New Topological Field Theories from Dimensional Reduction of Nonlinear Gauge Theories

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§1. Introduction

Purpose 1.

Construct new gauge theories and apply to physics and mathematics

A new gauge theory → \{new physics (Yang-Mills, gravity, ...) \text{new mathematics(...)}

Consistency of quantum field theories ⇐⇒ a Batalin-Vilkovisky Structure

\( (S, S) = 0 \) (classical) \( (S_q, S_q) - 2i\hbar\Delta S_q = 0 \) (quantum)
Purpose 2.

Unify (and classify) all geometries and algebras as (a kind of substructures of) (super) Poisson geometry

Super Poisson geometry ⊇ a Batalin-Vilkovisky Structure

\[(S, S') = 0\] (classical) geometry
\[(S_q, S_q) - 2i\hbar \Delta S_q = 0\] (quantum geometry)

Many geometric structures are constructed as a BV structure. 
*Lie, Poisson, complex, symplectic, Kähler, Calabi-Yau, Courant, Dirac, algebroid, generalized geometry, · · ·*
We consider a topological field theory with a BF type (Schwarz type) kinetic term $S = S_0 + S_1 = \int_{\Pi T X} B dA + S_1$, as an interesting playground.

A nonlinear gauge theory is a most general topological field theory with a BF type (Schwarz type) kinetic term $S = \int_{\Pi T X} B dA + S_1$ in any dimension, which includes the Poisson sigma model, $A$-model, $B$-model, · · ·, and describes a topological membrane.

A nonlinear gauge theory constructed from superfields with the nonnegative total degree has been analyzed. We know general $S_1$.  

N.I. ’01
This Talk

We construct topological field theories which include superfields with the negative total degree, consistent with dimensional reduction and deformation,

and consider an Application to a generalized complex structure (and complex, symplectic geometry).

Technique

· Deformation Theory
  
  Barnich, Henneaux '93, Barnich, Brandt, Henneaux, '95

· Dimensional Reduction
  
  Kaluza, '21, Klein, '26
Plan of Talk

- Review of the AKSZ formalism and nonlinear gauge theories
- Dimensional reduction from 3 dimensions to 2 dimensions
- The Poisson sigma model with a two-form
- Application to a generalized complex structure
\section*{\S 2. AKSZ Formalism of Nonlinear Gauge Theories}

\(X\): a manifold in \(n\) dimensions (worldsheet or worldvolume)
\(M\): a manifold in \(d\) dimensions (target space)
a map \(\phi : X \rightarrow M\)

The \textbf{AKSZ Formalism} of the Batalin-Vilkovisky formalism is formulated by three elements,

- \textbf{Supermanifold} (Superfield \(\Phi\))
- \textbf{P-structure} (Antibracket \((\ast, \ast)\))
- \textbf{Q-structure} (BV action \(S\))

Alexandrov, Kontsevich, Schwartz, Zaboronsky '97
• **Supermanifold** (Superfield)
  
  ○ $X$ to supermanifold

  $TX$: a tangent bundle

  $\Pi TX$: A *supermanifold* is a tangent bundle with reversed parity of the fiber.

  · local coordinates
  
  $\{\sigma^\mu\}$ on $X$, where $\mu = 1, 2, \cdots, n$

  $\{\theta^\mu\}$ on $T_\sigma X$, fermionic supercoordinate

  Def: **form degree** $\deg \sigma = 0$ and $\deg \theta = 1$

  We extend a smooth map $\phi : X \longrightarrow M$ to a map $\phi : \Pi TX \longrightarrow M$. 
This procedure introduces ghosts and antifields systematically. A function on $\Pi TX$ is called a superfield.

$$\phi^i = \phi^{(0)i} + \theta^{\mu_1} \phi_{\mu_1}^{(-1)i} + \frac{1}{2!} \theta^{\mu_1} \theta^{\mu_2} \phi_{\mu_1 \mu_2}^{(-2)i} + \cdots + \frac{1}{n!} \theta^{\mu_1} \cdots \theta^{\mu_n} \phi_{\mu_1 \cdots \mu_n}^{(-n)i}.$$ 

○ $M$ to supermanifold (a graded manifold)

We introduce superfield pairs with the nonnegative total degrees and with the sum of the total degree $n - 1$.

$$\{ A^{a_p} p, B_{n-p-1, a_p} \}, \ p = 0, 1, \cdots, \lfloor \frac{n-1}{2} \rfloor$$
1. $T^*M$: a cotangent bundle

$T^*[n-1]M$: A *graded cotangent bundle* is a cotangent bundle with the degree of the fiber $n-1$.

- local coordinates $\{\phi^i, B_{n-1,i}\}$
  - $\{\phi^i = A_0^i\}$: a map from $\Pi TX$ to $M$, where $i = 1, 2, \ldots, d$.
  - $\{B_{n-1,i}\}$: a superfield on $\Pi TX$, which take a value on $\phi^*(T^*[n-1]M)$

$p$ even $\implies \{B_{p,i}\}$ bosonic, $p$ odd $\implies \{B_{p,i}\}$ fermionic

**Def:** total degree: $|\phi| = 0$ and $|B_{n-1}| = n - 1$

**Def:** ghost number: $\text{gh} F = |F| - \deg F$, where $F$ is a superfield.
2. \( E \): a vector bundle on \( M \)
\( E_p[p] \): a vector bundle with the degree of the fiber \( p \)

We consider \( E_p[p] \oplus E_p^*[n-p-1] \): \( p \) is an integer with \( 1 \leq p \leq n-2 \).

