New Topological Field Theories from Dimensional Reduction of Nonlinear Gauge Theories

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§1. Introduction

Purpose 1.

Construct new gauge theories and apply to physics and mathematics

$$\text{A new gauge theory} \longrightarrow \begin{cases} \text{new physics (Yang-Mills, gravity}, \cdots) \\ \text{new mathematics}(\cdots) \end{cases}$$

Consistency of quantum field theories $\Longleftrightarrow$ a Batalin-Vilkovisky Structure

$(S, S) = 0$ (classical) $(S_q, S_q) - 2i\hbar\Delta S_q = 0$ (quantum)

Purpose 2.

Unify (and classify) all geometries and algebras as (a kind of substructures of) (super) Poisson geometry

Super Poisson geometry $\supset$ a Batalin-Vilkovisky Structure
$(S, S) = 0$ (classical) geometry
$(S_q, S_q) - 2i\hbar\Delta S_q = 0$ (quantum geometry)

Many geometric structures are constructed as a BV structure.
Lie, Poisson, complex, symplectic, Kähler, Calabi-Yau, Courant, Dirac, algebroid, generalized geometry, $\cdots$

We consider a topological field theory with a BF type (Schwarz type) kinetic term $S = S_0 + S_1 = \int_{\Pi TX} \boldsymbol{B} d\boldsymbol{A} + S_1$, as an interesting play ground.

A nonlinear gauge theory is a most general topological field theory with a BF type (Schwarz type) kinetic term $S = \int_{\Pi TX} \boldsymbol{B} d\boldsymbol{A} + S_1$ in any dimension,
which includes the Poisson sigma model, A-model, B-model, $\cdots$, and describes a topological membrane.

A nonlinear gauge theory constructed from superfields with the nonnegative total degree has been analyzed. We know general S_1.

N.I. '01

This Talk

We construct toplogical field theories which include superfields with the negative total degree, consistent with dimensional reduction and deformation,

and consider an **Application** to a generalized complex structure (and complex, symplectic geometry).

Technique

· Deformation Theory

Barnich, Henneaux '93, Barnich, Brandt, Henneaux, '95

· Dimensional Reduction Kaluza, '21, Klein, '26

Plan of Talk

- Review of the AKSZ formalism and nonlinear gauge theories
- Dimensional reduction from 3 dimensions to 2 dimensions
- The Poisson sigma model with a two-form
- Application to a generalized complex structure

§2. AKSZ Formalism of Nonlinear Gauge Theories

X: a manifold in n dimensions (worldsheet or worldvolume)
M: a manifold in d dimensions (target space)
a map $\phi : X \longrightarrow M$

The AKSZ Formalism of the Batalin-Vilkovisky formalism is formulated by three elements,

- Supermanifold (Superfield Φ)
- P-structure (Antibracket $(*, *)$)
- Q-structure (BV action S)

Alexandrov, Kontsevich, Schwartz, Zaboronsky '97

• Supermanifold (Superfield)

∘ X to supermanifold

TX: a tangent bundle
ΠTX: A *supermanifold* is a tangent bundle with reversed parity of the fiber.

· local coordinates
$\{\sigma^\mu\}$ on X, where $\mu = 1, 2, \cdots, n$
$\{\theta^\mu\}$ on $T_\sigma X$, fermionic supercoordinate

Def: **form degree** $\deg\sigma = 0$ and $\deg\theta = 1$

We extend a smooth map $\phi : X \longrightarrow M$ to a map $\boldsymbol{\phi} : \Pi TX \longrightarrow M$.

This procedure introduces ghosts and antifields systematically. A function on $\Pi T X$ is called a *superfield*.

$$\boldsymbol{\phi}^i = \phi^{(0)i} + \theta^{\mu_1}\phi^{(-1)i}_{\mu_1} + \frac{1}{2!}\theta^{\mu_1}\theta^{\mu_2}\phi^{(-2)i}_{\mu_1\mu_2} + \cdots + \frac{1}{n!}\theta^{\mu_1}\cdots\theta^{\mu_n}\phi^{(-n)i}_{\mu_1\cdots\mu_n}.$$

$\circ$ M to supermanifold (a graded manifold)

We introduce superfield pairs with the nonnegative total degrees and with the sum of the **total degree** $n-1$.

$$\{\boldsymbol{A}_p{}^{a_p},\ \boldsymbol{B}_{n-p-1,a_p}\},\ p = 0, 1, \cdots, \lfloor \tfrac{n-1}{2} \rfloor$$

1. T^*M: a cotangent bundle

$T^*[n-1]M$: A *graded cotangent bundle* is a cotangent bundle with the degree of the fiber $n-1$.

· local coordinates $\{\boldsymbol{\phi}^i, \boldsymbol{B}_{n-1,i}\}$

$\{\boldsymbol{\phi}^i = \boldsymbol{A}_0{}^i\}$: a map from ΠTX to M, where $i = 1, 2, \cdots, d$.

$\{\boldsymbol{B}_{n-1,i}\}$: a superfield on ΠTX, which take a value on $\boldsymbol{\phi}^*(T^*[n-1]M)$

p even $\Longrightarrow \{\boldsymbol{B}_{p,i}\}$ bosonic, p odd $\Longrightarrow \{\boldsymbol{B}_{p,i}\}$ fermionic

Def: **total degree**: $|\boldsymbol{\phi}| = 0$ and $|\boldsymbol{B}_{n-1}| = n-1$

Def: **ghost number**: $\mathrm{gh}F = |F| - \mathrm{deg}F$, where F is a superfield.

2. E: a vector bundle on M

$E_p[p]$: a vector bundle with the degree of the fiber p

We consider $E_p[p] \oplus E_p^*[n-p-1]$: p is an integer with $1 \leq p \leq n-2$.

· local coordinates $\{\boldsymbol{A}_p{}^{a_p}, \boldsymbol{B}_{n-p-1,a_p}\}$

$\boldsymbol{A}_p{}^{a_p}$: a total degree p superfield on ΠTX, which take a value on $\boldsymbol{\phi}^*(E[p])$

$\boldsymbol{B}_{n-p-1,a_p}$: a total degree $n-p-1$ superfield on ΠTX, which take a value on $\boldsymbol{\phi}^*(E^*[n-p-1])$

$\lfloor x \rfloor$: the floor function which gives the largest integer less than or equal to $x \in \boldsymbol{R}$.

