Sigma Models with Nontrivial Flux and Generalized Geometry

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1. Introduction

String theory (M theory) with
nonzero fluxes
especially
nonzero NS flux $H \neq 0$
becomes more important
in the context of →

∘ $\boxed{\text{flux compactification}}$ 10D→4D

NS flux $H \neq 0$

RR fluxes $F \neq 0$ vacua

provide new terms in

Kähler & super potentials

in 4D theory

→ new phenomelogical models

∘ direct generalization

of $\boxed{\text{noncommutative geometry}}$

in string theory

$$\text{NCG} \begin{cases} H = db = 0 \\ b \neq 0 \quad \text{NS B-field} \end{cases}$$

'99 Seiberg-Witten

↓

GG $H = db \neq 0$

In order to discuss $H \neq 0$,
We need new geometry

$H=0$		$H \neq 0$
complex	$\longrightarrow$	generalized complex
Kähler	$\longrightarrow$	generalized Kähler
Calabi-Yau	$\longrightarrow$	generalized Calabi-Yau
⋮		⋮

generalized geometry
'02 Hitchin
'04 Gualtieri

Purpose of the project

is to investigate

(mathematical / theoretical) aspects of

generalized geometry

in {string / M, F} theory

→ Classification of string vacua

Landscape

→ mathematical analysis

→ M theory

In this talk,
We are interested in

world sheet (world volume) descriptions

or

sigma model (membrane)

of generalized geometry

§2 $N=(2,2)$ SUSY sigma model

§3 definition of generalized complex structure

§4 topological sigma model
topological membrane

type II superstring   4 directions

	non-topological	topological
worldsheet 2D	$N=(2,2)$ SUSY σ-model with $H \neq 0$ '84 Gates, Hull, Roček '04 Gualtieri	topological twist → topological σ-model with generalized CY (generalized A-model " B-model '03 Kapstin, Li '04 Zucchini ↕ '05 Pestun, Witten
target space 10D	$N=2$ superstring (SUGRA) $H \neq 0$ '05 Grana, Minasian, Petrini, Tomasiello et.al. Review '06 Graña	6D '03 Hitchin Hitchin functional (generalized) with SU(3)-structure '04 Dijkgraaf, Gukov, Neitzke, Vafa '04 Nekrasov '05 Pestun, Witten

∘ M-theory N=1

	non-topological	topological
world volume	3D supermembrane	? 3D topological 2-brane with G_2 structure '05 Bonelli, Zabzine
	2D N=1 String with G_2 structure '91 Howe, Papadopoulus '94 Shatashvili, Vafa	? 2D topological G_2 string '05 de Boer, Naqvi, Shomer
target space	11D SUGRA G_{MN} $A_{MNR} \Leftrightarrow$ M2-brane, M5-brane	7D Hitchin functional with G_2 structure '04 DGNV '04 Nekrasov

2. 2D $N=(2,2)$ SUSY Sigma Model with $H \neq 0$

'84 Gates, Hull, Roček

First we write manifest $N=(1,1)$ SUSY σ-model

$$S=\frac{1}{2}\int_{\Sigma} d^2\sigma d^2\theta \left(g_{ij}(\Phi)+b_{ij}(\Phi)\right) D_+\Phi^i D_-\Phi^j$$

$\Phi^i \equiv \phi^i + \theta^+\psi_+^i + \theta^-\psi_-^i + \theta^-\theta^+F^i$

$N=(1,1)$ superfield

$\phi^i : \underset{\text{2D mfd}}{\Sigma} \longrightarrow \underset{\text{target sp.}}{M}$

$H_{ijk} \equiv \frac{1}{3!}\partial_{[i} b_{jk]}$ NS flux

$g_{ij}=g_{ji}$ metric

$b_{ij}=-b_{ji}$ NS b-field

$D_\pm \equiv \frac{\partial}{\partial\theta^\pm} + i\theta^\pm\partial_\pm$ superderivative

S has manifest $N=(1,1)$ SUSY

$$\delta_1 \Phi^i \equiv \epsilon_1^+ D_+ \Phi^i + \epsilon_1^- D_- \Phi^i$$

If S has another SUSY

$$\boxed{\delta_2 \Phi^i \equiv \epsilon_2^+ D_+ \Phi^j J_{+j}^{i}(\Phi) + \epsilon_2^- D_- \Phi^j J_{-j}^{i}(\Phi)}$$

$(g, b, J_\pm)$ must have some conditions

then S has $N=(2,2)$ SUSY.

