Supergeometry における QP Pair と ポアソン幾何、カレント代数への応用

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湯谷 2013

NI and Xiaomeng Xu, arXiv:1301.4805, arXiv:1308.0100.

Introduction

最近の Poisson 幾何および場の理論に現れる新しい幾何学構造を統一的に記述する新しい枠組みを提案する。

Supergeometry で考える。

- いろいろな幾何構造、代数構造、物理構造の統一的な記述
- 数学 $\longleftrightarrow$ supergeometry $\longleftrightarrow$ 物理

Introduction

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New Notion

QP pair (DG Symplectic Pair)

A tower of two differential graded symplectic manifolds constructed from twisting of a Lagrangian subsupermanifold by a **canonical transformation.**

Introduction I

This structure appears in many situations in mathematics and physics.

0, Complex, Symplectic, Poisson structures, etc.

1, quasi-Poisson, twisted Poisson structures, generalized geometry Alekseev, Kosmann-Schwarzbach, Meinrenken '02, Park '00, Klimcik, Strobl, '01, Severa, Weinstein '01

2, Courant algebroids, Dirac structures Courant '90, Liu, Weinstein, Xu '96

3, L_∞−structures Lada, Stasheff '92

4, Poisson functions Terashima '08, Kosmann-Schwarzbach '11

5, V-data Voronov '05

11, Geometry of BRST-BV-BFV formalisms in Gauge Theories Batalin, Vilkovisky '83, Batalin, Fradkin '83, Schwarz '92

12, Anomalies in Gauge Theories Wess, Zumino '71, Faddeev, Shatashvili '84

Introduction II

13, AKSZ sigma models witout and with boundaries.

Park '00, N.I. '01, Cattaneo, Mnev, Reshetikhin, '12, N.I. Xu '13-1

14, Symmetries in the BV master equations in String Field Theories

Hata, Zwiebach '93

15, **Generalized current algebras**

Alekseev, Strobl '05, Bonelli, Zabzine '05, N.I. Koizumi '11,

Purpose

- 微分作用 Q の homological な性質 $Q^2 = 0$ が破れているときの制御。Ex.) WZ terms, Maurer-Cartan equations, etc.
- Anomaly の supergeomety 的な理解
- 新しいカレント代数の発見
- moment map の理論の拡張

QP Manifold (DG Symplectic Manifold) I

Definition 1

A following triple $(\mathcal{M}, \omega, Q)$ is called a QP-manifold (a differential graded symplectic manifold) of degree n.

• $\mathcal{M}$: N-manifold (nonnegatively graded manifold)

A **graded manifold** $\mathcal{M}$ on a smooth manifold M is a ringed space $(M, \mathcal{O}_M)$, which structure sheaf $\mathcal{O}_M$ is **Z**–graded commutative algebras over M, locally isomorphic to $C^\infty(U) \otimes S^\cdot(V)$, where U is a local chart on M, V is a graded vector space and $S^\cdot(V)$ is a free graded commutative ring on V.

If degrees are nonnegative, a graded manifold is called a **N-manifold**.

QP Manifold (DG Symplectic Manifold) II

• ω: P-structure (graded Poisson bracket)
A graded symplectic form of degree n on $\mathcal{M}$.

• Q: Q-structure (a homological vector field)
A vector field of degree $+1$ such that $Q^2 = 0$, is a symplectic vector field, that is, $L_Q\omega = 0$.

Note: We assume that there exists a Hamiltoinan function (a homological function) $\Theta \in C^\infty(\mathcal{M})$ such that $Q(-) = \{\Theta, -\}$, $Q^2 = 0$ is $\{\Theta, \Theta\} = 0$.

Theorem 2

If $n \neq -1$, above Q is a Hamlitonian vector field, that is, there exists $\Theta \in C^\infty(\mathcal{M})$ such that $Q(-) = \{\Theta, -\}$.

QP Manifold (DG Symplectic Manifold) III

Example 3 ($n = 1$)

$\mathcal{M} = T^*[1]M$
$C^\infty(T^*[1]M) \simeq \Gamma(\wedge^\bullet TM)$, locally $(x^i, \xi_i) \simeq \left(x^i, \frac{\partial}{\partial x^i}\right)$.
ω defines the Schouten bracket $[-,-]_S$ on $\wedge^\bullet TM$.
$\Theta = \frac{1}{2}\pi^{ij}(x)\frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}$ satisfying $[\Theta, \Theta]_S = 0$ is a Poisson bivector field.

Theorem 4

QP-structure of degree 1 *on* $T^*[1]M \simeq$ *Poisson structure on* M

A Poisson bracket on M is defined as $\{F, G\}_{P.B.} = -\{\{F, \Theta\}, G\}$, where $F, G \in C^\infty(M)$. This double bracket is called the **derived bracket**.

QP Manifold (DG Symplectic Manifold) IV

Example 5 ($n = 2$)

$\mathcal{M} = T^*[2]E[1]$, is a N-manifold of degree 2, where E is a vector bundle on M. (Generally, $\mathcal{M}$ of degree 2 is not a vector bundle.).

Theorem 6

Roytenberg '99

A homological function Θ on a QP manifold of degree 2 induces a Courant algebroid structure on E.

E is recovered from a graded manifold $\mathcal{M}$ by a natural filtration of degree $\mathcal{M} \longrightarrow E[1] \longrightarrow M$.

Canonical Transformation and Canonical Function

Definition 7 (Canonical Transformation (Twisting))

Let $(\mathcal{M}, \omega, \Theta)$ be a QP manifold of degree n. Let $\alpha \in C^\infty(\mathcal{M})$ be a function of degree n. A **canonical transformation** e^{δ_α} is defined by $f' = e^{\delta_\alpha} f = f + \{f, \alpha\} + \frac{1}{2}\{\{f, \alpha\}, \alpha\} + \cdots$.
e^{δ_α} is also called twisting.

Let $\Theta' = e^{\delta_\alpha}\Theta$. If $\{\Theta, \Theta\} = 0$, $\{\Theta', \Theta'\} = e^{\delta_\alpha}\{\Theta, \Theta\} = 0$ for any twisting.

Definition 8 (Canonical Function)

Let $\mathcal{L}$ be a Lagrangian subspace of $\mathcal{M}$ with respect to ω. If twisting α satisfies $\Theta'|_{\mathcal{L}} = e^{\delta_\alpha}\Theta|_{\mathcal{L}} = 0$, α is called a **canonical function** of order n.

Derived QP Manifolds I

A bracket on $C^\infty(\mathcal{L})$ defined by the derived bracket,

$$\{-,-\}_s = \{\{-,\Theta\},-\}|_{\mathcal{L}}.$$

is graded Poisson. It defines a graded symplectic structure on $\mathcal{L}$ if $\{-,-\}_s$ is nondegenerate.