If $\lfloor \frac{n}{2} \rfloor \leq p \leq n-2$, we can identify $E[p] \oplus E^*[n-p-1]$ with the dual bundle $E^*[n-p-1] \oplus (E^*)^*[p]$,

We consider graded bundle,

$$T^*[n-1] \left(\sum_{p=1}^{\lfloor \frac{n-1}{2} \rfloor} E_p[p] \right),$$

$\{\boldsymbol{A}_p{}^{a_p},\ \boldsymbol{B}_{n-p-1,a_p}\}$, $p = 0, 1, \cdots, \lfloor \frac{n-1}{2} \rfloor$ in the local coordinate expression

$n = 2$

$$T^*[1]M.$$

• P-structure is *a graded Poisson structure* on a graded (or super) manifold $\tilde{N}$, whose bracket called the **antibracket** $(*, *)$ with the total degree $-p$ satisfies the following identities:

$$
\begin{aligned}
&(F, G) = -(-1)^{(|F|-p)(|G|-p)}(G, F), \\
&(F, GH) = (F, G)H + (-1)^{(|F|-p)|G|}G(F, H), \\
&(FG, H) = F(G, H) + (-1)^{|G|(|H|-p)}(F, H)G, \\
&(-1)^{(|F|-p)(|H|-p)}(F, (G, H)) + \text{cyclic permutations} = 0,
\end{aligned}
$$

1. $T^*[n-1]M$ has a natural P-structure induced from a natural Poisson (symplectic) structure on T^*M:

$$(F,G) \equiv F \frac{\overleftarrow{\partial}}{\partial \boldsymbol{\phi}^i} \frac{\overrightarrow{\partial}}{\partial \boldsymbol{B}_{n-1,i}} G - F \frac{\overleftarrow{\partial}}{\partial \boldsymbol{B}_{n-1,i}} \frac{\overrightarrow{\partial}}{\partial \boldsymbol{\phi}^i} G.$$

in a Darboux coordinate. The total degree is $-n+1$.

2. An antibracket on $E_p[p] \oplus E_p^*[n-p-1]$

$$(F,G) \equiv F \frac{\overleftarrow{\partial}}{\partial \boldsymbol{A}_p{}^{a_p}} \frac{\overrightarrow{\partial}}{\partial \boldsymbol{B}_{n-p-1,a_p}} G - (-1)^{p(n-p-1)} F \frac{\overleftarrow{\partial}}{\partial \boldsymbol{B}_{n-p-1,a_p}} \frac{\overrightarrow{\partial}}{\partial \boldsymbol{A}_p{}^{a_p}} G$$

in a Darboux coordinate. The total degree is $-n+1$.

• Q-structure called a BV *action* S is a function on a target graded bundle, and a functional on ΠTX, which satisfies the classical master equation $(S, S) = 0$.

We require $\mathrm{gh} S = 0$.

$\delta F = (S, F)$: a $BRST$ *transformation*, satisfies $\delta^2 = 0$.

§3. BV Action of Nonlinear Gauge Theories

• Q-structure (BV Action)

$$S = S_0 + S_1,$$

where

$$S_0 = \sum_{p=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^{n-p} \int_{\Pi TX} \boldsymbol{B}_{n-p-1,a_p} d\boldsymbol{A}_p{}^{a_p},$$

where $d = \theta^\mu \partial_\mu$, $\int_{\Pi TX} \equiv \int_{\Pi TX} d^n\theta d^n\sigma$.

The general solution $S_1 = \int_{\Pi TX} F(\mathbf{\Phi})$, where F is an arbitrary function of all superfields $\mathbf{\Phi}$ with the total degree n, which satisfies $(S_1, S_1) = 0$, up to total derivatives and BRST exact terms. Higher order deformations vanish.

Izawa, '00, N.I, '00 '01

Bizdadea, Ciobirca, Cioroianu, Saliu, Sararu '00 '01 '02 '03

We call a resulting field theory $S = S_0 + S_1$ a *nonlinear gauge theory* in n dimensions, a *topological sigma model* in n dimensions, or a *topological $(n-1)$-brane*.

Example.1. $n = 2$

- Supermanifold $T^*[1]M$
- Antibracket $(F, G) \equiv F \frac{\overleftarrow{\partial}}{\partial \phi^i} \frac{\overrightarrow{\partial}}{\partial B_{1,i}} G - F \frac{\overleftarrow{\partial}}{\partial B_{1,i}} \frac{\overrightarrow{\partial}}{\partial \phi^i} G.$
- BV Action is the Poisson sigma model

$$S_0 = \int_{\Pi TX} \boldsymbol{B}_{1i} d\boldsymbol{\phi}^i, \qquad S_1 = \int_{\Pi TX} \frac{1}{2} f^{ij}(\boldsymbol{\phi}) \boldsymbol{B}_{1i} \boldsymbol{B}_{1j},$$

$(S_1, S_1) = 0$ derives $f^{kl} \frac{\overrightarrow{\partial}}{\partial \phi^l} f^{ij} + f^{il} \frac{\overrightarrow{\partial}}{\partial \phi^l} f^{jk} + f^{jl} \frac{\overrightarrow{\partial}}{\partial \phi^l} f^{ki} = 0$

$\Longleftrightarrow \pi = -f^{ij} \frac{\partial}{\partial \phi^l} \wedge \frac{\partial}{\partial \phi^l}$ is a Poisson bivector field.

Example.2. $n = 3$

- Supermanifold $T^*[2]E[1]$

- Antibracket

$$(F, G) \equiv F \frac{\overleftarrow{\partial}}{\partial \boldsymbol{\phi}^i} \frac{\overrightarrow{\partial}}{\partial \boldsymbol{B}_{2,i}} G - F \frac{\overleftarrow{\partial}}{\partial \boldsymbol{B}_{2,i}} \frac{\overrightarrow{\partial}}{\partial \boldsymbol{\phi}^i} G + F \frac{\overleftarrow{\partial}}{\partial \boldsymbol{A}_1{}^a} \frac{\overrightarrow{\partial}}{\partial \boldsymbol{B}_{1,a}} G + F \frac{\overleftarrow{\partial}}{\partial \boldsymbol{B}_{1,a}} \frac{\overrightarrow{\partial}}{\partial \boldsymbol{A}_1{}^a} G$$

- BV Action

The Courant sigma model

$$S = S_0 + S_1,$$
$$S_0 = \int_{\Pi TX} \left[-\boldsymbol{B}_{2i} d\boldsymbol{\phi}^i + \boldsymbol{B}_{1a} d\boldsymbol{A}_1{}^a \right],$$

$$
\begin{aligned}
S_1 = \int_{\Pi TX} & [f_{1a}{}^i(\boldsymbol{\phi})\boldsymbol{B}_{2i}\boldsymbol{A}_1{}^a + f_2^{ib}(\boldsymbol{\phi})\boldsymbol{B}_{2i}\boldsymbol{B}_{1b} \\
& +\frac{1}{3!}f_{3abc}(\boldsymbol{\phi})\boldsymbol{A}_1{}^a\boldsymbol{A}_1{}^b\boldsymbol{A}_1{}^c + \frac{1}{2}f_{4ab}{}^c(\boldsymbol{\phi})\boldsymbol{A}_1{}^a\boldsymbol{A}_1{}^b\boldsymbol{B}_{1c} \\
& +\frac{1}{2}f_{5a}{}^{bc}(\boldsymbol{\phi})\boldsymbol{A}_1{}^a\boldsymbol{B}_{1b}\boldsymbol{B}_{1c} + \frac{1}{3!}f_6{}^{abc}(\boldsymbol{\phi})\boldsymbol{B}_{1a}\boldsymbol{B}_{1b}\boldsymbol{B}_{1c}], \qquad (1)
\end{aligned}
$$

$(S_1, S_1) = 0 \Longleftrightarrow$ six f's are structure functions of a Courant algebroid.