◎ Conditions

- If $H=0$

  M : Kähler $\Rightarrow$ $N=(2,2)$ SUSY

  $J_\pm$: complex structure

  $\omega_\pm \equiv g J_\pm$: Kähler form

- If $H \neq 0$

  ① $J_\pm$ are integrable almost complex str.

  ② g_{ij} is hermitian w.r.t J_+ & J_-

③ $\nabla^{(\pm)}_i J_{\pm}{}^j{}_k \equiv \partial_i J_{\pm}{}^j{}_k - \Gamma_{\pm}{}^j{}_{il} J_{\pm}{}^l{}_k - \Gamma_{\pm}{}^l{}_{ik} J_{\pm}{}^j{}_l = 0$

where

$$\Gamma_{\pm}{}^i{}_{jk} \equiv \Gamma^i_{jk} \pm \frac{1}{2} g^{il} H_{ljk}$$

covariant derivative with torsion

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①②③ is called

the bi-Hermitian structure.

Generally $\omega_{\pm} \equiv g J_{\pm}$ is not closed.

(non Kähler)

But this formulation is not so useful because the relation to the Kähler geometry is not manifest.

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bi-Hermitian structure ← '04 Gualtieri

$\Longleftrightarrow$ (twisted) generalized Kähler str.

3. Generalized Geometry
(generalized complex structure)

cf. We remember the definition of the complex structure on M

M : d-dim manifold

TM : tangent bundle

$J : TM \to TM$ map

$\left(J^i_j : X^i \to J^i_j X^j \right)$

s.t.

$$\boxed{J^2 = -1 \quad \text{—— ①}}$$

$\pm\sqrt{-1}$ are eigenvalues of J.

Let $\Pi_\pm$ be projections on $\pm\sqrt{-1}$ eigen bundle.

Integrable condition is

$$\boxed{\Pi_\mp [\Pi_\pm X, \Pi_\pm Y] = 0 \quad \text{—— ②}}$$

where $[\ ,\ ]$: Lie bracket

$X, Y \in TM$

◎ generalized complex structure

We replace TM to $TM \oplus T^*M$

$X, Y, \cdots$ vector field on TM

$\xi, \eta, \cdots$ 1-form on T^*M

$X+\xi$ formal sum $\in TM \oplus T^*M$

○ Inner product $\langle \ , \ \rangle$

$$\langle X+\xi, Y+\eta \rangle \equiv \frac{1}{2}(i_X \eta + i_Y \xi)$$

i_X, i_Y : interior product

i.e. $\left\langle X^i \frac{\partial}{\partial \phi^i} + \xi_i d\phi^i, Y^j \frac{\partial}{\partial \phi^j} + \eta_j d\phi^j \right\rangle = \frac{1}{2}(X^j \eta_j + \xi_i Y^i)$

$\{\phi^i\}$ local coordinate on M

○ signature (d, d)

○ on orthogonal coordinate

$$I = \begin{pmatrix} 0 & 1_d \\ 1_d & 0 \end{pmatrix}$$

◦ Courant bracket

$$[X+\xi, Y+\eta] \equiv [X,Y] + \mathcal{L}_X\eta - \mathcal{L}_Y\xi - \frac{1}{2}d_M(i_X\eta - i_Y\xi)$$

$\mathcal{L}_X, \mathcal{L}_Y$; Lie differential

d_M ; exterior derivative on M

◦ antisymmetric

◦ does not satisfy the Jacobi identity

Note) $\Leftrightarrow$ Dorfman bracket

$$(X+\xi)\circ(Y+\eta) \equiv [X,Y] + \mathcal{L}_X\eta - i_Y d\xi$$

◦ not antisymmetric

◦ satisfies the Jacobi identity

$$[X+\xi, Y+\eta] = (X+\xi)\circ(Y+\eta) - (Y+\eta)\circ(X+\xi)$$

<u>Def</u> generalized complex structure $\mathcal{J}$

$$\boxed{\mathcal{J} : TM \oplus T^*M \longrightarrow TM \oplus T^*M}$$

s.t.

