# ゲージ理論の変形理論とその応用

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## 1. Introduction

Utiyama '56
Yang-Mills '54
Nonabelian Gauge Theory

- ある条件の下で新しいゲージ理論を見つける
- ある条件の下でゲージ理論がつくれないことをいう。 no-go theorem
  - 場の種類
  - locality
  - unitarity
  - global invariance
  - polynomial etc.
- "Deformation Theory of Everything"
  '00 Kontsevich, Soibelman

Unification of deformation theory?

2D BF の変形 ↔ 変形量子化

B-model の変形 ↔ 複素構造の変形

$\boxed{\text{場の理論}}$ の変形 ↔ $\boxed{\phantom{xxxx}}$ の変形

# §2 ゲージ理論の変形理論

## コンセプト

例 abelian gauge theory (EM theory)

$$S_0 = -\frac{1}{4}\int d^4x\, F^{(0)a}_{\mu\nu} F^{(0)\mu\nu a} \qquad a=1,2,\cdots N$$

$$F^{(0)a}_{\mu\nu} \equiv \partial_\mu A_\nu^a - \partial_\nu A_\mu^a$$

$$\delta_0 A_\mu^a = \partial_\mu \epsilon^a \qquad U(1)^N$$

abelian ゲージ理論から nonabelian ゲージ理論を発見できるか?

action とゲージ変換の変形

$$\begin{cases} S[A_\mu^a] = S_0 + S_1 \\ \delta A_\mu^a = \delta_0 A_\mu^a + \delta_1 A_\mu^a \end{cases}$$

条件 $\delta S = 0$ ゲージ不変

$$[\delta_\varepsilon, \delta_{\varepsilon'}] = \delta_{[\varepsilon, \varepsilon']}$$

答　nonabelian gauge theory

$$S = -\frac{1}{4}\int d^4x\, F^a_{\mu\nu}F^{\mu\nu a}$$

$$F^a_{\mu\nu} \equiv \partial_\mu A_\nu{}^a - \partial_\nu A_\mu{}^a + g f^{abc} A_\mu{}^b A_\nu{}^c$$

$$\delta A_\mu{}^a = \partial_\mu \epsilon^a + g f^{abc} A_\mu{}^b \epsilon^c$$

ここで $f_{ab[c} f^a{}_{de]} = 0$　　$f_{abc}$: structure const.

$g$: 変形パラメータ

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◎ unique consistent deformation under the condition　　'95 Barnich, Brandt, Henneaux

- Lorentz 不変　　大局的対称性
- local
- (unitary)
- polynomial　　$S, \delta$ は有限項　~~Born Infeld~~
- $\boxed{\delta \text{ も } \delta_0}$
- 同値類 → local な場の再定義で一致する理論は同値

$$A_\mu{}^a \to A_\mu{}^a + f_\mu{}^a(A) + \cdots$$

ゲージ対称性の変形

一般に条件は

$$\boxed{\begin{aligned}&\text{I. } \delta S = 0\\ &\text{II. } [\delta_\varepsilon, \delta_{\varepsilon'}] = \delta_{[\varepsilon,\varepsilon']} + (\text{on shell})\end{aligned}}$$

でよい。よって

I. $S = S_0 + \underset{\sim}{S_1}$, II. $\delta = \delta_0 + \underset{\sim}{\delta_1}$

を同時に決める必要がある。

→ **Batalin-Vilkovisky 形式**

'81 Batalin-Vilkovisky

(システマティックな方法)

- ゲージパラメータ $\epsilon^a$ を FP ghost $c^a$ に変えてゲージ変換 →(classical) BRS 変換とする。
- $\Phi^A$: 基本場, FP ゴースト

  $\Phi^*_A$: antifield を導入

  s.t. $gh\,\Phi^A + gh\,\Phi^*_A = -1$
- Antibracket

$$(X, Y) \equiv \frac{X\overleftarrow{\delta}}{\delta\Phi^A}\frac{\overrightarrow{\delta}Y}{\delta\Phi^*_A} - \frac{X\overleftarrow{\delta}}{\delta\Phi^*_A}\frac{\overrightarrow{\delta}Y}{\delta\Phi^A}$$

性質

- $(F,G) = -(-1)^{(\mathrm{gh}F+1)(\mathrm{gh}G+1)}(G,F)$ symmetric
- $(F,GH) = (F,G)H + (-1)^{(\mathrm{gh}F+1)\,\mathrm{gh}G}\,G(F,H)$ Leibniz
- $(-1)^{(\mathrm{gh}F+1)(\mathrm{gh}H+1)}(F,(G,H)) + (\text{cyclic permutations}) = 0$ Jacobi

Batalin-Vilkovisky action

$$S_{BV} \equiv S[\Phi^A] + (-)^{\mathrm{gh}\Phi^A}\int \Phi^*_A\,\delta\Phi^A + O({\Phi^*_A}^2)$$

実際 $(S_{BV},\Phi^A)\big|_{\Phi^*_A=0} = \delta\Phi^A$

- BRS on BV formalism

$$\delta_{BV} F[\Phi,\Phi^*] \equiv (S_{BV}, F[\Phi,\Phi^*])$$

これから

$$\delta_{BV}^2 = 0 \Longleftrightarrow (S_{BV}, S_{BV}) = 0$$ classical master eq.

