Chiral Fermion in AdS(dS) Gravity

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§1. Introduction

- 一般相対論の Palatini formalism

vierbein $e_\mu{}^a$ と spin connection $\omega_\mu{}^{ab}$ を独立な場として扱う。

$$\mathcal{L}_{\mathrm{GRAV}} = -\frac{e}{16\pi G}\left(R + \Lambda\right).$$

$$R_{\mu\nu ab} \equiv \partial_\mu \omega_{\nu ab} - \partial_\nu \omega_{\mu ab} - \omega_{\mu ac}\omega_{\nu cb} + \omega_{\nu ac}\omega_{\mu cb}, \qquad e = \det(e_{\mu a}).$$

- 一般相対性理論のゲージ理論的構成

$(e_\mu{}^a, \omega_\mu{}^{ab})$ をPoincaré 群のゲージ場と考えて重力理論を構成する。

cf. Poincaré gauge theory, 3D Chern-Simons gravity, BF gravity, Ashtekar formalism, $\cdots$

- (Anti) de Sitter Gravity (MMSW Gravity)

MacDowell and Mansouri '77, West '78, Stelle and West '79, Fukuyama '83

${e_\mu}^a$ と ${\omega_\mu}^{ab}$ を同じ multiplet に組む。

$$ {\omega_\mu}^{AB} = \begin{cases} {\omega_\mu}^{ab} & \text{if } A = a, B = b, \\ {\omega_\mu}^{a5} \quad \sim {e_\mu}^a & \text{if } b = 5, \end{cases} $$

$$ A, B = 1, 2, 3, 4, 5, \; a, b = 1, 2, 3, 4. $$

${\omega_\mu}^{AB}$: $SO(2,3)$(anti de Sitter 群) または $SO(1,4)$(de Sitter 群) のゲージ場

次に、ゲージ群を $SO(1,3)$ に破って重力理論を導出

AdS(dS) gravity

4次元 $SO(2,3)$ or $SO(1,4)$ ゲージ理論 $\xrightarrow{\text{break}}$ Einstein 重力理論

- metric $g_{\mu\nu}$ の起源
- Cosmological Constant: $\Lambda \sim \frac{1}{l^2}$ （l: 破れのスケール）

$SO(2,3) \Longrightarrow$ negative, $SO(1,4) \Longrightarrow$ positive

問題

Weyl, Majorana fermion が作れない。

$SO(2,3)$, $SO(1,4)$ の表現には Weyl fermion が存在しない。

$SO(1,4)$ の表現には Majorana fermion が存在しない。$SO(2,3)$ Majorana fermion 条件は action と整合しない。

Kugo, Townsend '82

目的

4D AdS(dS) gravity に Weyl, Majorana fermion を入れる。

結果

4D AdS(dS) gravity に Weyl, Majorana fermion を導入できる。

$SO(2,3)$ or $SO(1,4)$ Dirac fermion で、破ったときにそれぞれ $SO(1,3)$ Weyl fermion, $SO(1,3)$ Majorana fermion となる場を構成した。

§2. (Anti) de Sitter Gravity in Four Dimensions

4D spacetime でゲージ群 $SO(2,3)$ or $SO(1,4)$ のゲージ場 $\omega_{\mu AB}$ のゲージ理論を作る。時空の metric は導入しない。

compensator field (Higgs 場) $Z_A = Z_A(x)$ と補助場 $\sigma(x)$ を導入し $SO(1,3)$ に破る。

$SO(2,3)$ (AdS)

A field strength $R_{\mu\nu AB}$ takes the form

$$R_{\mu\nu AB} = \partial_\mu \omega_{\nu AB} - \partial_\nu \omega_{\mu AB} - \omega_{\mu AC}\omega_{\nu CB} + \omega_{\nu AC}\omega_{\mu CB}.$$

We construct an $SO(2,3)$ invariant action

AdS Gravity

$$\begin{aligned} S_{\rm GRAV} &= \int d^4x \mathcal{L}_{\rm GRAV} \\ &= \int d^4x \epsilon^{ABCDE} \epsilon^{\mu\nu\rho\sigma} \left(\frac{Z_A}{il}\right) \left[\left(\frac{1}{16g^2}\right) R_{\mu\nu BC} R_{\lambda\rho DE} \right. \\ & \qquad \left. + \sigma(x) \left\{ \left(\frac{Z_F}{il}\right)^2 - 1 \right\} D_\mu Z_B D_\nu Z_C D_\rho Z_D D_\sigma Z_E \right], \end{aligned}$$

g is a coupling constant and l is a real constant.

