Deformation of Batalin-Vilkovisky structures

Noriaki Ikeda
Ritsumeikan Univ.

1, Introduction

Batalin-Vilkovisky formalism (BV)
is the most general method to treat
consistent quantum field theory (gauge theory)
'91 '93 Batalin-Vilkovisky

Topological Field Theory (TFT)
is a method to analyse a geometry
as a quantum field theory.

→ TFT + BV is a natural framework.
to analyse geometry by a quantum
field theory.

Deformation

is a powerful method to construct a new geometry.

ex. commutative → non commutative

Deformation + TFT + BV

↓

- Construction of Geometry as TFT
- Unification of Geometry as TFT
- Classification ……

○ Example :

'97 Kontsevich
Deformation quantization formula
on Poisson manifold

↓

'99 Cattaneo-Felder
have described as a TFT (topological string).

Its TFT is the Poisson sigma model

◎ sigma model

We introduce two manifolds X & M and a map $\phi : X \to M$.
We construct quantum field theory on X using ϕ (& auxially fields).
We analyse the geometry of M by this quantum field theory on X.

2 the Poisson sigma model

$$\Sigma \xrightarrow{\phi^i} M$$

Σ: 2D mfd disc

M: Poisson mfd

$\phi^i : \Sigma \to M$ (smooth) map

$B_i \in \Gamma(T^*\Sigma \otimes \phi^*(T^*M))$ 1-form

i, j, k : indices of local coordinate on M

d : exterior derivative on Σ

Action

$$\boxed{S = \int_\Sigma B_i \wedge d\phi^i + \frac{1}{2} P^{ij}(\phi) B_i \wedge B_j}$$

$P^{ij}(\phi)$: function of ϕ

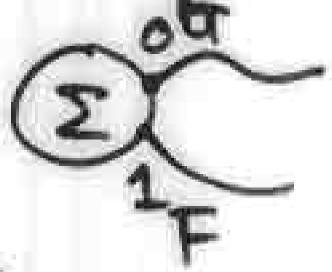
s.t. $\boxed{P^{kl}\dfrac{\partial P^{ij}}{\partial \phi^l} + (i, j, k \text{ cyclic}) = 0}$

i.e. $P^{ij}\dfrac{\partial}{\partial\phi^i}\wedge\dfrac{\partial}{\partial\phi^j}$ is a Poisson bi-vector quantize (path integral)

$$F \star G(x) = \int_{\phi(\infty)=x} \mathcal{D}\phi \mathcal{D}B \, F(\phi(1)) \, G(\phi(0)) \, e^{\frac{i}{\hbar}(S+S_{GF})}$$

$\phi(\infty) = x$ boundary condition

F, G : fns on M



$\hbar$: constant

S_{GF} : gauge fixing & ghosts term

We can obtain

$\star$: $*$-product on M

If we consider the only 1-st term

$$S_0 = \int_\Sigma B_i \wedge d\phi^i$$

the similar calculation produces

$$F(x)G(x) = \int_{\phi(\infty)=x} \mathcal{D}\phi \mathcal{D}B \, F(\phi(1)) G(\phi(0)) e^{\frac{i}{\hbar}(S_0 + S_{gf})}$$

↑

usual product

$$\text{product} \xrightarrow{\text{deformation}} *\text{-product}$$

$$S_0 \longrightarrow S$$

'99 Izawa

$$S_0 = \int B_i \wedge d\phi^i$$

$\downarrow$ deformation of TFT

$$S = \int B_i \wedge d\phi^i + \frac{1}{2} P^{ij}(\phi) B_i \wedge B_j$$

What deformation?

