Chiral Fermions in 4D (A)dS Gravity

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The (anti) de Sitter gravity (MacDowell-Mansouri-Stelle-West gravity) is the gauge theory of gravitation whose gauge group G is $SO(2,3)$ for anti de Sitter or $SO(1,4)$ for de Sitter. We define a special internal vector $Z_A = Z_A(x)$ such that

$$Z_1^2 + Z_2^2 + Z_3^2 + Z_4^2 + Z_5^2 = \mp l^2, \tag{1}$$

where the capital Latin letter $A = 1,2,3,4,5$ denotes internal indices. The signatures on the right-hand side of Ed. (1) are $-l^2$ for $SO(2,3)$ and $+l^2$ for $SO(1,4)$.

We consider the $SO(2,3)$ case. We consider a connection field $\omega_{\mu AB}$ and define the field strength

$$R_{\mu\nu AB} = \partial_\mu \omega_{\nu AB} - \partial_\nu \omega_{\mu AB} - \omega_{\mu AC}\omega_{\nu CB} + \omega_{\nu AC}\omega_{\mu CB}. \tag{2}$$

We construct an $SO(2,3)$ invariant Lagrangian

$$\mathcal{L}_{\rm grav} = \epsilon^{ABCDE}\epsilon^{\mu\nu\rho\sigma}\left(\frac{Z_A}{il}\right)\left[\left(\frac{1}{16g^2}\right)R_{\mu\nu BC}R_{\lambda\rho DE} + \sigma(x)\left\{\left(\frac{Z_F}{il}\right)^2 - 1\right\}D_\mu Z_B D_\nu Z_C D_\rho Z_D D_\sigma Z_E\right], \tag{3}$$

where $\sigma(x)$ is an auxiliary field.

We break the $SO(2,3)$ group to the local Lorentz group $SO(1,3)$ as $Z_A = (0,0,0,0,il)$. This breaking derives the vierbein $e_{\mu a}$, $D_\mu Z_A = (\partial_\mu \delta_{AB} - \omega_{\mu AB})Z_B = \begin{cases} -i\omega_{\mu a5} l \equiv e_{\mu a} & \text{if } A = a, \\ 0 & \text{if } A = 5, \end{cases}$ where the small Latin letters are $a = 1,2,3,4$. The field strength is $R_{\mu\nu ab} = \mathring{R}_{\mu\nu ab} + \frac{1}{l^2}e_{[\mu a}e_{\nu]b}$, where $\mathring{R}_{\mu\nu ab} = \partial_\mu\omega_{\nu ab} - \partial_\nu\omega_{\mu ab} - \omega_{\mu ac}\omega_{\nu cb} + \omega_{\nu ac}\omega_{\mu cb}$ is nothing but the gravitational Riemann tensor. $\mathcal{L}_{\rm grav}$ takes the Einstein gravity form

$$\mathcal{L}_{\rm grav} = \partial_\mu \mathcal{C}^\mu - \frac{e}{16\pi G}\left(\mathring{R} + \frac{6}{l^2}\right). \tag{4}$$

Here, $\partial_\mu \mathcal{C}^\mu$ is the topological Gauss-Bonnet term. $e = \det(e_{\mu a})$ and G is the gravitational constant derived from $16\pi G = g^2 l^2$. The cosmological constant is a negative term $-\left(+\frac{6}{l^2}\right)$ in the action.

In the $SO(1,4)$ case, we can construct the Lagrangian in a similar manner.

1, Dirac Let ψ be an $SO(2,3)(SO(1,4))$ Dirac fermion.

First, we consider the AdS ($SO(2,3)$) gravity. An $SO(2,3)$ invariant Dirac spinor action is defined as

$$\mathcal{L}_{\rm Dirac} = \epsilon^{ABCDE}\epsilon^{\mu\nu\rho\sigma}\bar{\psi}\left(iS_{AB}\frac{\overleftrightarrow{D}_\mu}{3!} - i\lambda\frac{Z_A}{il}\frac{D_\mu Z_B}{4!}\right)\psi D_\nu Z_C D_\rho Z_D D_\sigma Z_E, \tag{5}$$

where $\bar{\psi} = \psi^\dagger {\gamma^{(AdS)}}^5 {\gamma^{(AdS)}}^4$ and $S_{AB} = \frac{1}{4i}[{\gamma^{(AdS)}}_A, {\gamma^{(AdS)}}_B]$. By the symmetry breaking (**??**) ($Z^A = (0,0,0,0,il)$) from $SO(2,3)$ to $SO(1,3)$, $\mathcal{L}_{\rm Dirac}$ reduces to the Dirac action in the four-dimensional curved spacetime

$$\mathcal{L}_{\rm Dirac} = -e\bar{\psi}\left(\gamma_a e^{\mu a}\overleftrightarrow{D}_\mu + \lambda\right)\psi, = -e\bar{\psi}\left(\frac{1}{2}e^{\mu a}\left(\gamma_a\overrightarrow{D}_\mu - \overleftarrow{D}_\mu\gamma_a\right) + \lambda\right)\psi, \tag{6}$$

where

$$\gamma^{(AdS)}{}_a \equiv -i\gamma_5\gamma_a, \qquad \gamma^{(AdS)}{}_5 \equiv \gamma_5, \tag{7}$$

and $\bar{\psi} = \psi^\dagger\gamma^4$.