The equation of motion for Z_A is

$$(Z_A)^2 = -l^2.$$

If we take a solution breaking the $SO(2,3)$ symmetry

$$Z_A = (0, 0, 0, 0, il),$$

this breaking derives the vierbein $e_{\mu a}$,

$$D_\mu Z_A \equiv (\partial_\mu \delta_{AB} - \omega_{\mu AB}) Z_B = \begin{cases} -i\omega_{\mu a 5} l \equiv e_{\mu a} & \text{if} A = a, \\ 0 & \text{if} A = 5, \end{cases}$$

$\mathcal{L}_{\mathrm{GRAV}}$ takes the Einstein gravity form

$$\mathcal{L}_{\mathrm{GRAV}} = \partial_\mu \mathcal{C}^\mu - \frac{e}{16\pi G}\left(\mathring{R} + \frac{6}{l^2}\right).$$

Here, $\partial_\mu \mathcal{C}^\mu$ is the topological Gauss-Bonnet term. G is the gravitational constant derived from $16\pi G = g^2 l^2$.

$SO(1,4)$ **(dS)**

We construct an $SO(1,4)$ invariant action

dS Gravity

$$
\begin{aligned}
S_{\mathrm{GRAV}} &= -\int d^4x \mathcal{L}_{\mathrm{GRAV}} \\
&= -\int d^4x \epsilon^{ABCDE} \epsilon^{\mu\nu\rho\sigma} \left(\frac{Z_A}{l}\right) \left[\left(\frac{1}{16g^2}\right) R_{\mu\nu BC} R_{\lambda\rho DE} \right. \\
&\quad \left. + \sigma(x) \left\{ \left(\frac{Z_F}{l}\right)^2 - 1 \right\} D_\mu Z_B D_\nu Z_C D_\rho Z_D D_\sigma Z_E \right].
\end{aligned}
$$

The equation of motion for Z_A is $(Z_A)^2 = l^2$. We break the $SO(1,4)$ group to the local Lorentz group $SO(1,3)$ as

$$Z_A = (0,0,0,0,l).$$

This breaking leads to

$$D_\mu Z_A = (\partial_\mu \delta_{AB} - \omega_{\mu AB}) Z_B = \begin{cases} -\omega_{\mu a 5} l \equiv e_{\mu a} & \text{if} A = a. \\ 0 & \text{if} A = 5. \end{cases}$$

$\mathcal{L}_{\mathrm{GRAV}}$ takes the form

$$\mathcal{L}_{\mathrm{GRAV}} = \partial_\mu \mathcal{C}^\mu - \frac{e}{16\pi G}\left(\mathring{R} - \frac{6}{l^2}\right).$$

§3. Gamma Matrix

Gamma Matrix Γ_A は $SO(1,3)$ の γ_A, $SO(2,3)$ の ${\gamma^{(AdS)}}_A$, $SO(1,4)$ の ${\gamma^{(dS)}}_A$ でそれぞれ別のものにしておく。（あとで関係づける）

すべて

$$\{\Gamma_A, \Gamma_B\} = 2\delta_{AB},$$
$${\Gamma_A}^\dagger = \Gamma_A.$$

を満たす。

Dirac (Pauli) basis では、

$$\gamma_A^T = \begin{cases} \gamma_A & \text{if} \quad A = 2,4,5, \\ -\gamma_A & \text{if} \quad A = 1,3. \end{cases}$$

§4. Dirac Fermion

Fukuyama '83

Let ψ be an $SO(2,3)$($SO(1,4)$) Dirac fermion.

