Generalized Complex Structure and Topological Field Theory

Noriaki Ikeda
Ritsumeikan Univ.

1. Introduction

Purpose

topological M theory

'04 Dijkgraaf, Gukov, Neitzke, Vafa

'04 Grassi, Vanhove
'04 Nekrasov

What is the topological M theory?

@ Conjecture

Topological M theory will be described by the Hitchin functional.

(target space description)

Hitchin has analyzed the stable forms and constructed the actions in 6D & 7D which includes 3-form flux H. (and in 8D)

→ Hitchin functional '03 Hitchin

→ generalized geometry
(generalized complex structure)

Here,

worldsheet description of topological M theory
(worldvolume)

As a first step,

topological σ-model with
generalized geometry
(generalized complex structure)

@ Example

2D $N=(2,2)$ supersymmetric sigma model

First $N=(1,1)$ SUSY σ-model

$\phi^i : \Sigma \longrightarrow M$

worldsheet target sp.

$$S = \int_\Sigma d^2\sigma\, d^2\theta\, D_+\Phi^i D_-\Phi^j \left(g_{ij}(\Phi) + b_{ij}(\Phi)\right)$$

Φ^i : $N=(1,1)$ superfield

$D_\pm$: super derivative

g_{ij} : metric on M

$b_{ij} = -b_{ji}$: B-field 2-form

$\boxed{H = d_M b \neq 0}$: 3-form on M

d_M : derivative on M

SUSY,

$$\delta_1 \Phi^i = \epsilon_1^+ D_+ \Phi^i + \epsilon_1^- D_- \Phi^i$$

another SUSY

$$\delta_2 \Phi^i = \epsilon_2^+ D_+ \Phi^j J_+{}^i{}_j + \epsilon_2^- D_- \Phi^j J_-{}^i{}_j$$

We consider the condition to be $N=(2,2)$.

the condition of $(g, b, J_\pm)$ is

- $H=0$

  $N=(2,2) \Leftarrow M$: Kähler $J_\pm$: complex str.

- $H \neq 0$

  $N=(2,2) \Leftrightarrow M$; bi-Hermitian

  Gates, Hull, Roček '84

  $\Updownarrow$

  M : generalized Kähler str.

  Gualtieri '04

$$\boxed{N=2 \text{ SUSY} + H\text{-flux}}$$

$\Rightarrow$ generalized geometry

$(SU(3) \times SU(3)$ str$)$

nontopological

- $N=(2,2)$ supersymmetric sigma model Gates, Hull, Roček '84

with b-field s.t. $db = H \neq 0$

- type II superstring with SUSY SU(3) structure Grana, Minasian, Petrini, Tomasiello et.al. '04

topological

- Hitchin functional
  stable form Hitchin '03
  3-form H in 6D & 7D
- topological M-theory
- mirror symmetry
  complex $\leftrightarrow$ symplectic
- 3-form flux $H \neq 0$
- special holonomy
- non commutative geometry

2. Generalized Complex Structure

cf. complex structure J

d-dim manifold M

$J : TM \to TM$

s.t. $J^2 = -1$ —— ①

$\Pi_{\mp}[\Pi_{\pm} X, \Pi_{\pm} Y] = 0$ —— ②

where

X, Y vector field on TM

$\Pi_{\pm}$: projection on $\pm\sqrt{-1}$ eigen bundle

$[\ ,\ ]$: Lie bracket

Generalized Complex Structure

$$\mathcal{J} : TM \oplus T^*M \to TM \oplus T^*M$$

s.t. preserve $\langle\, , \rangle$ —— (0)

$\mathcal{J}^2 = -1$ —— (1)

$\Pi_{\mp}[\Pi_{\pm}(X+\xi), \Pi_{\pm}(Y+\eta)] = 0$ —— (2)

$$\mathcal{J} = \begin{array}{cc} \begin{pmatrix} J & P \\ Q & -J^* \end{pmatrix} & \begin{array}{c} TM \\ T^*M \end{array} \\ \begin{array}{cc} TM & T^*M \end{array} & \end{array}$$