II Jacobi $\Updownarrow$

$$\delta S_{BV} = 0$$ I'

I, II

今 I. $\delta S = 0$

II. $\delta^2 = $ (on shell) $= \frac{S_0 \overleftarrow{\delta}}{\delta \Phi_B} F^{AB}[\Phi]$

$\Phi^{*2}$ term

とする。

$$S_{BV} = S_0 + S_1 + S_2 + \cdots$$

$S_1 = \int (-)^{gh \Phi} \Phi^*_A \delta \Phi^A$

$S_2 + \cdots$: on shell effect

$$0 = (S_{BV}, S_{BV})$$
$$= (S_0 + S_1 + S_2 + \cdots, S_0 + S_1 + S_2 + \cdots)$$
$$= (S_0, S_0) + 2(S_0, S_1) + (S_1, S_1) + 2(S_0, S_2) + 2(S_1, S_1) + (S_2, S_2) + \cdots$$
$$= 0 + 2(-)^{gh\Phi^A} \delta S_0 + 2(-)^{gh\Phi^A} (\delta^2 \Phi^A) \Phi^*_A + 2 \frac{S_0 \overleftarrow{\delta}}{\delta \Phi^A} \frac{\overrightarrow{\delta} S_2}{\delta \Phi^*_A} + \cdots$$
$$= 2 \frac{S_0 \overleftarrow{\delta}}{\delta \Phi^A} \left( (-)^{gh\Phi^A} F^{BA}[\Phi] \Phi^*_B + \frac{\overrightarrow{\delta} S_2}{\delta \Phi^*_A} \right) + \cdots$$

よって

$$S_2 = -\int (-)^{gh\Phi^A} \Phi^*_A F^{BA}[\Phi] \Phi^*_B$$

on shell close

$S_3, S_4, \cdots$

# Batalin-Vilkovisky formalism

**I** $(\cdot, \cdot)$ antibracket degree +1

odd Poisson bracket **P-str.**

**II** $\Phi^A, \Phi^*_A$ field-antifield pair

canonical conjugate on $(\cdot, \cdot)$

**III** $S_{BV}$ BV action

$$\delta_{BV}(*) = (S_{BV}, *)$$

s.t. $\delta_{BV}^2 = 0 \iff (S_{BV}, S_{BV}) = 0$

classical master equation

$\updownarrow$

$Q_{BRS}^2 = 0$ **Q-str.**

$\delta_{BV}$: vector field on field sp. $S_{BV}$: Hamiltonian

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以下 $S_{BV}$ を $S$, $\delta_{BV}$ を $\delta$ と書く。

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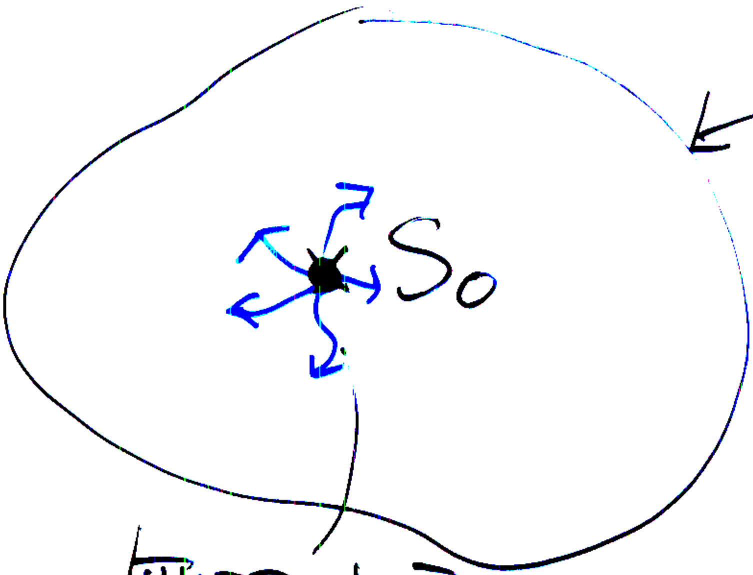
o ゲージ理論の変形

$\iff$ Q, P-構造の変形

$Q$: BRS, $Q^2 = 0$

$P$: antibracket = odd Poisson $(\cdot, \cdot)$




← ゲージ理論全体の空間

無限小変形の moduli

$S_0$ : known gauge theory (通常 abelian 理論)

$$S = S_0 + gS_1 + g^2 S_2 + \cdots$$

s.t. $(S, S) = 0$

条件

- Lorentz不変 〜 global symmetry
- local 〜 $S = \int \mathcal{L}$ local Lagrangian

(○ unitary) 〜 Kugo-Ojima or 微分の数

- $\delta \neq \hat{\delta}_0$

$\Leftrightarrow$ $S_i$ $i=1,2,\cdots$ は各項に少なくとも1つ $\Phi^*_A$ を含む

○同値類

$$S' = S + g^n \delta F \Rightarrow S' \sim S$$ BRS exact

$\because)$ $S'[\Phi^A, \Phi^{*\prime}_A] = S[\Phi^{\prime A}, \Phi^*_A]$ $g^n$ 1次

$$\Phi'^A = \Phi^A + g \frac{\vec{\delta} F}{\delta \Phi^*_A}$$

$$\Phi'^*_A = \Phi^*_A - g \frac{\vec{\delta} F}{\delta \Phi^A}$$

場のlocalな再定義

↓

BRS cohomology class

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Coleman-Mandula
-(Haag-Łopuzański-Sohnius)
の定理

Poincaré不変な理論の物理的S行列の(連続的)対称性
- Poincaré対称性 $P_\mu, M_{\mu\nu}$
- スカラーcharge $Q$
- 1階のスピノルcharge $Q_\alpha$

のみ

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local gauge 対称性 $\delta(*)$

$\downarrow$

global gauge 対称性 $Q^{a_1 \cdots a_n}$

あまり変なものは作れない

~~high spin~~ $\cancel{>2}$

○ Yang-Mills type

$S_0 = \int F^{(0)} \wedge {}^*F^{(0)}$ EM

→ 4 dim ではYang-Mills のみ
'94 Barnich-Brandt-Henneaux

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○ 反対称テンソル場 $A_{\mu_1 \cdots \mu_p}{}^a$ (p-form)

$H_{\mu_1 \cdots \mu_{p+1}}{}^a \equiv \partial_{[\mu_1} A_{\mu_2 \cdots \mu_{p+1}]}{}^a$

$S_0 = \int H \wedge {}^*H$

→ $n = p+2$ 次元のとき
Freedman-Townsend-model $f_{abc} A^a \wedge {}^*H^b \wedge {}^*H^c$

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○ 3D Chern-Simons 理論

'96 Henneaux
Knaepen
Schomblond

$S_0 = \int A^a \wedge dA^a$ $A^a = A_\mu{}^a dx^\mu$

→ 通常の nonabelian Chern-Simons 理論
'93 Barnich
Henneaux

○ self-dual p-form (chiral boson)