Most general Deformation to preserve the Batalin-Vilkovisky structure

$\downarrow$

Deformation of BV structures

Purpose

- Moduli of deformation
  (all the deformations)
- New geometry
  from topological σ-model
- Unification & Classification
  of geometry by TFT

& to apply to string, M theory

2 Batalin–Vilkovisky Structures

2.1. General Framework (modern, mathematical)

by Alexandrov, Kontsevich, Schwarz, Zaboronsky '95 (AKSZ formalism)

◎ A base manifold is $\Pi T\Sigma$.
(& $\Pi T^*\Sigma$)

where $\Pi T\Sigma = T[1]\Sigma$ is
a tangent bundle with reversed parity of the fiber.
i.e. supermanifold.

ϕ^i, B_i

$\boldsymbol{\phi}^i : \Pi T\Sigma \longrightarrow M$

$\mathbb{B}_i \in \Gamma(\Pi T^*\Sigma \otimes \phi^* \Pi T^* M)$

◎ the odd Poisson bracket on ΠT^*M

$(* \ , *)$ antibracket (BV bracket)

P-structure

$$(\Phi^i, \mathbb{B}_j) = \delta^i{}_j$$

② S_{BV}: BV action

functional of Φ^i & $\mathbb{B}_i$

$(S_{BV}, S_{BV}) = 0$ classical master eq.

s.t. · $S_{BV}|_{\Sigma} = S$

· compatible with $(*, *)$

$$(S_{BV}(F, G)) = ((S_{BV} F), G) \pm (F, (S_{BV} G))$$

· $(S_{BV}, *)$ defines the gauge transf.

Q-structure

i.e. Batalin–Vilkovisky structure (classical)

= QP-structure on supermanifold

Data ① $\Pi T \Sigma$

&

② S_{BV}

$$S_0 = \int_\Sigma B_i \wedge d\phi^i \qquad \text{cf. } p_i \wedge dq^i$$

(2D abelian BF theory)

Its BV-formalism is uniquely (up to diffeo) constructed

P
$$\boxed{(F, G) = \int_\Sigma \frac{F\overleftarrow{\delta}}{\delta \Phi^i} \frac{\overrightarrow{\delta} G}{\delta \mathbb{B}_i} - \frac{F\overleftarrow{\delta}}{\delta \mathbb{B}_i} \frac{\overrightarrow{\delta} G}{\delta \Phi^i}}$$

F, G : functions of superfields

Q
$$\boxed{S_{0BV} = \int_\Sigma \mathbb{B}_i \wedge d\Phi^i}$$

2-2 traditional construction
(in the textbook of physics)

$$S_0 = \int_\Sigma B_i \wedge d\phi^i$$

has a gauge symmetry $U(1)^d$
$d = \dim M$

$$\begin{cases} \delta_0 \phi^i = 0 \\ \delta_0 B_i = -dC_i \end{cases}$$

$C_i \in \Gamma(\Sigma \otimes \Pi T^*M)$
FP ghost (gauge parameter)

$\Phi = $	ϕ^i	B_i	C_i
form degree $\deg \Phi$	0	1	0
ghost number $\mathrm{gh}\,\Phi$	0	0	1
total degree $\|\Phi\| = \deg\Phi + \mathrm{gh}\,\Phi$	0	1	1

◎ We introduce the antifield Φ^* for each field Φ.

We assign the degrees as

$$\mathrm{gh}\,\Phi + \mathrm{gh}\,\Phi^* = -1$$

$$\deg \Phi + \deg \Phi^* = 2$$

i.e.

$$|\Phi| + |\Phi^*| = 1$$

$\Phi^* =$	(ϕ_i^*,	B^{*i},	C^{*i})
$\deg \Phi^*$	2	1	2
$\mathrm{gh}\,\Phi^*$	-1	-1	-2
$\lvert\Phi^*\rvert$	1	0	0

◎ antibracket is introduced as

$$\boxed{(F,G) = \int_\Sigma \frac{F\overleftarrow{\delta}}{\delta\Phi} \wedge \frac{\overrightarrow{\delta}G}{\delta\Phi^*} - (-1)^{\deg\Phi} \frac{F\overleftarrow{\delta}}{\delta\Phi^*} \wedge \frac{\overrightarrow{\delta}G}{\delta\Phi}}$$