◎ local coordinate expression

$$\mathcal{J} = \begin{array}{cc} \begin{pmatrix} J & P \\ Q & -J^* \end{pmatrix} & \begin{array}{l} TM \\ T^*M \end{array} \\ \begin{array}{cc} TM & T^*M \end{array} & \end{array}$$

①② $J^i{}_k J^k{}_j + P^{ik} Q_{kj} + \delta^i_j = 0$

$J^i{}_k P^{kj} + J^j{}_k P^{ki} = 0$

$Q_{ik} J^k{}_j + Q_{jk} J^k{}_i = 0$

$P^{ij} + P^{ji} = 0, \quad Q_{ij} + Q_{ji} = 0$

③ $A^{ijk} = B_i{}^{jk} = C_{ij}{}^k = D_{ijk} = 0$

$A^{ijk} = P^{il} \partial_l P^{jk} + (ijk \text{ cyclic})$

$B_i{}^{jk} = J^l{}_i \partial_l P^{jk} + P^{jl}(\partial_i J^k{}_l - \partial_l J^k{}_i)$
$\quad + P^{kl} \partial_l J^j{}_i - \partial_i J^j{}_l P^{lk}$

$C_{ij}{}^k = J^l{}_i \partial_l J^k{}_j - J^l{}_j \partial_l J^k{}_i - J^k{}_l \partial_i J^l{}_j$
$\quad + J^k{}_l \partial_j J^l{}_i + P^{kl}(\partial_l Q_{ij} + \partial_i Q_{jl}$
$\quad + \partial_j Q_{li})$

$D_{ijk} = J^l{}_i (\partial_l Q_{jk} + \partial_k Q_{lj}) + J^l{}_i (\partial_l Q_{ki}$
$\quad + \partial_i Q_{lk}) + J^l{}_k (\partial_l Q_{ij} + \partial_j Q_{li})$
$\quad - Q_{jl} \partial_i J^l{}_k - Q_{kl} \partial_j J^l{}_i - Q_{il} \partial_k J^l{}_j$

example

1. $\mathcal{J} = \begin{pmatrix} J & 0 \\ 0 & -J^* \end{pmatrix}$

$\mathcal{J}$ is GCS $\Leftrightarrow$ J is complex str.

2. $\mathcal{J} = \begin{pmatrix} 0 & -Q^{-1} \\ Q & 0 \end{pmatrix}$

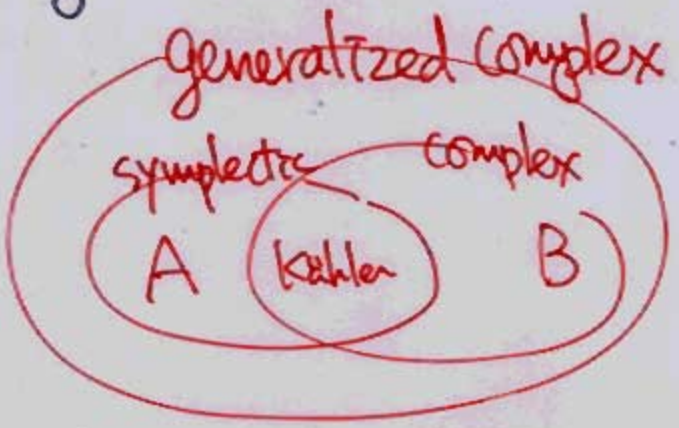
$\mathcal{J}$ is GCS $\Leftrightarrow$ Q is symplectic str.

Q twisted Generalized Complex structure

We can generalize GCS by a closed 3-form H

$$[X+\xi, Y+\eta]_H = [X+\xi, Y+\eta] + i_X i_Y H$$

In integrability condition ②, $[\ ,\ ]$ is replaced by $[\ ,\ ]_H$



◎ Automorphism on GCS

semi-direct product of

Diff(M) & b-transformation

(b-field transf.)