N.I. '02, Hofman, Park '02, Roytenberg '01, '06

Example.3. $n =$general

has a structure to the n-algebroid.

§4. Dimensional Reduction

Dimensional reduction of X in n dimensions to Σ in m dimensions ($n > m$). Reduction of each superfield $\mathbf{\Phi}$ is

$$\mathbf{\Phi}(\sigma^1, \cdots, \sigma^n) = \tilde{\mathbf{\Phi}}_{|\Phi|}(\sigma^1, \cdots, \sigma^m)$$
$$+ \sum_{p=1, \mu_q=n-m+1, \cdots, n}^{n-m} \theta^{\mu_1} \cdots \theta^{\mu_p} \tilde{\phi}_{|\Phi|-p, \mu_1 \cdots \mu_p}(\sigma^1, \cdots, \sigma^m),$$

where $\deg \theta = 1$, $|\theta| = 1$, $\mathrm{gh}\theta = 0$.

This procedure produces a new topological field theory on Σ, including a superfield with the negative total degree.

§5. Dimensional Reduction of the Courant Sigma Model in 3 Dimensions to 2 Dimensions

Reduction X to Σ, $(\sigma^1, \sigma^2, \sigma^3) \longrightarrow (\sigma^1, \sigma^2)$,

Procedure 1. Reduction of Superfields

$$
\begin{aligned}
\boldsymbol{\phi}^i(\sigma^\mu, \theta^\mu) &= \tilde{\boldsymbol{\phi}}^i(\sigma^1, \sigma^2, \theta^1, \theta^2) + \theta^3 \tilde{\boldsymbol{\phi}}_{\mathbf{-1}}{}^i(\sigma^1, \sigma^2, \theta^1, \theta^2), \\
\boldsymbol{B}_{2i}(\sigma^\mu, \theta^\mu) &= \tilde{\boldsymbol{B}}_{\mathbf{2}i}(\sigma^1, \sigma^2, \theta^1, \theta^2) + \theta^3 \tilde{\boldsymbol{\beta}}_{\mathbf{1}i}(\sigma^1, \sigma^2, \theta^1, \theta^2), \\
\boldsymbol{A}_1{}^a(\sigma^\mu, \theta^\mu) &= \tilde{\boldsymbol{A}}_{\mathbf{1}}{}^a(\sigma^1, \sigma^2, \theta^1, \theta^2) + \theta^3 \tilde{\boldsymbol{\alpha}}_{\mathbf{0}}{}^a(\sigma^1, \sigma^2, \theta^1, \theta^2), \\
\boldsymbol{B}_{1a}(\sigma^\mu, \theta^\mu) &= \tilde{\boldsymbol{B}}_{\mathbf{1}a}(\sigma^1, \sigma^2, \theta^1, \theta^2) + \theta^3 \tilde{\boldsymbol{\beta}}_{\mathbf{0}a}(\sigma^1, \sigma^2, \theta^1, \theta^2),
\end{aligned}
\qquad (2)
$$

All these superfields do not depend on σ^3 and θ^3. $|\tilde{\boldsymbol{\phi}}^i| = |\tilde{\boldsymbol{\alpha}}_{\mathbf{0}}{}^a| =$

$|\tilde{\boldsymbol{\beta}}_{\mathbf{0}a}| = 0$, $|\tilde{\boldsymbol{\phi}}_{\mathbf{-1}}{}^i| = -1$, $|\tilde{\boldsymbol{A}}_{\mathbf{1}}{}^a| = |\tilde{\boldsymbol{B}}_{\mathbf{1}a}| = |\tilde{\boldsymbol{\beta}}_{\mathbf{1}i}| = 1$, $|\tilde{\boldsymbol{B}}_{\mathbf{2}i}| = 2$.

Procedure 2. Substitute to 3D Action

do not derive the correct AKSZ action in 2 dimensions.

$(\theta^3)^2 = 0$ drops some terms in three dimensions but generally the extra terms appear in a reduced action.

The existence of the negative total degree superfield $\tilde{\boldsymbol{\phi}}_{\mathbf{-1}}{}^i$ complexifies the dimensional reduction in the AKSZ formalism.

In order to derive the correct AKSZ action, first we should consider the dimensional reduction via the non-BV formalism.

$$
\begin{array}{ccc}
3D \text{ AKSZ Action} & \longrightarrow & 3D \text{ non-BV Action} \\
 & & \downarrow \\
2D \text{ AKSZ Action} & \longleftarrow & 2D \text{ non-BV Action}
\end{array}
$$

Procedure 1. Expansion of superfields by the ghost numbers

$$
\begin{aligned}
\boldsymbol{\phi}^i &= \phi^i + \phi^{(-1)i} + \phi^{(-2)i} + \phi^{(-3)i}, \\
\boldsymbol{B}_{2,i} &= B_{2,i}^{(2)} + B_{2,i}^{(1)} + B_{2,i} + B_{2,i}^{(-1)}, \\
\boldsymbol{B}_{1a} &= B_{1,a}^{(1)} + B_{1,a} + B_{1,a}^{(-1)} + B_{1,a}^{(-2)}, \\
\boldsymbol{A}_1{}^a &= A_1^{(1)a} + A_1^a + A_1^{(-1)a} + A_1^{(-2)a},
\end{aligned}
\tag{3}
$$

where $\phi^i = \phi^{(0)i}$, etc.

Procedure 2. Setting $\Phi^{(p)} = 0$ for $p \neq 0$ in the action (antifield & ghost$= 0$) produces the 3D non-BV action.

Procedure 3. Dimensional Reduction

$$\begin{aligned}
\phi^i(\sigma^\mu, \theta^\mu) &= \tilde{\phi}^i(\sigma^1, \sigma^2, \theta^1, \theta^2), \\
B_{2i}(\sigma^\mu, \theta^\mu) &= \tilde{B}_{2i}(\sigma^1, \sigma^2, \theta^1, \theta^2) + \theta^3 \tilde{\beta}_{1i}(\sigma^1, \sigma^2, \theta^1, \theta^2), \\
A_1^a(\sigma^\mu, \theta^\mu) &= \tilde{A_1}^a(\sigma^1, \sigma^2, \theta^1, \theta^2) + \theta^3 \tilde{\alpha_0}^a(\sigma^1, \sigma^2, \theta^1, \theta^2), \\
B_{1a}(\sigma^\mu, \theta^\mu) &= \tilde{B}_{1a}(\sigma^1, \sigma^2, \theta^1, \theta^2) + \theta^3 \tilde{\beta}_{0a}(\sigma^1, \sigma^2, \theta^1, \theta^2),
\end{aligned} \qquad (4)$$