$\mathcal{J}$ preserves $\langle\, ,\, \rangle \Leftrightarrow \mathcal{J}^* I \mathcal{J} = I$ — ⓪

$\mathcal{J}^2 = -1$ —— ①

$\Pi_{\mp}\left[\Pi_{\pm}(X+\xi), \Pi_{\pm}(Y+\eta)\right] = 0$ — ②

- $\Pi_{\pm} \equiv \frac{1}{2}(1 \mp \sqrt{-1}\,\mathcal{J})$
  projections on $\pm\sqrt{-1}$ eigen bundles
- $[\ ,\ ]$ Courant bracket

We represent 2×2 matrix

$$\mathcal{J} = \begin{array}{c} \begin{pmatrix} J^i{}_j & P^{ij} \\ Q_{ij} & -J^*{}_i{}^j \end{pmatrix} \begin{array}{c} TM \\ T^*M \end{array} \\ \begin{array}{cc} TM & T^*M \end{array} \end{array}$$

◎ local coordinate expression

$$\mathcal{J} = \begin{array}{cc} \begin{pmatrix} J & P \\ Q & -J^* \end{pmatrix} & \begin{array}{c} TM \\ T^*M \end{array} \\ \begin{array}{cc} TM & T^*M \end{array} & \end{array}$$

◎ ① $J^i{}_k J^k{}_j + P^{ik} Q_{kj} + \delta^i{}_j = 0$

$J^i{}_k P^{kj} + J^j{}_k P^{ki} = 0$

$Q_{ik} J^k{}_j + Q_{jk} J^k{}_i = 0$

$P^{ij} + P^{ji} = 0, \ Q_{ij} + Q_{ji} = 0$

② $A^{ijk} = B_i{}^{jk} = C_{ij}{}^k = D_{ijk} = 0$

$A^{ijk} = P^{il} \partial_l P^{jk} + (ijk \text{ cyclic})$

$B_i{}^{jk} = J^l{}_i \partial_l P^{jk} + P^{jl}(\partial_i J^k{}_l - \partial_l J^k{}_i)$
$+ P^{kl} \partial_l J^j{}_i - \partial_i J^j{}_l P^{lk}$

$C_{ij}{}^k = J^l{}_i \partial_l J^k{}_j - J^l{}_j \partial_l J^k{}_i - J^k{}_l \partial_i J^l{}_j$
$+ J^k{}_l \partial_j J^l{}_i + P^{kl}(\partial_l Q_{ij} + \partial_i Q_{jl}$
$+ \partial_j Q_{li})$

$D_{ijk} = J^l{}_i(\partial_l Q_{jk} + \partial_k Q_{lj}) + J^l{}_i(\partial_l Q_{ki}$
$+ \partial_i Q_{lk}) + J^l{}_k(\partial_l Q_{ij} + \partial_j Q_{li})$
$- Q_{jl} \partial_i J^l{}_k - Q_{kl} \partial_j J^l{}_i - Q_{il} \partial_k J^l{}_j$

- example

1. $\mathcal{J} = \begin{pmatrix} J & 0 \\ 0 & -J^* \end{pmatrix}$

   $\mathcal{J}$ is GCS $\Leftrightarrow$ J is complex str.