<u>Def</u> b-transformation

for 2-form $b = b_{ij} d\phi^i d\phi^j$

$$\exp(b)(X+\xi) \equiv X + \xi + i_X b$$

◎ If $d_M b = 0 \Rightarrow$

$$[\exp(b)(X+\xi), \exp(b)(Y+\eta)]$$
$$= \exp(b)[X+\xi, Y+\eta] \quad \text{covariant}$$

2. $\hat{\mathcal{J}} \equiv \exp(-b)\, \mathcal{J} \exp(b)$ adjoint

∘ local coordinate expression

$$\begin{cases} \hat{J}^i{}_j = J^i{}_j - P^{ik} b_{kj} \\ \hat{P}^{ij} = P^{ij} \\ \hat{Q}_{ij} = Q_{ij} + b_{ik} J^k{}_j - b_{jk} J^k{}_i + P^{kl} b_{ki} b_{lj} \end{cases}$$

3. Topological Sigma Model with GCS

We construct topological σ-model s.t. GCS = BRST structure

- worldsheet description of Hitchin functional (?)

<u>Hint</u> 2D model

1. N=(2,2) SUSY σ-model with $H = d_M b + 0$

$\downarrow$ topological twist on generalized CY mfd

TFT

Kapustin'03 Kapustin, Li'04

$$Q_B \sim g_{ij}\partial_+\phi^i(H\sqrt{-1}J_+)\psi_+^j + g_{ij}\partial_-\phi^i(H\sqrt{-1}J_-)\psi_-^j$$

2. topological σ-model on symplectic mfd

$$S = \int_\Sigma \frac{1}{2}\Omega_{ij}\, d\phi^i d\phi^j$$

3. topological σ-model on Poisson mfd

$$S = \int_\Sigma B_i d\phi^i + \frac{1}{2}P^{ij}B_i B_j$$

$1+2+3 \longrightarrow$

$\phi^i : \Sigma \to M$

$B_i \in T^*\Sigma \otimes \phi^*(TM)$ 1-form

d : exterior derivative on Σ

Zucchini '04

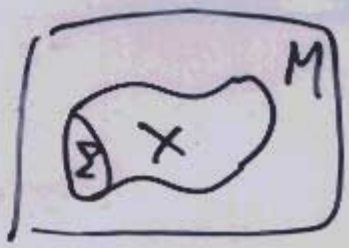
$$\boxed{S_Z = \int_\Sigma B_i d\phi^i + J^i{}_j B_i d\phi^j + \frac{1}{2} P^{ij} B_i B_j + \frac{1}{2} Q_{ij} d\phi^i d\phi^j}$$

Hitchin σ-model (Zucchini model)

J, P, Q is GCS $\underset{\not\Leftarrow}{\Rightarrow}$ S_Z is BRST inv.

- twisted GCS

X : 3D mfd s.t. $\partial X = \Sigma$



$$S_Z' = S_Z + \int_X \phi^* H \qquad \text{WZ-term}$$

J, P, Q_{iH} is twisted GCS $\underset{\not\Leftarrow}{\Rightarrow}$ S_Z' : BRST inv.

- Problems
  - GCS is <u>not</u> equivalent to BRST str.
  - Model is <u>not</u> invariant under b-transformation.

b-transformation

$$\begin{cases} \hat{\phi}^i = \phi^i \\ \hat{B}_i = B_i + b_{ij} d\phi^j \end{cases}$$

then $\hat{S}_Z = S_Z - \int_\Sigma b_{ij} d\phi^i d\phi^j$

Q Solution

3D world volume is more natural.

because

Courant bracket $[X+\xi, Y+\eta]$

The algebra constructed from the Courant bracket is Courant algebroid

and

Moduli of Courant algebroids

$\cong$ 3D Schwarz type TFT moduli N.I'a
(BF theory)

cf.) 2D Schwarz type TFT moduli

$\cong$ Lie algebroid moduli

$$\text{Courant algebroid} \otimes \mathbb{C} \xrightarrow{\mathcal{J}=\text{INFI projection}} \text{GCS}$$

therefore

3D Schwarz type TFT (Courant σ-model)