F, G : function of Φ & Φ^*

② Construction of S_{OBV} BV action from S_0

s.t.: $S_{OBV}|_{\Phi^*=0} = S_0$

. $(S_{OBV}, \Phi) = \delta_0 \Phi$ gauge sym

. $(S_{OBV}, S_{OBV}) = 0$

uniquely

$$\boxed{S_{OBV} = \int_{\Sigma} B_i \wedge d\phi^i - B^{*i} \wedge dC_i}$$

$gh\, S_{OBV} = 0$, $\deg S_{OBV} = 2 \Rightarrow |S_{OBV}| = 2$

We reconstruct this traditional BV action as a theory on supermanifold $\Pi T\Sigma$ $\longrightarrow$

15

$$\Phi = (\phi^i, B_i, C_i) \qquad |\Phi| = 1$$

$$\Phi^* = (\phi^*_i, B^{*i}, C^{*i}) \qquad |\Phi| = 0$$

We define superfield

$$\boldsymbol{\phi}^i = \phi^i + B^{*i} + C^{*i} \qquad |\boldsymbol{\phi}^i| = 0$$

$$\mathbb{B}_i = C_i + B_i + \phi^*_i \qquad |\mathbb{B}_i| = 1$$

form degree 0 1 2

◎ super product

$$F \cdot G \equiv (-)^{gh F \deg G} F G$$

from now on, products of superfields are always superproducts.

◎ antibracket is reconstructed as super antibracket

$$\boxed{(F, G) = \int_\Sigma \frac{F \overleftarrow{\delta}}{\delta \boldsymbol{\phi}^i} \frac{\overrightarrow{\delta} G}{\delta \mathbb{B}_i} - \frac{F \overleftarrow{\delta}}{\delta \mathbb{B}_i} \frac{\overrightarrow{\delta} G}{\delta \boldsymbol{\phi}^i}}$$

same with AKSZ

property

- $(F,G) = -(-)^{(|F|+1)(|G|+1)}(G,F)$
- $(F,GH) = (F,G)H + (-)^{(|F|+1)|G|}G(F,H)$
- $(-)^{(|F|+1)(|H|+1)}(F,(G,H)) + (\text{cyclic}) = 0$

◎ traditional BV action is rewritten by super-fields

$$\boxed{S_{0BV} = \int_{\Sigma} \mathbb{B}_i d\Phi^i + (\text{total derivative})}$$

same with AKSZ

Integration is defined as zero except for the 2-form part.

◎ gauge transformation (BRST transf.)

$$\delta_0 \Phi^i = (S_{0BV}, \Phi^i) = d\Phi^i$$

$$\delta_0 \mathbb{B}_i = (S_{0BV}, \mathbb{B}_i) = d\mathbb{B}_i$$

$$\delta_0^2 = 0$$

3 Deformation Theory

We consider infinitesimal deformations of S_{0BV}

$$S = S_{0BV} + g S_1 + g^2 S_2 + \cdots$$

s.t. $(S, S) = 0$

preserve QP-str.

g: deformation parameter

up to

$$S' = S + \underbrace{\delta_0(*)}_{\text{BRST exact term}}$$

$\Rightarrow$ S' is equivalent to S

---

first $|S| = 2$ case

$$S = S_{0BV} + g\underbrace{(S_1 + g S_2 + \cdots)}_{= S_1'}$$

$$S = S_{0BV} + g S_1'$$

$(S, S) = 0$

$$0 = (S_{OBV}, S_{OBV}) + 2g\,(S_{OBV}, S_1') + g^2\,(S_1', S_1')$$

g^0 : $(S_{OBV}, S_{OBV}) = 0$
is already satisfied.