$A_{\mu_1 \cdots \mu_p}{}^a$ $\qquad p = \frac{n}{2}$ in $n$ dim

$H_{\mu_1 \cdots \mu_{p+1}}{}^a \equiv \partial_{[\mu_1} A_{\mu_2 \cdots \mu_{p+1}]}{}^a$

$$\begin{cases} S_0 = \int H \wedge * H \\ H = * H \end{cases}$$

in $n = 4k+2$ $\quad (k=0,1,\cdots)$ Lorentzian

in $n = 4k$ $\quad (k=0,1,\cdots)$ Euclidian

→ 変形できない $\qquad$ '01 Bekaert, Henneaux, Sevrin

where $A_p{}^a \equiv A_{\mu_1 \cdots \mu_p}{}^a \, dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}$

$H_{p+1}{}^a \equiv H_{\mu_1 \cdots \mu_{p+1}}{}^a \, dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p} \wedge dx^{\mu_{p+1}}$

○ 現在の問題

① nonpolynomial の場合

( polynomial をはずす
or 場が無限個 )

例) 重力, 弦理論 $g_{\mu\nu}$

locality? 量子化?

○ massless / massive } spin 2 field $h_{\mu\nu}$

Pauli-Fierz action

$$S_0 = -\frac{1}{2}\int d^n x \Big[ \big( \partial_\mu h_{\nu\rho} \partial^\mu h^{\nu\rho} - 2\partial_\mu h^\mu{}_\nu \partial_\rho h^{\rho\nu} + 2\partial_\nu h^\mu{}_\mu \partial_\rho h^{\rho\nu} - \partial_\mu h^\nu{}_\nu \partial^\mu h^\rho{}_\rho \big) + m^2 \big( h_{\mu\nu} h^{\mu\nu} - h^\mu{}_\mu h^\nu{}_\nu \big) \Big]$$

Many authors have analyzed the conditions to derive Einstein gravity from the massless Pauli-Fierz action.

- infrared limit $p_\mu \to 0$ で coupling が消えないとき massless spin 2 field は graviton

'81 Kugo–Uehara

◎ A trial to nonpolynomiality

noncommutative Yang–Mills 理論

$$S_{NCYM} = -\frac{1}{4}\int d^n x\, F_{\mu\nu}{}^a \star F^{\mu\nu a}$$

$$F_{\mu\nu}{}^a = \partial_\mu A_\nu{}^a - \partial_\nu A_\mu{}^a + [A_\mu, A_\nu]_\star^a$$

$\star$ : Weyl–Moyal star product

$$\underset{\text{equivalent}}{S_{NCYM} \sim S_{YM}}$$

'99 Seiberg–Witten

'01 Barnich, Brandt, Grigoriev

where

$$F \star G = F \exp\left(\overleftarrow{\partial_\mu} \frac{i}{2} \theta^{\mu\nu} \overrightarrow{\partial_\nu}\right) G$$

# ② Topological Field Theory

Coleman-Mandulaの定理に抵触しない。

$\delta$ : local → $Q$ : global

trivial on S-matrix

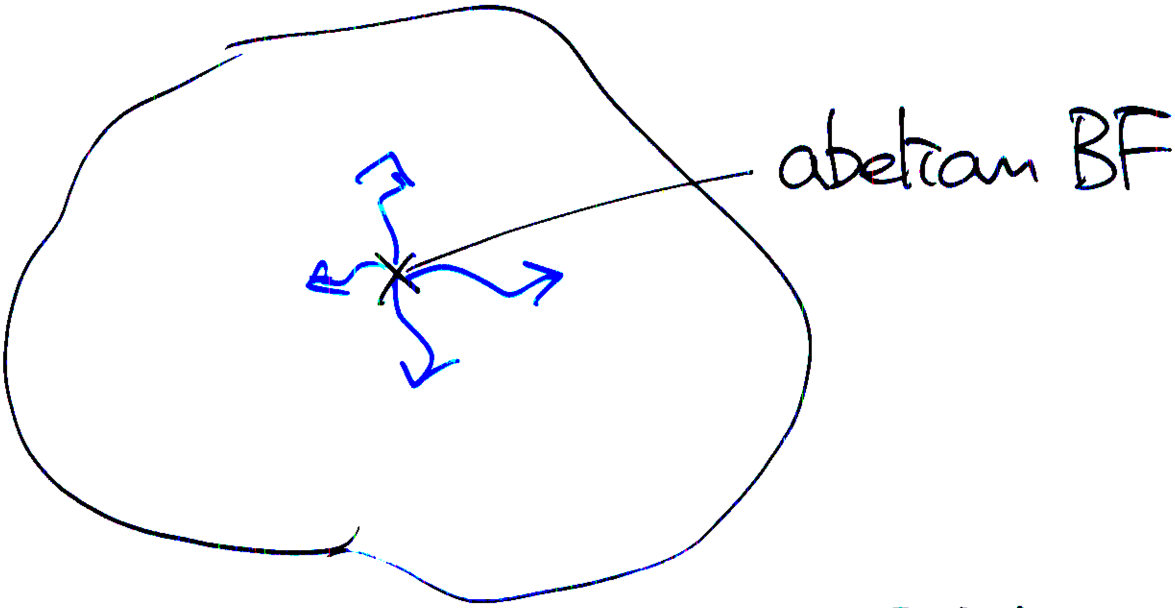
豊富な変形自由度

nontrivialなmanifold上ではゲージ対称性の違いがみえる。

- Schwarz type
  - BF
  - Chern-Simons
- Witten type

→ applications
- mathematical
- physical

# §3 BF理論の変形




## n次元 abelian BF 理論

$$S_0 = \sum_{p=0}^{[\frac{n}{2}]} \int_X (-)^{n-p} B_{n-p-1,a_p}\, dA_p{}^{a_p}$$

$X$ : n次元多様体

$A_p{}^{a_p}$ : p-form gauge field

$A_p = A_{\mu_1 \cdots \mu_n} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_n}$