$SO(2,3)$ (AdS)

An $SO(2,3)$ invariant Dirac spinor action is defined as

$$\mathcal{L}_{\rm DIRAC} = \epsilon^{ABCDE}\epsilon^{\mu\nu\rho\sigma}\bar{\psi}\left(iS_{AB}\frac{\overleftrightarrow{D}_\mu}{3!} - i\lambda\frac{Z_A}{il}\frac{D_\mu Z_B}{4!}\right)\psi D_\nu Z_C D_\rho Z_D D_\sigma Z_E$$

where $S_{AB} \equiv \frac{1}{4i}[\gamma^{(AdS)}{}_A, \gamma^{(AdS)}{}_B]$, and λ is a mass.

$\bar{\psi} \equiv \psi^\dagger \gamma^{(AdS)}{}_5 \gamma^{(AdS)}{}_4$

By the symmetry breaking $Z^A = (0,0,0,0,il)$ from $SO(2,3)$ to $SO(1,3)$, $\mathcal{L}_{\rm DIRAC}$ reduces to the Dirac action in the four-dimensional curved spacetime

$$\mathcal{L}_{\rm DIRAC} = -e\bar{\psi}\left(\gamma_a e^{\mu a}\overleftrightarrow{D}_\mu + \lambda\right)\psi, = -e\bar{\psi}\left(\frac{1}{2}e^{\mu a}\left(\gamma_a\overrightarrow{D}_\mu - \overleftarrow{D}_\mu\gamma_a\right) + \lambda\right)\psi,$$

$\bar{\psi} = \psi^\dagger\gamma_4$.

where $\gamma_a \equiv i\gamma^{(AdS)}{}_5\gamma^{(AdS)}{}_a$, $\gamma_5 \equiv \gamma^{(AdS)}{}_5$.

$$\gamma^{(AdS)}{}_a \equiv -i\gamma_5\gamma_a,$$
$$\gamma^{(AdS)}{}_5 \equiv \gamma_5.$$

$SO(1,4)$ **(dS)**

In the dS gravity, we consider an $SO(1,4)$ invariant Dirac spinor action

$$\mathcal{L}_{\text{DIRAC}} = -\epsilon^{ABCDE}\epsilon^{\mu\nu\rho\sigma}\bar{\psi}\left(\frac{Z_A}{l}{\gamma^{(dS)}}_B\frac{\overleftrightarrow{D}_\mu}{3!}+\lambda\frac{Z_A}{l}\frac{D_\mu Z_B}{4!}\right)\psi D_\nu Z_C D_\rho Z_D D_\sigma Z_E$$

which is a slightly different form from the $SO(2,3)$ case. Here, $\bar{\psi}=\psi^\dagger {\gamma^{(dS)}}_4$.

By the symmetry breaking $Z^A=(0,0,0,0,l)$ from $SO(1,4)$ to $SO(1,3)$, $\mathcal{L}_{\text{DIRAC}}$

reduces to the Dirac action in the four-dimensional curved spacetime

$$\mathcal{L}_{\rm DIRAC} \;=\; -e\bar{\psi}\left(\gamma_a e^{\mu a}\overleftrightarrow{D}_\mu + \lambda\right)\psi, = -e\bar{\psi}\left(\frac{1}{2}e^{\mu a}\left(\gamma_a\overrightarrow{D}_\mu - \overleftarrow{D}_\mu\gamma_a\right) + \lambda\right)\psi,$$

where $\bar{\psi} = \psi^\dagger\gamma_4$ and

$$\gamma^{(dS)}{}_A \equiv \gamma_A.$$

§5. Weyl Fermion

symmetry を破ったときに 4D Weyl fermion となる $SO(2,3)$ または $SO(1,4)$ spinor を作る。

1, SO(2,3)(SO(1,4)) covariant
2, 破ったとき chiral projections $\frac{1\pm\gamma_5}{2}$ になる
operator $P_\pm$ を作る。

$SO(2,3)$ (AdS)

