d + d i_G)\alpha' + \langle E - dK, u' \rangle, \quad \bar{u} = i_G du' + \langle F, u' \rangle, \\
&\bar{G} = [G, G'], \\
&\bar{K} = i_G dK' - i_{G'} dK + i_{G'} E + \langle F, K' \rangle, \\
&\bar{F} = i_G dF' - i_{G'} dF + \langle F, F' \rangle, \\
&\bar{B} = (d i_G + i_G d) B' - i_{G'} dB + \langle E, E' \rangle + \langle K', dE \rangle - \langle dK, E' \rangle + i_{G'} i_G H, \\
&\bar{E} = (d i_G + i_G d) E' - i_{G'} dE + \langle E, F' \rangle - \langle E', F \rangle + \langle dF, K' \rangle - \langle dK, F' \rangle,
\end{aligned}
$$

where all the terms are evaluated by σ'. Here $[-,-]$ is a Lie bracket on TM, i_G is an interior product with respect to a vector field G and $\langle -,- \rangle$ is the graded bilinear form on the fiber of E with respect to the metric k^{AB}.

The condition vanishing anomalous terms, $G^I\alpha'_I - k^{AB}K_Au'_B = 0$, $G^JB'_{JI} + G'^JB_{JI} + k^{AB}(K_AE'_{BI} + E_{AI}K'_B) = 0$ and $G^IE'_{AI} + G'^IE_{AI} + k^{BC}(K_BF'_{AC} + F_{AC}K'_B) = 0$ is equivalent that $J_{(i)}$'s are commutative.

The classical current algebra is the ghost number zero component of superfields:

$$\{\boldsymbol{J}_{(0)(f)}|_{cl}(\epsilon), \boldsymbol{J}_{(0)(f')}|_{cl}(\epsilon')\}_{P.B.} = 0,$$

$$\{\boldsymbol{J}_{(1)(u,a)}|_{cl}(\epsilon), \boldsymbol{J}_{(0)(f')}|_{cl}(\epsilon')\}_{P.B.} = 0,$$

$$\{\boldsymbol{J}_{(2)(G,K,F,H,E)}|_{cl}(\epsilon), \boldsymbol{J}_{(0)(f')}|_{cl}(\epsilon')\}_{P.B.} = -G^I \frac{\partial \boldsymbol{J}_{(0)(f')}|_{cl}}{\partial x^I}(\epsilon\epsilon'),$$

$$\{\boldsymbol{J}_{(1)(u,a)}|_{cl}(\epsilon), \boldsymbol{J}_{(1)(u',a')}|_{cl}(\epsilon')\}_{P.B.} = -\int_{\Sigma_2} {\epsilon_{cl}}^{(1)}_{(1)} \wedge {\epsilon_{cl}}^{(1)\prime}_{(1)} \mathrm{ev}^* k^{AB} u_A u'_B|_{cl},$$

$$\{\boldsymbol{J}_{(2)(G,K,F,B,E)}|_{cl}(\epsilon), \boldsymbol{J}_{(1)(u',a')}|_{cl}(\epsilon')\}_{P.B.}$$
$$= -\boldsymbol{J}_{(1)(\bar{u},\bar{\alpha})}|_{cl}(\epsilon\epsilon') - \int_{\Sigma_2} d\epsilon_{cl(0)} \wedge {\epsilon_{cl}}^{(1)\prime}_{(1)} (G^I \alpha'_I - k^{AB} K_A u'_B)|_{cl},$$

$$\{\boldsymbol{J}_{(2)(G,K,F,B,E)}|_{cl}(\epsilon), \boldsymbol{J}_{(2)(G',K',F',B',E')}|_{cl}(\epsilon')\}_{P.B.} = -\boldsymbol{J}_{(2)(\bar{G},\bar{K},\bar{F},\bar{B},\bar{E})}|_{cl}(\epsilon\epsilon')$$
$$-\int_{\Sigma_2} d\epsilon_{cl(0)} {\epsilon_{cl}}'_{(0)} \left[(G^J B'_{JI} + G'^J B_{JI} + k^{AB}(K_A E'_{BI} + E_{AI} K'_B))|_{cl} dx^I \right.$$
$$\left. + (G^I E'_{AI} + G'^I E_{AI} + k^{BC}(K_B F'_{AC} + F_{AC} K'_B))|_{cl} q^A \right],$$

where ${\epsilon_{cl}}^{(i)}_{(i)} = \frac{1}{i!} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_i} \epsilon_{(i)\mu_1 \cdots \mu_i}(\sigma)$ and $q^A = dx^\mu q^A_\mu$.

Note: This current algebra includes the Chern-Simons theory with matter, the Courant sigma model, etc.

§. Conclusions and Future Outlook

We have proposed a new formulation of current algebras based on the BFV-AKSZ formalism. A **derived bracket** and a **canonical transformation** on a **Lagrangian submanifold of a graded symplectic manifold** derive a current algebra. .

Outlook

- Generalizations to currents with higher derivatives Ekstrand, Zabzine '11
- Nonassociative generalizations, Nambu-Poisson structure
- Applications to string theory and M-theory

• Quantization problem

Deformation, Geometric, Path integral $\cdots$ anomaly terms in current algebras.

• Poisson vertex algebras and vertex algebras Li '02, Sole, Kac, Wakimoto '10, Ekstrand, Heluani, Zabzine '11, Hekmati, Mathai '12

• higher structure WZW models?

• BV Algebras in String Field Theory Hata, Zwiebach '93

• Higher groupoids and higher category

Thank you!