g^1 : $(S_{OBV}, S_1') = 0$ — deformation freedom

since $\delta_0 \Phi^i = (S_{OBV}, \Phi^i) = d\Phi^i$

$\delta_0 \mathbb{B}_i = (S_{OBV}, \mathbb{B}_i) = d\mathbb{B}_i$

$$S_1' = \int_\Sigma \left(\text{arbitrary functions of } \Phi^i, \mathbb{B}_i \text{ with degree 2} \right)$$

$$= \int_\Sigma \frac{1}{2} \underset{0}{P^{ij}(\Phi)}\, \underset{1}{\mathbb{B}_i}\, \underset{1}{\mathbb{B}_j} + \underset{0}{J^i{}_j(\Phi)}\, \underset{1}{\mathbb{B}_i}\, \underset{1}{d\Phi^j} + \frac{1}{2} \underset{0}{Q_{ij}(\Phi)}\, \underset{1}{d\Phi^i}\, \underset{1}{d\Phi^j}$$

P, J, Q : functions of Φ^i

$J^i{}_j B_i d\Phi^j$ & $Q_{ij} d\Phi^i d\Phi^j$

are BRST exact (upto total derivative)

$$J^i{}_j(\Phi) B_i d\Phi^j \Big|_{\text{form deg}=2}$$

$$= J^i{}_j(\phi) B_i d\phi^j + \frac{\partial J^i{}_j}{\partial \phi^k} B^{*k} C_i d\phi^j - J^i{}_j(\phi) C_i dB^{*j}$$

$$= \delta_0\left(J^i{}_j B_i B^{*j} + \frac{\partial J^i{}_j}{\partial \phi^k} B^{*k} C_i B^{*j} \right) + d\left(-J^i{}_j C_i C^{*j}\right)$$

$$Q_{ij}(\Phi) d\Phi^i d\Phi^j \Big|_{\text{form deg}=2}$$

$$= Q_{ij}(\phi) d\phi^i d\phi^j$$

$$= \delta_0\left(Q_{ij} B^{*i} d\phi^j \right)$$

$\boxed{\text{Thm}}$ If a term includes at least one d, the term is BRST exact.

$$S_1' = \int \frac{1}{2} P^{ij}(\Phi)\, \mathbb{B}_i \mathbb{B}_j$$

g^2: $(S_1', S_1') = 0$ **obstruction**

$$\Longleftrightarrow \boxed{P^{il} \frac{\partial P^{jk}}{\partial \phi^l} + (i, j, k \text{ cyclic}) = 0} \quad (\star)$$

i.e., all the possible deformations are

$$S = \int_\Sigma \mathbb{B}_i d\Phi^i + \frac{g}{2} P^{ij}(\Phi)\, \mathbb{B}_i \mathbb{B}_j$$

with $(\star)$

i.e. the Poisson sigma model

note) $|S|$ general case

We obtain the topological σ-model for polyvectors in Cattaneo-Felder.

Kontsevich formula is universal in the aspect of BV-str.

note) $J^i{}_j B_i d\phi^j$, $Q_{ij} d\phi^i d\phi^j$

are important ?

'04 Zucchini Hitchin sigma model

under consideration

realizes generalize complex structure

4. Lie algebroid and observables

<u>Def</u> Lie algebroid

- vector bundle $\mathcal{E} \to M$
- for section $e_1, e_2 \in \Gamma(\mathcal{E})$
  the bracket $[e_1, e_2]$ is defined
  with a Lie algebra str.
- bundle map (anchor)
  $\rho : \mathcal{E} \to TM$

s.t.
(1) for $\forall e_1, e_2 \in \Gamma(\mathcal{E})$

$$[\rho(e_1), \rho(e_2)] = \rho([e_1, e_2])$$

(2) for $\forall e_1, e_2 \in \Gamma(\mathcal{E})$, $F \in C^\infty(M)$

$$[e_1, Fe_2] = F[e_1, e_2] + (\rho(e_1)F)e_2$$

If we define

$$[e_1, e_2] = ((S, e_1), e_2)$$

$$\rho(e) F(\phi) = (e, (S, F(\phi)))$$

where S : Poisson sigma model

$\mathcal{E}$ is Lie algebroid
$\Leftrightarrow (S, S) = 0$

where $\mathcal{M} = \{\phi^i : \Sigma \to M\}$
$\mathcal{E} = T^* \mathcal{M}$

Batalin-Vilkovisky structure of total degree 2 topological σ-model
$\cong$ Lie algebroid on T^*M