· local coordinates \( \{ A_p^{a_p}, B_{n-p-1,a_p} \} \)

\( A_p^{a_p} \): a total degree \( p \) superfield on \( \Pi TX \), which take a value on \( \phi^*(E[p]) \)

\( B_{n-p-1,a_p} \): a total degree \( n-p-1 \) superfield on \( \Pi TX \), which take a value on \( \phi^*(E^*[n-p-1]) \)

\( \lfloor x \rfloor \): the floor function which gives the largest integer less than or equal to \( x \in \mathbb{R} \).
If $\lfloor \frac{n}{2} \rfloor \leq p \leq n - 2$, we can identify $E[p] \oplus E^*[n - p - 1]$ with the dual bundle $E^*[n - p - 1] \oplus (E^*)^*[p]$.

We consider graded bundle,

$$T^*[n - 1] \left( \sum_{p=1}^{\lfloor \frac{n-1}{2} \rfloor} E_p[p] \right),$$

$$\{ A^{\alpha_p}_p, B_{n-p-1,\alpha_p} \}, \ p = 0, 1, \ldots, \lfloor \frac{n-1}{2} \rfloor$$ in the local coordinate expression.
\[ n = 2 \]

\[ T^*[1]M. \]
• **P-structure** is a graded Poisson structure on a graded (or super) manifold $\tilde{\mathcal{N}}$, whose bracket called the **antibracket** $(\ast, \ast)$ with the total degree $-p$ satisfies the following identities:

\[
(F, G) = -(-1)^{|F|-p}(|G|-p)(G, F),
\]

\[
(F, GH) = (F, G)H + (-1)^{|F|-p}|G|G(F, H),
\]

\[
(FG, H) = F(G, H) + (-1)^{|G|(|H|-p)}(F, H)G,
\]

\[
(-1)^{|F|-p}(|H|-p)(F, (G, H)) + \text{cyclic permutations} = 0,
\]
1. $T^*[n-1]M$ has a natural P-structure induced from a natural Poisson (symplectic) structure on $T^*M$:

$$(F, G) \equiv F \frac{\partial}{\partial \phi^i} \frac{\partial}{\partial B_{n-1,i}} G - F \frac{\partial}{\partial B_{n-1,i}} \frac{\partial}{\partial \phi^i} G.$$ 

in a Darboux coordinate. The total degree is $-n + 1$.

2. An antibracket on $E_p[p] \oplus E_p^*[n-p-1]$

$$(F, G) \equiv F \frac{\partial}{\partial A_p^a p} \frac{\partial}{\partial B_{n-p-1,a_p}} G - (-1)^{p(n-p-1)} F \frac{\partial}{\partial B_{n-p-1,a_p}} \frac{\partial}{\partial A_p^a p} G$$

in a Darboux coordinate. The total degree is $-n + 1$. 

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• **Q-structure** called a *BV action* $S$ is a function on a target graded bundle, and a functional on $\Pi T^* X$, which satisfies the classical master equation $(S, S) = 0$.

We require $\hbar hS = 0$.

$\delta F = (S, F)$: a *BRST transformation*, satisfies $\delta^2 = 0$.  


§3. BV Action of Nonlinear Gauge Theories

- **Q-structure (BV Action)**

\[ S = S_0 + S_1, \]

where

\[ S_0 = \sum_{p=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^{n-p} \int_{\Pi T X} B_{n-p-1,a_p} dA_p^a_p, \]

where \( d = \theta^\mu \partial_\mu, \int_{\Pi T X} \equiv \int_{\Pi T X} d^n \theta d^n \sigma. \)
The general solution $S_1 = \int_{\Pi T X} F(\Phi)$, where $F$ is an arbitrary function of all superfields $\Phi$ with the total degree $n$, which satisfies $(S_1, S_1) = 0$, up to total derivatives and BRST exact terms. Higher order deformations vanish.

Izawa, ’00, N.I, ’00 ’01
Bizdadea, Ciobirca, Cioroianu, Saliu, Sararu ’00 ’01 ’02 ’03

We call a resulting field theory $S = S_0 + S_1$ a nonlinear gauge theory in $n$ dimensions, a topological sigma model in $n$ dimensions, or a topological $(n - 1)$-brane.
Example 1. \( n = 2 \)
- **Supermanifold** \( T^*[1]M \)
- **Antibracket** \( (F, G) \equiv F \leftarrow \frac{\partial}{\partial \phi^i} \frac{\partial}{\partial B_{1,i}} G - F \frac{\partial}{\partial B_{1,i}} \frac{\partial}{\partial \phi^i} G. \)
- **BV Action** is the Poisson sigma model

\[
S_0 = \int_{\Pi T X} B_{1i} d\phi^i, \quad S_1 = \int_{\Pi T X} \frac{1}{2} f^{ij}(\phi) B_{1i} B_{1j},
\]