$B_{n-p-1,a_p}$ : n-p-1 form 補助場

$a_p$ : target sp indices

ゲージ対称性

$$\begin{cases} \delta_0 A_p{}^{a_p} = dC^{(p)}_{p-1}{}^{a_p} \\ \delta_0 B_{n-p-1,a_p} = dt^{(n-p-1)}_{n-p-2,a_p} \end{cases} \quad \text{abelian}$$

Towers of gauge symmetries

$$\hat{\delta}_0 A_p{}^{a_p} = d C^{(p)\,a_p}_{p-1} \qquad \hat{\delta}_0 B_{n-p-1,a_p} = d t^{(n-p-1)}_{n-p-2,a_p}$$

$$\hat{\delta}_0 C^{(p)\,a_p}_{p-1} = d C^{(p)\,a_p}_{p-2} \qquad \hat{\delta}_0 t^{(n-p-1)}_{n-p-2,a_p} = d t^{(n-p-1)}_{n-p-3,a_p}$$

$$\vdots \qquad\qquad \vdots$$

$$\hat{\delta}_0 t^{(n-p-1)}_1 = d t^{(n-p-1)}_0$$

$$\hat{\delta}_0 C^{(p)\,a_p}_1 = d C^{(p)\,a_p}_0$$

Batalin-Vilkovisky 形式

。$C$, $t$ を FP ghost, $\hat{\delta}_0$ BRS 変換

$$gh(\hat{\delta}_0) = +1$$

$$\hat{\delta}_0^2 = 0$$

。場 $\Phi^A$ に対して antifield $\Phi^\dagger_A$ を導入

ここで $gh(\Phi^A) + gh(\Phi^\dagger_A) = -1$

$$\deg(\Phi^A) + \deg(\Phi^\dagger_A) = n$$

$gh$ : ゴースト数　$\deg$ : form degree

$A \leftrightarrow A^\dagger$　　$B \leftrightarrow B^\dagger$

$C \leftrightarrow C^\dagger$　　$t \leftrightarrow t^\dagger$

- BV action をつくる。

$$S_{0BV} = S_0 + (-)^{gh\Phi^A}\int \Phi^{\dagger}_A\,\delta\Phi^A$$

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◎ Abelian BF 理論の superfield 形式

total degree

$$|\Phi^A| \equiv gh(\Phi^A) + deg(\Phi^A)$$

とすると、

$$|\Phi^A| + |\Phi^{\dagger}_A| = n-1$$

たとえば、

$$|A_p^{a_p}| + |A^{\dagger a_p}_{n-p}| = n-1$$

$$|A^{\dagger}| = n-1-|A_p^{a_p}| = n-1-p$$

$$= |B_{n-p-1}|$$

同様に

degree $p$ | antifield | degree $n-p-1$

$A_p, c$ | | $B_{n-p-1}, t$

$B^{\dagger}, t^{\dagger}$ | | $A^{\dagger}, c^{\dagger}$

$\mathbb{A}_p$ | | $\mathbb{B}_{n-p-1}$

## degree $p$ superfield

$$A_p{}^{a_p} = C_0^{(p)a_p} + \cdots + C_{p-1}^{(p)a_p} + A_p{}^{a_p}$$
$$+ B_{p+1}^{+(n-p-1)a_p} + t_{p+2}^{+(n-p-1)a_p} + \cdots + t_n^{+(n-p-1)a_p}$$

## degree $n-p-1$ superfield

$$\mathbb{B}_{n-p-1,a_p} = t_{0,a_p}^{(n-p-1)} + \cdots + t_{n-p-2,a_p}^{(n-p-1)} + B_{n-p-1,a_p}$$
$$+ A_{n-p,a_p}^{+(p)} + C_{n-p+1,a_p}^{+(p)} + \cdots + C_{n,a_p}^{+(p)}$$

。superfield 上の積

$$F \cdot G \equiv (-)^{gh(F)\deg(G)} FG$$

とすると、

$$S_{0BV} = \sum_{p=0}^{\left[\frac{n-1}{2}\right]} (-)^{n-p} \int_X \mathbb{B}_{n-p-1,a_p} \cdot d\mathbb{A}_p{}^{a_p}$$

- BV-antibracket on superfields

$$(F, G) = \sum_{p=0}^{\left[\frac{n-1}{2}\right]} F \cdot \frac{\overleftarrow{\partial}}{\partial A_p{}^{a_p}} \cdot \frac{\overrightarrow{\partial}}{\partial \mathbb{B}_{n-p-1, a_p}} \cdot G$$

$$- (-)^{np} F \cdot \frac{\overleftarrow{\partial}}{\partial \mathbb{B}_{n-p-1, a_p}} \cdot \frac{\overrightarrow{\partial}}{\partial A_p{}^{a_p}} \cdot G$$

$A_p{}^{a_p} \Leftrightarrow \mathbb{B}_{n-p-1, a_p}$ conjugate

- BRS

$$\delta_0 A_p{}^{a_p} = (S_{0BV}, A_p{}^{a_p}) = dA_p{}^{a_p}$$

$$\delta_0 \mathbb{B}_{n-p-1, a_p} = (S_{0BV}, \mathbb{B}_{n-p-1, a_p}) = d\mathbb{B}_{n-p-1, a_p}$$

$\delta_0 \sim d$

以下 $S_{0BV}$ を $S_0$ と書く。

# abelian BF理論の変形

$S = S_0 + gS_1 + g^2 S_2 + \cdots$

classical master equation

$(S, S) = 0$

$$0 = (S_0 + gS_1 + g^2 S_2 + \cdots, S_0 + gS_1 + g^2 S_2 + \cdots)$$
$$= (S_0, S_0)$$
$$+ 2g (S_0, S_1)$$
$$+ g^2 [(S_1, S_1) + 2(S_0, S_2)] + \cdots$$

