Hamiltonian Lie algebroids: physical applications and cohomological descriptions

1. Introduction

- Momentum maps

(M, ω) symplectic manifold

Lie group $G \curvearrowright M$ induces $\rho : \mathfrak{g} \to TM$

$\mathfrak{g}$: Lie algebra of G

map $\mu : M \to \mathfrak{g}^*$ is called a momentum map

1. $d\mu(e) = -\iota_{\rho(e)}\omega$ $\qquad \langle \mu, e \rangle = \mu(e)$

$\langle \, , \rangle$ pairing of $\mathfrak{g}$ & $\mathfrak{g}^*$ $\qquad e \in \mathfrak{g}$

$\rho : \mathfrak{g} \to TM$: infinitesimal action of $\mathfrak{g}$

2. equivariant

$$\langle \mu(g \cdot p), e \rangle = \langle \mu(p), Ad_{g^{-1}} \cdot e \rangle$$

for $\forall p \in M$, $\forall e \in \mathfrak{g}$

infinitesimally

$$\mathcal{L}_{\rho(e_1)} \mu(e_2) = \mu(ad_{e_1}(e_2)) = \mu([e_1, e_2])$$

for $\forall e_1, e_2 \in \mathfrak{g}$

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1. Generalization to Lie algebroid
   Lie gpd
2. $E = M \times \mathfrak{g}$

$$\mu \in \Gamma(E^*) = \Gamma(M \times \mathfrak{g}^*)$$

$\to$ generalization to A : Lie algebroid

HLA Blohmann, Weinstein '18(24) Kotov-Strobl '16

2 Preliminary

Def A Lie algebroid E is a vector bundle over a smooth manifold M, equipped with a bundle map

$\rho : E \to TM$

and a Lie bracket,

$[-,-] : \Gamma(E) \times \Gamma(E) \to \Gamma(E)$

satisfying the Leibniz rule

$[e_1, fe_2] = f[e_1, e_2] + (\rho(e_1)f)e_2$

for all $e_1, e_2 \in \Gamma(E)$, $f \in C^\infty(M)$.

$(A, \rho, [-,-])$ is called a Lie algebroid.

Ex. $G \curvearrowright M$ Lie group action

induces $\rho : M \times \mathfrak{g} \to TM$ infinitesimal Lie algebra action

s.t. $[\rho(e_1), \rho(e_2)] = \rho([e_1, e_2])$ $e_1, e_2 \in \mathfrak{g}$ id,

$(E = M \times \mathfrak{g}\ [-,-]\ \rho)$ is called an action Lie algebroid

Def A Lie algebroid differential (E-differential) ${}^{E}d$ on $\Gamma(\Lambda^{\bullet}E^*)$

For $\eta \in \Gamma(\Lambda^m E^*)$

$\mathcal{L}_{\rho(e_i)}\eta$

$$ {}^{E}d\eta(e_1, \cdots, e_{m+1}) := \sum_{i=1}^{m+1} (-1)^{i-1} \rho(e_i)\eta(e_1 \cdots \check{e}_i \cdots e_{m+1}) + \sum_{1\le i<j\le m+1} (-1)^{i+j} \eta([e_i, e_j], e_1 \cdots \check{e}_i \cdots \check{e}_j \cdots e_{m+1}) $$

$({}^{E}d)^2 = 0$

Def A connection on E is a $\mathbb{R}$-linear map $\nabla : \Gamma(E) \to \Gamma(E \otimes T^*M)$ satisfying

$$\nabla(fe) = f(\nabla e) + df \otimes e$$

for $e \in \Gamma(E)$, $f \in C^\infty(M)$.

Def Let E be a Lie algebroid.
An E-connection on E' is a $\mathbb{R}$-linear map
$^E\nabla : \Gamma(E') \to \Gamma(E' \otimes E^*)$ satisfying
$^E\nabla_e(fe') = f(^E\nabla_e e') + (\rho(e)f)e'$
$e \in \Gamma(E)$, $e' \in \Gamma(E')$ $f \in C^\infty(M)$

Remark The connection ∇ is TM-connection $\nabla = {}^{TM}\nabla$.