\((S_1, S_1) = 0\) derives
\[
f^{kl} \frac{\partial}{\partial \phi^l} f^{ij} + f^{il} \frac{\partial}{\partial \phi^l} f^{jk} + f^{jl} \frac{\partial}{\partial \phi^l} f^{ki} = 0
\]
\(\iff \pi = - f^{ij} \frac{\partial}{\partial \phi^l} \wedge \frac{\partial}{\partial \phi^l}\) is a Poisson bivector field.
Example 2. \( n = 3 \)

- **Supermanifold** \( T^*[2]E[1] \)

- **Antibracket**

\[
(F, G) \equiv F \frac{\bar{\partial}}{\partial \phi^i} \frac{\partial}{\partial B_{2,i}} G - F \frac{\bar{\partial}}{\partial B_{2,i}} \frac{\partial}{\partial \phi^i} G + F \frac{\bar{\partial}}{\partial A_1^a} \frac{\partial}{\partial B_{1,a}} G + F \frac{\bar{\partial}}{\partial B_{1,a}} \frac{\partial}{\partial A_1^a} G
\]

- **BV Action**

The Courant sigma model

\[
S = S_0 + S_1,
\]

\[
S_0 = \int_{\Pi TX} \left[ -B_{2i} d\phi^i + B_{1a} dA_1^a \right],
\]
\[ S_1 = \int_{\Pi T X} \left[ f_1^a i(\phi) B_{2i} A_1^a + f_2^{ib}(\phi) B_{2i} B_{1b} ight. \\
+ \frac{1}{3!} f_{3abc}(\phi) A_1^a A_1^b A_1^c + \frac{1}{2} f_{4ab^c}(\phi) A_1^a A_1^b B_{1c} \\
+ \frac{1}{2} f_{5ab^c}(\phi) A_1^a B_{1b} B_{1c} + \frac{1}{3!} f_{6abc}(\phi) B_{1a} B_{1b} B_{1c}] , \quad (1) \]

\((S_1, S_1) = 0 \iff \) six \( f \)'s are structure functions of a Courant algebroid.

N.I. ’02, Hofman, Park ’02, Roytenberg ’01, ’06

Example 3. \( n = \text{general} \)

has a structure to the \( n \)-algebroid.
§4. Dimensional Reduction

Dimensional reduction of $X$ in $n$ dimensions to $\Sigma$ in $m$ dimensions ($n > m$). Reduction of each superfield $\Phi$ is

$$\Phi(\sigma^1, \ldots, \sigma^n) = \tilde{\Phi}|_{\Phi}(\sigma^1, \ldots, \sigma^m)$$

$$+ \sum_{p=1, \mu_q=n-m+1, \ldots, n}^{n-m} \theta^{\mu_1} \cdots \theta^{\mu_p} \tilde{\phi}|_{\Phi_{-p, \mu_1\cdots\mu_p}}(\sigma^1, \ldots, \sigma^m),$$

where $\deg \theta = 1$, $|\theta| = 1$, $gh\theta = 0$.

This procedure produces a new topological field theory on $\Sigma$, including a superfield with the **negative** total degree.
§5. Dimensional Reduction of the Courant Sigma Model in 3 Dimensions to 2 Dimensions

Reduction $X$ to $\Sigma$, $(\sigma^1, \sigma^2, \sigma^3) \rightarrow (\sigma^1, \sigma^2)$,

**Procedure 1.** Reduction of Superfields

\begin{align*}
\phi^i(\sigma^\mu, \theta^\mu) &= \tilde{\phi}^i(\sigma^1, \sigma^2, \theta^1, \theta^2) + \theta^3 \tilde{\phi}_{-1}^i(\sigma^1, \sigma^2, \theta^1, \theta^2), \\
B_{2i}(\sigma^\mu, \theta^\mu) &= \tilde{B}_{2i}(\sigma^1, \sigma^2, \theta^1, \theta^2) + \theta^3 \tilde{\beta}_{1i}(\sigma^1, \sigma^2, \theta^1, \theta^2), \\
A_1^a(\sigma^\mu, \theta^\mu) &= \tilde{A}_1^a(\sigma^1, \sigma^2, \theta^1, \theta^2) + \theta^3 \tilde{\alpha}_0^a(\sigma^1, \sigma^2, \theta^1, \theta^2), \\
B_{1a}(\sigma^\mu, \theta^\mu) &= \tilde{B}_{1a}(\sigma^1, \sigma^2, \theta^1, \theta^2) + \theta^3 \tilde{\beta}_0^a(\sigma^1, \sigma^2, \theta^1, \theta^2),
\end{align*}

(2)

All these superfields do not depend on $\sigma^3$ and $\theta^3$. $|\tilde{\phi}^i| = |\tilde{\alpha}_0^a| =$
\[ |\tilde{\beta}_{0a}| = 0, |\tilde{\phi}_{-1}^i| = -1, |\tilde{A}_1^a| = |\tilde{B}_{1a}| = |\tilde{\beta}_{1i}| = 1, |\tilde{B}_{2i}| = 2. \]

**Procedure 2.** Substitute to 3D Action

do not derive the correct AKSZ action in 2 dimensions.

\((\theta^3)^2 = 0\) drops some terms in three dimensions but generally the extra terms appear in a reduced action.