Let ψ be an $SO(2,3)$ Dirac spinor. We introduce a projection operator,

$$P_\pm \equiv \frac{1}{2}\left(1 \pm \sqrt{-\frac{l^2}{Z^2}}\frac{Z_A \gamma^{(AdS)}{}_A}{il}\right),$$

which is $P_\pm^2 = P_\pm$ and $P_+ P_- = 0$. We define

$$\psi_\pm \equiv P_\pm \psi.$$

If we break the $SO(2,3)$ symmetry

$$Z^A = (0, 0, 0, 0, il),$$

$P_\pm$ reduces to the chiral projections $\mathring{P}_\pm$

$$P_\pm \longrightarrow \mathring{P}_\pm = \frac{1 \pm \gamma^{(AdS)}{}_5}{2} = \frac{1 \pm \gamma_5}{2}.$$

Then, $\psi_\pm$ becomes Weyl spinors $\mathring{\psi}_\pm$

$$\psi_\pm \longrightarrow \mathring{\psi}_\pm = \mathring{P}_\pm \psi,$$

respectively, which have definite chirality. We can construct an $SO(2,3)$ invariant action by modifying the action for a Dirac fermion,

$$\mathcal{L}_{\mathrm{WEYL}} = \epsilon^{ABCDE}\epsilon^{\mu\nu\rho\sigma}\bar{\psi}_+ \left(iS_{AB}\frac{\overleftrightarrow{D}_\mu}{3!} - i\lambda\frac{Z_A}{il}\frac{D_\mu Z_B}{4!} \right) \psi_+ D_\nu Z_C D_\rho Z_D D_\sigma Z_E$$

The action becomes a $SO(1,3)$ massless Weyl fermion action by breaking the symmetry

$$\mathcal{L}_{\mathrm{WEYL}} = -e\bar{\mathring{\psi}}_+ \left(\gamma_a e^{\mu a}\overleftrightarrow{D}_\mu + \lambda\right)\mathring{\psi}_+ = -e\bar{\mathring{\psi}}_+ \left(\gamma_a e^{\mu a}\overleftrightarrow{\mathring{D}}_\mu\right)\mathring{\psi}_+,$$

$SO(1,4)$ **(dS)**

Let ψ be an $SO(1,4)$ Dirac spinor. In the $SO(1,4)$ case, we introduce

$$P_{\pm} \equiv \frac{1}{2}\left(1 \pm \sqrt{\frac{l^2}{Z^2}}\frac{Z_A \gamma^{(dS)}{}_A}{l}\right),$$

which is $P_{\pm}^2 = P_{\pm}$ and $P_+ P_- = 0$. We define

$$\psi_{\pm} \equiv P_{\pm}\psi.$$

If we break the $SO(1,4)$ symmetry as

$$Z^A = (0,0,0,0,l),$$

$P_\pm$ reduces to chiral projections $\mathring{P}_\pm$

$$P_\pm \longrightarrow \mathring{P}_\pm = \frac{1 \pm \gamma^{(dS)}{}_5}{2} = \frac{1 \pm \gamma_5}{2}.$$

Then $\psi_\pm$ becomes Weyl fermions $\mathring{\psi}_\pm$,

$$\psi_\pm \longrightarrow \mathring{\psi}_\pm = \mathring{P}_\pm \psi,$$

respectively, which have definite chirality.

We can construct $SO(1,4)$ invariant action by modifying the Dirac action

$$\mathcal{L}_{\text{WEYL}} = -\epsilon^{ABCDE}\epsilon^{\mu\nu\rho\sigma}\bar{\psi}_{+}\left(\frac{Z_A}{l}\gamma^{(dS)}{}_{B}\frac{\overleftrightarrow{D}_{\mu}}{3!}+\lambda\frac{Z_A}{l}\frac{D_\mu Z_B}{4!}\right)\psi_{+} \times D_\nu Z_C D_\rho Z_D D_\sigma Z_E.$$

The action becomes an $SO(1,3)$ massless Weyl fermion action by breaking the symmetry

$$\mathcal{L}_{\text{WEYL}} = -e\bar{\mathring{\psi}}_{+}\left(\gamma_a e^{\mu a}\overleftrightarrow{D}_{\mu}+\lambda\right)\mathring{\psi}_{+} = -e\bar{\mathring{\psi}}_{+}\left(\gamma_a e^{\mu a}\overleftrightarrow{\mathring{D}}_{\mu}\right)\mathring{\psi}_{+}.$$

§6. Majorana Fermion

$SO(1,3)$

4D Majorana fermion ψ_M

$$\psi_M = \psi_M^c \equiv C\bar{\psi}_M^T,$$

C is the charge conjugation in $SO(1,3)$. If we take the Dirac (Pauli) basis, C is

$$C = \gamma_2\gamma_4.$$

However, C is not covariant under either $SO(2,3)$ or $SO(1,4)$. ψ_M is not consistent with the $SO(2,3)$ ($SO(1,4)$) covariance.