Def Let E be a Lie algebroid equipped with a connection ∇, E'
The basic E-connection on TM is
$$^E\nabla^{bas}_e v := \mathcal{L}_{\rho(e)} v + \rho(\nabla_v e) = [\rho(e), v] + \rho(\nabla_v e)$$

Remark The basic E-connection on E
$$^E\nabla^{bas}_e e' := \nabla_{\rho(e')} e + [e, e']$$

2. Hamiltonian Lie algebroids

<u>Def</u> Let (M, ω) be a pre-symplectic manifold.

$\omega \in \Omega^2(M)$, $d\omega = 0$

Let $E \to M$ be a Lie algebroid $([-,-], \rho)$

(S1) E is pre-symplectically anchored if

$${}^E\nabla \omega = 0$$

(S2) A section $\Gamma(A^*)$ is a momentum section if it satisfies

$$(\nabla \mu)(e) = - \iota_{\rho(e)} \omega$$

(S3) μ is called bracket compatible if

$$({}^E d \mu)(e_1, e_2) = - \iota_{\rho(e_1)} \iota_{\rho(e_2)} \omega$$

for $e, e_1, e_2 \in \Gamma(E)$

If (E, M, ω) has (∇, μ) satisfying (S1)(S2)(S3), E is called <u>Hamiltonian</u>.

<u>Prop</u> Let $E = M \times \mathfrak{g}$ be an action Lie algebroid. Then momentum section μ is a momentum map. Hamiltonian Lie algebroid is a Ham. $\mathfrak{g}$-space

Proof We can choose $\nabla = d$ deRham differential

$(S1) \iff d\omega = 0$

$(S2) \iff d\mu(e) = -\iota_{\rho(e)}\omega$

$\mu(e)$ hamiltonian

$(S3)$ under $(S2) \iff \mathcal{L}_{\rho(e_1)}\mu(e_2) = \mu([e_1, e_2])$

equivariant

$\square$

(S3)

$(\boxed{\mathrm{E}} d\mu)(e_1, e_2)$

$= \mathcal{L}_{\rho(e_1)}\mu(e_2) - \mathcal{L}_{\rho(e_2)}\mu(e_1) - \mu([e_1, e_2])$

$= -\iota_{\rho(e_1)}\iota_{\rho(e_2)}\omega$ —— ①

$(S2) \times \iota_\rho$

$\mathcal{L}_{\rho(e_1)}\mu(e_2) = -\iota_{\rho(e_1)}\iota_{\rho(e_2)}\omega$ —— ②

① + ②

$\mathcal{L}_{\rho(e_2)}\mu(e_1) = -\mu([e_1, e_2])$

$= \mu([e_2, e_1])$ //

3. Examples in physics

__Example__ Hamiltonian mechanics

M : smooth manifold $\qquad g$: metric

$T^*M \quad (x^i, p_i) \qquad \omega = dx^i \wedge dp_i$

ω_{can}

E vector bundle on M

$$\begin{array}{c} E \\ \downarrow \\ M \end{array}$$

$$\begin{tikzcd} pr^*E \arrow[r, "pr^*"] \arrow[d] & E \arrow[d] \\ T^*M \arrow[r, "pr"'] & M \end{tikzcd}$$

Hamiltonian $H := \frac{1}{2} g^{ij}(x) p_i p_j$

$g^{ij}(x)$: g^{-1}

constraints $\quad G = G_a(x,p) e^a \in \Gamma(pr^*E^*)$

$G_a := \rho_a^i(x) p_i \qquad e^a \in \Gamma(E^*)$

consistency conditions

① $\{H, H\} = 0$

② $\{H, G_a\} = \gamma_a^b(x,p) G_b$

③ $\{G_a, G_b\} = C_{ab}^c(x) G_c$

$\Big\} \; ☆$

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③ We obtain two operation on E

$\rho := \rho_a^i(x) e^a \partial_i$: $E \to TM \qquad$ anchor

$[e_a, e_b] := C_{ab}^c(x) e_c \qquad e_a \in \Gamma(E)$

③ gives $[\rho(e_1), \rho(e_2)] = \rho([e_1, e_2])$

E is an almost Lie algebroid

Jacobi of ③ $\Longleftrightarrow$ Jacobiator of $[-,-] \in \mathrm{Ker}(\mathrm{Im}\,\rho)$

bilinear bracket

$\Longleftarrow$ E is a Lie algebroid
Courant algebroid
(L_∞ -algebroid) Leibniz algebroid

Assume that E is a Lie algebroid.
Then $[-,-]$ is a Lie bracket.