The existence of the negative total degree superfield \(\tilde{\phi}_{-1}^i\) complexifies the dimensional reduction in the AKSZ formalism.

In order to derive the correct AKSZ action, first we should consider the dimensional reduction via the non-BV formalism.
Procedure 1. Expansion of superfields by the ghost numbers

\[ \phi^i = \phi^i + \phi^{(-1)i} + \phi^{(-2)i} + \phi^{(-3)i}, \]
\[ B_{2,i} = B_{2,i}^{(2)} + B_{2,i}^{(1)} + B_{2,i} + B_{2,i}^{(-1)}, \]
\[ B_{1a} = B_{1,a}^{(1)} + B_{1,a} + B_{1,a}^{(-1)} + B_{1,a}^{(-2)}, \]
\[ A_1^a = A_1^{(1)a} + A_1^a + A_1^{(-1)a} + A_1^{(-2)a}, \]  \quad (3)

where \( \phi^i = \phi^{(0)i}, \) etc.
Procedure 2. Setting $\Phi^{(p)} = 0$ for $p \neq 0$ in the action (antifield & ghost $= 0$) produces the 3D non-BV action.

Procedure 3. Dimensional Reduction

$$
\begin{align*}
\phi^i(\sigma^\mu, \theta^\mu) &= \tilde{\phi}^i(\sigma^1, \sigma^2, \theta^1, \theta^2), \\
B_{2i}(\sigma^\mu, \theta^\mu) &= \tilde{B}_{2i}(\sigma^1, \sigma^2, \theta^1, \theta^2) + \theta^3 \tilde{\beta}_{1i}(\sigma^1, \sigma^2, \theta^1, \theta^2), \\
A_{1a}(\sigma^\mu, \theta^\mu) &= \tilde{A}_{1a}(\sigma^1, \sigma^2, \theta^1, \theta^2) + \theta^3 \tilde{\alpha}_{0a}(\sigma^1, \sigma^2, \theta^1, \theta^2), \\
B_{1a}(\sigma^\mu, \theta^\mu) &= \tilde{B}_{1a}(\sigma^1, \sigma^2, \theta^1, \theta^2) + \theta^3 \tilde{\beta}_{0a}(\sigma^1, \sigma^2, \theta^1, \theta^2),
\end{align*}
$$

(4)

3D action to 2D action

$$
S = S_0 + S_1
$$
\[ S_0 = \int d\theta^3 \theta^3 d\sigma^3 \int_{\Pi T \Sigma} \left( \tilde{\beta}_1^i \wedge d\tilde{\phi}^i + \tilde{\Lambda}_1^a \wedge d\tilde{\beta}_0^a + \tilde{\beta}_1^a \wedge d\tilde{\alpha}_0^a \right) \]

\[ S_1 = \int d\theta^3 \theta^3 d\sigma^3 \int_{\Pi T \Sigma} f_{1a}^i \tilde{\Lambda}_1^a \tilde{\beta}_1^i + f_{2}^{ib} \tilde{B}_{1b} \tilde{\beta}_1^i \]

\[ + \frac{1}{2} (f_{3abc} \tilde{\alpha}_0^c + f_{4ab}^c \tilde{\beta}_0^c) \tilde{\Lambda}_1^a \tilde{\Lambda}_1^b + (-f_{4ab}^c \tilde{\alpha}_0^b + f_{5a}^{cb} \tilde{\beta}_0^b) \tilde{\Lambda}_1^a \tilde{B}_{1c} \]

\[ + \frac{1}{2} (f_{5a}^{bc} \tilde{\alpha}_0^a + f_{6}^{abc} \tilde{\beta}_0^a) \tilde{B}_{1b} \tilde{B}_{1c} + (f_{1b}^i \tilde{\alpha}_0^b + f_{2}^{ia} \tilde{\beta}_0^a) \tilde{B}_{2i}, \]

\[ (5) \]

Procedure 4. 2D AKSZ Action

We need a new theory.
§6. The Poisson sigma model with a 2-form

\[ S_t = \int_{\Pi T \Sigma} \tilde{B}_{1A} d\tilde{\Phi}^A + \frac{1}{2} F^{AB}(\tilde{\Phi}) \tilde{B}_{1A} \tilde{B}_{1B} + G^A(\tilde{\Phi}) \tilde{B}_{2A}, \tag{6} \]

where \( \tilde{B}_{2A} \) is a total degree 2 superfield. The reduced action (5) is obtained by setting

\[ \tilde{\Phi}^A = (\tilde{\phi}^i, \tilde{\beta}_{0a}, \tilde{\alpha}_0^b), \tilde{B}_{1A} = (\tilde{\beta}_{1i}, \tilde{A}_1^a, \tilde{B}_{1b}), \tilde{B}_{2A} = (\tilde{B}_{2i}, 0, 0), \]

\[ F^{AB} = \begin{pmatrix} 0 & -f_1 c^i & -f_2 i^d \\ f_1 a^j & f_{3ac} e_0^e + f_{4ae} e_0^e & -f_{4ae} e_0^e + f_{5a} d^e \tilde{\beta}_{0e} \\ f_2 b^c & f_{4be} c_0^e - f_{5b} c_0^e & f_{5e} b^d \tilde{\alpha}_0^e + f_{6} b^d e_0^e \end{pmatrix}, \]

\[ G^A = (f_{1b} i^b \tilde{\alpha}_0^b + f_{2a} i^a \tilde{\beta}_{0a}, 0, 0), \tag{7} \]
We take $A = (i, a, b)$ indices on $M$, $E^\phi_\tilde{\phi}$ and $E^\sim_\tilde{\phi}$. The action (5) is a particular case of the action (6) on $E \oplus E^\ast$.