Levin, Olshanetsky

③ observable ~ invariant of structure

$$
\begin{aligned}
&F \star G(x) \\
&= \int_{\phi(\infty)=x} \mathcal{D}\phi\,\mathcal{D}B\; \underbrace{F(\phi(1))}\,\underbrace{G(\phi(0))}\, e^{\frac{i}{\hbar}(S+S_{GF})}
\end{aligned}
$$

observables on boundary

$$\Delta F \equiv \frac{\vec{\delta}}{\delta \phi^i} \frac{\vec{\delta}}{\delta B_i} F \qquad \text{BV-Laplacian}$$

classical master eq. $\quad (S, S) = 0$

is modified by quantization

to quantum master eq.

$$(S, S) - 2i\hbar \Delta S = 0$$

and F observable

$$\Leftrightarrow \boxed{(S, F) - i\hbar \Delta F = 0}$$

$$F = F(\phi^i)_{|\partial\Sigma,\, \phi(\infty)=x}$$

function of ϕ^i on boundary

(Vertex operator)

$*$-product

$$f*g = fg + \hbar \overset{\{f,g\}}{B_1(f,g)} + \hbar B_2(f,g) + \cdots$$

where B_i: bi-differential op

$$(f*g)*h = f*(g*h)$$

associative

$$f \mapsto f + \hbar D_1 f + \hbar^2 D_2 f + \cdots$$

· equivalent $\qquad$ D_i: differential op.

classical limit
Poisson bivector P^{ij}

1-pt fn

$\Leftrightarrow$ master eq $(S, S) = 0$

$\Leftrightarrow$ $S' = S + \delta(*)$
BRST exact

5. 3 dimensions

We generalize the theory of total degree $|S_0|=|S|=2$ case

to

total degree $|S_0|=|S|=3$ case.

Q We consider 3D manifold X

& $\phi^i : X \to M$

Batalin-Vilkovisky algorism

We define superfields on $T[1]X$

$\boldsymbol{\phi}^i$ $\quad |\boldsymbol{\phi}^i|=0$

$\boldsymbol{B}_{2i}$ $\quad |\boldsymbol{B}_{2i}|=2$

super anti bracket $(*,*)$ $\quad$ P

s.t. $(\boldsymbol{\phi}^i, \boldsymbol{B}_{2j}) = \delta^i{}_j$, other $=0$

BV action $\quad$ Q

$$S_{0BV} = \int_X -\underset{2}{\boldsymbol{B}_{2i}}\, d\underset{0}{\boldsymbol{\phi}^i}$$

In 3 dimensions
We can consider the term

$$\int_X \mathbb{B}_{1a} d\mathbb{A}^a$$
$\quad 1 \quad 1$

where $|\mathbb{A}^a| = |\mathbb{B}_{1a}| = 1$

Setting

$\phi^i : \Pi TX \to M$

$\mathbb{B}_{2i} \in \Gamma(\Lambda^2 \Pi T^*X \otimes \phi^*(T^*[2]M))$

$\mathbb{A}^a \in \Gamma(\Pi T^*X \otimes \phi^* E[1])$

$\mathbb{B}_{1a} \in \Gamma(\Pi T^*X \otimes \phi^* E^*[1])$

E : vector bundle over M

i,j,k : indices on M

a,b,c : indices on the fiber of E

$$S_{0BV} = \int_X -\mathbb{B}_{2i} d\phi^i + \mathbb{B}_{1a} d\mathbb{A}^a$$
$\quad 2 \quad 0 \quad 1 \quad 1$

