Generalized Complex Structure and Topological Field Theory
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1. Introduction

**Purpose**
topological M theory

by D. Gravin, N. Itoh, N. Nakato, N. \( \hat{N} \)

by G. V. N.

by G. V. N. H. T.

What is the topological M theory?

**Conjecture**
Topological M theory will be described by the Hitchin functional.
(target space description)
Hitchin has analyzed the stable forms and constructed the actions in 6D & 7D which includes 3-form flux $H$. (and in 8D)

→ Hitchin functional

→ generalized geometry

(generalized complex structure)

Here, worldsheet description of topological (worldvolume) M theory

As a first step, topological $\sigma$-model with

generalized geometry

(generalized complex structure)
Example

2D \( N=(2,2) \) supersymmetric sigma model

First \( N=(1,1) \) SUSY \( \sigma \)-model

\( \phi^i: \Sigma \rightarrow M \)

worldsheet \ target \ space

\[
S = \int_{\Sigma} d^2 \sigma d^2 \theta D_{\pm} \phi D_{\pm} \phi^* (g_{ij}(\hat{\sigma}) + b_{ij}(\hat{\tau}))
\]

\( \Phi^i : N=(1,1) \) superfield

\( D_{\pm} : \) super derivative

\( g_{ij} : \) metric on \( M \)

\( b_{ij} = -b_{ji} : \) B-field 2-form

Hadamard: \( 3 \)-form on \( M \)

\( \delta \Phi^i = \epsilon^i D_{\pm} \phi^* + \epsilon \mp D_{\pm} \phi^* \)

SUSY
We consider the condition to be \( N = (2, 2) \).

1. \( H = 0 \)
   - \( N = (2, 2) \leq M \) if Kähler T: complex structure.

2. \( H \neq 0 \)
   - \( N = (2, 2) \equiv M \) if bi-Hermitian.
   
   \[ \Downarrow \]

- \( M \) : generalized Kähler structure.

\[ N = 2 \text{ SUSY} + H \cdot \text{flux} \]

\[ \Rightarrow \text{generalized geometry} \]

\[ (SU(3) \times SU(3) \text{ str}) \]
N=(2,2) supersymmetric sigma model with b-field st. \( db = H = 0 \)

- Type II superstring with SUSY SU(3) structure
- Hitchin functional stable form
- 3-form \( H \) in 6D & 7D
- Topological M-theory
- Mirror symmetry complex \( \leftrightarrow \) symplectic
- 3-form flux \( H = 0 \)
- Special holonomy
- Noncommutative geometry
2. Generalized Complex Structure

cf. complex structure $J$

d-dim manifold $M$

$J : TM \to TM$

such that $J^2 = -1$ \hspace{1cm} (1)

$T_{\tau}[T_{\tau}X, T_{\tau}Y] = 0$ \hspace{1cm} (2)

where $X, Y$ vector fields on $TM$

$T_{\tau}$ : projection on $\pm \sqrt{-1}$-eigen bundle

$[\cdot, \cdot]$ : Lie bracket
Q: Generalized Complex Structure

\[ J : TM \oplus T^*M \rightarrow TM \oplus T^*M \]

\[ J^2 = -1 \]

\[ \mathcal{F} \{ TT_{\pm}(X \pm 1), TT_{\pm}(Y \pm 1) \} = 0 \]

\[ J = \begin{pmatrix} J & P \\ Q & -J^* \end{pmatrix} \]

\[ \begin{array}{c}
\text{T}^M \\
\text{T}^*M \\
\end{array} \]
\[ j = \left( \begin{array}{c} T \\ \mathbf{p} \\ \mathbf{Q} \end{array} \right) \]

\[ \mathbf{T}^{\ast} \mathbf{M} \]

1. \( j_{i}^k \mathbf{J}_{j}^k + P_{i}^k \mathbf{Q}_{j}^k + S_{i}^{\ast} = 0 \)

2. \( j_{i}^k \mathbf{P}_{j}^k + J_{j}^k \mathbf{P}_{i}^k = 0 \)

3. \( j_{i}^k \mathbf{Q}_{j}^k + Q_{j}^k \mathbf{J}_{i}^k = 0 \)

4. \( P_{i}^{\ast} + P_{j}^{\ast} = 0 \)