The action has the following gauge symmetry:

$$\delta \tilde{\Phi}^A = -F^{AB}c_B,$$

$$\delta \tilde{B}_1^A = dc_A + \frac{\partial F^{BC}}{\partial \tilde{\Phi}^A} \tilde{B}_1^B \tilde{B}_1^C - \frac{\partial G^B}{\partial \tilde{\Phi}^A} t_B,$$

$$\delta \tilde{B}_2^A = dt_A + U^{BC}_A (\tilde{B}_1^B t_C - \tilde{B}_2^C c_B) + \frac{1}{2} X^{BCD}_A \tilde{B}_1^B \tilde{B}_1^C c_D,$$

where $c_A$ is a gauge parameter with the total degree 1 and $t_A$ is a gauge parameter with the total degree 2. $X^{BCD}_A$ is completely antisymmetric with respect to the indices $BCD$. 
In fact, the action $S$ is gauge invariant

$$\delta S_t = \int_{\Pi T \Sigma} d(c_A d\Phi_A + G^A t_A),$$

if and only if $F$, $G$, $U$ and $X$ satisfy the identities

$$F^D[A \partial F^{BC}] = G^D X_D^{ABC},$$

$$F^{AB} \frac{\partial G^C}{\partial \Phi_A} + U_A^{BC} G^A = 0,$$

Deformation of the Poisson bivector by the 3-forms $G^D X_D^{ABC}$.

cf. WZ Poisson sigma model

Klimcik, Strobl ’01
§7. AKSZ action of the Poisson sigma model with a 2-form

- Fields $\rightarrow$ Superfields

\[
\begin{align*}
\tilde{B}_{1A} &= \tilde{B}_{1A}^{(1)} + \tilde{B}_{1A} + \tilde{B}_{1A}^{(-1)}, \\
\tilde{\Phi}^A &= \Phi^A + \Phi^{(-1)A} + \Phi^{(-2)A}, \\
\tilde{B}_{2A} &= \tilde{B}_{2i}^{(2)} + \tilde{B}_{2i}^{(1)} + \tilde{B}_{2i} \\
\tilde{\phi}_{-1}^A &= \tilde{\phi}_{-1}^{(-1)i} + \tilde{\phi}_{-1}^{(-2)i} + \tilde{\phi}_{-1}^{(-3)i}
\end{align*}
\]

where $|\tilde{\phi}_{-1}^A| = -1$. 

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• **P-structure** (Antibracket)

\[
(F, G) = F \frac{\partial}{\partial \tilde{\Phi}^A} G - F \frac{\partial}{\partial \tilde{B}_1^A} \frac{\partial}{\partial \tilde{\Phi}^A} G
\]

\[
+ F \frac{\partial}{\partial \tilde{\phi}_{-1}^A} \frac{\partial}{\partial \tilde{B}_2^A} G, - F \frac{\partial}{\partial \tilde{B}_2^A} \frac{\partial}{\partial \tilde{\phi}_{-1}^A} G
\]

• **BV Action** \( S = S_0 + S_1 \)

\[
S_0 = \int_{\Pi T \Sigma} \tilde{B}_1^A d\tilde{\Phi}^A - \tilde{B}_2^A d\tilde{\phi}_{-1}^A
\]
Procedure to obtain $S_1$

1. Consider most general (consistent) deformation of $S_0 \rightarrow S$ in the sense of BBH. (Write down all the possible terms)
2. The BV action of $S_t$ is a particular case of the resulting action $S$.
3. If $S|_{ghS=0} = S_0 + S_1|_{ghS=0} = S_t$, it is the correct AKSZ action.

Def: $\text{neg}(F)$: a \emph{negative total degree} of a superfield $F$

$\text{neg}(\tilde{\phi}_-^A) = 1$ and $\text{neg} (\text{other superfields}) = 0$
We expand $S_1$ by the negative total degrees:

$$S_1 = \sum_{p=0}^{\infty} S_1^{(p)} = \sum_{p=0}^{\dim M} \int_{\Pi T \Sigma} \mathcal{L}_1^{(p)} = \sum_{p} \int_{\Pi T \Sigma} \tilde{\phi}^{-i_1} \cdots \tilde{\phi}^{-i_p} \mathcal{L}_{i_1 \ldots i_p}^{(p)}(\tilde{\Phi}, \tilde{B}_1, \tilde{B}_2)$$

$$(S_0, S_1) = 0 \implies \mathcal{L}_1^{(0)} = \frac{1}{2} f_{1AB}(\tilde{\Phi}) \tilde{B}_1^A \tilde{B}_1^B + f_{2A}^{A}(\tilde{\Phi}) \tilde{B}_2^A,$$

$$\mathcal{L}_1^{(1)} = \frac{1}{3!} f_{3A BCD}(\tilde{\Phi}) \tilde{\phi}^{-A} \tilde{B}_1^B \tilde{B}_1^C \tilde{B}_1^D$$

$$+ f_{4A}^{BC}(\tilde{\Phi}) \tilde{\phi}^{-A} \tilde{B}_1^B \tilde{B}_2^C,$$

$$\mathcal{L}_1^{(2)} = \cdots,$$
and so on, where $f_i(\tilde{\Phi})$ is a function of $\tilde{\Phi}^A$.

$$(S_1, S_1) = \sum_p (S_1, S_1)^{(p)} = 0,$$

(9)

determines the identities of $f_i(\Phi)$ recursively, where $(S_1, S_1)^{(p)}$ is the negative total degree $p$ part of $(S_1, S_1)$.