3D action to 2D action

$$S = S_0 + S_1$$

$$S_0 = \int d\theta^3 \theta^3 d\sigma^3 \int_{\Pi T \Sigma} \left(\tilde{\beta}_{1i} \wedge d\tilde{\phi}^i + \tilde{A}_1{}^a \wedge d\tilde{\beta}_{0a} + \tilde{\beta}_{1a} \wedge d\tilde{\alpha}_0{}^a \right)$$

$$S_1 = \int d\theta^3 \theta^3 d\sigma^3 \int_{\Pi T \Sigma} f_{1a}{}^i \tilde{A}_1{}^a \tilde{\beta}_{1i} + f_2{}^{ib} \tilde{B}_{1b} \tilde{\beta}_{1i}$$
$$+ \frac{1}{2}(f_{3abc} \tilde{\alpha}_0{}^c + f_{4ab}{}^c \tilde{\beta}_{0c}) \tilde{A}_1{}^a \tilde{A}_1{}^b + (-f_{4ab}{}^c \tilde{\alpha}_0{}^b + f_{5a}{}^{cb} \tilde{\beta}_{0b}) \tilde{A}_1{}^a \tilde{B}_{1c}$$
$$+ \frac{1}{2}(f_{5a}{}^{bc} \tilde{\alpha}_0{}^a + f_6{}^{abc} \tilde{\beta}_{0a}) \tilde{B}_{1b} \tilde{B}_{1c} + (f_{1b}{}^i \tilde{\alpha}_0{}^b + f_2{}^{ia} \tilde{\beta}_{0a}) \tilde{B}_{2i}, \quad (5)$$

Procedure 4. 2D AKSZ Action

We need a new theory.

§6. The Poisson sigma model with a 2-form

$$S_t = \int_{\Pi T\Sigma} \tilde{B}_{1A} d\tilde{\Phi}^A + \frac{1}{2} F^{AB}(\tilde{\Phi}) \tilde{B}_{1A} \tilde{B}_{1B} + G^A(\tilde{\Phi}) \tilde{B}_{2A}, \qquad (6)$$

where $\tilde{B}_{2A}$ is a total degree 2 superfield. The reduced action (5) is obtained by setting

$$\tilde{\Phi}^A = (\tilde{\phi}^i, \tilde{\beta}_{0a}, \tilde{\alpha_0}^b), \tilde{B}_{1A} = (\tilde{\beta}_{1i}, \tilde{A_1}^a, \tilde{B}_{1b}), \tilde{B}_{2A} = (\tilde{B}_{2i}, 0, 0),$$
$$F^{AB} = \begin{pmatrix} 0 & -f_{1c}{}^i & -f_2{}^{id} \\ f_{1a}{}^j & f_{3ace}\tilde{\alpha_0}^e + f_{4ac}{}^e \tilde{\beta}_{0e} & -f_{4ae}{}^d \tilde{\alpha_0}^e + f_{5a}{}^{de}\tilde{\beta}_{0e} \\ f_2{}^{jb} & f_{4be}{}^c \tilde{\alpha_0}^e - f_{5b}{}^{ce}\tilde{\beta}_{0e} & f_{5e}{}^{bd}\tilde{\alpha_0}^e + f_6^{bde}\tilde{\beta}_{0e} \end{pmatrix},$$
$$G^A = (f_{1b}{}^i \tilde{\alpha_0}^b + f_2{}^{ia}\tilde{\beta}_{0a}, 0, 0), \qquad (7)$$

We take $^{A} = (^{i},\ {}_{a}\ ,\ ^{b})$ indices on M, $E^*_{\tilde{\phi}}$ and $E_{\tilde{\phi}}$. The action (5) is a particular case of the action (6) on $E \oplus E^*$.

The action has the following gauge symmetry:

$$\delta\tilde{\Phi}^A = -F^{AB} c_B,$$

$$\delta\tilde{B}_{1A} = dc_A + \frac{\partial F^{BC}}{\partial \Phi^A}\tilde{B}_{1B}\tilde{B}_{1C} - \frac{\partial G^B}{\partial \Phi^A} t_B,$$

$$\delta\tilde{B}_{2A} = dt_A + U_A{}^{BC}(\tilde{B}_{1B} t_C - \tilde{B}_{2C} c_B) + \frac{1}{2} X_A{}^{BCD}\tilde{B}_{1B}\tilde{B}_{1C} c_D,$$

where c_A is a gauge parameter with the total degree 1 and t_A is a gauge parameter with the total degree 2. $X_A{}^{BCD}$ is completely antisymmetric with respect to the indices BCD.

In fact, the action S is gauge invariant

$$\delta S_t = \int_{\Pi T\Sigma} d(c_A d\Phi_A + G^A t_A),$$

if and only if F, G, U and X satisfy the identities

$$F^{D[A}\frac{\partial F^{BC]}}{\partial \Phi^D} = G^D X_D{}^{ABC}.$$

$$F^{AB}\frac{\partial G^C}{\partial \Phi^A} + U_A{}^{BC} G^A = 0,$$

Deformation of the Poisson bivector by the 3-forms $G^D X_D{}^{ABC}$.

cf. WZ Poisson sigma model Klimcik, Strobl '01

§7. AKSZ action of the Poisson sigma model with a 2-form

- Fields $\longrightarrow$ Superfields

$$
\begin{aligned}
\tilde{\boldsymbol{B}}_{\boldsymbol{1}A} &= \tilde{B}_{1A}^{(1)} + \tilde{B}_{1A} + \tilde{B}_{1A}^{(-1)}, \\
\tilde{\boldsymbol{\Phi}}^{A} &= \Phi^{A} + \Phi^{(-1)A} + \Phi^{(-2)A}, \\
\tilde{\boldsymbol{B}}_{\boldsymbol{2}A} &= \tilde{B}_{2i}^{(2)} + \tilde{B}_{2i}^{(1)} + \tilde{B}_{2i} \\
\tilde{\boldsymbol{\phi}}_{\boldsymbol{-1}}{}^{A} &= \tilde{\phi}_{-1}{}^{(-1)i} + \tilde{\phi}_{-1}{}^{(-2)i} + \tilde{\phi}_{-1}{}^{(-3)i}
\end{aligned}
$$

where $|\tilde{\boldsymbol{\phi}}_{\boldsymbol{-1}}{}^{A}| = -1$.

- P-structure (Antibracket)

$$(F,G) = F\frac{\overleftarrow{\partial}}{\partial\tilde{\mathbf{\Phi}}^{A}}\frac{\overrightarrow{\partial}}{\partial\tilde{\boldsymbol{B}}_{\mathbf{1}A}}G - F\frac{\overleftarrow{\partial}}{\partial\tilde{\boldsymbol{B}}_{\mathbf{1}A}}\frac{\overrightarrow{\partial}}{\partial\tilde{\mathbf{\Phi}}^{A}}G$$
$$+F\frac{\overleftarrow{\partial}}{\partial\tilde{\boldsymbol{\phi}}_{\mathbf{-1}}{}^{A}}\frac{\overrightarrow{\partial}}{\partial\tilde{\boldsymbol{B}}_{\mathbf{2}A}}G, -F\frac{\overleftarrow{\partial}}{\partial\tilde{\boldsymbol{B}}_{\mathbf{2}A}}\frac{\overrightarrow{\partial}}{\partial\tilde{\boldsymbol{\phi}}_{\mathbf{-1}}{}^{A}}G$$

- BV Action $S = S_0 + S_1$

$$S_0 = \int_{\Pi T\Sigma} \tilde{\boldsymbol{B}}_{\mathbf{1}A} d\tilde{\mathbf{\Phi}}^{A} - \tilde{\boldsymbol{B}}_{\mathbf{2}A} d\tilde{\boldsymbol{\phi}}_{\mathbf{-1}}{}^{A},$$

Procedure to obtain S_1

1. Consider most general (consistent) deformation of $S_0 \longrightarrow S$ in the sense of BBH. (Write down all the possible terms)
2. The BV action of S_t is a particular case of the resulting action S.
3. If $S|_{\mathrm{gh} S=0} = S_0 + S_1|_{\mathrm{gh} S=0} = S_t$, it is the correct AKSZ action.