2. $\mathcal{J} = \begin{pmatrix} 0 & -Q^{-1} \\ Q & 0 \end{pmatrix}$

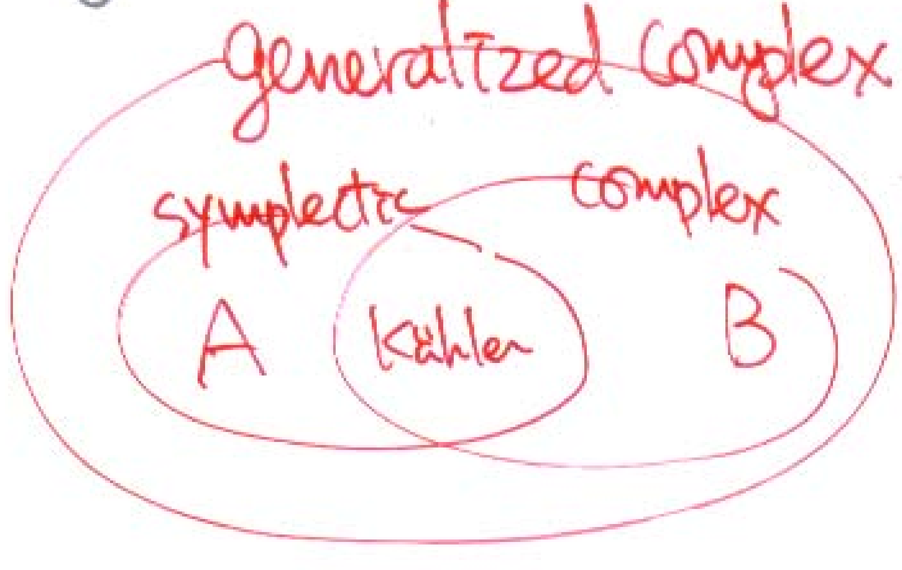
   $\mathcal{J}$ is GCS $\Leftrightarrow$ Q is symplectic str.

Q twisted Generalized Complex structure

We can generalize GCS by a closed 3-form H

$$[X+\xi, Y+\eta]_H \equiv [X+\xi, Y+\eta] + i_X i_Y H$$

In integrability condition 2, $[\ ,\ ]$ is replaced by $[\ ,\ ]_H$



then

(2) is modified

$$A_H{}^{ijk} = A^{ijk}$$

$$B_{Hi}{}^{jk} = B_i{}^{jk} + P^{il} P^{km} H_{ilm}$$

$$C_{Hij}{}^{k} = C_{ij}{}^{k} - J^l{}_i P^{km} H_{jlm} + J^l{}_j P^{km} H_{ilm}$$

$$\mathcal{D}_{Hijk} = \mathcal{D}_{ijk} - H_{ijk} + J^l{}_i J^m{}_j H_{klm} + J^l{}_j J^m{}_k H_{ilm} + J^l{}_k J^m{}_i H_{jlm}$$

$$A_H = B_H = C_H = \mathcal{D}_H = 0$$

② Automorphism on GCS

semi-direct product of

Diff(M) & b-transformation

(b-field transf.)

<u>Def</u> b-transformation $\left(\mathrm{Diff}(M) \ltimes \Omega^2_{\mathrm{closed}}(M)\right)$

for 2-form $b = b_{ij} d\phi^i d\phi^j$

$$\exp(b)(X+\xi) \equiv X + \xi + i_X b$$

1. If $d_M b = 0 \Rightarrow$

$$[\exp(b)(X+\xi), \exp(b)(Y+\eta)] = \exp(b)[X+\xi, Y+\eta]$$ covariant

2. $\hat{\mathcal{J}} \equiv \exp(-b)\, \mathcal{J} \exp(b)$ adjoint

- local coordinate expression

$$\begin{cases} \hat{J}^i{}_j = J^i{}_j - P^{ik} b_{kj} \\ \hat{P}^{ij} = P^{ij} \\ \hat{Q}_{ij} = Q_{ij} + b_{ik} J^k{}_j - b_{jk} J^k{}_i + P^{kl} b_{ki} b_{lj} \end{cases}$$

b-transformation $\begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}$

∘ If $H = d_M b \neq 0$

$[\exp(b)(X+\xi), \exp(b)(Y+\eta)]_{H+d_M b}$

$= \exp(b)[X+\xi, Y+\eta]_H$

$$H \xrightarrow{b\text{-transf}} \hat{H} = H - d_M b$$

+GCS is defined for $H \in H^3(M)$

② generalized Kähler structure

$\mathcal{J}_1, \mathcal{J}_2$: two (twisted) GCS

generalized Kähler

$\Leftrightarrow$ $G \equiv -\mathcal{J}_1\mathcal{J}_2$ is a positive definite metric on M.

example) Kähler manifold (g, J, ω)

$$\mathcal{J}_1 = \begin{pmatrix} J & 0 \\ 0 & -J^* \end{pmatrix}, \quad \mathcal{J}_2 = \begin{pmatrix} 0 & \omega^{-1} \\ \omega & 0 \end{pmatrix}$$

then $G = \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix}$ $\quad g = \omega J$

is metric.