In this setting, if a canonical function satisfies
$\{\alpha,\alpha\}_s = \{\{\alpha,\Theta\},\alpha\}|_{\mathcal{L}} = \{\{\Theta,\alpha\},\alpha\}|_{\mathcal{L}} = 0,$ we have

Theorem 9

If $\{-,-\}_s$ is nondegenerate, $(\mathcal{L},\{-,-\}_s,\alpha)$ is a QP manifold of degree $n-1$.

We call this QP manifold a **derived QP manifold**.
Conversely,

Derived QP Manifolds II

Theorem 10 (N.I.,Xu '13)

For any QP manifold of degree $n-1$, $(\mathcal{M}, \{-,-\}_{\mathcal{M}}, \alpha)$, *there exists a canonical QP manifold* $(T^*[n]\mathcal{M}, \omega, \Theta, \alpha)$, *in which graded manifold* α *is a canonical function.*

Proof.

Take a canonical symplectic form on $T^*[n]\mathcal{M}$ as ω.
Let $\mathcal{L}$ be a Lagrangian submanifold of $\mathcal{M}$ with respect to $\omega_{\mathcal{M}}$. The differential $d_{\mathcal{L}}$ on $\mathcal{L}$ is lifted to a vector field Q on $T^*[1]\mathcal{L}$. Θ is defined as a Hamiltonian for Q with respect to ω.
This satisfies $\{-,-\}_{\mathcal{M}} = \{\{-,\Theta\},-\}$. □

Twisted QP Manifolds

Generally α is not homological for $\{-,-\}_s$ since

$$\begin{aligned} \{\alpha,\alpha\}_s &= -\{\{\Theta,\alpha\},\alpha\}|_{\mathcal{L}} \\ &= 2\left(\Theta + \{\Theta,\alpha\} + \frac{1}{3!}\{\{\{\Theta,\alpha\},\alpha\},\alpha\} + \cdots\right)|_{\mathcal{L}}. \end{aligned}$$

Definition 11

Let $\mathcal{L}$ be a N-manifold with a Poisson bracket of degree $n-1$ $\{-,-\}_s$ and α be a function of degree n.
$(\mathcal{L}, \{-,-\}_s, \alpha)$ is called a **twisted QP manifold** of degree $n-1$ if there exists a QP manifold $(T^*[n]\mathcal{L}, \omega, \Theta)$ such that $\{-,-\}_s$ is given by the derived bracket $\{\{-,\Theta\},-\}|_{\mathcal{L}}$ and α is a canonical function on $T^*[n]\mathcal{L}$.

QP Pair

N.I., Xu '13

Definition 12 (QP Pair)

A pair of $(\mathcal{M} = T^*[n]\mathcal{L}, \omega_b, \Theta, \alpha)$ and $(\mathcal{L}, \{-,-\}_s, \alpha)$ is called a (twisted) **QP pair** if $(\mathcal{M}, \omega_b, \Theta, \alpha)$ is a QP manifold and $(\mathcal{L}, \{-,-\}_s, \alpha)$ is its twisted QP manifold.
We call $(\mathcal{M}, \omega_b, \Theta)$ is a **big QP manifold** and $(\mathcal{L}, \{-,-\}_s, \alpha)$ a **small (twisted) QP manifold**.

Example: $(n = 2)$ I

Terashima '08, Kosmann-Schwarzbach '11

$\mathcal{M} = T^*[2](T^*[1]M \times \mathfrak{g}^*[1])$: a QP manifold of degree 2, where M is a manifold and $\mathfrak{g}$ is a Lie algebra.

(x^i, p_i, v_a): Local coordinates on M and the fiber of $T^*[1]M$, and $\mathfrak{g}^*[1]$ of degree $(0, 1, 1)$.

(ξ_i, q^i, u^a): Conjugate coordinates of the fiber $T^*[2]$ of degree $(2, 1, 1)$.

Θ of degree 3:

$$
\begin{aligned}
\Theta &= \Theta_M + \Theta_C + \Theta_R + \Theta_H \\
&= \xi_i q^i + \frac{1}{2} C_{ab}{}^c u^a u^b v_c + \frac{1}{3!} R^{abc}(x) v_a v_b v_c + \frac{1}{3!} H_{ijk}(x) q^i q^j q^k,
\end{aligned}
$$

Example: $(n=2)$ II

which is a homological function if $\{\Theta_M, \Theta_H\} = 0$ and $\{\Theta_C, \Theta_R\} = 0$. This means H is a closed 3-form on M. and R is a closed 3-form associated to Lie algebra cohomology.

Take the Lagrangian submanifold,
$\mathcal{L} = T^*[1]M \times \mathfrak{g}^*[1] = \{\xi_i = q^i = u^a = 0\}$.

Suppose a degree 2 function
$\alpha = \pi + \rho = \frac{1}{2}\pi^{ij}(x)p_i p_j + \rho^j{}_a(x)u^a p_j$ is a canonical function.

Derived QP manifold
Take $\Theta_C = \Theta_R = \Theta_H = 0$ and $\rho = 0$.
Then $(\mathcal{L}, \{-,-\}_s, \alpha)$ is a derived QP manifold of degree 1.
$e^{\delta_\alpha}\Theta|_{\mathcal{L}} = -\{\{\Theta, \pi\}, \pi\}|_{\mathcal{L}} = [\pi, \pi]_S = 0$ which defines a **Poisson structure** on M.

Example: $(n = 2)$ III

Twisted QP manifold

Generally, $(\mathcal{L}, \{-,-\}_s, \alpha)$ is a twisted QP manifold.

If $\rho = 0$, the canonical function equation $e^{\delta_\alpha}\Theta|_{\mathcal{L}} = 0$ defines a **twisted-Poisson structure**, $[\pi, \pi]_S = \wedge^3 \pi^\# H$, where H is a 3-form on M defined by $H_{ijk}(x)$.

Ševera, Weinstein '01

If $\Theta_H = 0$, $e^{\delta_\alpha}\Theta|_{\mathcal{L}} = 0$ defines a **quasi-Poisson structure**, $[\pi, \pi]_S = \wedge^3 \rho^\# R$.