② $\gamma^b{}_a(x,p) = \omega_a^{bi}(x) p_i$ by order counting

$G'_a = M_a^b(x) G_b$

$M_a^b(x)$ transition function of E

$\omega^b_{ai} := g_{ij} \omega_a^{bj}$ $\quad \omega := \omega^b_{ai} dx^i$

$$\omega' = M^{-1} \omega M + M^{-1} dM$$

is a connection 1-form $\quad \nabla = d + \omega$

${}^E\nabla$: basic E-connection on TM

② Jacobi $\quad {}^E\nabla g = 0$

Th If $E \to M$ is a Lie algebroid with ∇ and satisfies ${}^E\nabla g = 0$

⊛ is satisfied.

Remark $\quad E \to (M, g)$ $\quad$ Killing Lie algebroids

3-2 EM fields $\quad$ $U(1)$-connection $\begin{array}{c} U(1)\to L \\ \downarrow \\ M \end{array}$

$$A = A_i(x)dx^i \quad U(1)\text{-connection 1-form on } M$$

$$p'_i = p_i - A_i$$

$$H' = \frac{1}{2} g^{ij}(p_i - A_i)(p_j - A_j) \overset{+V(x)}{} = \frac{1}{2} g^{ij} p'_i p'_j + V(x)$$

$$G'_a = \rho^i{}_a(x)(p_i - A_i) + \mu_a(x) = \rho_a^i p'_i + \mu_a$$

$$\mu = \mu_a(x) e^a \in \Gamma(E^*)$$

Then
$$\{x^i, x^j\} = 0$$
$$\{x^i, p'_j\} = \delta^i{}_j$$
$$\{p'_i, p'_j\} = F_{ij}(x)$$

$F = dA \in \Omega^2(M)$
obviously $dF = 0$

F pre-symplectic form on M

We impose,

$$(\ast\ast)\begin{cases} \{H', H'\} = 0 \\ \{H', G'_a\} = g^{ij} \omega^b_{aj}(x) p'_j G'_b \\ \{G'_a, G'_b\} = C^c_{ab}(x) G'_c \end{cases}$$

Assume that E is a Lie algebroid $\quad E$

$\omega^b_{ai} dx^i$ connection 1-form $\quad \nabla$

under this assumption.

new ones $\quad \mu, \ F = dA$

Th [NI'19]

Let E is a Killing Lie algebroid.

(E Lie algebroid + ${}^E\nabla g = 0$)

Then Iff E is Hamiltonian Lie algebroid

($\mu \in \Gamma(E^*)$ is a momentum section)

(S1)(S2)(S3)

$\bigstar\bigstar$ is satisfied, where $\omega = F$

4. Cohomological description of HLA

BFV formalism

$N = T^*E[1]$ with $\omega_{N\,can}$ $|\omega_{N\,can}| = 0$

(x^i, p_i, c^a, b_a) $= \delta x^i \wedge \delta p_i + \delta c^a \wedge \delta b_a$

$\quad 0 \quad 0 \quad 1 \quad -1$

BFV charge & BFV Ham degree $(1,0)$

S_{BFV} $\qquad H_{BFV}$

s.t. (※) $\{S_{BFV}, S_{BFV}\} = \{S_{BFV}, H_{BFV}\} = \{H_{BFV}, H_{BFV}\} = 0$

$$S_{BFV} = \underbrace{c^a G_{Ta} - \frac{1}{2} C_{ab}^{c} c^a c^b b_b}_{\text{Lie algebroid}} + \cdots$$

$$H_{BFV} = H + \underbrace{\omega_a^{bi} p_i}_{\gamma} c^a b_b + \cdots$$

$p^\nabla_i := p_i - \omega_{ai}^{b} c^a b_b$ covariantized momentum

$$= \frac{1}{2} g^{ij} p^\nabla_i p^\nabla_j + \frac{1}{2} U_{ab}^{cd}(x) c^a c^b b_c b_d$$

Def $U \in \Gamma(\wedge^2 E^* \otimes \wedge^2 E)$ (2,2) tensor

Th If U satisfies

(NI-stab) $\iota_S \langle U, \rho \rangle = ES$

$E\nabla U + ETU = 0$

(※) holds.