$$P(F,G) = \int_X \frac{F\overleftarrow{\delta}}{\delta \Phi^i}\frac{\overrightarrow{\delta}G}{\delta \mathbb{B}_{2i}} - \frac{F\overleftarrow{\delta}}{\delta \mathbb{B}_{2i}}\frac{\overrightarrow{\delta}G}{\delta \Phi^i} + \frac{F\overleftarrow{\delta}}{\delta \mathbb{A}^a}\frac{\overrightarrow{\delta}G}{\delta \mathbb{B}_{1a}} + \frac{F\overleftarrow{\delta}}{\delta \mathbb{B}_{1a}}\frac{\overrightarrow{\delta}G}{\delta \mathbb{A}^a} \quad \text{Lo}$$

$$Q \qquad S_{\text{QBV}} = \int_X -\underset{2}{\mathbb{B}_{2i}}\, d\underset{0}{\Phi^i} + \underset{1}{\mathbb{B}_{1a}}\, d\underset{1}{\mathbb{A}^a}$$

We consider deformations

$$S = S_{0BV} + g S_1'$$

answer

$$S_1' = \int_X \Big(f_{1a}{}^{i}(\Phi) A_1{}^a \mathbb{B}_{2i} + f_2{}^{ib}(\Phi) \mathbb{B}_{2i} \mathbb{B}_{1b}$$
$$+ \frac{1}{3!} f_{3abc}(\Phi) A_1{}^a A_1{}^b A_1{}^c + \frac{1}{2} f_{4ab}{}^{c}(\Phi) A_1{}^a A_1{}^b \mathbb{B}_{1c}$$
$$+ \frac{1}{2} f_{5a}{}^{bc}(\Phi) A_1{}^a \mathbb{B}_{1b} \mathbb{B}_{1c} + \frac{1}{3!} f_6{}^{abc}(\Phi) \mathbb{B}_{1a} \mathbb{B}_{1b} \mathbb{B}_{1c} \Big)$$

where $f_1 \sim f_6$ satisfy

$(S_1', S_1') = 0$
$\downarrow$ obstruction

① $f_{1e}{}^{i} f_2{}^{je} + f_2{}^{ie} f_{1e}{}^{j} = 0$

② $-\dfrac{\partial f_{1c}{}^{i}}{\partial \Phi^j} f_{1b}{}^{j} + \dfrac{\partial f_{1b}{}^{i}}{\partial \Phi^j} f_{1c}{}^{j} + f_{1e}{}^{i} f_{4bc}{}^{e} + f_2{}^{ie} f_{3ebc} = 0$

③ $f_{1b}{}^{j} \dfrac{\partial f_2{}^{ic}}{\partial \Phi^j} - f_2{}^{jc} \dfrac{\partial f_{1b}{}^{i}}{\partial \Phi^j} + f_{1e}{}^{i} f_{5b}{}^{ec} - f_2{}^{ie} f_{4eb}{}^{c} = 0$

④ $-f_2{}^{jb} \dfrac{\partial f_2{}^{ic}}{\partial \Phi^j} + f_2{}^{jc} \dfrac{\partial f_2{}^{ib}}{\partial \Phi^j} + f_{1e}{}^{i} f_6{}^{ebc} + f_2{}^{ie} f_{5e}{}^{bc} = 0$

⑤ $-f_{1[a}{}^{j} \dfrac{\partial f_{4bc]}{}^{d}}{\partial \Phi^j} + f_2{}^{jd} \dfrac{\partial f_{3abc}}{\partial \Phi^j} + f_{4e[a}{}^{d} f_{4bc]}{}^{e}$
$\qquad + f_{3[ab} f_{5c]}{}^{de} = 0$