Batalin, Marnelius '01, N.I., Izawa '04
0-th order:

\[(S_1, S_1)^{(0)} = 0 \implies f_1^{D[A} \frac{\partial f_1^{BC]}{\partial \Phi^D} - f_2^D f_3^{ABC} = 0,\]

\[f_1^{AB} \frac{\partial f_2^C}{\partial \Phi^A} + f_4^{A}{}^{BC} f_2^A = 0.\]

\[f_1^{AB} = F^{AB}, \quad f_2^A = G^A,\]

\[f_3^{A}{}^{BCD} = X_A^{BCD}, \quad f_4^{A}{}^{BC} = U_A^{BC},\]

\[S|_{g_S=0} = S_t.\]

The action obtained by deformation is the correct action compatible with dimensional reduction. Substitute (7).
§8. Application – Generalized Complex Structure

3D topological sigma model with a generalized complex structure on $TM \oplus T^*M$ ($H = 0$):

\[ \mathcal{S} = \int_{\pi TX} -\frac{1}{2}\langle 0 + B_2, d(\phi + 0) \rangle + \frac{1}{4}\langle A_1 + B_1, d(A_1 + B_1) \rangle \\
- \langle 0 + B_2, \mathcal{J}(A_1 + B_1) \rangle - \frac{1}{2}\langle A_1 + B_1, A_1^i \frac{\partial \mathcal{J}}{\partial \phi^i}(A_1 + B_1) \rangle \\
= \int_{\pi TX} -\frac{1}{2}B_2i d\phi^i + \frac{1}{2}B_1i dA_1^i - J^i_j B_{2i} A_j^j - P^{ij} B_{2i} B_{1j} \\
+ \frac{1}{2} \frac{\partial Q_{jk}}{\partial \phi^i} A^i_1 A^j_1 A^k_1 + \frac{1}{2} \left( -\frac{\partial J^k_j}{\partial \phi^i} + \frac{\partial J^i_k}{\partial \phi^j} \right) A^i_1 A^j_1 B_{1k} + \frac{1}{2} \frac{\partial P^{jk}}{\partial \phi^i} A^i_1 B_{1j} B_{1k}. \]
is derived by setting,

\begin{align*}
 f_{1j}^i &= -J^i_j, \
 f_{2ij}^j &= -P^{ij}, \
 f_{3ijk} &= \frac{\partial Q^k_j}{\partial \phi^i} + \frac{\partial Q^i_k}{\partial \phi^j} + \frac{\partial Q^i_j}{\partial \phi^k}, \
 f_{4ij}^k &= -\frac{\partial J^k_j}{\partial \phi^i} + \frac{\partial J^k_i}{\partial \phi^j}, \
 f_{5ijk} &= \frac{\partial P^k_j}{\partial \phi^i}, \
 f_{6ijk} &= 0,
\end{align*}

in the Courant sigma model, where $\mathcal{J}$ is a generalized complex structure:

$$
\mathcal{J} = \begin{pmatrix} J & P \\ Q & -^tJ \end{pmatrix}.
$$

$(S, S) = 0 \iff \mathcal{J}$ is a generalized complex structure.
This action derives the Zucchini model (the Hitchin sigma model) as a boundary action if $\Sigma = \partial X$:

$$S_Z = \int_{\Pi T \Sigma} \tilde{B}_1 dx + J^i \tilde{B}_1 d\tilde{\phi}^i + \frac{1}{2} P_{ij} \tilde{B}_1 \tilde{B}_1 + \frac{1}{2} Q_{ij} d\tilde{\phi}^i d\tilde{\phi}^j$$

- Dimensional reduction provides a new 2D action with GCS

$$S_0 = \int_{\Pi T \Sigma} \frac{1}{2} \left( \tilde{\beta}_1 i d\tilde{\phi}^i - \tilde{B}_2 i d\tilde{\phi}^i - \tilde{B}_1 i d\tilde{a}_0^i + \tilde{A}_1 i d\tilde{\beta}_0^i \right)$$

$$S_1^{(0)} = \int_{\Pi T \Sigma} -J^i j \tilde{A}_1 \tilde{\beta}_1 i + P_{ij} \tilde{B}_1 i \tilde{\beta}_1 j$$