Def: $\mathrm{neg}(F)$: a *negative total degree* of a superfield F

$$\mathrm{neg}(\tilde{\boldsymbol{\phi}}_{\boldsymbol{-1}}{}^A) = 1 \text{ and } \mathrm{neg} \text{ (other superfields)} = 0$$

We expand S_1 by the negative total degrees:

$$S_1 = \sum_{p=0}^{\infty} S_1^{(p)} = \sum_{p=0}^{\dim M} \int_{\Pi T\Sigma} \mathcal{L}_1^{(p)} = \sum_p \int_{\Pi T\Sigma} \tilde{\boldsymbol{\phi}}_{\boldsymbol{-1}}{}^{i_1} \cdots \tilde{\boldsymbol{\phi}}_{\boldsymbol{-1}}{}^{i_p} \mathcal{L}^{(p)}_{i_1 \cdots i_p}(\tilde{\boldsymbol{\Phi}}, \tilde{\boldsymbol{B}}_{\boldsymbol{1}}, \tilde{\boldsymbol{B}}_{\boldsymbol{2}})$$

$$\begin{aligned}
(S_0, S_1) = 0 \quad \Longrightarrow \quad \mathcal{L}_1^{(0)} &= \frac{1}{2} f_1^{AB}(\tilde{\boldsymbol{\Phi}}) \tilde{\boldsymbol{B}}_{\boldsymbol{1}A} \tilde{\boldsymbol{B}}_{\boldsymbol{1}B} + f_2^{A}(\tilde{\boldsymbol{\Phi}}) \tilde{\boldsymbol{B}}_{\boldsymbol{2}A}, \\
\mathcal{L}_1^{(1)} &= \frac{1}{3!} f_{3A}{}^{BCD}(\tilde{\boldsymbol{\Phi}}) \tilde{\boldsymbol{\phi}}_{\boldsymbol{-1}}{}^{A} \tilde{\boldsymbol{B}}_{\boldsymbol{1}B} \tilde{\boldsymbol{B}}_{\boldsymbol{1}C} \tilde{\boldsymbol{B}}_{\boldsymbol{1}D} \\
&\qquad + f_{4A}{}^{BC}(\tilde{\boldsymbol{\Phi}}) \tilde{\boldsymbol{\phi}}_{\boldsymbol{-1}}{}^{A} \tilde{\boldsymbol{B}}_{\boldsymbol{1}B} \tilde{\boldsymbol{B}}_{\boldsymbol{2}C}, \\
\mathcal{L}_1^{(2)} &= \cdots,
\end{aligned} \tag{8}$$

and so on, where $f_i(\tilde{\mathbf{\Phi}})$ is a function of $\tilde{\mathbf{\Phi}}^A$.

$$(S_1, S_1) = \sum_p (S_1, S_1)^{(p)} = 0, \tag{9}$$

determines the identities of $f_i(\Phi)$ recursively, where $(S_1, S_1)^{(p)}$ is the negative total degree p part of (S_1, S_1).

Batalin, Marnelius '01, N.I., Izawa '04

0-th order:

$$(S_1, S_1)^{(0)} = 0 \Longrightarrow \qquad f_1^{D[A} \frac{\partial f_1^{BC]}}{\partial \Phi^D} - f_2^D f_{3D}{}^{ABC} = 0,$$

$$f_1^{AB} \frac{\partial f_2^C}{\partial \Phi^A} + f_{4A}{}^{BC} f_2^A = 0.$$

$$f_1^{AB} = F^{AB}, \qquad f_2^A = G^A,$$

$$f_{3A}{}^{BCD} = X_A{}^{BCD}, \qquad f_{4A}{}^{BC} = U_A{}^{BC},$$

$$S|_{\mathrm{gh} S=0} = S_t.$$

The action obtained by deformation is the correct aciton compatible with dimensional reduction. Substitute (7).

§8. Application – Generalized Complex Structure

3D topological sigma model with a generalized complex structure on $TM \oplus T^*M$ ($H = 0$): N.I. '04

$$
\begin{aligned}
S &= \int_{\Pi TX} -\frac{1}{2}\langle 0 + \boldsymbol{B}_2, d(\boldsymbol{\phi} + 0)\rangle + \frac{1}{4}\langle \boldsymbol{A}_1 + \boldsymbol{B}_1, d(\boldsymbol{A}_1 + \boldsymbol{B}_1)\rangle \\
&\quad -\langle 0 + \boldsymbol{B}_2, \mathcal{J}(\boldsymbol{A}_1 + \boldsymbol{B}_1)\rangle - \frac{1}{2}\langle \boldsymbol{A}_1 + \boldsymbol{B}_1, \boldsymbol{A}_1^i \frac{\partial \mathcal{J}}{\partial \boldsymbol{\phi}^i}(\boldsymbol{A}_1 + \boldsymbol{B}_1)\rangle \\
&= \int_{\Pi TX} -\frac{1}{2}\boldsymbol{B}_{2i} d\boldsymbol{\phi}^i + \frac{1}{2}\boldsymbol{B}_{1i} d\boldsymbol{A}_1^i - J^i{}_j \boldsymbol{B}_{2i}\boldsymbol{A}_1^j - P^{ij}\boldsymbol{B}_{2i}\boldsymbol{B}_{1j} \\
&\quad + \frac{1}{2}\frac{\partial Q_{jk}}{\partial \boldsymbol{\phi}^i}\boldsymbol{A}_1^i \boldsymbol{A}_1^j \boldsymbol{A}_1^k + \frac{1}{2}\left(-\frac{\partial J^k{}_j}{\partial \boldsymbol{\phi}^i} + \frac{\partial J^k{}_i}{\partial \boldsymbol{\phi}^j}\right)\boldsymbol{A}_1^i \boldsymbol{A}_1^j \boldsymbol{B}_{1k} + \frac{1}{2}\frac{\partial P^{jk}}{\partial \boldsymbol{\phi}^i}\boldsymbol{A}_1^i \boldsymbol{B}_{1j}\boldsymbol{B}_{1k}.
\end{aligned}
$$

is derived by setting,

$$f_{1j}{}^{i} = -J^{i}{}_{j}, \quad f_{2}{}^{ij} = -P^{ij}, \quad f_{3ijk} = \frac{\partial Q_{jk}}{\partial \phi^i} + \frac{\partial Q_{ki}}{\partial \phi^j} + \frac{\partial Q_{ij}}{\partial \phi^k},$$

$$f_{4ij}{}^{k} = -\frac{\partial J^{k}{}_{j}}{\partial \phi^i} + \frac{\partial J^{k}{}_{i}}{\partial \phi^j}, \quad f_{5i}{}^{jk} = \frac{\partial P^{jk}}{\partial \phi^i}, \quad f_{6}{}^{ijk} = 0, \qquad (10)$$

in the Courant sigma model, where $\mathcal{J}$ is a generalized complex structure:

$$\mathcal{J} = \begin{pmatrix} J & P \\ Q & -{}^tJ \end{pmatrix}.$$

$(S, S) = 0 \Longleftrightarrow \mathcal{J}$ is a generalized complex structure.