⑥
$$-f_{1[a}{}^{\dot\delta}\frac{\partial f_{5\,b]}{}^{cd}}{\partial\phi^{\dot\delta}}+f_2{}^{\dot\delta[c}\frac{\partial f_{4ab}{}^{d]}}{\partial\phi^{\dot\delta}}$$
$$+f_{3eab}f_6{}^{ecd}+f_{4e[a}{}^{[d}f_{5b]}{}^{c]e}+f_{4ab}{}^{e}f_{5e}{}^{cd}=0$$

⑦
$$-f_{1a}{}^{\dot\delta}\frac{\partial f_6{}^{bcd}}{\partial\phi^{\dot\delta}}+f_2{}^{\dot\delta[b}\frac{\partial f_{5a}{}^{cd]}}{\partial\phi^{\dot\delta}}$$
$$+f_{4ea}{}^{[b}f_6{}^{cd]e}+f_{5e}{}^{[bc}f_{5a}{}^{d]e}=0$$

⑧
$$-f_2{}^{\dot\delta[a}\frac{\partial f_6{}^{bcd]}}{\partial\phi^{\dot\delta}}+f_6{}^{e[ab}f_{5e}{}^{cd]}=0$$

⑨
$$-f_{1[a}{}^{\dot\delta}\frac{\partial f_{3bcd]}}{\partial\phi^{\dot\delta}}+f_{4[ab}{}^{e}f_{3cd]e}=0$$

What is the structure of ①~⑨ ?

Courant algebroid

'90 Courant
'97 Liu, Weinstein, Xu

vectorbundle $\mathcal{E} \to M$

with $\langle \cdot , \cdot \rangle$: (graded) symmetric bilinear form

$\circ$: bilinear form (Dorfman bracket)

$\rho : \mathcal{E} \to TM$ anchor

s.t.

1. $e_1 \circ (e_2 \circ e_3) = (e_1 \circ e_2) \circ e_3 + e_2 \circ (e_1 \circ e_3)$
2. $\rho(e_1 \circ e_2) = [\rho(e_1), \rho(e_2)]$
3. $e_1 \circ F e_2 = F(e_1 \circ e_2) + (\rho(e_1) F) e_2$
4. $e_1 \circ e_2 = \frac{1}{2} \mathcal{D} \langle e_1, e_2 \rangle$
5. $\rho(e_1) \langle e_2, e_3 \rangle = \langle e_1 \circ e_2, e_3 \rangle + \langle e_2, e_1 \circ e_3 \rangle$

where $e_1, e_2, e_3 \in \Gamma(\mathcal{E})$

$F \in C^\infty(M)$

$\mathcal{D} : M \to \Gamma(\mathcal{E})$ s.t. $\langle \mathcal{D} F, e \rangle = \rho(e) F$

- We take

$\mathcal{M} = \{\phi : X \to M\}$

fiber of $\mathcal{E}$ $\quad V \oplus V^*$ $\quad$ bases $\{A^a, B_a\}$

V : vector sp.

If we set

$$\begin{cases} \langle e_1, e_2 \rangle \equiv (e_1, e_2) & \text{antibracket} \\ e_1 \circ e_2 \equiv ((S, e_1), e_2) & \\ \rho(e) F(\phi) \equiv (e, (S, F(\phi))) & \left.\right\} \text{defined as derived bracket} \\ \mathcal{D}(*) \equiv (S, *) & \end{cases}$$

then

$$\boxed{\begin{array}{l} \langle\ ,\ \rangle, \circ, \rho, \mathcal{D} \text{ is Courant algebroid} \\ \Longleftrightarrow (S, S) = 0 \quad \text{master eq} \end{array}}$$