$\underline{g^0|}$ $(S_0, S_0) = \tilde{\delta}_0 S_0 = 0$

$\underline{g^1|}$ $(S_0, S_1) = \tilde{\delta}_0 S_1 = 0$

$\Phi = A_0{}^{a_0}$

解| $S_1 = \int \mathcal{L}_1$

$$= \int \sum_{p(1)\cdots p(k), q(1)\cdots q(\ell), p_1 + \cdots + p_k + q_1 + \cdots + q_\ell = n} F_{p(1)\cdots p(k) q(1)\cdots q(\ell) a_{p(1)} \cdots a_{p_k}}{}^{b_{q_1} \cdots b_{q(\ell)}}(\Phi)$$

$$A_{p_1}{}^{a_{p(1)}} \cdots A_{p_k}{}^{a_{p(k)}} B_{q_1 b_{q(1)}} \cdots B_{q_\ell b_{q(\ell)}}$$

$g^2 \lrcorner \ (S_1, S_1) + (S_0, S_2) = 0$

$\delta_0 S_2$

今 全てのlocalな場 $\mathcal{L}$ に対して $\delta_0 \mathcal{L} = d(*)$

よって $\delta_0 S_2 = \delta_0 \int \mathcal{L}_2 = 0$

$\therefore$ $(S_1, S_1) = 0$

これは F の identity を導出する。

解

$S = S_0 + g S_1$

$(S, S) = 0 \Leftrightarrow 2\delta_0 S_1 + g(S_1, S_1) = 0$

Maurer-Cartan eq.

"flat condition"

◎ 2次元 '99 Izawa

$$S = S_0 + g S_1$$

$$S_0 = \int_X \underset{1}{\mathbb{B}_{1a}} \cdot d\underset{0}{\phi^a} \qquad \phi^a = A_0{}^a$$

$$S_1 = \int_{X_1} \frac{1}{2} f^{ab}(\phi) \cdot \mathbb{B}_{1a} \cdot \mathbb{B}_{1b}$$

$$(S_1, S_1) = 0 \qquad \text{extended Jacobi}$$

$$\Longleftrightarrow \quad f^{cd} \frac{\partial f^{ab}}{\partial \phi^d} + f^{ad} \frac{\partial f^{bc}}{\partial \phi^d} + f^{bd} \frac{\partial f^{ca}}{\partial \phi^d} = 0$$

nonlinear gauge theory ('93 Izawa-N.I., '94 Schaller-Strobl)
(Poisson σ-model)
$\cup$
2D $R^2$ 重力

$f^{ab}(\phi) = f^{abc} \phi_c$ のとき YM 的

今

$$\{F(\phi), G(\phi)\} \equiv -f^{ab}(\phi) \frac{\partial F}{\partial \phi^a} \frac{\partial G}{\partial \phi^b}$$

とすると,

$\{\cdot, \cdot\}$ が Poisson かっこ

$$\Leftrightarrow f^{cd} \frac{\partial f^{ab}}{\partial \phi^d} + f^{ad} \frac{\partial f^{bc}}{\partial \phi^d} + f^{bd} \frac{\partial f^{ca}}{\partial \phi^d} = 0$$

$$\Leftrightarrow (S, S) = 0$$

実際

$$\{F(\phi), G(\phi)\} = ((S, F(\phi)), G(\phi))$$

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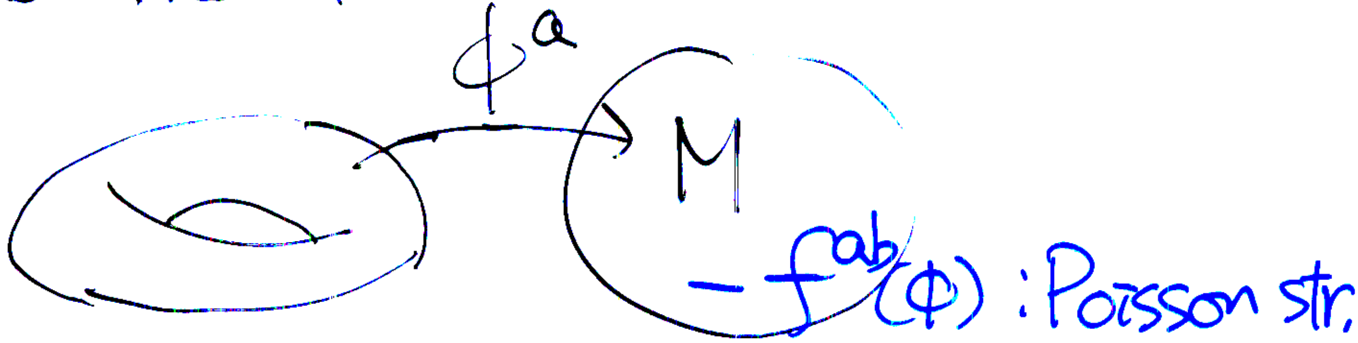
J-model として




2D

変形自由度 = Poisson str. の自由度

the Poisson sigma-model

# Lie Algebroid

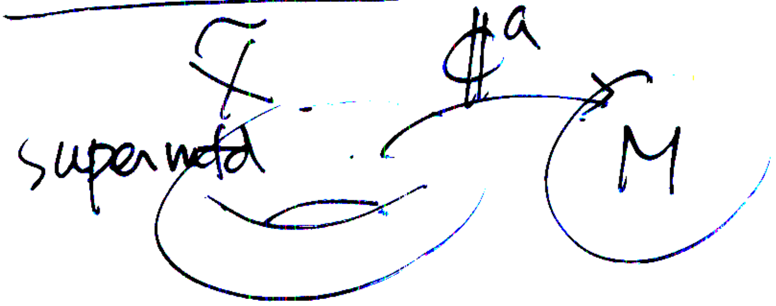
vector bundle $\mathcal{E} \to M$

$e_1, e_2 \in \Gamma(\mathcal{E})$, $F \in C^\infty(M)$

$\rho : \mathcal{E} \to TM$ anchor, $[\cdot,\cdot]$ Lie bracket

s.t.