We return to $N=(2,2)$ σ-model

$(g, b, J_\pm)$: bi-Hermitian str.

If We define

$$\mathcal{J}_{1/2} \equiv \frac{1}{2}\begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}\begin{pmatrix} J_+ \pm J_- & -(\omega_+^{-1} \mp \omega_-^{-1}) \\ \omega_+ \mp \omega_- & -({}^tJ_+ \pm {}^tJ_-) \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -b & 1 \end{pmatrix}$$

then $\quad \omega_\pm \equiv g J_\pm$

$$\boxed{\begin{array}{l} (g, b, J_\pm) \text{ bi-Hermitian} \\ \quad\Leftrightarrow \mathcal{J}_1, \mathcal{J}_2 \text{ generalized Kähler} \end{array}}$$

4. topological sigma model

We can construct a topological σ-model corresponding to $N(2,2)$ σ-model
in order to analyze
only a vacuum (geometry).

Q topological twist '91 Witten '03 Kapstin '04 Kapstin Li

changes in $N=(2,2)$ σ-model

$$\left\{\begin{array}{lll} P_1 & P_2 & \text{translation} \\ M & & \text{Lorentz} \\ Q_\pm, & \bar{Q}_\pm & \text{SUSY} \\ I_V & & \text{vector R-sym} \\ I_A & & \text{axial R-sym} \end{array}\right.$$

we replace Lorentz generator M to

$$M \to M' = M + \frac{1}{2} I_V \quad \text{(generalized) A-twist}$$

then BRST $\quad Q_B = \dfrac{Q_+ + \bar{Q}_+}{2} + \dfrac{Q_- - \bar{Q}_-}{2}$

$$M \to M' = M + \frac{1}{2} I_A \quad \text{(generalized) B-twist}$$

BRST $\quad Q_B = \dfrac{Q_+ + \bar{Q}_+}{2} + \dfrac{Q_- + \bar{Q}_-}{2}$

In order to make topological twist
I_V or I_A cannot have any anomaly

$\pm\sqrt{-1}$ eigenbundles w.r.t J_+
$T_+^{1,0} \oplus T_+^{0,1}$

$\pm\sqrt{-1}$ eigenbundles w.r.t J_-
$T_-^{1,0} \oplus T_-^{0,1}$

Anomaly cancellation condition

$U(1)_V$: $c_1(T_-^{1,0}) - c_1(T_+^{1,0}) = 0$

$U(1)_A$: $c_1(T_-^{1,0}) + c_1(T_+^{1,0}) = 0$

$\Longleftrightarrow$ generalized Calabi-Yau condition

where $c_1(T)$: 1-st Chern class

is a generalization of CY condition

Note) $b=0$
If $J_+ = J_-$ & A-twist $\to$ A-model
B-twist $\to$ B-model
'91 Witten

If $b=0$

A-twist (A-model) Kähler

- need not CY condition
- depends only on $\mathcal{J}=\begin{pmatrix} 0 & \omega^{-1} \\ \omega & 0 \end{pmatrix}$

B-twist (B-model) Complex

- does not have quantum correction
- depends only on $\mathcal{J}=\begin{pmatrix} J & 0 \\ 0 & -J^* \end{pmatrix}$

$\boxed{\text{But}}$

If $H=db\neq 0$

generalized A & B twist

do not have special properties

- depends on

$$\mathcal{J}=\begin{pmatrix} J & P \\ Q & -J^* \end{pmatrix}$$

Let us construct the topological twisted action in the case of $H \neq 0$

Procedure

- Change fermion statistics
  (A or B-twist)

  $\downarrow$

- Rewrite the action

$$S = S_{top} + \delta_B(*)$$

by decoupling the BRST exact terms

'97 Alexandrov, Kontsevich, Schwarz, Zaboronsky
'06 Bredthauer, Lindström, Persson, Zabzine

The answer is the Zucchini action
'04 Zucchini

$$\boxed{\begin{aligned} S_{top} &= \int_\Sigma B_i d\phi^i + J^i{}_j(\phi) B_i d\phi^j \\ &\quad + \frac{1}{2} P^{ij}(\phi) B_i B_j + \frac{1}{2} Q_{ij}(\phi) d\phi^i d\phi^j \\ &= \int B_i d\phi^i + \frac{1}{2}\int (d\phi^i \; B_i)\, \mathcal{J} \begin{pmatrix} d\phi^j \\ B_j \end{pmatrix} \end{aligned}}$$