$\downarrow \mathcal{J}=$ INFI projection

3D generalized complex σ-model

Generalized Complex Sigma model in 3D

$\phi^i : X \to M \qquad X : 3D$

$A^i \in T^*X \otimes \phi^*(T^*M) \qquad$ 1-form

$B_{1i} \in T^*X \otimes \phi^*(TM) \qquad$ 1-form

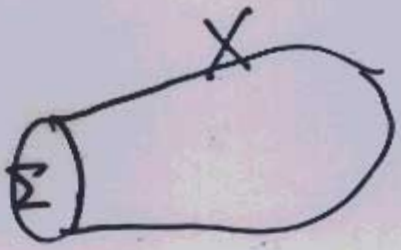
B_{2i} : 2-form

$$S_{GC} = \int_X -B_{2i} d\phi^i + B_{1i} dA^i - J^i{}_j B_{2i} A^j - P^{ij} B_{2i} B_{1j}$$
$$+ \frac{1}{2}\left(J^l{}_i H_{jkl} + \frac{\partial Q_{jk}}{\partial \phi^i}\right) A^i A^j A^k$$
$$+ \frac{1}{2}\left(-P^{kl} H_{ijl} - \frac{\partial J^k{}_j}{\partial \phi^i} + \frac{\partial J^k{}_i}{\partial \phi^j}\right) A^i A^j B_{1k}$$
$$+ \frac{1}{2} \frac{\partial P^{jk}}{\partial \phi^i} A^i B_{1j} B_{1k}$$

- S_{GC} is BRST inv. $\Leftrightarrow$ J, P, Q, H is (twisted) GCS
- S_{GC} is b-transformation invariant.

4. Zucchini Model as a boundary action

$\Sigma = \partial X$



Σ : 2D, X : 3D

We can derive 2D Zucchini model as a boundary action on Σ of 3D GC sigma model.

<u>H=0 case</u>

$$S_{GC}(X) = S_Z(\Sigma) + S_a(X)$$

- $S_Z(\Sigma)$: Zucchini action $\quad B_{1i} = B_i$
- $S_a(X)$: topological term independent of GCS

$$S_a(X) = \int_X -Y'_{2i}\, d\phi^i + dB_{1i} Z'^i + Y_{2i} A^i + B_{2i} Z'^i + B_{2i} A^i$$

Y'_{2i}, Z'^i : auxiliary fields

$S_{GC}(X)$ & $S_Z(\Sigma)$ is based on the same GCS, so $S_Z(\Sigma)$ is regarded as a boundary action of $S_{GC}(X)$

$H \neq 0$ case

We can obtain

$$S'_{GC}(X) = S'_Z(\Sigma) + S_a(X)$$

$$S'_Z(\Sigma) = S_Z(\Sigma) + \frac{1}{2}\int J^l{}_i H_{jkl}\, d\phi^i d\phi^j d\phi^k - p^{kl} H_{ijl} B_{ik} d\phi^i d\phi^j$$

b-inv. WZ term

Zucchini action is not b-invariant

$$S_{GC} = S_Z(\Sigma) + S_a(X)$$

$\Downarrow$ b-transformation

$$\hat{S}_{GC} = \hat{S}_Z(\Sigma) + \hat{S}_a(X)$$

$$\hat{S}_Z(\Sigma) = S_Z(\Sigma) - \frac{1}{2}\int_\Sigma b_{ij}\, d\phi^i d\phi^j$$

$H = -d_M b$

$$= S_Z(\Sigma) + \frac{1}{2}\int_X H$$ = Zucchini model with H

5 Summary

3D topological σ-model
with twisted GCS on X

$\downarrow \partial X$ boundary

2D topological σ-model
with twisted GCS on ∂X

- BRST $\Leftrightarrow$ tGCS
- b-transformation invariant

- worldsheet description $\leftrightarrow$ spacetime description
  (volume)

  Pestun, Witten '05

- (topological) σ-model with H-flux

  6D generalized $SU(3)$
  7D " G_2
  8D " $Spin(7)$

  '05 Witt

related to Hitchin
generalized geometry
(topological) M, F theory

- brane
  σ-model with boundary