1. $[\rho(e_1), \rho(e_2)] = \rho([e_1, e_2])$
2. $[e_1, F e_2] = F[e_1, e_2] + (\rho(e_1)F)\, e_2$




$\mathcal{M} = \{\phi^a : \hat{X} \to M\}$ fiber $\mathcal{E} = T^*M$

$[e_1, e_2] = ((S, e_1), e_2)$

$\rho(e) F(\phi) = (e, (S, F(\phi)))$

とすると、

Lie Algebroid condition 1, 2

$\Leftrightarrow (S, S) = 0$

"associativity"を保って変形する自由度

$S_0 \to S$

2次元Disc X


量子化

observable

$$F(\Phi) \star G(\Phi) = \int \mathcal{D}\Phi \mathcal{D}B_1 \times F(\Phi(1))\, G(\Phi(0))\, e^{\frac{i}{\hbar}(S+S_{gf})}$$

$\ast$-product formula = correlation fns. of $S$

'99 Cattaneo-Felder

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$$\begin{array}{ccc} S_0 & \xrightarrow{\text{deformation}} & S \\ \wr & & \wr \\ F(\Phi)G(\Phi) & \xrightarrow{\text{deformation quantization}} & F(\Phi)\star G(\Phi) \end{array}$$

◎ 3次元

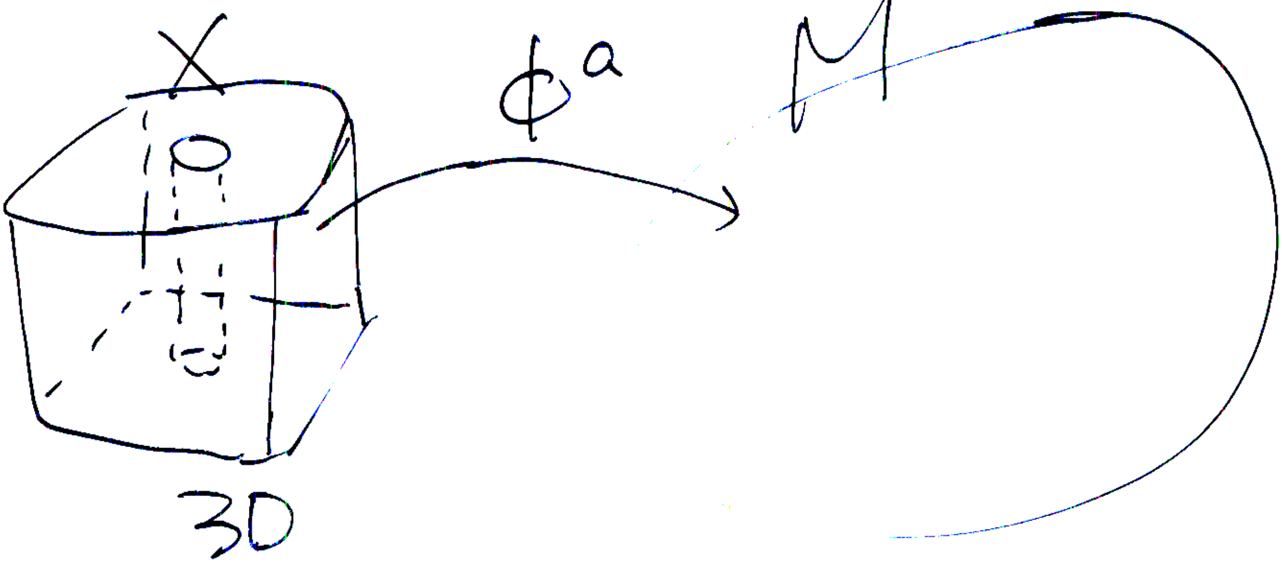
$$S = S_0 + g S_1$$

$$S_0 = \int_X \Big[ -\underset{2}{B_{2i}} \cdot d\underset{0}{\phi^i} + \underset{1}{B_{1a}} \cdot d\underset{1}{A_1^a} \Big]$$

$$S_1 = \int_X \Big[ f_{1a}{}^{i}(\phi) A_1^a B_{2i} + f_2{}^{ib}(\phi) B_{2i} B_{1b} + \frac{1}{3!} f_{3abc}(\phi) A_1^a A_1^b A_1^c + \frac{1}{2} f_{4ab}{}^{c}(\phi) A_1^a A_1^b B_{1c}$$
$$+ \frac{1}{2} f_{5a}{}^{bc}(\phi) A_1^a B_{1b} B_{1c} + \frac{1}{3!} f_6{}^{abc}(\phi) B_{1a} B_{1b} B_{1c} \Big]$$

$$(S_1, S_1) = 0$$

$\Rightarrow f_1 \sim f_6$ の identity



- $f_{1e}{}^{i} f_2{}^{je} + f_2{}^{ie} f_{1e}{}^{j} = 0$
- $\dfrac{\partial f_{1c}{}^{i}}{\partial \phi^j} f_{1b}{}^{j} - \dfrac{\partial f_{1b}{}^{i}}{\partial \phi^j} f_{1c}{}^{j} + f_{1e}{}^{i} f_{4bc}{}^{e} + f_2{}^{ie} f_{3ebc} = 0$
- $-f_{1b}{}^{j} \dfrac{\partial f_2{}^{ic}}{\partial \phi^j} + f_2{}^{jc} \dfrac{\partial f_{1b}{}^{i}}{\partial \phi^j} + f_{1e}{}^{i} f_{5e}{}^{bc} - f_2{}^{ie} f_{4eb}{}^{c} = 0$
- $f_2{}^{jb} \dfrac{\partial f_2{}^{ic}}{\partial \phi^j} - f_2{}^{jc} \dfrac{\partial f_2{}^{ib}}{\partial \phi^j} + f_{1e}{}^{i} f_6{}^{ebc} + f_2{}^{ie} f_{5e}{}^{bc} = 0$
- $f_{1[a}{}^{j} \dfrac{\partial f_{4bc]}{}^{d}}{\partial \phi^j} - f_2{}^{jd} \dfrac{\partial f_{3abc}}{\partial \phi^j} + f_{4e[a}{}^{d} f_{4bc]}{}^{e} + f_{3e[ab} f_{5c]}{}^{de} = 0$
- $f_{1[a}{}^{j} \dfrac{\partial f_{5b]}{}^{cd}}{\partial \phi^j} + f_2{}^{j[c} \dfrac{\partial f_{4ab}{}^{d]}}{\partial \phi^j} + f_{3eab} f_6{}^{ecd} + f_{4e[a}{}^{[d} f_{5b]}{}^{c]e} + f_{4ab}{}^{e} f_{5e}{}^{cd} = 0$
- $f_{1a}{}^{j} \dfrac{\partial f_6{}^{bcd}}{\partial \phi^j} - f_2{}^{j[b} \dfrac{\partial f_{5a}{}^{cd]}}{\partial \phi^j} + f_{4ea}{}^{[b} f_6{}^{cd]e} + f_{5e}{}^{[bc} f_{5a}{}^{d]e} = 0$