This action derives the Zucchini model (the Hitchin sigma model) as a boundary action if $\Sigma = \partial X$: Zucchini '04

$$S_Z = \int_{\Pi T\Sigma} \tilde{\boldsymbol{B}}_{\boldsymbol{1}i} d\tilde{\boldsymbol{\phi}}^i + J^i{}_j \tilde{\boldsymbol{B}}_{\boldsymbol{1}i} d\tilde{\boldsymbol{\phi}}^j + \frac{1}{2} P^{ij} \tilde{\boldsymbol{B}}_{\boldsymbol{1}i} \tilde{\boldsymbol{B}}_{\boldsymbol{1}j} + \frac{1}{2} Q_{ij} d\tilde{\boldsymbol{\phi}}^i d\tilde{\boldsymbol{\phi}}^j$$

- Dimensional reduction provides a new 2D action with GCS

$$\begin{aligned} S_0 &= \int_{\Pi T\Sigma} \frac{1}{2} \left(\tilde{\boldsymbol{\beta}}_{\boldsymbol{1}i} d\tilde{\boldsymbol{\phi}}^i - \tilde{\boldsymbol{B}}_{\boldsymbol{2}i} d\tilde{\boldsymbol{\phi}}_{\boldsymbol{-1}}{}^i + \tilde{\boldsymbol{B}}_{\boldsymbol{1}i} d\tilde{\boldsymbol{\alpha}}_{\boldsymbol{0}}{}^i + \tilde{\boldsymbol{A}}_{\boldsymbol{1}}{}^i d\tilde{\boldsymbol{\beta}}_{\boldsymbol{0}i} \right) \\ S_1^{(0)} &= \int_{\Pi T\Sigma} -J^i{}_j \tilde{\boldsymbol{A}}_{\boldsymbol{1}}{}^j \tilde{\boldsymbol{\beta}}_{\boldsymbol{1}i} + P^{ij} \tilde{\boldsymbol{B}}_{\boldsymbol{1}i} \tilde{\boldsymbol{\beta}}_{\boldsymbol{1}j} \\ &\quad + \frac{1}{2} \left(\left(\frac{\partial Q_{jk}}{\partial \tilde{\boldsymbol{\phi}}^i} + \frac{\partial Q_{ij}}{\partial \tilde{\boldsymbol{\phi}}^k} + \frac{\partial Q_{ki}}{\partial \tilde{\boldsymbol{\phi}}^j} \right) \tilde{\boldsymbol{\alpha}}_{\boldsymbol{0}}{}^k + \left(-\frac{\partial J^k{}_j}{\partial \tilde{\boldsymbol{\phi}}^i} + \frac{\partial J^k{}_i}{\partial \tilde{\boldsymbol{\phi}}^j} \right) \tilde{\boldsymbol{\beta}}_{\boldsymbol{0}k} \right) \end{aligned}$$

$$
\begin{aligned}
& \times \tilde{\boldsymbol{A}}_{\boldsymbol{1}}{}^{i} \tilde{\boldsymbol{A}}_{\boldsymbol{1}}{}^{j} + \left(\left(\frac{\partial J^{k}{}_{j}}{\partial \tilde{\boldsymbol{\phi}}^{i}} - \frac{\partial J^{k}{}_{i}}{\partial \tilde{\boldsymbol{\phi}}^{j}} \right) \tilde{\boldsymbol{\alpha}}_{\boldsymbol{0}}{}^{j} - \frac{\partial P^{jk}}{\partial \tilde{\boldsymbol{\phi}}^{i}} \tilde{\boldsymbol{\beta}}_{\boldsymbol{0}j} \right) \tilde{\boldsymbol{A}}_{\boldsymbol{1}}{}^{i} \tilde{\boldsymbol{B}}_{\boldsymbol{1}k} \\
& + \frac{1}{2} \left(\frac{\partial P^{jk}}{\partial \tilde{\boldsymbol{\phi}}^{i}} \tilde{\boldsymbol{\alpha}}_{\boldsymbol{0}}{}^{i} \right) \tilde{\boldsymbol{B}}_{\boldsymbol{1}j} \tilde{\boldsymbol{B}}_{\boldsymbol{1}k} - \left(J^{i}{}_{j} \tilde{\boldsymbol{\alpha}}_{\boldsymbol{0}}{}^{j} + P^{ij} \tilde{\boldsymbol{\beta}}_{\boldsymbol{0}j} \right) \tilde{\boldsymbol{B}}_{\boldsymbol{2}i},
\end{aligned}
$$

$$
\begin{aligned}
S_1^{(1)} &= \int_{\Pi T \Sigma} \tilde{\boldsymbol{\phi}}_{\boldsymbol{-1}}{}^{l} \Bigg[\frac{\partial J^{i}{}_{j}}{\partial \tilde{\boldsymbol{\phi}}^{l}} \tilde{\boldsymbol{B}}_{\boldsymbol{2}i} \tilde{\boldsymbol{A}}_{\boldsymbol{1}}{}^{j} + \frac{\partial P^{ij}}{\partial \tilde{\boldsymbol{\phi}}^{l}} \tilde{\boldsymbol{B}}_{\boldsymbol{2}i} \tilde{\boldsymbol{B}}_{\boldsymbol{1}j} - \frac{1}{2} \frac{\partial^2 Q_{jk}}{\partial \tilde{\boldsymbol{\phi}}^{i} \partial \tilde{\boldsymbol{\phi}}^{l}} \tilde{\boldsymbol{A}}_{\boldsymbol{1}}{}^{i} \tilde{\boldsymbol{A}}_{\boldsymbol{1}}{}^{j} \tilde{\boldsymbol{A}}_{\boldsymbol{1}}{}^{k} \\
& \quad - \frac{1}{2} \frac{\partial}{\partial \tilde{\boldsymbol{\phi}}^{l}} \left(- \frac{\partial J^{k}{}_{j}}{\partial \tilde{\boldsymbol{\phi}}^{i}} + \frac{\partial J^{k}{}_{i}}{\partial \tilde{\boldsymbol{\phi}}^{j}} \right) \tilde{\boldsymbol{A}}_{\boldsymbol{1}}{}^{i} \tilde{\boldsymbol{A}}_{\boldsymbol{1}}{}^{j} \tilde{\boldsymbol{B}}_{\boldsymbol{1}k} - \frac{1}{2} \frac{\partial^2 P^{jk}}{\partial \tilde{\boldsymbol{\phi}}^{i} \partial \tilde{\boldsymbol{\phi}}^{l}} \tilde{\boldsymbol{A}}_{\boldsymbol{1}}{}^{i} \tilde{\boldsymbol{B}}_{\boldsymbol{1}j} \tilde{\boldsymbol{B}}_{\boldsymbol{1}k} \Bigg] . \\
S_1^{(2)} &= \cdots ,
\end{aligned}
$$