Alekseev, Kosmann-Schwarzbach, Meinrenken '02

The Poisson bracket on M is obtained by the derived-derived bracket,

$$\{-,-\}_s = \{\{-,\Theta\},-\}|_{\mathcal{L}},$$
$$\{-,-\}_{P.B.} = \{\{-,\alpha\}_s,-\}_s.$$

Example: Nambu-Poisson Structures I

A **Nambu-Poisson bracket** of order n (≥ 3) on M is a skew symmetric linear map $\{\cdot, \cdots, \cdot\} : C^\infty(M)^{\otimes n} \longrightarrow C^\infty(M)$ such that

$$
\begin{aligned}
&(1) \quad \{f_{\sigma(1)}, f_{\sigma(2)}, \cdots, f_{\sigma(n)}\} = (-1)^{\epsilon(\sigma)}\{f_1, f_2, \cdots, f_n\},\\
&(2) \quad \{f_1 g_1, f_2, \cdots, f_n\} = f_1\{g_1, f_2, \cdots, f_n\} + g_1\{f_1, f_2, \cdots, f_n\},\\
&(3) \quad \{f_1, f_2, \cdots, f_{n-1}, \{g_1, g_2, \cdots, g_n\}\}\\
&= \sum_{k=1}^{n}\{g_1, \cdots, g_k, \{f_1, f_2, \cdots, f_{n-1}, g_k\}, g_{k+1}, \cdots, g_n\}.
\end{aligned}
$$

The Nambu-Poisson tensor field is the n-vector field $\pi \in \wedge^n TM$ which is defined as $\pi(df_1, df_2, \cdots, df_n) = \{f_1, f_2, \cdots, f_n\}$.

Let us assume 'decomposability' of the Nambu-Poisson tensor, $\pi^{[i_1 \cdots i_n}\pi^{j_1] \cdots j_n} = 0$.

Example: Nambu-Poisson Structures II

$\mathcal{M} = T^*[n](T^*[n-1]E[1])$, where M be a manifold and $E = \wedge^{n-1} T^*M$.

Local coordinates on $T^*[n-1]E[1]$ are denoted by (x^i, v_I, p_i, w^I) of degree $(0, 1, n-1, n-2)$ and conjugate local coordinates of the fiber are (ξ_i, u^I, q^i, z_I) of degree $(n, n-1, 1, 2)$, respectively, where $I = (i_1, i_2, \cdots, i_{n-1})$.

A graded symplectic structure of degree n is $\omega = \delta x^i \wedge \delta \xi_i + \delta v_I \wedge \delta u^I + \delta p_i \wedge \delta q^i + \delta w^I \wedge \delta z_I$.

$$\Theta = -q^i \xi_i + \frac{1}{(n-1)!} z_I (u^I - q^{i_1} \cdots q^{i_{n-1}}),$$

which trivially satisfies $\{\Theta, \Theta\} = 0$. Θ defines the Dorfman bracket on $TM \oplus \wedge^{n-1} T^*M$ by the derived bracket $[-, -]_D = \{\{-, \Theta\}, -\}$.

Example: Nambu-Poisson Structures III

We take a function α as

$$\alpha = -\frac{1}{(n-1)!}\pi^{i_1\cdots i_{n-1}i_n}(x)v_{i_1\cdots i_{n-1}}p_{i_n}.$$

Note that $\{\alpha, \alpha\} = 0$.

Proposition 0.1

Let $\mathcal{M}$, Θ and α be the above ones. Let $\mathcal{L} = T^*[n-1]E[1]$ be the Lagrangian submanifold of $\mathcal{M}$. Then α is a canonical function with respect to Θ and $\mathcal{L}$, i.e., $e^{\delta_\alpha}\Theta|_{\mathcal{L}} = 0$ if and only if π is a decomposable Nambu-Poisson tensor.

Bouwknegt, Jurčo '10

Current Algebras from QP Pairs

Current algebras $\sim$ Twisted Poisson algebra on a mapping space $\sim$ 'Moment Map' up to homotopy.

big QP manifold $\mathcal{M}$ **small QP manifold $\mathcal{L}$**

$$
\begin{array}{ccc}
\text{Poisson algebra of functions on } \mathcal{M} & \xleftarrow{\text{Uplift}} & \text{Target space algebra on } \mathcal{L} \\
\text{AKSZ Construction} \downarrow & & \downarrow \text{(sigma model)} \\
\text{Poisson algebra on } \mathrm{Map}(\mathcal{X}, \mathcal{M}) & & \text{Symmetry algebra on } \mathrm{Map}(\mathcal{X}, \mathcal{L}) \\
\text{twisting by } S_s \downarrow & & \downarrow \text{Quantization} \\
\text{Poisson algebra}' \text{ on } \mathrm{Map}(\mathcal{X}, \mathcal{M}) & \xrightarrow{\text{Lag.}} & \text{Current algebra on } \mathrm{Map}(\mathcal{X}, \mathcal{L})
\end{array}
$$

Functions and Poisson Algebras on QP Pairs I

$(\mathcal{M} = T^*[n]\mathcal{L}, \mathcal{L})$: a QP pair of degree n.

Poisson algebra on big bracket (Seed of Current Algebras)

$C_{n-1}(\mathcal{M}) = \{f \in C^\infty(\mathcal{M}) | |f| \leq n-1\}$: A space of functions of degree equals or less than $n-1$ on $T^*[n]\mathcal{L}$.

$C_{n-1}(\mathcal{M})$ is an algebra not only under the big Poisson bracket $\{-,-\}_b$, but also under the derived bracket $\{\{-,\Theta\}_b,-\}_b$.

However the derived bracket is not necessarily the graded Poisson bracket.

not skew:

$$\begin{aligned}\{\{f,\Theta\}_b, g\}_b &= -(-1)^{(|f|-n+1)(|g|-n+1)}\{\{g,\Theta\}_b, f\}_b \\ &\quad -(-1)^{(|f|-n+1)}\{\Theta, \{f,g\}_b\}_b.\end{aligned}$$

Functions and Poisson Algebras on QP Pairs II

not Leibniz:

$$\begin{aligned}\{\{fg,\Theta\}_b,h\}_b &= \{f\{g,\Theta\}_b+(-1)^{|g|}\{f,\Theta\}_b g,h\}_b\\ &= f\{\{g,\Theta\}_b,h\}_b+(-1)^{|g|(|h|+1-n)}\{\{f,\Theta\}_b,h\}_b g\\ &\quad +(-1)^{|g|}\{f,\Theta\}_b\{g,h\}_b+(-1)^{(|g|+1)(|h|-n)}\{f,h\}_b\{g,\Theta\}_b.\end{aligned}$$

These terms are the origin of the anomaly terms.

Poisson algebra on mapping space is Current algebra

Let Σ_{n-1} be a (compact) manifold in $n-1$ dimensions.
$X_n = \mathbf{R} \times \Sigma_{n-1}$ is a manifold in n dimensions, which is regarded as a spacetime.

Assume a small Poisson bracket (a derived bracket) is nondegenerate.

Definition 13

A **current algebra** is a (twisted) Poisson algebra on a small (twisted) QP manifold, $\mathrm{Map}(T[1]\Sigma_{n-1}, \mathcal{L})$.

The **AKSZ construction** induces a Poisson algebra on a big QP manifold, $\mathrm{Map}(\mathcal{X} = T[1]\Sigma_{n-1}, \mathcal{M} = T^*[n]\mathcal{L})$.

AKSZ Construction I

Alexandrov, Kontsevich, Schwartz, Zaboronsky '97

The AKSZ construction induces a QP structure (dg symplectic structure) on a mapping space $\mathrm{Map}(\mathcal{X}, \mathcal{M})$ from the following data.