If a 'charge conjugation' is defined, a Majorana fermion can be defined.

Conditions for $SO(2,3)$ or $SO(1,4)$ 'charge conjugation' $\tilde{C}$

1. $\tilde{C}^{-1}\gamma_A\tilde{C}$ is covariant under the symmetry to be consistent with the action.

$$\tilde{C}^{-1}\gamma_A\tilde{C} = \pm\gamma_A^T,$$

is sufficient where the signatures are the same for all A.

2. B defined by $B\psi_M^* = \tilde{C}\bar{\psi}_M^T$ must satisfy

$$B^*B = 1,$$

since a charge conjugation has a Z_2 symmetry. $(B = \gamma_2$ for $SO(1,3).)$

3. $\tilde{C}$ reduces to $C = \gamma_2\gamma_4$ by breaking the symmetry.

$SO(2,3)$ **(AdS)**

$$\tilde{C} = \gamma^{(AdS)}{}_2 \gamma^{(AdS)}{}_4 .$$

$SO(1,4)$ **(dS)**

$$\tilde{C} \equiv \left(\frac{Z_A \gamma^{(dS)}{}_A}{l} + \left| \sqrt{\frac{Z^2 - l^2}{l^2}} \right| i \right) \gamma^{(dS)}{}_2 \gamma^{(dS)}{}_4 \gamma^{(dS)}{}_5 .$$

$SO(2,3)$ **(AdS)**

The $SO(2,3)$ gamma matrices ${\gamma^{(AdS)}}_A$ are constructed as

$$
\begin{aligned}
{\gamma^{(AdS)}}_a &\equiv -i\gamma_5\gamma_a, \\
{\gamma^{(AdS)}}_5 &\equiv \gamma_5,
\end{aligned}
$$

From the condition 1, we have two candidates

$$
\begin{aligned}
C_1 &= {\gamma^{(AdS)}}_1{\gamma^{(AdS)}}_3{\gamma^{(AdS)}}_5, \\
C_2 &= {\gamma^{(AdS)}}_2{\gamma^{(AdS)}}_4.
\end{aligned}
$$

$C_2 = {\gamma^{(AdS)}}_2{\gamma^{(AdS)}}_4 = \gamma_2\gamma_4$ is equal to the $SO(1,3)$ charge conjugation Therefore C_2 satisfies the condition 2 and 3. Note that $C = C_2$ is not a charge conjugation in the $SO(2,3)$ representation. Therefore AdS 'Majorana' fermion ψ_M is defined by

$$\psi_M = \tilde{C}\bar{\psi}_M^T = C_2\bar{\psi}_M^T.$$

$SO(2,3)$ invariant AdS 'Majorana' fermion action

$$\mathcal{L}_{\text{MAJORANA}} \\ = \epsilon^{ABCDE}\epsilon^{\mu\nu\rho\sigma}\bar{\psi}_M\left(iS_{AB}\frac{\overleftrightarrow{D}_\mu}{3!} - i\lambda\frac{Z_A}{il}\frac{D_\mu Z_B}{4!}\right)\psi_M D_\nu Z_C D_\rho Z_D D_\sigma Z_E$$

Let us investigate the consistency of this action. Substituting $\psi_M = C_2\bar{\psi}_M^T$, to the right-hand of the action, we obtain

$$\epsilon^{ABCDE}\epsilon^{\mu\nu\rho\sigma}\left(\psi_M^T(\tilde{C}^T)^{-1}\right)\left(iS_{AB}\frac{\overleftrightarrow{D}_\mu}{3!}-i\lambda\frac{Z_A}{il}\frac{D_\mu Z_B}{4!}\right)\left(\tilde{C}\bar{\psi}_M^T\right)D_\nu Z_C D_\rho Z_D D_\sigma Z_E.$$

We can easily check that

$$= \mathcal{L}_{\text{MAJORANA}}.$$

Thus, the definition of the charge conjugation is consistent with the action.