$$+ \frac{1}{2} \left( \left( \frac{\partial Q_{jk}}{\partial \tilde{\phi}^i} + \frac{\partial Q_{ij}}{\partial \tilde{\phi}^k} + \frac{\partial Q_{ki}}{\partial \tilde{\phi}^j} \right) \tilde{a}_0^k + \left( -\frac{\partial J_{jk}}{\partial \tilde{\phi}^i} + \frac{\partial J_{ij}}{\partial \tilde{\phi}^k} \right) \tilde{\beta}_0^k \right)$$
\[ \times \tilde{A}_1^i \tilde{A}_1^j + \left( \left( \frac{\partial J^k_j}{\partial \tilde{\phi}^i} - \frac{\partial J^k_i}{\partial \tilde{\phi}^j} \right) \tilde{\alpha}_0^j - \frac{\partial P^{jk}_i}{\partial \tilde{\phi}^l} \tilde{\beta}_0^j \right) \tilde{A}_1^i \tilde{B}_{1k} \]

\[ + \frac{1}{2} \left( \frac{\partial P^{jk}_i}{\partial \tilde{\phi}^l} \tilde{\alpha}_0^i \right) \tilde{B}_{1j} \tilde{B}_{1k} - \left( J^i_j \tilde{\alpha}_0^j + P^{ij} \tilde{\beta}_0^j \right) \tilde{B}_{2i}, \]

\[ S_{1(1)}^{(1)} = \int_{\Pi T \Sigma} l \left[ \frac{\partial J^i_j}{\partial \tilde{\phi}} \tilde{B}_{2i} \tilde{A}_1^j + \frac{\partial P^{ij}_l}{\partial \tilde{\phi}} \tilde{B}_{2j} \tilde{B}_{1i} - \frac{1}{2} \frac{\partial^2 Q_{jk}}{\partial \tilde{\phi}^i \partial \tilde{\phi}^j} \tilde{A}_1^i \tilde{A}_1^j \tilde{A}_1^k \right. \]

\[ - \frac{1}{2} \frac{\partial}{\partial \tilde{\phi}^l} \left( - \frac{\partial J^k_j}{\partial \tilde{\phi}^i} + \frac{\partial J^k_i}{\partial \tilde{\phi}^j} \right) \tilde{A}_1^i \tilde{A}_1^j \tilde{B}_{1k} - \frac{1}{2} \frac{\partial^2 P^{jk}_i}{\partial \tilde{\phi}^i \partial \tilde{\phi}^l} \tilde{A}_1^i \tilde{B}_{1j} \tilde{B}_{1k} \right]. \]

\[ S_{1(2)}^{(2)} = \ldots, \]
§9. Two Special Reductions to Complex Geometry and Symplectic Geometry

• $P = Q = 0$ (Complex geometry)

$(S, S') = 0 \iff J = \text{complex structure}$

This condition is invariant under the redefinition ($\lambda$: constant):

\[
\begin{align*}
\tilde{\phi}^i &= \tilde{\phi}^i, \\
\tilde{\phi}_{-1}^i &= \lambda \tilde{\phi}_{-1}'^i, \\
\tilde{B}_{2i} &= \lambda \tilde{B}_{2i}', \\
\tilde{\beta}_1^i &= \frac{1}{2} \tilde{\beta}_1'^i, \\
\tilde{A}_1^i &= \frac{1}{2} \tilde{A}_1'^i, \\
\tilde{\alpha}_0^i &= \lambda \tilde{\alpha}_0'^i, \\
\tilde{B}_{1i} &= \lambda \tilde{B}_{1i}', \\
\tilde{\beta}_0^i &= -\tilde{\beta}_0'^i;
\end{align*}
\] (11)
\[ S_0 = \int_{\Pi T \Sigma} \frac{1}{4} \left( \beta_1 d\phi^i - \tilde{A}_1^i d\beta_0^i \right) + \frac{\lambda^2}{2} \left( - \tilde{B}_2 d\phi_{-1}^i + \tilde{B}_1 d\alpha_0^i \right), \]

\[ S_{1}^{(0)} = \int_{\Pi T \Sigma} \frac{1}{4} \left( J_{ij} \beta_{1i} \tilde{A}_{1j} + \frac{\partial J_{j}}{\tilde{\phi}_i} \beta_{0k} \tilde{A}_{1i} \tilde{A}_{1j} \right) \]

\[ \quad + \lambda^2 \left( \frac{1}{2} \frac{\partial J_{k}}{\tilde{\phi}_i} \alpha_{0j} \tilde{A}_{1i} \tilde{B}_{1k} - J_{ij} \alpha_{0j} \tilde{B}_{2i} \right), \]

\[ S_{1}^{(1)} = \int_{\Pi T \Sigma} \lambda \tilde{\phi}_{-1} \left[ \frac{\lambda \partial J_{ij}}{2 \tilde{\phi}_i} \tilde{B}_{2i} \tilde{A}_{1j} + \frac{\lambda \partial^2 J_{ij}}{4 \tilde{\phi}_i \tilde{\phi}_j} \tilde{A}_{1i} \tilde{A}_{1j} \tilde{B}_{1k} \right], \]

\[ S_{1}^{(p)} \sim \lambda^p. \]
If $\lambda \to 0$, $S_1^{(p)} \to 0$ for $p > 0$. The 2D action is

$$S_J = \frac{1}{4} \int_{\Pi T \Sigma} \tilde{\beta}_i d\tilde{\phi}^i - \tilde{A}_1 i d\tilde{\beta}_0 + J^i j \tilde{\beta}_1 i \tilde{A}_1 j + \frac{\partial J^i k}{\partial \tilde{\phi}^j} \tilde{\beta}_0 i \tilde{A}_1 j \tilde{A}_1 k.$$

This action is nothing but the $B$-model action on $T^*[1](T^*M)$ in AKSZ.