- $$f_2{}^{j[a}\frac{\partial f_6{}^{bcd]}}{\partial\phi^j} + f_6{}^{e[ab}f_{5e}{}^{cd]} = 0$$
- $$f_{1[a}{}^{j}\frac{\partial f_{3\,bcd]}}{\partial\phi^j} + f_{4[ab}{}^{e}f_{3\,cd]e} = 0$$

# Courant Algebroid

'90 Courant

vector bundle $\mathcal{E} \to \mathcal{M}$

with $\langle \cdot , \cdot \rangle$ (graded) symmetric bilinear form

$\circ$ bilinear form

$\rho : \mathcal{E} \to T\mathcal{M}$ the anchor

s.t.

1. $e_1 \circ (e_2 \circ e_3) = (e_1 \circ e_2) \circ e_3 + e_2 \circ (e_1 \circ e_3)$
2. $\rho(e_1 \circ e_2) = [\rho(e_1), \rho(e_2)]$
3. $e_1 \circ F e_2 = F(e_1 \circ e_2) + (\rho(e_1) F) e_2$
4. $e_1 \circ e_2 = \frac{1}{2} \mathcal{D} \langle e_1, e_2 \rangle$
5. $\rho(e_1) \langle e_2, e_3 \rangle = \langle e_1 \circ e_2, e_3 \rangle + \langle e_2, e_1 \circ e_3 \rangle$

$e_1, e_2, e_3 \in \Gamma(\mathcal{E})$

$F$ ; $\mathcal{M}$ 上の fn

$\mathcal{D} : \mathcal{M} \to \Gamma(\mathcal{E})$ s.t. $\langle \mathcal{D} F, e \rangle = \rho(e) F$