If we break the $SO(2,3)$ symmetry by $Z_A = (0,0,0,0,il)$, the action reduces to an $SO(1,3)$ Majorana fermion action in the Einstein gravitational theory in four dimensions

$$\mathcal{L}_{\text{MAJORANA}} = -e\bar{\psi}_M\left(\gamma_a e^{\mu a}\overleftrightarrow{D}_\mu + \lambda\right)\psi_M.$$

$SO(1,4)$ **(dS)**

$$\gamma^{(dS)}{}_A \equiv \gamma_A$$

From the condition 1, we obtain two candidates

$$C_3 \equiv \gamma^{(dS)}{}_1 \gamma^{(dS)}{}_3,$$
$$C_4 \equiv \gamma^{(dS)}{}_2 \gamma^{(dS)}{}_4 \gamma^{(dS)}{}_5.$$

Condition 2

$B^*B = -1$: Neither C_3 nor C_4 can be defined as a consistent charge conjugation.

Now, we consider a third candidate:

$$C_5 \equiv \left(\frac{Z_A \gamma^{(dS)}{}_A}{l} + \left| \sqrt{\frac{Z^2 - l^2}{l^2}} \right| i \right) \gamma^{(dS)}{}_2 \gamma^{(dS)}{}_4 \gamma^{(dS)}{}_5 .$$

This satisfies the condition 1.

Condition 2. $B_5^* B_5 = 1$ $\left(B_5 = -\left(\frac{Z_A \gamma^{(dS)}{}_A}{l} + \left| \sqrt{\frac{Z^2 - l^2}{l^2}} \right| i \right) \gamma^{(dS)}{}_2 \gamma^{(dS)}{}_5 \right)$

Condition 3. $C_5 \longrightarrow \gamma^{(dS)}{}_2 \gamma^{(dS)}{}_4 = \gamma_2 \gamma_4 = C.$

A dS 'Majorana' spinor

$$\psi_M = C_5 \bar{\psi}_M^T .$$

$SO(1,4)$ invariant dS 'Majorana' fermion action

$$\mathcal{L}_{\text{MAJORANA}} = -\epsilon^{ABCDE}\epsilon^{\mu\nu\rho\sigma}\bar{\psi}_M\left(\frac{Z_A}{l}\gamma^{(dS)}{}_B\frac{\overleftrightarrow{D}_\mu}{3!}+\lambda\frac{Z_A}{l}\frac{D_\mu Z_B}{4!}\right)\psi_M \times D_\nu Z_C D_\rho Z_D D_\sigma Z_E$$

We can prove the consistency of the action for the charge conjugation C_5 similar to $SO(2,3)$ case.

$$-\epsilon^{ABCDE}\epsilon^{\mu\nu\rho\sigma}\left(\psi_M^T(C^T)^{-1}\right)\left(\frac{Z_A}{l}\gamma^{(dS)}{}_B\frac{\overleftrightarrow{D}_\mu}{3!}+\lambda\frac{Z_A}{l}\frac{D_\mu Z_B}{4!}\right)\left(C\bar{\psi}_M^T\right) \times D_\nu Z_C D_\rho Z_D D_\sigma Z_E.$$

We can easily check that

$$= \mathcal{L}_{\text{MAJORANA}}.$$

If we break the $SO(1,4)$ symmetry by $Z_A = (0,0,0,0,l)$, the action becomes the Majorana fermion action in the Einstein gravitational theory in four dimensions

$$\mathcal{L}_{\text{MAJORANA}} = -e\bar{\psi}_M \left(\gamma_a e^{\mu a} \overleftrightarrow{D}_\mu + \lambda \right) \psi_M,$$

§7. Summary and Discussion

対称性を破って重力理論になったとき、Weyl, Majorana fermion action となる AdS (dS) Gravity の action を作った。

New mechanism to derive a chiral fermion from a nonchiral fermion

Chiral symmetry and chiral anomaly

Z_A を dynamical にする？