$(\mathcal{X}, D, \mu)$: $\mathcal{X}$ is a dg manifold with a D-invariant nondegenerate measure μ. D is a differential on $\mathcal{X}$.

$(\mathcal{M}, \omega, Q)$: A QP-manifold of degree n

An *evaluation map* $\mathrm{ev} : \mathcal{X} \times \mathcal{M}^{\mathcal{X}} \longrightarrow \mathcal{M}$ is defined as $\mathrm{ev} : (z, \Phi) \longmapsto \Phi(z)$, where $z \in \mathcal{X}$ and $\Phi \in \mathcal{M}^{\mathcal{X}}$.

A *chain map* $\mu_* : \Omega^\bullet(\mathcal{X} \times \mathcal{M}^{\mathcal{X}}) \longrightarrow \Omega^\bullet(\mathcal{M}^{\mathcal{X}})$ is defined as $\mu_* F = \int_{\mathcal{X}} \mu\, F$ where $F \in \Omega^\bullet(\mathcal{X} \times \mathcal{M}^{\mathcal{X}})$ and $\int_{\mathcal{X}} \mu$ is a Berezin integration on $\mathcal{X}$.

$\mu_* \mathrm{ev}^* : \Omega^\bullet(\mathcal{M}) \longrightarrow \Omega^\bullet(\mathcal{M}^{\mathcal{X}})$ is called a *transgression map*.

AKSZ Construction II

•P-structure (graded symplectic structure)

Theorem 14

For a graded symplectic form ω *on* $\mathcal{M}$, $\boldsymbol{\omega} = \mu_* \mathrm{ev}^* \omega$ *is a graded symplectic form on* $\mathrm{Map}(\mathcal{X}, \mathcal{M})$.

• Q-structure (Homological function)

Theorem 15

Let ($S_{b0} := \iota_{\hat{D}} \mu_* \mathrm{ev}^* \vartheta_b$ *with* $\omega_b = -\delta \vartheta_b$*, and)* $S_{b1} := \mu_* \mathrm{ev}^* \Theta$*. Then* $S_b = (S_{b0} +) S_{b1}$ *is a homological function on* $\mathrm{Map}(\mathcal{X}, \mathcal{M})$*, that is,*

$$\{S_b, S_b\} = 0,$$

Degree

• $\mathrm{Map}(T[1]\Sigma_{n-1}, \mathcal{M} = T^*[n]\mathcal{L})$ is a QP manifold of degree 1.

Poisson algebra on big mapping space

Take a function $J \in C_{n-1}(\mathcal{M})$. The ASKZ construction induces pullbacks J to $\mathrm{Map}(T[1]\Sigma_{n-1}, \mathcal{M})$.
It is denoted by $\mathcal{J}(\epsilon) = \mu_* \epsilon\, \mathrm{ev}^* J$, where ϵ is a test function on $T[1]\Sigma_{n-1}$ of degree $n-1-|J|$.

$$\mathcal{CA}_{n-1}(\mathcal{M}) = \mathcal{CA}_{n-1}(\Sigma_{n-1}, \mathcal{M})$$
$$= \{\mathcal{J} = \mu_* \epsilon\, \mathrm{ev}^* J \in C^\infty(\mathrm{Map}(T[1]\Sigma_{n-1}, \mathcal{M})) | J \in C_{n-1}(\mathcal{M})\},$$

is a Poisson algebra. Moreover, because of $|\mathcal{J}| = 0$, $\mathcal{CA}_{n-1}(\mathcal{M})$ is closed under the derived bracket $\{-,-\}_s = \{\{-, S_{b1}\}_b, -\}_b$.

Note

The bracket $\{-,-\}_s = \{\{-, S_{b1}\}_b, -\}_b|_{\mathrm{Map}(T[1]\Sigma_{n-1}, \mathcal{L})}$ is of degree 0.
Therefore it is the usual Poisson bracket $\{-,-\}_{P.B}$ on $\mathrm{Map}(T[1]\Sigma_{n-1}, \mathcal{L})$.

Twisting by small canonical 1-form

The derived bracket on $\mathcal{CA}_{n-1}(T^*[n]\mathcal{L})|_{\mathcal{L}}$ does not have an anomalous term, which gives a trivial current. In order to define physical currents, we introduce twisting called the **twisted pullback**.

Given the small symplectic structure ω_s on $\mathcal{L}$, we have a pullback to $\mathcal{M}$ with respect to a projection $\pi : \mathcal{M} \longrightarrow \mathcal{L}$. We define a special function S_s of degree 1 on $\mathrm{Map}(T[1]\Sigma_{n-1}, T^*[n]\mathcal{L})$,

$$S_s = S_{s0} = \iota_{\hat{D}} \mu_* \mathrm{ev}^* \vartheta_s,$$

where ϑ_s is the canonical 1-form for ω_s such that $\omega_s = -\delta\vartheta_s$.

Functions on small mapping space

For any function J on $T^*[n]\mathcal{L}$, a twisted pullback of J to $\mathrm{Map}(T[1]\Sigma_{n-1}, T^*[n]\mathcal{L})$ is a canonical transformation (twisting) by S_s:

$$e^{\delta_{S_s}}\mathcal{J} = e^{\delta_{S_s}}\mu_*\epsilon\,\mathrm{ev}^* J.$$

Definition 16

A current $\mathbf{J}(\epsilon)$ on $\mathrm{Map}(T[1]\Sigma_{n-1}, \mathcal{L})$ respect to a current function J on $T^*[n]\mathcal{L}$ is defined by
$\mathbf{J}(\epsilon) := e^{\delta_{S_s}}\mathcal{J}|_{\mathrm{Map}(T[1]\Sigma_{n-1},\mathcal{L})} = e^{\delta_{S_s}}\mu_*\epsilon\,\mathrm{ev}^* J|_{\mathrm{Map}(T[1]\Sigma_{n-1},\mathcal{L})}$.

The Poisson bracket of two currents $\mathbf{J}_1$, $\mathbf{J}_2$ is induced from the Poisson bracket of J_1 and J_2 in $C_{n-1}(\mathcal{M})$.