In the dS $SO(1,4)$ gravity, we consider an $SO(1,4)$ invariant Dirac spinor action

$$\mathcal{L}_{Dirac} = -\epsilon^{ABCDE}\epsilon^{\mu\nu\rho\sigma}\bar{\psi}\left(\frac{Z_A}{l}\gamma^{(dS)}{}_B\frac{\overleftrightarrow{D}_\mu}{3!} + \lambda\frac{Z_A}{l}\frac{D_\mu Z_B}{4!}\right)\psi D_\nu Z_C D_\rho Z_D D_\sigma Z_E, \tag{8}$$

which is a slightly different form from the $SO(2,3)$ case. Here, $\bar{\psi} = \psi^\dagger\gamma^{(dS)4}$ and $\bar{\psi}\gamma^{(dS)}{}_B\overleftrightarrow{D}_\mu\psi = \frac{1}{2}(\bar{\psi}\gamma^{(dS)}{}_B D_\mu\psi - \bar{\psi}\overleftarrow{D}_\mu\gamma^{(dS)}{}_B\psi)$. By the symmetry breaking from $SO(1,4)$ to $SO(1,3)$, $\mathcal{L}_{\mathrm{Dirac}}$ reduces to the Dirac action in the four-dimensional curved spacetime. if we set

$$\gamma^{(dS)}{}_A = \gamma_A, \tag{9}$$

2, Weyl Let ψ be an $SO(2,3)$ Dirac spinor. We introduce a projection operator,

$$P_\pm \equiv \frac{1}{2}\left(1 \pm \sqrt{-\frac{l^2}{Z^2}}\frac{Z_A\gamma^{(AdS)}{}_A}{il}\right), \tag{10}$$

and define $\psi_\pm \equiv P_\pm\psi$. Since $P_\pm$ is $SO(2,3)$ covariant, $\psi_\pm$ is a covariant spinor. We can construct an $SO(2,3)$ invariant action by modifying (5),

$$\mathcal{L}_{\mathrm{Weyl}} = \epsilon^{ABCDE}\epsilon^{\mu\nu\rho\sigma}\bar{\psi}_+\left(iS_{AB}\frac{\overleftrightarrow{D}_\mu}{3!} - i\lambda\frac{Z_A}{il}\frac{D_\mu Z_B}{4!}\right)\psi_+ D_\nu Z_C D_\rho Z_D D_\sigma Z_E. \tag{11}$$

If we break the $SO(2,3)$ symmetry $P_\pm$ reduces to the chiral projections $P_\pm \longrightarrow \mathring{P}_\pm = \frac{1\pm\gamma^{(AdS)}{}_5}{2} = \frac{1\pm\gamma_5}{2}$. Then, $\psi_\pm$ becomes Weyl spinors $\psi_\pm \longrightarrow \mathring{\psi}_\pm = \mathring{P}_\pm\psi$, respectively, which have definite chirality. The action (11) becomes a $SO(1,3)$ massless Weyl fermion action

$$\mathcal{L}_{\mathrm{Weyl}} = -e\bar{\mathring{\psi}}_+\left(\gamma_a e^{\mu a}\overleftrightarrow{D}_\mu + \lambda\right)\mathring{\psi}_+ = -e\bar{\mathring{\psi}}_+\left(\gamma_a e^{\mu a}\overleftrightarrow{\mathring{D}}_\mu\right)\mathring{\psi}_+, \tag{12}$$

In the $SO(1,4)$ case, we can construct the Lagrangian in a similar manner.

3, Majorana A Majorana fermion ψ_M in four-dimensional spacetime with the local Lorentz symmetry is defined by $\psi_M = \psi_M^c \equiv C\bar{\psi}_M^T$, where C is the charge conjugation in four-dimensional spacetime. If we take the Dirac (Pauli) basis, C is $C = \gamma_2\gamma_4$. However, generally C is not covariant under either $SO(2,3)$ or $SO(1,4)$. ψ_M is not consistent with the $SO(2,3)$ ($SO(1,4)$) covariance.

The condition of the $SO(2,3)$ or $SO(1,4)$ 'charge conjugation' $\tilde{C}$ is following:

1, $\tilde{C}^{-1}\gamma_A\tilde{C}$ is covariant under the symmetry $\tilde{C}^{-1}\gamma_A\tilde{C} = \pm\gamma_A^T$, in order to be consistent with the action.

2, B defined by $B\psi_M^* = \tilde{C}\bar{\psi}_M^T$ must satisfy $B^*B = 1$, since a charge conjugation has a Z_2 symmetry. ($B = \gamma_2$ for $SO(1,3)$.)

3, $\tilde{C}$ reduces to $C = \gamma_2\gamma_4$ by breaking the symmetry.