by Roytenberg

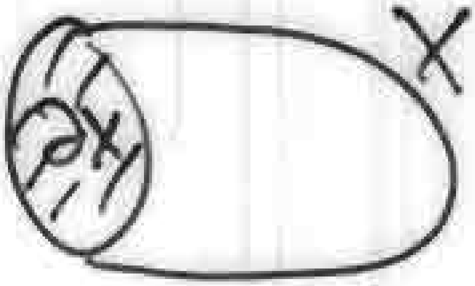
S is $|S| = 3$

Therefore We call

S : <u>Courant sigma model</u>

A generalization of deformation quantization

X : 3D mfd with boundary



@ observable on boundary

$\mathcal{O}$

$(S, \mathcal{O}) - i\hbar \Delta \mathcal{O} = 0$

two kind

$\mathcal{O}_1 = F(\Phi^i)|_{\partial X}$

function of Φ^i on boundary

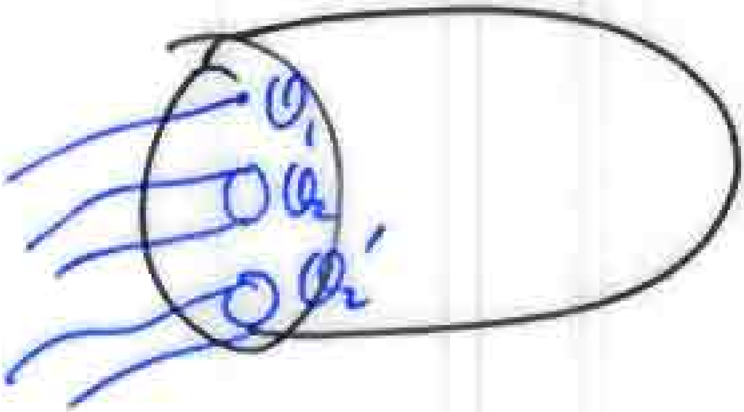
$\mathcal{O}_2 = \int_L F(\Phi^i, A^a)$

$L \subset \partial X$ loop on ∂X

quantization

$$\mathcal{O} \star \mathcal{O}' = \int \mathcal{D}\phi_0 \mathcal{D}B_2 \mathcal{D}A_0 \mathcal{D}B_1 \, \mathcal{O}\mathcal{O}' \times e^{\frac{i}{\hbar}(S+S_{GF})}$$

"deformation quantization" of loops?



7. n dimensions

$\dim X = n$ $\qquad |S| = n$

$$S_{0BV} = \int_X \underset{n-1}{(-1)^{n-1}\mathbb{B}_{n-1}}\, d\underset{0}{\phi} + \underset{n-2}{(-1)^{n-2}\mathbb{B}_{n-2}}\, d\underset{1}{\mathbb{A}_1}$$

$$+ \underset{n-3}{(-)^{n-3}\mathbb{B}_{n-3}}\, d\underset{2}{\mathbb{A}_2} + \cdots$$

$$= \sum_{p=0}^{[\frac{n-1}{2}]} (-)^{n-p} \int_X \mathbb{B}_{n-p-1}\, d\mathbb{A}_p$$

n-dim abelian BF theory

$\downarrow$ deformation

$$S = S_{0BV} + g S_1'$$

$$(S, S) = 0$$

S is called nonlinear gauge theory.

n-dim nonlinear gauge theory
defines an algebroid on M
defined from the master eq.

$$\boxed{(S,S)=0}$$

We name n-algebroid

n-algebroid	classical	quantum
2 Lie algebroid	Poisson	$*$-product on Poisson
3 Courant	Courant algebroid	?
n n-algebroid	n-algebroid	?

n	classical	quantum
2	Poisson	*-product
	complex B	B-model
	Kähler A	A-model
	contact (Jacobi) →	
	symplectic →	
	...	
3	Courant →	corresponding quantum geometry
	Dirac →	
	generalized geometry →	
	twisted Poisson →	
	topological M →	
⋮		
n	topological membrane →	

The list includes many geometries but have not be completed yet.

Problems

- Characterization of n-algebroid
  classicaly $(n \geq 4)$
  quantum $(n \geq 3)$

- $|S|=2$ case
  $J B d\varphi$, $Q d\varphi d\varphi$ term
  reduction
  generalized complex structure
  Zucchini

  $|S|=3$ origin
  from Courant σ-model
  N.I.