§9. Two Special Reductions to Complex Geometry and Symplectic Geometry

• $P = Q = 0$ (Complex geometry)

$(S, S) = 0 \Longleftrightarrow J =$ complex structure
This condition is invariant under the redefinition (λ: constant):

$$\begin{aligned}
\tilde{\boldsymbol{\phi}}^{i} &= \tilde{\boldsymbol{\phi}}'^{i}, & \tilde{\boldsymbol{\phi}}_{\mathbf{-1}}{}^{i} &= \lambda \tilde{\boldsymbol{\phi}}'_{\mathbf{-1}}{}^{i}, \\
\tilde{\boldsymbol{B}}_{\mathbf{2}i} &= \lambda \tilde{\boldsymbol{B}}'_{\mathbf{2}i}, & \tilde{\boldsymbol{\beta}}_{\mathbf{1}i} &= \frac{1}{2}\tilde{\boldsymbol{\beta}}'_{\mathbf{1}i}, \\
\tilde{\boldsymbol{A}}_{\mathbf{1}}{}^{i} &= \frac{1}{2}\tilde{\boldsymbol{A}}'_{\mathbf{1}}{}^{i}, & \tilde{\boldsymbol{\alpha}}_{\mathbf{0}}{}^{i} &= \lambda \tilde{\boldsymbol{\alpha}}'_{\mathbf{0}}{}^{i}, \\
\tilde{\boldsymbol{B}}_{\mathbf{1}i} &= \lambda \tilde{\boldsymbol{B}}'_{\mathbf{1}i}, & \tilde{\boldsymbol{\beta}}_{\mathbf{0}i} &= -\tilde{\boldsymbol{\beta}}'_{\mathbf{0}i},
\end{aligned} \tag{11}$$

$$
\begin{aligned}
S_0 &= \int_{\Pi T\Sigma} \frac{1}{4}\left(\tilde{\boldsymbol{\beta}}'_{\mathbf{1}i} d\tilde{\boldsymbol{\phi}}^i - \tilde{\boldsymbol{A}}'^{\,i}_{\mathbf{1}} d\tilde{\boldsymbol{\beta}}'_{\mathbf{0}i}\right) + \frac{\lambda^2}{2}\left(-\tilde{\boldsymbol{B}}'_{\mathbf{2}i} d\tilde{\boldsymbol{\phi}}'^{\,i}_{\mathbf{-1}} + \tilde{\boldsymbol{B}}'_{\mathbf{1}i} d\tilde{\boldsymbol{\alpha}}'^{\,i}_{\mathbf{0}}\right), \\
S_1^{(0)} &= \int_{\Pi T\Sigma} \frac{1}{4}\left(J^i{}_j \tilde{\boldsymbol{\beta}}'_{\mathbf{1}i} \tilde{\boldsymbol{A}}'^{\,j}_{\mathbf{1}} + \frac{\partial J^k{}_j}{\partial \tilde{\boldsymbol{\phi}}^i} \tilde{\boldsymbol{\beta}}'_{\mathbf{0}k} \tilde{\boldsymbol{A}}'^{\,i}_{\mathbf{1}} \tilde{\boldsymbol{A}}'^{\,j}_{\mathbf{1}}\right) \\
&\quad + \lambda^2 \left(\frac{1}{2} \frac{\partial J^k{}_j}{\partial \tilde{\boldsymbol{\phi}}^i} \tilde{\boldsymbol{\alpha}}'^{\,j}_{\mathbf{0}} \tilde{\boldsymbol{A}}'^{\,i}_{\mathbf{1}} \tilde{\boldsymbol{B}}'_{\mathbf{1}k} - J^i{}_j \tilde{\boldsymbol{\alpha}}'^{\,j}_{\mathbf{0}} \tilde{\boldsymbol{B}}'_{\mathbf{2}i}\right), \\
S_1^{(1)} &= \int_{\Pi T\Sigma} \lambda \tilde{\boldsymbol{\phi}}^{\,l}_{\mathbf{-1}} \left[\frac{\lambda}{2} \frac{\partial J^i{}_j}{\partial \tilde{\boldsymbol{\phi}}^l} \tilde{\boldsymbol{B}}'_{\mathbf{2}i} \tilde{\boldsymbol{A}}'^{\,j}_{\mathbf{1}} + \frac{\lambda}{4} \frac{\partial^2 J^k{}_j}{\partial \tilde{\boldsymbol{\phi}}^l \partial \tilde{\boldsymbol{\phi}}^i} \tilde{\boldsymbol{A}}'^{\,i}_{\mathbf{1}} \tilde{\boldsymbol{A}}'^{\,j}_{\mathbf{1}} \tilde{\boldsymbol{B}}'_{\mathbf{1}k}\right],
\end{aligned}
$$

$S_1^{(p)} \sim \lambda^p.$

If $\lambda \longrightarrow 0$, $S_1^{(p)} \longrightarrow 0$ for $p > 0$. The 2D action is

$$S_J \;=\; \frac{1}{4}\int_{\Pi T\Sigma} \tilde{\boldsymbol{\beta}}_{\mathbf{1}i} d\tilde{\boldsymbol{\phi}}^i - \tilde{\boldsymbol{A}}_{\mathbf{1}}{}^i d\tilde{\boldsymbol{\beta}}_{\mathbf{0}i} + J^i{}_j \tilde{\boldsymbol{\beta}}_{\mathbf{1}i} \tilde{\boldsymbol{A}}_{\mathbf{1}}{}^j + \frac{\partial J^i{}_k}{\partial \tilde{\boldsymbol{\phi}}^j} \tilde{\boldsymbol{\beta}}_{\mathbf{0}i} \tilde{\boldsymbol{A}}_{\mathbf{1}}{}^j \tilde{\boldsymbol{A}}_{\mathbf{1}}{}^k.$$

This action is nothing but the B-model action on $T^*[1](T^*M)$ in AKSZ.

$(S_J, S_J) = 0 \Longleftrightarrow J =$ complex structure.