$(S_J, S_J) = 0 \iff J = \text{complex structure.}$
• $J = 0$ (Symplectic(Poisson) geometry)

$(S, S') = 0 \iff Q = P^{-1} = \text{symplectic structure}$

The condition is invariant under the redefinition: ($\mu$: constant.)

\[ \begin{align*}
\tilde{\phi}^i &= \phi^i, \quad \tilde{\phi}^{-1}_i = \mu \phi^{-1}_i, \\
\tilde{B}_{2i} &= \mu \tilde{B}'_{2i}, \quad \tilde{\beta}_1 i = \frac{1}{2} \tilde{\beta}'_{1i}, \\
\tilde{A}_1^i &= \mu \tilde{A}'_1^i, \quad \tilde{\alpha}_0^i = \tilde{\alpha}'_0^i, \\
\tilde{B}_{1i} &= \frac{1}{2} \tilde{B}'_{1i}, \quad \tilde{\beta}_0 i = -\mu \tilde{\beta}'_{0i},
\end{align*} \] (12)
\begin{align*}
S_0 &= \int_{\Pi T \Sigma} \frac{1}{4} \left( \tilde{\beta}'_1 d\tilde{\phi}^i + \tilde{B}'_1 d\tilde{\alpha}'_0 \right) + \frac{\mu^2}{2} \left( -\tilde{B}'_2 d\tilde{\phi}'_1 - \tilde{A}'_1 d\tilde{\beta}'_0 \right)
S_1^{(0)} &= \int_{\Pi T \Sigma} \frac{1}{4} \left( P^{ij} \tilde{B}'_{1i} \tilde{\beta}'_{1j} + \frac{1}{2} \frac{\partial P^{jk}}{\partial \tilde{\phi}} \tilde{\alpha}'_0 \tilde{B}'_{1j} \tilde{B}'_{1k} \right) - \frac{1}{2} \frac{\partial P^{jk}}{\partial \tilde{\phi}} \tilde{\beta}'_{0j} \tilde{A}'_1 \tilde{B}'_{1k}
&\quad + \mu^2 \left( \frac{1}{2} \left( \frac{\partial Q_{jk}}{\partial \tilde{\phi}} + \frac{\partial Q_{ij}}{\partial \tilde{\phi}} + \frac{\partial Q_{ki}}{\partial \tilde{\phi}} \right) \tilde{\alpha}'_0 \tilde{A}'_1 \tilde{A}'_1 + P^{ij} \tilde{\beta}'_{0j} \tilde{B}'_{2i} \right),
S_1^{(1)} &= \int_{\Pi T \Sigma} \mu \tilde{\phi}_{-1} \left[ \frac{\mu}{2} \frac{\partial P^{ij}}{\partial \tilde{\phi}} \tilde{B}'_{2i} \tilde{B}'_{1j} - \frac{\mu^3}{2} \frac{\partial^2 Q_{jk}}{\partial \tilde{\phi} \partial \tilde{\phi}} \tilde{A}'_1 \tilde{A}'_1 \tilde{A}'_1 \tilde{A}'_1 \right.
&\quad - \frac{\mu}{8} \frac{\partial^2 P^{jk}}{\partial \tilde{\phi} \partial \tilde{\phi}} \tilde{A}'_1 \tilde{B}'_{1j} \tilde{B}'_{1k} \right],
\end{align*}
\[ S_1^{(p)} \sim \mu^p. \]

If \( \mu \to 0 \), \( S_1^{(p)} \to 0 \) for \( p > 0 \).

After taking the limit \( \mu \to 0 \) with preserving the symplectic structure, \( S_1^{(p)} \) for \( p > 0 \) reduces to zero, and the 2D action is

\[
S_P = \frac{1}{4} \int_{\Pi T \Sigma} \bar{\beta}_1 d\bar{\phi}^i + \bar{B}_1 d\bar{\alpha}_0^i + P^{ij} \bar{B}_1 \bar{\beta}_1^j + \frac{1}{2} \frac{\partial P^{jk}}{\partial \bar{\phi}^i} \bar{\alpha}_0^i \bar{B}_1 \bar{B}_1^k. \tag{13}
\]

\[(S_P, S_P) = 0 \iff P^{ij} = \text{Poisson}\]

This action on \( T^*[1](TM) \) is a different realization of a Poisson structure from \( A \)-model (the Poisson sigma model).
§10. **Summary and Outlook**

- We have obtained a systematic method to construct a new topological field theory including superfields with the negative total degree by the AKSZ formalism, which is compatible with dimensional reduction.

- We have obtained one-parameter families of topological sigma models with a complex structure or a symplectic structure.

- Dimensional reduction via the AKSZ formalism (or the BV formalism) does not work. We need a general formalism.

- Quantization is direct and clear as a physical theory but
the calculation is complicated and the mathematical structure is unknown.

- A general theory in general dimensions is not constructed.