ϕ : scalar superfield

$B_i \sim (g_{ij} + b_{ij}) d\phi^j$: auxiliary 1-form superfield

However

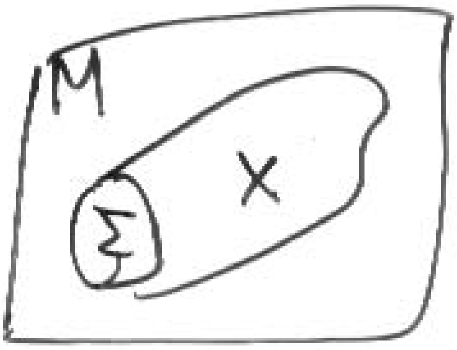
J.P.Q is GCS $\overset{\Longrightarrow}{\not\Longleftarrow}$ S_{top} is BRST inv.

@ If $H \neq 0$ twisted GCS

We add the WZ term

X : 3D mfd

s.t. $\partial X = \Sigma$



$$S'_{top} = S_{top} + \int_X H_{ijk}\, d\phi^i d\phi^j d\phi^k$$

J.P.Q, H is twisted GCS

$\overset{\Longrightarrow}{\not\Longleftarrow}$ S'_{top} is BRST inv.

Moreover

b-transformation

$$\begin{cases} \hat{\phi}^i = \phi^i \\ \hat{B}_i = B_i + b_{ij}\, d\phi^i d\phi^j \end{cases}$$

$$\hat{S}_{top} = S_{top} - \int_\Sigma b_{ij}\, d\phi^i d\phi^j$$

not b-inv.

② Our Purpose & Program

- Stop is not b-transformation inv.
  → b-transf. invariant formulation

- topological string → topological M theory   (string ↓ 2-brane)

- 「Courant bracket」 is not familiar to physicists. →
  The algebra closed by the Courant bracket is the Courant algebroid.

Lie bracket → Lie algebra → gauge theory
Courant bracket → Courant algebroid →

-

2D	
WZW model	Chern-Simons theory
Zucchini Stop	

Courant algebroid

'90 Courant
'97 Liu, Weinstein, Xu

vectorbundle $\mathcal{E} \to \mathcal{M}$

with $\langle \cdot , \cdot \rangle$: (graded) symmetric bilinear form

$\circ$: bilinear form (Dorfman bracket)

$\rho : \mathcal{E} \to T\mathcal{M}$ anchor

s.t.

1. $e_1 \circ (e_2 \circ e_3) = (e_1 \circ e_2) \circ e_3 + e_2 \circ (e_1 \circ e_3)$
2. $\rho(e_1 \circ e_2) = [\rho(e_1), \rho(e_2)]$
3. $e_1 \circ F e_2 = F(e_1 \circ e_2) + (\rho(e_1) F) e_2$
4. $e_1 \circ e_2 = \frac{1}{2} \mathcal{D} \langle e_1 , e_2 \rangle$
5. $\rho(e_1) \langle e_2 , e_3 \rangle = \langle e_1 \circ e_2 , e_3 \rangle + \langle e_2 , e_1 \circ e_3 \rangle$

where $e_1, e_2, e_3 \in \Gamma(\mathcal{E})$

$F \in C^\infty(\mathcal{M})$

$\mathcal{D} : \mathcal{M} \to \Gamma(\mathcal{E})$ s.t. $\langle \mathcal{D} F, e \rangle = \rho(e) F$

We consider topological 2-brane
$X \to M$

<u>4-superfields</u>

$\phi^i : X \to M$ scalar superfield

$A^i \in T^*X \otimes \phi^*(TM)$ 1-form superfield

$B_{1i} \in T^*X \otimes \phi^*(T^*M)$ 1-form "

$B_{2i} \in \Lambda^2 T^*X \otimes \phi^*(T^*M)$ 2-form "

We consider BF type kinetic term

$$S_0 = \int_X B_{2i} d\phi^i + B_{1i} dA^i$$

Most general interaction term S_1

$$S = S_0 + g S_1$$

is

Courant Sigma Model

'02 N.I.