AKSZ-BFV (Supergeometric) Formalism of Current Algebras I

Theorem 17 (N.I. Xu '13)

For currents $\mathbf{J}_1$ *and* $\mathbf{J}_2$ *associated to current functions* J_1, J_2 $\in C_{n-1}(\mathcal{M})$ *respectively, the commutation relation is given by*

$$
\begin{aligned}
\{\mathbf{J}_1(\epsilon_1), \mathbf{J}_2(\epsilon_2)\}_{P.B} &= \Big(-e^{\delta_{S_s}} \mu_* \epsilon_1 \epsilon_2 \mathrm{ev}^* \{\{J_1, \Theta\}_b, J_2\}_b \\
&\quad - e^{\delta_{S_s}} \iota_{\hat{D}} \mu_* (d\epsilon_1) \epsilon_2 \mathrm{ev}^* \{J_1, J_2\}_b \Big) \Big|_{\mathrm{Map}(T[1]\Sigma_{n-1}, \mathcal{L})} \\
&= -\mathbf{J}_{[J_1, J_2]_D}(\epsilon_1 \epsilon_2) \\
&\quad - e^{\delta_{S_s}} \iota_{\hat{D}} \mu_* (d\epsilon_1) \epsilon_2 \mathrm{ev}^* \{J_1, J_2\}_b \big|_{\mathrm{Map}(T[1]\Sigma_{n-1}, \mathcal{L})},
\end{aligned}
$$

where ϵ_i *are test functions for* J_i *on* $\mathrm{Map}(T[1]\Sigma_{n-1}, T^*[n]\mathcal{L})$ *and* $[J_1, J_2]_D$ *is the bracket defined from the drived bracket on* $C_{n-1}(\mathcal{M})$.

AKSZ-BFV (Supergeometric) Formalism of Current Algebras II

Due to the second term in the equation, it fails to be a Poisson algebra. The anomalous terms vanish if J_1 and J_2 commute, $\{J_1, J_2\}_b = 0$. So we have

Corollary 18

Any consistent current algebra is isomorphic to a Poisson algebra $(Comm, \{\{-, \Theta\}_b, -\}_b)$*, where* $Comm$ *is a commutative subspace of* $C_{n-1}(T^*[n]\mathcal{L})$ *under the Poisson bracket* $\{-,-\}_b$ *on* $T^*[n]\mathcal{L}$*.*

Derivation of Physical Currents

Introduce the second degree, the **form degree** $\deg f$ for a general superfield of definite degree, $f \in C^\infty(\mathrm{Map}(T[1]\Sigma_{n-1}, T^*[n]\mathcal{L})$. It is, by definition, zero on Σ_{n-1} and one on the $T[1]$ direction.
$\mathrm{gh}\ f = |f| - \deg f$ is called the **ghost number**. We denote $f_{cl} = f|_{\mathrm{gh} f=0}$.

Theorem 19

The **ghost number zero components** *of superfields of the supergeometric current algebra gives the physical current algebra:*

$$\begin{aligned}\{\mathbf{J}_{J_1}|_{cl}(\epsilon_1), \mathbf{J}_{J_2}|_{cl}(\epsilon_2)\}_{P.B} &= \Big(-\mathbf{J}_{[J_1,J_2]_D}(\epsilon_1\epsilon_2) \\ &\quad -e^{\delta_{S_s}}\iota_{\hat{D}}\mu_*(d\epsilon_1)\epsilon_2 \mathrm{ev}^*\{J_1, J_2\}_b|_{\mathrm{Map}(T[1]\Sigma_n,\mathcal{L})}\Big)\,|_{cl},\end{aligned}$$

Our Results

NI, Xu 13-2

- Known current algebras are included in our formulation, such as Lie algebras (gauge currents), Kac-Moody algebras, Alekseev-Strobl types, topological membranes, L_∞-algebra, etc.
- New current algebras of homotopy Lie-n algebroids are discoverd.
- Anomaly cancellation conditions are characterized in terms of supergeometry.
- Current algebras in the AKSZ sigma models are characterized mathematically. (They are constructed from twisting by general canonical functions.)

Example: $n = 2$ I

Twisted Poisson Structures and Current Algebras of Alekseev-Strobl Type

$(\mathcal{M} = T^*[2]T^*[1]M, \omega_b, \Theta)$: a big QP-manifold of degree 2, where M is a usual smooth manifold.
$\mathcal{L} = T^*[1]M$: Lagrangian manifold of degree 1.

Local coordinate

(x^I, p_I, q^I, ξ_I) of degree $(0, 1, 1, 2)$, where (x^I, p_I) is the $\mathcal{L}$ component.

$\omega_b = \delta x^I \wedge \delta \xi_I + \delta p_I \wedge \delta q^I$: graded symplectic structure.

Example: $n = 2$ II

We choose a homological function of degree 3,

$$\Theta = \xi_I q^I + \frac{1}{3!} H_{IJK}(x) q^I q^J q^K.$$

$\{\Theta, \Theta\}_b = 0$ if H is a closed 3-form on M, where
$H = \frac{1}{3!} H_{IJK}(x) dx^I \wedge dx^J \wedge dx^K$.

cf.) The twisted Poisson structure

Θ defines a small Poisson bracket (a symplectic structure) on $\mathcal{L}$ by the derived bracket $\{-, -\}_s = \{\{-, \Theta\}_b, -\}_b|_{\mathcal{L}}$. $\{x^I, p_J\}_s = \delta^I{}_J$.

Poisson Algebra on $\mathcal{M}$ and $\mathcal{L}$

Let us consider a space of functions
$C_1(T^*[2]T^*[1]M) = \{f \in C^\infty(T^*[2]T^*[1]M) || f| \leq 1\}$.

Example: $n = 2$ III

Elements are a function of degree 0, $J_{(0)f} = f(x)$ and a functions of degree 1, $J_{(1)(u,a)} = a_I(x)q^I + u^I(x)p_I$.
$J_{(1)(u,a)}$ is regarded as a section of $TM \oplus T^*M$, since $a_I(x)dx^I + u^I(x)\frac{\partial}{\partial x^I} \in \Gamma(TM \oplus T^*M)$ can be identified as $a_I(x)q^I + u^I(x)p_I$.

Big Poisson bracket

$$
\begin{aligned}
\{J_{(0)(f)}, J'_{(0)(g)}\}_b &= 0, \\
\{J_{(1)(u,a)}, J'_{(0)(g)}\}_b &= 0. \\
\{J_{(1)(u,a)}, J'_{(1)(v,b)}\}_b &= a_I v^I + u^I b_I = \langle (u,a), (v,b) \rangle .
\end{aligned}
$$

where $J'_{(0)(v)} = g(x)$ and $J'_{(1)(v,b)} = b_I(x) q^I + v^I(x) p_I$. $\langle (u,a), (v,b) \rangle$ is the inner product on $TM \oplus T^*M$. Therefore the commutative subspace $Comm_1(T^*[2]\mathcal{L})$ is defined by $J_{(i)}$'s with $\langle (u,a), (v,b) \rangle = 0$.