The $SO(2,3)$ charge conjugation $\tilde{C}$ which satisfies the condition 1 is

$$C_1 = \gamma^{(AdS)}{}_1\gamma^{(AdS)}{}_3\gamma^{(AdS)}{}_5, \qquad C_2 = \gamma^{(AdS)}{}_2\gamma^{(AdS)}{}_4. \tag{13}$$

from the properties of $SO(2,3)$ gamma matrices $\gamma^{(AdS)}{}_A$, Since $C_2 = \gamma^{(AdS)}{}_2\gamma^{(AdS)}{}_4 = \gamma_2\gamma_4$ is equal to the $SO(1,3)$ charge conjugation, C_2 satisfies the condition 2 and 3. Therefore, we can take $\tilde{C} = C_2$ as the $SO(2,3)$ charge conjugation. Note that C_2 is not the same as the charge conjugation in the $SO(2,3)$ spacetime symmetry in five dimensions. AdS 'Majorana' fermion ψ_M is defined by $\psi_M = \tilde{C}\bar{\psi}_M^T = C_2\bar{\psi}_M^T$.

We propose a $SO(2,3)$ invariant AdS 'Majorana' fermion action by replacing a Dirac spinor to an AdS 'Majorana' spinor in the action (5)

$$\mathcal{L}_{\text{Majorana}} = \epsilon^{ABCDE}\epsilon^{\mu\nu\rho\sigma}\bar{\psi}_M\left(iS_{AB}\frac{\overleftrightarrow{D}_\mu}{3!} - i\lambda\frac{Z_A}{il}\frac{D_\mu Z_B}{4!}\right)\psi_M D_\nu Z_C D_\rho Z_D D_\sigma Z_E. \tag{14}$$

Let us investigate the consistency of this action. Substituting the condition to the right-hand of (14), we obtain $\epsilon^{ABCDE}\epsilon^{\mu\nu\rho\sigma}\left(\psi_M^T(\tilde{C}^T)^{-1}\right)\left(iS_{AB}\frac{\overleftrightarrow{D}_\mu}{3!} - i\lambda\frac{Z_A}{il}\frac{D_\mu Z_B}{4!}\right)\left(\tilde{C}\bar{\psi}_M^T\right)D_\nu Z_C D_\rho Z_D D_\sigma Z_E$. We can easily check that this equation is equal to (14).

If we break the $SO(2,3)$ symmetry by $Z_A = (0,0,0,0,il)$, (14) reduces to an $SO(1,3)$ Majorana fermion action in the Einstein gravitational theory in four dimensions

$$\mathcal{L}_{\text{Majorana}} = -e\bar{\psi}_M\left(\gamma_a e^{\mu a}\overleftrightarrow{D}_\mu + \lambda\right)\psi_M. \tag{15}$$

Let us take two candidates for the 'charge conjugation' from the condition 1, from the $SO(1,4)$ covariance of ψ_M and $C\bar{\psi}_M^T$,

$$C_3 \equiv \gamma^{(dS)}{}_1\gamma^{(dS)}{}_3, \qquad C_4 \equiv \gamma^{(dS)}{}_2\gamma^{(dS)}{}_4\gamma^{(dS)}{}_5. \tag{16}$$

Since B constructed from both C_3 and C_4 satisfy $B^*B = -1$, neither C_3 nor C_4 can be defined as a consistent charge conjugation.

Now, we consider a third candidate:

$$C_5 \equiv \left(\frac{Z_A\gamma^{(dS)}{}_A}{l} + \left|\sqrt{\frac{Z^2-l^2}{l^2}}\right| i\right)\gamma^{(dS)}{}_2\gamma^{(dS)}{}_4\gamma^{(dS)}{}_5. \tag{17}$$

C_5 satisfies the condition 1, 2 and 3. We define a dS 'Majorana' spinor $\psi_M = \tilde{C}\bar{\psi}_M^T = C_5\bar{\psi}_M^T$. We propose an $SO(1,4)$ invariant dS 'Majorana' fermion action by replacing a Dirac spinor to a dS 'Majorana' spinor in the action (8)

$$\mathcal{L}_{\text{Majorana}} = -\epsilon^{ABCDE}\epsilon^{\mu\nu\rho\sigma}\bar{\psi}_M\left(\frac{Z_A}{l}\gamma^{(dS)}{}_B\frac{\overleftrightarrow{D}_\mu}{3!} + \lambda\frac{Z_A}{l}\frac{D_\mu Z_B}{4!}\right)\psi_M D_\nu Z_C D_\rho Z_D D_\sigma Z_E. \tag{18}$$

We can prove the consistency of the action (18) for the charge conjugation C_5 similar to $SO(2,3)$ case.

If we break the $SO(1,4)$ symmetry by $Z_A = (0,0,0,0,l)$, (18) becomes the Majorana fermion action in the Einstein gravitational theory in four dimensions.

References N. Ikeda and T. Fukuyama, "Fermions in (Anti) de Sitter Gravity in Four Dimensions," Prog. Theor. Phys. **122** (2009) 339 [arXiv:0904.1936 [hep-th]].