$$S = S_0 + g S_1$$

answer

$$S_1' = \int_X \Big(f_{1a}{}^{i}(\phi) A_1{}^a B_{2i} + f_2{}^{ib}(\phi) B_{2i} B_{1b}$$
$$+ \frac{1}{3!} f_{3abc}(\phi) A_1{}^a A_1{}^b A_1{}^c + \frac{1}{2} f_{4ab}{}^{c}(\phi) A_1{}^a A_1{}^b B_{1c}$$
$$+ \frac{1}{2} f_{5a}{}^{bc}(\phi) A_1{}^a B_{1b} B_{1c} + \frac{1}{3!} f_6{}^{abc}(\phi) B_{1a} B_{1b} B_{1c} \Big)$$

where $f_1 \sim f_6$ satisfy

$(S_1', S_1') = 0$ ↓ obstruction

① $f_{1e}{}^{i} f_2{}^{je} + f_2{}^{ie} f_{1e}{}^{j} = 0$

② $-\frac{\partial f_{1c}{}^{i}}{\partial \phi^j} f_{1b}{}^{j} + \frac{\partial f_{1b}{}^{i}}{\partial \phi^j} f_{1c}{}^{j} + f_{1e}{}^{i} f_{4bc}{}^{e} + f_2{}^{ie} f_{3ebc} = 0$

③ $f_{1b}{}^{j} \frac{\partial f_2{}^{ic}}{\partial \phi^j} - f_2{}^{jc} \frac{\partial f_{1b}{}^{i}}{\partial \phi^j} + f_{1e}{}^{i} f_{5b}{}^{ec} - f_2{}^{ie} f_{4eb}{}^{c} = 0$

④ $-f_2{}^{jb} \frac{\partial f_2{}^{ic}}{\partial \phi^j} + f_2{}^{jc} \frac{\partial f_2{}^{ib}}{\partial \phi^j} + f_{1e}{}^{i} f_6{}^{ebc} + f_2{}^{ie} f_{5e}{}^{bc} = 0$

⑤ $-f_{1[a}{}^{j} \frac{\partial f_{4bc]}{}^{d}}{\partial \phi^j} + f_2{}^{jd} \frac{\partial f_{3abc}}{\partial \phi^j} + f_{4e[a}{}^{d} f_{4bc]}{}^{e}$
$\qquad + f_{3[ab} f_{5c]}{}^{de} = 0$

⑥
$$-f_{1[a}{}^{j}\frac{\partial f_{5b]}{}^{cd}}{\partial\phi^{j}}+f_{2}{}^{j[c}\frac{\partial f_{4ab}{}^{d]}}{\partial\phi^{j}}$$
$$+f_{3eab}f_{6}{}^{ecd}+f_{4e[a}{}^{[c}f_{5b]}{}^{d]e}+f_{4ab}{}^{e}f_{5e}{}^{cd}=0$$

⑦
$$-f_{1a}{}^{j}\frac{\partial f_{6}{}^{bcd}}{\partial\phi^{j}}+f_{2}{}^{j[b}\frac{\partial f_{5a}{}^{cd]}}{\partial\phi^{j}}$$
$$+f_{4ea}{}^{[b}f_{6}{}^{cd]e}+f_{5e}{}^{[bc}f_{5a}{}^{d]e}=0$$

⑧
$$-f_{2}{}^{j[a}\frac{\partial f_{6}{}^{bcd]}}{\partial\phi^{j}}+f_{6}{}^{e[ab}f_{5e}{}^{cd]}=0$$

⑨
$$-f_{1[a}{}^{j}\frac{\partial f_{3bcd]}}{\partial\phi^{j}}+f_{4[ab}{}^{e}f_{3cd]e}=0$$

$\Leftrightarrow Q_B^2=0 \Leftrightarrow$ gauge symmetry is the Courant algebroid.