Derived and small Poisson bracket I

$$\{\{J_{(0)(f)}, \Theta\}_b, J'_{(0)(g)}\}_b = 0,$$

$$\{\{J_{(1)(u,a)}, \Theta\}_b, J'_{(0)(g)}\}_b = -u^I \frac{\partial J'_{(0)(g)}}{\partial x^I},$$

$$\{\{J_{(1)(u,a)}, \Theta\}_b, J'_{(1)(v,b)}\}_b$$

$$= -\left[\left(u^J \frac{\partial v^I}{\partial x^J} - v^J \frac{\partial u^I}{\partial x^J}\right) p_I\right.$$

$$\left. + \left(u^J \frac{\partial b_I}{\partial x^J} - v^J \frac{\partial a_I}{\partial x^J} + v^J \frac{\partial a_J}{\partial x^I} + b_J \frac{\partial u^J}{\partial x^I} + H_{JKI} u^J v^K\right) q^I\right]$$

$$= -J_{(1)([(u,a),(v,b)]_D)}.$$

Derived and small Poisson bracket II

Here $[(u,a),(v,b)]_D$ is the Dorfman bracket on $TM \oplus T^*M$ defined by

$$[(u,a),(v,b)]_D = [u,v] + L_u b - \iota_v da + \iota_u \iota_v H,$$

for $u, v \in \Gamma(TM)$, $a, b \in \Gamma(T^*M)$, where $[u,v]$ is a Lie bracket of the vector fields, H is a closed 3-form, L_u is a Lie derivative and ι_v is the interior product. This induces the commutation relations on the small P-manifold $\mathcal{L}$:

$$\{J_{(0)(f)}, J'_{(0)(g)}\}_s = 0,$$
$$\{J_{(1)(u,\alpha)}, J'_{(0)(g)}\}_s = -u^I \frac{\partial J'_{(0)(g)}}{\partial x^I},$$
$$\{J_{(1)(u,a)}, J'_{(1)(v,b)}\}_s = -J_{(1)([u,v],0)}.$$

Twisting by Canonical Function

Let us assume the canonical function α such that
$-\alpha = -\frac{1}{2}\pi^{IJ}(x)p_I p_J$.
$e^{-\delta_\alpha}\Theta|_{\mathcal{L}} = 0$ is equivalent to the twisted Poisson structure on M:

$$\frac{\partial \pi^{IJ}}{\partial x^L}\pi^{LK} + (IJK \text{ cyclic}) = \pi^{IL}\pi^{JM}\pi^{KN}H_{LMN}.$$

Analyze a twisted Poisson algebra for $\{\{-, e^{-\delta\alpha}\Theta\}_b, -\}_b$.
$\{\{f_1, e^{-\delta\alpha}\Theta\}_b, f_2\}_b$ is equivalent to $\{\{e^{\delta\alpha}f_1, \Theta\}_b, e^{\delta\alpha}f_2\}_b$ by a canonical transformation.

Twisted Poisson algebra on small QP Manifold I

Let us take the basis $K^I_{(0)} = x^I$ and $K^I_{(1)} = q^I$ for $B_1(T^*[2]T^*[1]M, 0) = \{f | f \in C_1(T^*[2]T^*[1]M), f|_{T^*[2]} = 0\}$. They are commutative and their derived brackets are zero.
Next we make twisting by α, $B_1(T^*[2]T^*[1]M, \alpha)$. The basis change to

$$
\begin{aligned}
J^I_{(0)} &= e^{\delta_\alpha} x^I = 0, \\
J^I := J^I_{(1)} &= e^{\delta_\alpha} q^I = q^I + \pi^{IJ}(x) p_J,
\end{aligned}
$$

J^I's are commutative, $\{J^I, J^J\}_b = 0$, and the derived bracket of the big Poisson bracket is

$$
\{\{J^I, \Theta\}_b, J^J\}_b = -\left(\frac{\partial \pi^{IJ}}{\partial x^K} + \pi^{IL}\pi^{JM} H_{LMK}\right) J^K,
$$

Twisted Poisson algebra on small QP Manifold II

This is a Poisson algebra associated to twisting by the canonical function α. Moreover we obtain

$$\{J^I, J^J\}_s = \{\{J^I, \Theta\}_b, J^J\}_b|_{\mathcal{L}} = -\left(\frac{\partial \pi^{IJ}}{\partial x^K} + \pi^{IL}\pi^{JM} H_{LMK}\right) J^K|_{\mathcal{L}},$$

where $J^K|_{\mathcal{L}} = \pi^{KL} p_L$.

Note

This derives a current algebra of the twisted Poisson sigma model by the AKSZ construction.

Klimcik, Strobl, '01

Big mapping space

Let us take $\Sigma_1 = S^1$. Take $\mathcal{X} = T[1]S^1$ and a local coordinate (σ, θ). The Berezin measure is $\mu = \mu_{T[1]S^1} = d\sigma d\theta$.

The AKSZ construction induces a QP structure on $\mathrm{Map}(T[1]S^1, T^*[2]T^*[1]M)$.

In the local coordinate superfields, the symplectic structure on $\mathrm{Map}(T[1]S^1, T^*[2]T^*[1]M)$ is

$$\boldsymbol{\omega}_b \;=\; \mu_* \mathrm{ev}^* \omega_b = \int_{T[1]S^1} \mu \, \left(\delta \mathbf{x}^I \wedge \delta \boldsymbol{\xi}_I + \delta \mathbf{p}_I \wedge \delta \mathbf{q}^I\right),$$

where the boldface is the superfields corresponding to local coordinates on $T^*[2]T^*[1]M$. $\mathbf{x} : T[1]S^1 \to M$, $\mathbf{p} \in \Gamma(T[1]S^1 \otimes \mathbf{x}^*(T^*[1]M))$, $\mathbf{q} \in \Gamma(T[1]S^1 \otimes \mathbf{x}^*(T^*[2]T^*[1]M))$ and $\boldsymbol{\xi} \in \Gamma(T[1]S^1 \otimes \mathbf{x}^*(T^*[2]M))$.

S_s and Twisted Pullback I

ω_s is

$$\boldsymbol{\omega}_b = \int_{T[1]S^1} \mu \ \left(\delta \mathbf{x}^I \wedge \delta \mathbf{p}_I\right),$$

Therefore The small canonical 1-from is locally

$$S_s \ = \ \iota_{\hat{D}} \mu_* \mathrm{ev}^* \vartheta_s = \int_{T[1]S^1} \mu \ \mathbf{p}_I \mathbf{dx}^I.$$

Current functions are one of degree 0 and one of degree 1:

$$J_{(0)(f)} = f(x),$$
$$J_{(1)(u,a)} = a_I(x) q^I + u^I(x) p_I,$$

on $C_1(T^*[2]T^*[1]M)$.