② Geometric quantities

$v, v' \in \mathfrak{X}(M)$, $e, e' \in \Gamma(E)$

curvature $R \in \Omega^2(M, E \otimes E')$

$$R(v,v') := [\nabla_v, \nabla_{v'}] - \nabla_{[v,v']}$$

(E-curvature ${}^E R \in \Gamma(\Lambda^2 E^* \otimes E \otimes E')$

$${}^E R(e,e') := [{}^E\nabla_e, {}^E\nabla_{e'}] - {}^E\nabla_{[e,e']} \quad)$$

E-torsion ${}^E T \in \Gamma(\Lambda^2 E^* \otimes E)$

$${}^E T(e,e') := \nabla_{\rho(e)} e' - \nabla_{\rho(e')} e - [e,e']$$

basic curvature ${}^E S \in \Omega^1(M, \Lambda^2 E^* \otimes E)$

$$ {}^E S(e,e') := [e, \nabla e'] - [e', \nabla e] - \nabla [e,e'] - \nabla_{\rho(\nabla e)} e' + \nabla_{\rho(\nabla e')} e$$

$$= (\nabla^{A}T + 2\,\mathrm{Alt}\, \iota_\rho R)(e,e')(v)$$

$$= [e, \nabla_v e'] - [e', \nabla_v e] - \nabla_v [e,e'] - \nabla_{{}^E\nabla^{bas}_e v} e' + \nabla_{{}^E\nabla^{bas}_{e'} v} e$$

Properties

$$\rho \circ {}^E\nabla^{bas} = {}^E\nabla^{bas} \circ \rho$$

$$ {}^E R = \rho \circ {}^E S$$

$$ {}^E\nabla^{bas}\, {}^E S = 0 \qquad \text{"Bianchi identity"}$$

$$[\nabla, {}^E\nabla]\, \alpha \sim \langle {}^E S, \alpha \rangle$$

<u>Cor</u>, If $^{E}S = 0$ i.e. E is a Cartan LA,
$U = 0$ is a solution
(★★★) has a solution.

BV AKSZ

$\mathcal{M} = \mathrm{Map}(T[1]\mathbb{R}, N)$ $T[1]\mathbb{R} \ni (t, \theta)$

$$\omega_{BV} = \int_{T[1]\mathbb{R}} dt d\theta \, \omega_{BFV}$$

$$S_{BV} = \int_{T[1]\mathbb{R}} dt d\theta \left(p_i dx^i + b_a dc^a - (S_{BFV} + \theta H_{BFV}) \right)$$

Grigoriev–Damgard AKSZ
BFV to BV

$$(S_{BV}, S_{BV})_{\mathcal{M}} = 0$$

<u>Remark</u> HLA

$$\langle {}^{E}S, \mu \rangle = 0$$

Other examples

1. 2DGNLSM with boundary
2. 2D g PSM

HLA on Poisson Blohmann Ronchi Weinstein '23

HLA on pre multi-symp Hirota-NI '22

n D GNLSM with WZ term Hull-Spence '89

Callies, Fregier, Rogers Zambon '16

HLA on CA NI '21

Dirac '23

HLA over multisy Hirota, NI '22

A momentum map is a Poisson map

$$\mu : M \to \mathfrak{g}^*$$

$$\pi \qquad \pi_{KKS}$$

A momentum section is not a Poisson map.

$$\mu : M \to A^*$$

$$\pi_M \qquad \tilde{\pi}_M + \pi_A$$

$\Longrightarrow$ generalizations Hirota NI '24