Courant algebroid $\otimes \mathbb{C}$ $\xrightarrow{\quad}$ GCS

$\underbrace{\qquad\qquad}_{J=I\sqrt{-1}\text{ projection}}$

therefore

3D Schwarz type TFT (Courant σ-model)
2-brane

$\downarrow J=I\sqrt{-1}$ projection

3D generalized complex σ-model

Generalized Complex Sigma model in 3D

$\phi^i : X \to M$ $\qquad X : 3D$

$A^i \in T^*X \otimes \phi^*(TM)$ 1-form

$B_{1i} \in T^*X \otimes \phi^*(T^*M)$ 1-form

$B_{2i} \in \Lambda^2 T^*X \otimes \phi^*(T^*M)$ 2-form

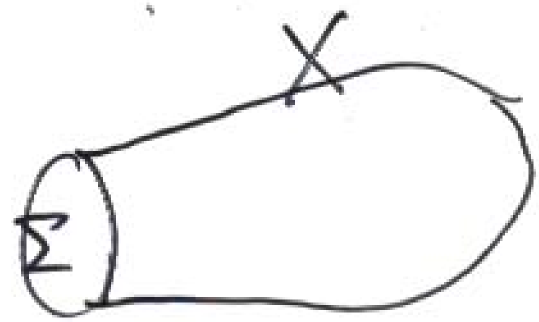
$$S_{GC} = \int_X - B_{2i} d\phi^i + B_{1i} dA^i$$
$$- J^i{}_j B_{2i} A^j - P^{ij} B_{2i} B_{1j}$$
$$+ \frac{1}{2}\left(J^l{}_i H_{jkl} + \frac{\partial Q_{jk}}{\partial \phi^i}\right) A^i A^j A^k$$
$$+ \frac{1}{2}\left(-P^{kl} H_{ijl} - \frac{\partial J^k{}_j}{\partial \phi^i} + \frac{\partial J^k{}_i}{\partial \phi^j}\right) A^i A^j B_{1k}$$
$$+ \frac{1}{2} \frac{\partial P^{jk}}{\partial \phi^i} A^i B_{1j} B_{1k}$$

- S_{GC} is BRST inv. $\Leftrightarrow$ J, P, Q, H is (twisted) GCS
- S_{GC} is b-transformation invariant.

6. Zucchini Model as a boundary action

$\Sigma = \partial X$

$\Sigma : 2D,\ X : 3D$



We can derive 2D Zucchini model as a boundary action on Σ of 3D GC sigma model.

<u>H=0 case</u>

$$S_{GC}(X) = S_Z(\Sigma) + S_a(X)$$

- $S_Z(\Sigma)$: Zucchini action $\quad B_{1i} = B_i$
- $S_a(X)$: topological term independent of GCS

$$S_a(X) = \int_X -Y_{2i}' d\phi^i + dB_{1i} Z'^i + Y_{2i} A^i + B_{2i} Z'^i + B_{2i} A^i$$

Y_{2i}', Z'^i : auxiliary fields

$S_{GC}(X)$ & $S_Z(\Sigma)$ is based on the same GCS, so $S_Z(\Sigma)$ is regarded as a boundary action of $S_{GC}(X)$

$H \neq 0$ case

We can obtain

$$S'_{GC}(X) = S'_Z(\Sigma) + S_a(X)$$

$$S'_Z(\Sigma) = S_Z(\Sigma) + \frac{1}{2}\int J^l{}_i H_{jkl}\, d\phi^i d\phi^j d\phi^k - p^{kl} H_{jil} B_{ik}\, d\phi^i d\phi^j$$

b-inv. WZ term

Zucchini action is not b-invariant

$$S_{GC} = S_Z(\Sigma) + S_a(X)$$

$$\Big\Downarrow \text{b-transformation}$$

$$\hat{S}_{GC} = \hat{S}_Z(\Sigma) + \hat{S}_a(X)$$

$$\hat{S}_Z(\Sigma) = S_Z(\Sigma) - \frac{1}{2}\int_\Sigma b_{ij}\, d\phi^i d\phi^j$$

$H = -d_M b$

$$= S_Z(\Sigma) + \frac{1}{2}\int_X H \quad = \text{Zucchini model with } H$$

Summary

3D topological σ-model
with twisted GCS on X

$\downarrow$ ∂X boundary

2D topological σ-model
with twisted GCS on ∂X

- BRST $\Leftrightarrow$ tGCS
- b-transformation invariant

Future direction $H \neq 0$

	nontopological	topological
worldsheet	topological twist quantization CFT?	
correspondence		
targetsp		
M theory	super membrane	topological membrane

relation?