5. \( Q_{i}^{\ast} + Q_{j}^{\ast} = 0 \)

\[ \mathbf{A}_{i}^{jk} = B_{i}^{jk} = C_{i}^{jk} = \mathbf{D}_{i}^{jk} = 0 \]

\[ \mathbf{A}_{i}^{jk} = \mathbf{P}_{i}^{jk} \mathbf{Q}_{i}^{jk} + (i \leftrightarrow j \text{ cyclic}) \]

\[ \mathbf{B}_{i}^{jk} = \mathbf{J}_{i}^{jk} \mathbf{P}_{i}^{jk} + \mathbf{P}_{i}^{jk} (\mathbf{Q}_{i}^{jk} - \mathbf{J}_{i}^{jk}) \]

\[ \mathbf{C}_{i}^{jk} = \mathbf{J}_{i}^{jk} \mathbf{J}_{i}^{jk} - \mathbf{J}_{i}^{jk} \mathbf{Q}_{i}^{jk} - \mathbf{J}_{i}^{jk} \mathbf{J}_{i}^{jk} \]

\[ \mathbf{D}_{i}^{jk} = \mathbf{J}_{i}^{jk} (\mathbf{Q}_{i}^{jk} + \mathbf{J}_{i}^{jk}) + \mathbf{J}_{i}^{jk} (\mathbf{Q}_{i}^{jk} + \mathbf{J}_{i}^{jk}) \]

- \( \mathbf{Q}_{i}^{\ast} \mathbf{J}_{i}^{\ast} - \mathbf{Q}_{i}^{\ast} \mathbf{J}_{i}^{\ast} - \mathbf{Q}_{i}^{\ast} \mathbf{J}_{i}^{\ast} \)
1. $J = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

$J$ is GCS $\iff$ $J$ is complex str.

2. $J = \begin{pmatrix} 0 & -2^{-1} \\ 2^{-1} & 0 \end{pmatrix}$

$J$ is GCS $\iff$ $Q$ is symplectic str.

A twisted Generalized Complex structure

We can generalize GCS by a closed 3-form $H$

$[X+\delta, Y+\eta]_H = [X+\delta, Y+\eta] + i\chi(X, Y, H)$

In integrability condition $[X, [Y, Z]]$ is replaced by $C, F, J_H$

Diagram:

```
+----------------+  +----------------+
|                |    |                |
|complex         |    |complex         |
|                |    |                |
|sympetctic      |    |sympetctic      |
|                |    |                |
+----------------+  +----------------+
   A         B

generalized complex
```
2. Automorphism on GCS
   semi-direct product of
   \[ \text{Diff}(M) \times b \text{-transformation} \]
   \[ (b = b \text{-field transf.)} \]

Def. b-transformation
for 2-form \( b = b_{ij} \delta^i \delta^j \)
\[ \exp(b)(X+\xi) = X + \xi^i b_i \]

1. If \( \partial X b = 0 \Rightarrow \]
\[ [\exp(b)(X+\xi), \exp(b)(Y+\eta)] = \exp(b) [X+\xi, Y+\eta] \]
   covariant

2. \[ J \equiv \exp(-b) J \exp(b) \]
   adjoint

Local coordinate expression
\[ \tilde{J}^i_j = J^i_j - \mathcal{P}^i_k b_k j \]
\[ \tilde{b}^i_j = b^i_j \]
\[ \tilde{\Omega}^i_j = \Omega^i_j + b^k_j \mathcal{P}^i_k b^k_j \]
3. Topological Sigma Model with GCS
We construct topological σ-model with GCS = BRST structure.

1. N=(2,2) SUSY σ-model with H-symmetry
   → topological twist on generalized CF-field

2. topological σ-model on symplectic mild

3. topological σ-model on Poisson mild

1+2+3 →
\( \phi : \Sigma \to M \)

\( B \in \mathcal{E}T^*\Sigma \otimes \phi^*(TM) \) 1-form

\( d \circ \text{exterior derivative on } \Sigma \)

\( Z_{\phi,\Sigma} = B_2 \wedge d\phi + \sum_{i,j} B_i \wedge d\phi^j + \frac{1}{2} P_i^j B_i \wedge d\phi^j \)

\( H = \frac{1}{2} \epsilon_{ijk} d\phi^i \wedge d\phi^j \wedge d\phi^k \)

Hitchin 0-model (Zucchini model)

\( J, P, Q \) is GCS \( \Rightarrow \) \( S_2 \) is BRST m.v.

- twisted GCS

\( \chi : \Sigma \text{ 3D mfd s.t., } \forall x \in \Sigma \)

\( S_2' = S_2 + \sum_i \phi^i H \)

\( WZ \)-term

\( T, P, Q, H \) is twisted GCS \( \Rightarrow \) \( S_2' \) \text{ BRST m.v.} \leftarrow

- Problems

- GCS is not equivalent to BRST str.