S_s and Twisted Pullback II

The corresponding currents are

$$\mathbf{J}_{(0)(f)} = \int_{T[1]S^1} \mu\epsilon_{(1)} f(\mathbf{x}),$$

$$\mathbf{J}_{(1)(u,a)} = \int_{T[1]S^1} \mu\epsilon_{(0)}(a_I(\mathbf{x})\mathbf{dx}^I + u^I(\mathbf{x})\mathbf{p}_I),$$

where $\epsilon_{(i)}$ is a test function of degree i. Let
$\mathbf{J}'_{(0)(g)} = \int_{T[1]S^1} \mu\epsilon_{(1)} g(\mathbf{x})$,
$\mathbf{J}'_{(1)(v,b)} = \int_{T[1]S^1} \mu\epsilon_{(0)}(b_I(\mathbf{x})\mathbf{dx}^I + v^I(\mathbf{x})\mathbf{p}_I)$.
The derived brackets of these currents are computed as follows:

$$\{\mathbf{J}_{(0)(f)}(\epsilon), \mathbf{J}'_{(0)(g)}(\epsilon')\}_{P.B.} = 0,$$

Current Algebras I

$$
\begin{aligned}
&\{\mathbf{J}_{(1)(u,a)}(\epsilon), \mathbf{J}'_{(0)(g)}(\epsilon')\}_{P.B.} \\
&= \Big(-e^{\delta_{S_s}}\mu_* \epsilon\epsilon' \mathrm{ev}^* \{\{J_{(1)(u,a)}, \Theta\}_b, J'_{(0)(g)}\}_b \\
&-e^{\delta_{S_s}}\iota_{\hat{D}}\mu_*(d\epsilon)\epsilon' \mathrm{ev}^* \{J_{(1)(u,a)}, J_{(0)(g)}\}_b\Big)\Big|_{\mathrm{Map}(T[1]S^1,\mathcal{M})} \\
&= -u^I \frac{\partial \mathbf{J}'_{(0)(g)}}{\partial \mathbf{x}^I}(\epsilon\epsilon'), \\
&\{\mathbf{J}_{(1)(u,a)}(\epsilon), \mathbf{J}_{(1)(v,b)}(\epsilon')\}_{P.B.} \\
&= \Big(-e^{\delta_{S_s}}\mu_* \epsilon\epsilon' \mathrm{ev}^* \{\{J_{(1)(u,a)}, \Theta\}_b, J'_{(1)(v,b)}\}_b \\
&\quad -e^{\delta_{S_s}}\iota_{\hat{D}}\mu_*(d\epsilon)\epsilon' \mathrm{ev}^* \{J_{(1)(u,a)}, J_{(1)(v,b)}\}_b\Big)\Big|_{\mathrm{Map}(T[1]S^1,\mathcal{M})} \\
&= -\mathbf{J}_{(1)([(u,a),(v,b)]_D)}(\epsilon\epsilon') \\
&\quad -\int_{T[1]S^1} \mu(\mathbf{d}\epsilon_{(0)}\epsilon'_{(0)}\langle (a_I(\mathbf{x}), u^I(\mathbf{x})), (b_I(\mathbf{x}), v^I(\mathbf{x}))\rangle,
\end{aligned}
$$

Current Algebras II

The classical current algebra is the ghost number zero components of superfields in the equations:

$$\begin{aligned}
\{\mathbf{J}_{(0)(f)}|_{cl}(\epsilon), \mathbf{J}'_{(0)(g)}|_{cl}(\epsilon')\}_{P.B.} &= 0, \\
\{\mathbf{J}_{(1)(u,a)}|_{cl}(\epsilon), \mathbf{J}'_{(0)(g)}|_{cl}(\epsilon')\}_{P.B.} &= -u^I(\sigma)\frac{\partial \mathbf{J}'_{(0)(g)}|_{cl}}{\partial x^I}(\epsilon\epsilon'), \\
\{\mathbf{J}_{(1)(u,a)}|_{cl}(\epsilon), \mathbf{J}_{(1)(v,b)}|_{cl}(\epsilon')\}_{P.B.} &= -\mathbf{J}_{(1)([(u,a),(v,b)]_D)}|_{cl}(\epsilon\epsilon') \\
&\quad - \int_{S^1} d\sigma(\partial_\sigma\epsilon_{(0)}\epsilon'_{(0)}\langle(a_I(x(\sigma)), u^I(x(\sigma))), (b_I(x(\sigma)), v^I(x(\sigma)))\rangle,
\end{aligned}$$

where $\mathbf{J}_{(0)(f)}|_{cl} = \int_{S^1} d\sigma \epsilon_{cl(1)}(\sigma) f(x(\sigma))$, $\epsilon_{(1)}(\sigma,\theta) = \theta\epsilon_{cl(1)}(\sigma)$,
$\mathbf{J}_{(1)(u,a)}|_{cl} = \int_{S^1} d\sigma \epsilon_{(0)}(\sigma)(a_I(x(\sigma))\partial_\sigma x^I(\sigma) + u^I(x(\sigma))p_I(\sigma))$.
This coincides with the generalized current algebra in Alkseev-Strobl.

Future Outlook I

• Supergeometry of the QP pair
Twisted Poisson geometry, the Dirac Structure, Lie n-algebroids, Higher category, etc.

• Physical theories,
BV-BFV formalism, TQFT and topological membranes, AKSZ sigma models, bulk-boundary correspondences, Instanton counting

• Quantization
Canonical, Path integral, Deformation, Geometric, $\cdots$
Anomalies in current algebras, Index theory, From commutative to noncommutative geometry, Intergration from algebroid to groupoid.

• S- and T-duality, Localication

• Poisson Vertex Algebra, Operads

Future Outlook II

Super symplectic geometry has rich contents and should be analyzed!

Thank you!

Appendix, Example: $n = 3$ I

N.I. Koizumi '12

$\mathbf{R} \times \Sigma_2$ is a spacetime in three dimensions, where Σ_2 is a (compact) manifold in two dimensions.

Let us construct the current algebras on $\mathrm{Map}(T[1]\Sigma_2, T^*[2]E[1])$, where E is a vector bundle over a manifold M and $T[1]\Sigma_2$ is a supermanifold with a differential D and the compatible Berezin measure $\mu = \mu_{T[1]\Sigma_2}$.

$\mathcal{M} = T^*[3]\mathcal{L} = T^*[3]T^*[2]E[1]$ is a big graded symplectic manifold of degree 3 as the auxiliary phase space.

Take local coordinates (x^I, q^A, p_I) of degree $(0, 1, 2)$ on $T^*[2]E[1]$, and conjugate Darboux coordinates (ξ_I, η^A, χ^I) of degree $(3, 2, 1)$ on $T^*[3]$.

Appendix, Example: $n = 3$ II

A big graded symplectic structure on $\mathcal{M}$ is
$\omega_b = \delta x^I \wedge \delta \xi_I + k_{AB} \delta q^A \wedge \delta \eta^B + \delta p_I \wedge \delta \chi^I$, where k_{AB} is a fiber metric on E.

A Q-structure function is

$$\Theta = \chi^I \xi_I + \frac{1}{2} k_{AB} \eta^A \eta^B + \frac{1}{4!} H_{IJKL}(x) \chi^I \chi^J \chi^K \chi^L.$$

Θ is homological if H is a closed 4-form.