---

cf. $TM \oplus T^*M$ 上

$$(X + \xi) \circ (Y + \eta) = [X, Y] + (L_X \eta - i_Y d\xi)$$

$\mathcal{M} = \{ \phi^a ; \tilde{X} \to M \}$

fiber $V[1] \oplus V^*[1] = A_i^a, B_{ia}$ で張られるベクトル空間

$\langle e_1, e_2 \rangle \equiv (e_1, e_2)$ antibracket

$e_1 \circ e_2 \equiv ((S, e_1), e_2)$ not symmetric

$\rho(e) F(\phi) \equiv (e, (S, F(\phi)))$

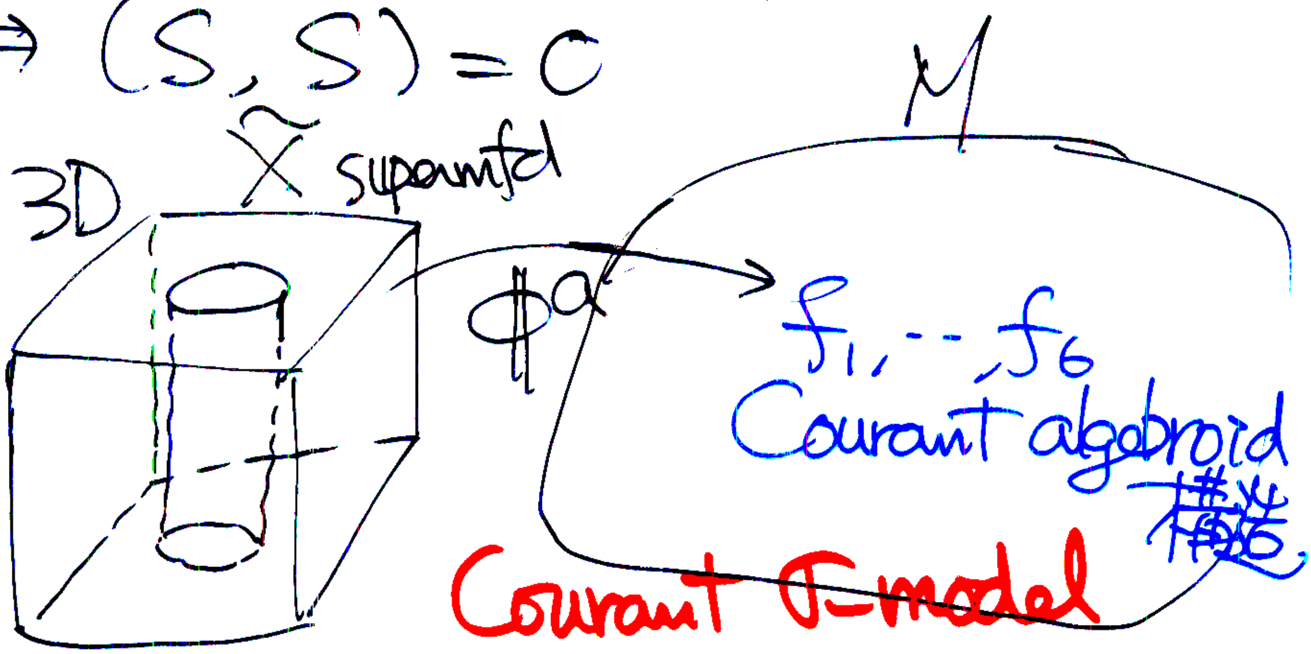
$\mathcal{d}(*) = (S, *)$ BRS

とすると、

Courant algebroid condition 1〜5

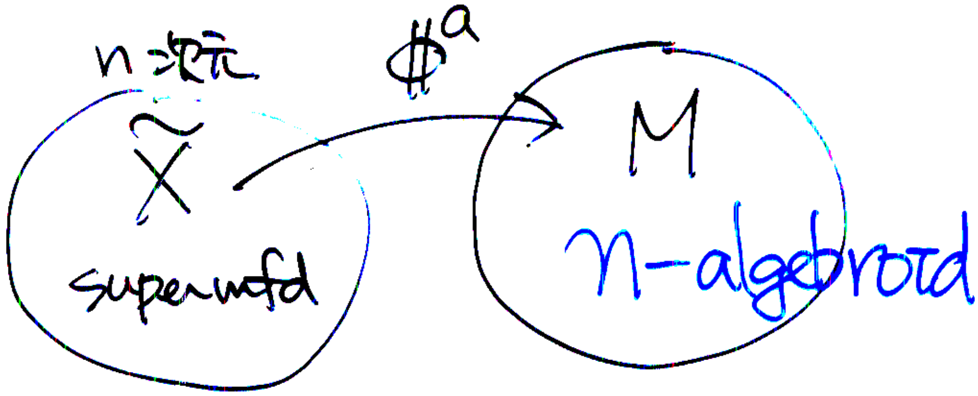
($\Leftrightarrow$ $f_1 \sim f_6$ の identity)

$\Leftrightarrow (S, S) = 0$



○ n次元

$\mathcal{M} = \{\phi^a : \tilde{X} \to \mathcal{M}\}$




$\mathbb{A}_p{}^{a_p}, \mathbb{B}_{n-p-1, a_p}$ $(p \neq 0)$ で張られる

vector space $\bigoplus_{p=1}^{[\frac{n-1}{2}]} \left(V_p[p] \oplus V_p^*[n-p-1]\right)$

をfiberとするvector bundle $\mathcal{E}$

$\mathcal{E} \oplus T^*M$ 上に graded symmetric

な $\langle \cdot , \cdot \rangle$ がdef される

$= ( \cdot , \cdot )$ antibracket

$E_1, E_2 \in \Gamma(\mathcal{E})$ 又は $C^\infty(M)$

として

$$\begin{cases} \langle E_1, E_2 \rangle \equiv (E_1, E_2) \\ \tau(E_1, E_2) \equiv ((S, E_1), E_2) \\ \mathcal{D}(*) \equiv (S, *) \end{cases}$$

$S$ : n dim BF deformed

$E_1 \in C^\infty(M)$ のとき $\tau(E_1, E_2)$ はanchor

$$\begin{pmatrix} \langle \cdot , \cdot \rangle \\ \tau(\cdot , \cdot) \\ S \end{pmatrix}$$ の $\left\{\begin{array}{l}\text{代数} \\ \text{幾何学}\end{array}\right.$ $\Longleftrightarrow$ BV algebra $(S, S) = 0$

n-algebroid $\Longleftrightarrow$ $S$ : n dim deformed BF

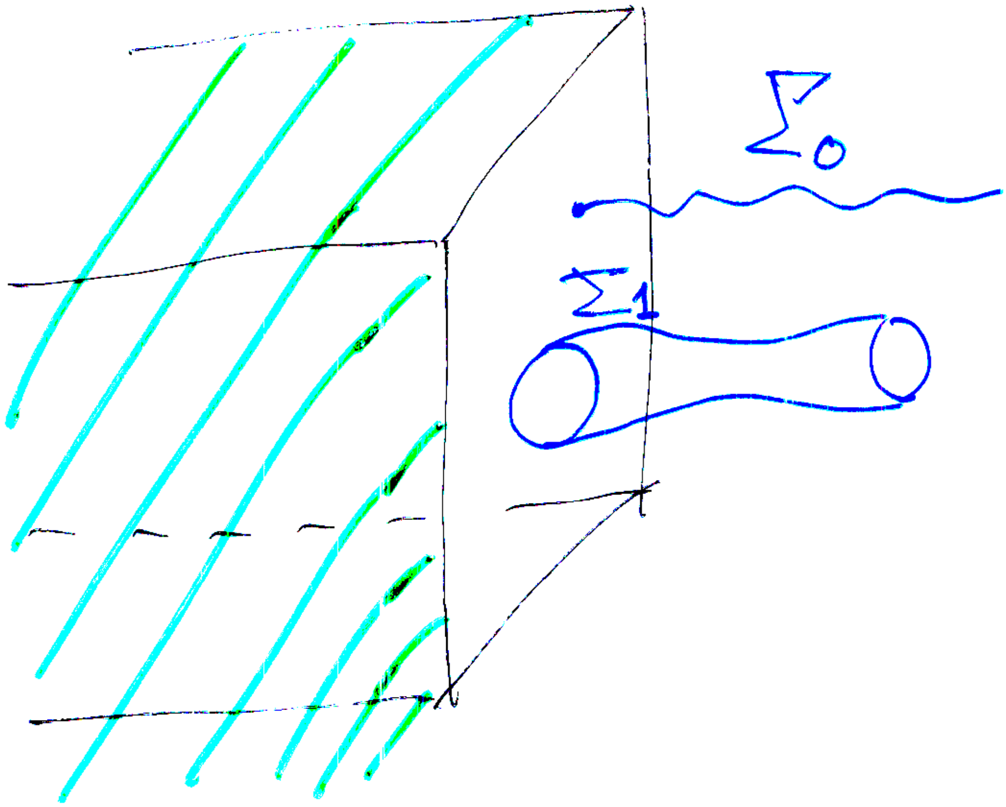
2-algebroid $\Longleftrightarrow$ Poisson

3-algebroid $\Longleftrightarrow$ Courant

n-algebroid $\underset{1対1}{\Longleftrightarrow}$ n dim BF の変形

It will become important to analyze such a new type of "algebras" in mathematics & physics.

# Topological Open Membrane




observables

$\mathcal{O}[\Sigma_0, \Sigma_1]$

$\Sigma_0 = F(\phi)$ vertex

$\Sigma_1 = \int A_1^a$ or $\int B_1^a$ loop

$\langle \mathcal{O}_1 \mathcal{O}_2 \cdots \mathcal{O}_k \rangle = \cdots\cdots$

"quantum" Courant algebroid

- $L_\infty$- algebra — Stasheff
  Hochshild cohomology
- $d$- algebra — Kontsevich
  Operad
- Gerbe
- M -theory
  M2, M5-brane
- quasi Hopf algebra — Drinfeld