- Model is not invariant under \( \beta \)-transformation.
\[ b - transformation \]
\[ \phi^i = \phi^i \]
\[ \vec{B}_i = B_i + b_i \phi^i \]

then \( \hat{S}_2 = S_2 - \sum_b b_i \phi^i \phi^i \)

**Solution**

3D world volume is more natural because

Courant bracket \([X + \phi, Y + \eta]\)

The algebra constructed from the Courant bracket is **Courant algebroid**

and **Moduli of Courant algebroids**

\[ \equiv 3D \text{ Schwarz type TFT moduli } \text{Nil'2} \]

(cf.) 2D Schwarz type TFT moduli

\[ \equiv \text{Lie algebroid moduli} \]
Courant algebroid $\otimes C \rightarrow GCS$  
\[ \text{f is Inf projection} \]

Therefore

3D Schwenz type TFT (Courant model)  
\[ \text{f is Inf projection} \]

3D generalized complex G-model

Generalized Complex Sigma model in 3D
\[ \phi^i : X \rightarrow M \times \mathbb{R}^3 \]
\[ A^i \in T^* \times \otimes \phi^*(TM) \quad 1\text{-form} \]
\[ B_{ij} \in T^* \times \otimes \phi^*(TM) \quad 1\text{-form} \]
\[ B_{ij} : 2\text{-form} \]

\[ S_{GC} = \sum_X -B_{ij}d\phi^i + B_{ij}dA^i 
- \int_X B_{ij}B_{kl} - \int_X B_{ij}B_{kl} 
+ \frac{1}{2} \left( \frac{\partial^2}{\partial \phi^i \partial \phi^j} + \frac{\partial^2}{\partial x^i \partial x^j} \right) A^i A^j A^k 
+ \frac{1}{2} \left( -\frac{\partial^2}{\partial \phi^i \partial \phi^j} + \frac{\partial^2}{\partial x^i \partial x^j} \right) A^i A^j B^k 
+ \frac{1}{2} \left( \frac{\partial^2}{\partial x^i \partial x^j} - \frac{\partial^2}{\partial \phi^i \partial \phi^j} \right) A^i B^j B^k \right] 
\]
\( \text{SG} \) is BRST inv. \( \Theta \), \( J.P.Q.H \) is (twisted) GCS

\( \text{SG} \) is b-transformation invariant.
Zucchini Model as a boundary action

\[ \Sigma = \mathcal{E}X \]
\[ \Sigma: 2D, X: 3D \]

We can derive the Zucchini model as a boundary action on \( \Sigma \) of 3D GC sigma model.

**H=0 case**

\[ S_{tc}(X) = S_\mathcal{E}(\Sigma) + S_a(X) \]

- \( S_\mathcal{E}(\Sigma) \): Zucchini action
  - \( B_i = B_i \)
- \( S_a(X) \): topological term independent of GCS

\[ S_a(X) = \int -y_{2i} d\phi^i + dB_{2i}Z^i + y_{2 \epsilon} A^i \]

- \( y_{2i}, Z^i \): auxiliary fields
$S_{gc}(x) \& S_{z}(\Sigma)$ is based on the same GCS, so $S_{z}(\Sigma)$ is regarded as a boundary action of $S_{gc}(x)$

**H \neq 0 case**

We can obtain

$S'_{gc}(x) = S'_{z}(\Sigma) + S_{a}(x)$

$S'_{z}(\Sigma) = S_{z}(\Sigma) + \frac{1}{2} \sum_{x \in H} b_{g} \phi_{g} \alpha \phi_{g}$

$H = \text{dub}$

$S_{z}(\Sigma) = S_{z}(\Sigma) + \frac{1}{2} \sum_{x \in H} b_{g} \phi_{g} \alpha \phi_{g}$

$\text{Zucchini model with } H$

*b-inv. WZ term*

Zucchini action is not $b$-invariant

$S_{gc} = S_{z}(\Sigma) + S_{a}(x)$

$b$-transformation

$S_{gc} = \hat{S}_{z}(\Sigma) + \hat{S}_{a}(x)$

$S_{z}(\Sigma) = S_{z}(\Sigma) - \frac{1}{2} \sum_{x \in H} b_{g} \phi_{g} \alpha \phi_{g}$

$\text{Zucchini model with } H$
Summary

3D topological σ-model
with twisted GCS on X
↓
Ex boundary

2D topological σ-model
with twisted GCS on Ex

- BRST ↔ tGCS
- b-transformation invariant
worldsheet description & spacetime description (volume)  
Postum, Witten '05

(topological) O-model with H-flux
6D generalized SU(3)
7D " G2
8D " Spin(7)
'05 Witten

related to Hitchin
  generalized geometry
  (topological) M, F theory

brane O-model with boundary