• $J = 0$ (Symplectic(Poisson) geometry)

$(S, S) = 0 \Longleftrightarrow Q = P^{-1} =$ symplectic structure

The condition is invariant under the redefinition: (μ: constant.)

$$
\begin{aligned}
\tilde{\boldsymbol{\phi}}^i &= \tilde{\boldsymbol{\phi}}^i, & \tilde{\boldsymbol{\phi}}_{\mathbf{-1}}{}^i &= \mu \tilde{\boldsymbol{\phi}}'_{\mathbf{-1}}{}^i, \\
\tilde{\boldsymbol{B}}_{\mathbf{2}i} &= \mu \tilde{\boldsymbol{B}}'_{\mathbf{2}i}, & \tilde{\boldsymbol{\beta}}_{\mathbf{1}i} &= \frac{1}{2} \tilde{\boldsymbol{\beta}}'_{\mathbf{1}i}, \\
\tilde{\boldsymbol{A}}_{\mathbf{1}}{}^i &= \mu \tilde{\boldsymbol{A}}'_{\mathbf{1}}{}^i, & \tilde{\boldsymbol{\alpha}}_{\mathbf{0}}{}^i &= \tilde{\boldsymbol{\alpha}}'_{\mathbf{0}}{}^i, \\
\tilde{\boldsymbol{B}}_{\mathbf{1}i} &= \frac{1}{2} \tilde{\boldsymbol{B}}'_{\mathbf{1}i}, & \tilde{\boldsymbol{\beta}}_{\mathbf{0}i} &= -\mu \tilde{\boldsymbol{\beta}}'_{\mathbf{0}i},
\end{aligned} \tag{12}
$$

$$
\begin{aligned}
S_0 &= \int_{\Pi T\Sigma} \frac{1}{4}\left(\tilde{\boldsymbol{\beta}}'_{\mathbf{1}i} d\tilde{\boldsymbol{\phi}}^i + \tilde{\boldsymbol{B}}'_{\mathbf{1}i} d\tilde{\boldsymbol{\alpha}}'^{\,i}_{\mathbf{0}}\right) + \frac{\mu^2}{2}\left(-\tilde{\boldsymbol{B}}'_{\mathbf{2}i} d\tilde{\boldsymbol{\phi}}'^{\,i}_{\mathbf{-1}} - \tilde{\boldsymbol{A}}'^{\,i}_{\mathbf{1}} d\tilde{\boldsymbol{\beta}}'_{\mathbf{0}i}\right) \\
S_1^{(0)} &= \int_{\Pi T\Sigma} \frac{1}{4}\left(P^{ij}\tilde{\boldsymbol{B}}'_{\mathbf{1}i}\tilde{\boldsymbol{\beta}}'_{\mathbf{1}j} + \frac{1}{2}\frac{\partial P^{jk}}{\partial \tilde{\boldsymbol{\phi}}^i}\tilde{\boldsymbol{\alpha}}'^{\,i}_{\mathbf{0}}\tilde{\boldsymbol{B}}'_{\mathbf{1}j}\tilde{\boldsymbol{B}}'_{\mathbf{1}k}\right) - \frac{1}{2}\frac{\partial P^{jk}}{\partial \tilde{\boldsymbol{\phi}}^i}\tilde{\boldsymbol{\beta}}'_{\mathbf{0}j}\tilde{\boldsymbol{A}}'^{\,i}_{\mathbf{1}}\tilde{\boldsymbol{B}}'_{\mathbf{1}k} \\
&\quad + \mu^2\left(\frac{1}{2}\left(\frac{\partial Q_{jk}}{\partial \tilde{\boldsymbol{\phi}}^i} + \frac{\partial Q_{ij}}{\partial \tilde{\boldsymbol{\phi}}^k} + \frac{\partial Q_{ki}}{\partial \tilde{\boldsymbol{\phi}}^j}\right)\tilde{\boldsymbol{\alpha}}'^{\,k}_{\mathbf{0}}\tilde{\boldsymbol{A}}'^{\,i}_{\mathbf{1}}\tilde{\boldsymbol{A}}'^{\,j}_{\mathbf{1}} + P^{ij}\tilde{\boldsymbol{\beta}}'_{\mathbf{0}j}\tilde{\boldsymbol{B}}'_{\mathbf{2}i}\right), \\
S_1^{(1)} &= \int_{\Pi T\Sigma} \mu\tilde{\boldsymbol{\phi}}^{\,l}_{\mathbf{-1}}\left[\frac{\mu}{2}\frac{\partial P^{ij}}{\partial \tilde{\boldsymbol{\phi}}^l}\tilde{\boldsymbol{B}}_{\mathbf{2}i}\tilde{\boldsymbol{B}}_{\mathbf{1}j} - \frac{\mu^3}{2}\frac{\partial^2 Q_{jk}}{\partial \tilde{\boldsymbol{\phi}}^i\partial \tilde{\boldsymbol{\phi}}^l}\tilde{\boldsymbol{A}}'^{\,i}_{\mathbf{1}}\tilde{\boldsymbol{A}}'^{\,j}_{\mathbf{1}}\tilde{\boldsymbol{A}}'^{\,k}_{\mathbf{1}}\right. \\
&\quad \left. - \frac{\mu}{8}\frac{\partial^2 P^{jk}}{\partial \tilde{\boldsymbol{\phi}}^i\partial \tilde{\boldsymbol{\phi}}^l}\tilde{\boldsymbol{A}}'^{\,i}_{\mathbf{1}}\tilde{\boldsymbol{B}}'_{\mathbf{1}j}\tilde{\boldsymbol{B}}'_{\mathbf{1}k}\right],
\end{aligned}
$$

$S_1^{(p)} \sim \mu^p$.

If $\mu \longrightarrow 0$, $S_1^{(p)} \longrightarrow 0$ for $p > 0$.

After taking the limit $\mu \longrightarrow 0$ with preserving the symplectic structure, $S_1^{(p)}$ for $p > 0$ reduces to zero, and the 2D action is

$$S_P = \frac{1}{4} \int_{\Pi T \Sigma} \tilde{\boldsymbol{\beta}}_{\mathbf{1}i} d\tilde{\boldsymbol{\phi}}^i + \tilde{\boldsymbol{B}}_{\mathbf{1}i} d\tilde{\boldsymbol{\alpha}}_{\mathbf{0}}{}^i + P^{ij} \tilde{\boldsymbol{B}}_{\mathbf{1}i} \tilde{\boldsymbol{\beta}}_{\mathbf{1}j} + \frac{1}{2} \frac{\partial P^{jk}}{\partial \tilde{\boldsymbol{\phi}}^i} \tilde{\boldsymbol{\alpha}}_{\mathbf{0}}{}^i \tilde{\boldsymbol{B}}_{\mathbf{1}j} \tilde{\boldsymbol{B}}_{\mathbf{1}k}. \qquad (13)$$

$(S_P, S_P) = 0 \Longleftrightarrow P^{ij} =$ Poisson

This action on $T^*[1](TM)$ is a different realization of a Poisson structure from A-model (the Poisson sigma model).

§10. Summary and Outlook

· We have obtained a systematic method to construct a new topological field theory including superfields with the negative total degree by the AKSZ formalism, which is compatible with dimensional reduction.

· We have obtained one-parameter families of topological sigma models with a complex structure or a symplectic structure.

· Dimensional reduction via the AKSZ formalism (or the BV formalism) does not work. We need a general formalism.

· Quantization is direct and clear as a physical theory but the calculation is complicated and the mathematical structure is unknown.

· A general theory in general dimensions is not constructed.