This defines a Lie algebroid up to homtopy on E. N.I. Uchino '11

Poisson algebra on the big QP manifold I

Let us consider the space of functions on the big QP manifold, $C_2(T^*[3]T^*[2]E[1]) = \{f \in C^\infty(T^*[3]T^*[2]E[1]) || f| \leq 2\}$.
Functions of $C_2(T^*[3]T^*[2]E[1])$ of degree 0, 1 and 2 are generally described by

$$
\begin{aligned}
J_{(0)(f)} &= f(x), \\
J_{(1)(a,u)} &= a_I(x)\chi^I + u_A(x)q^A, \\
J_{(2)(G,K,F,B,E)} &= G^I(x)p_I + K_A(x)\eta^A + \frac{1}{2}F_{AB}(x)q^A q^B \\
&\quad + \frac{1}{2}B_{IJ}(x)\chi^I\chi^J + E_{AI}(x)\chi^I q^A.
\end{aligned}
$$

Here all coefficients are some local functions of x.

Poisson algebra on the big mapping space I

Let $\mathbf{x}^I$ be a smooth map from $T[1]\Sigma_2$ to M, $\mathbf{q}^A \in \Gamma(T^*[1]\Sigma_2 \otimes \mathbf{x}^*(E[1]))$ and $\mathbf{p}_I \in \Gamma(T^*[1]\Sigma_2 \otimes \mathbf{x}^*(T^*[2]M))$ be superfields of degree 1 and 2. A graded symplectic form $\boldsymbol{\omega}_b$ of degree 1 is defined as

$$\boldsymbol{\omega}_b = \mu_* \mathrm{ev}^* \omega_b = \int_{T[1]\Sigma_2} \mu \left(\delta \mathbf{x}^I \wedge \delta \boldsymbol{\xi}_I + k_{AB} \delta \mathbf{q}^A \wedge \delta \boldsymbol{\eta}^B + \delta \mathbf{p}_I \wedge \delta \boldsymbol{\chi}^I\right),$$

where $\boldsymbol{\xi}_I(\sigma, \theta) \in \Gamma(T^*[1]\Sigma_2 \otimes \mathbf{x}^*(T^*[3]M))$, $\boldsymbol{\eta}^A(\sigma, \theta) \in \Gamma(T^*[1]\Sigma_2 \otimes \mathbf{x}^*(T^*[3]E[1]))$, and $\boldsymbol{\chi}^I(\sigma, \theta) \in \Gamma(T^*[1]\Sigma_2 \otimes \mathbf{x}^*(T^*[3]T^*[2]M))$.

Twisted Pullback I

The canonical 1-form is

$$\begin{aligned} S_s &= \iota_{\hat{D}} \mu_* \mathrm{ev}^* \vartheta_s \\ &= \int_{T[1]\Sigma_2} \mu \left(-\mathbf{p}_I \mathbf{dx}^I + \frac{1}{2} k_{AB} \mathbf{q}^A \mathbf{dq}^B \right). \end{aligned}$$

Next we take the space of the twisted pullback $\mathcal{C}\mathcal{A}_2'(T^*[3]T^*[2]E[1])) = \{\mathbf{J} \in C^\infty(\mathrm{Map}(T[1]S^1, T^*[2]E[1]))| J \in C_2(T^*[3]T^*[2]E[1])\}$, where $\mathbf{J} = e^{\delta_{S_s}} \mathcal{J}|_{\mathrm{Map}(T[1]\Sigma_2, T^*[2]E[1])} = e^{\delta_{S_s}} \mu_* \epsilon\, \mathrm{ev}^* J|_{\mathrm{Map}(T[1]S^1, T^*[2]E[1])}$.

Currents I

Currents of degree 0, 1 and 2 are

$$
\begin{aligned}
\mathbf{J}_{(0)(f)} &= \int_{T[1]\Sigma_2} \mu \epsilon_{(2)} f(\mathbf{x}), \\
\mathbf{J}_{(1)(a,u)} &= \int_{T[1]\Sigma_2} \mu \epsilon_{(1)} (a_I(\mathbf{x}) \mathbf{dx}^I + u_A(\mathbf{x}) \mathbf{q}^A), \\
\mathbf{J}_{(2)(G,K,F,B,E)}(\sigma, \theta) &= \int_{T[1]\Sigma_2} \mu \epsilon_{(0)} (G^I(\mathbf{x}) \mathbf{p}_I + K_A(\mathbf{x}) \mathbf{dq}^A \\
&\quad + \frac{1}{2} F_{AB}(\mathbf{x}) \mathbf{q}^A \mathbf{q}^B + \frac{1}{2} B_{IJ}(\mathbf{x}) \mathbf{dx}^I \mathbf{dx}^J \\
&\quad + E_{AI}(\mathbf{x}) \mathbf{dx}^I \mathbf{q}^A).
\end{aligned}
$$

Current Algebra I

The derived brackets produce the current algebra as follows:

$$
\begin{aligned}
&\{\mathbf{J}_{(0)(f)}(\epsilon), \mathbf{J}_{(0)(f')}(\epsilon')\}_{P.B.} = 0,\\
&\{\mathbf{J}_{(1)(u,a)}(\epsilon), \mathbf{J}_{(0)(f')}(\epsilon')\}_{P.B.} = 0,\\
&\{\mathbf{J}_{(2)(G,K,F,H,E)}(\epsilon), \mathbf{J}_{(0)(f')}(\epsilon')\}_{P.B.} = -G^I \frac{\partial \mathbf{J}_{(0)(f')}}{\partial \mathbf{x}^I}(\epsilon\epsilon'),\\
&\{\mathbf{J}_{(1)(u,a)}(\epsilon), \mathbf{J}_{(1)(u',a')}(\epsilon')\}_{P.B.} = -\int_{T[1]\Sigma_2} \mu \epsilon_{(1)} \epsilon'_{(1)} \mathrm{ev}^* k^{AB} u_A u'_B,\\
&\{\mathbf{J}_{(2)(G,K,F,B,E)}(\epsilon), \mathbf{J}_{(1)(u',a')}(\epsilon')\}_{P.B.}\\
&= -\mathbf{J}_{(1)(\bar{u},\bar{\alpha})}(\epsilon\epsilon') - \int_{T[1]\Sigma_2} \mu (\mathbf{d}\epsilon_{(0)}) \epsilon'_{(1)} (G^I \alpha'_I - k^{AB} K_A u'_B),
\end{aligned}
$$

Current Algebra II

$$\{\mathbf{J}_{(2)(G,K,F,B,E)}(\epsilon), \mathbf{J}_{(2)(G',K',F',B',E')}(\epsilon')\}_{P.B.} = -\mathbf{J}_{(2)(\bar{G},\bar{K},\bar{F},\bar{B},\bar{E})}(\epsilon\epsilon')$$
$$-\int_{T[1]\Sigma_2} \mu(\mathbf{d}\epsilon_{(0)})\epsilon'_{(0)} \left[(G^J B'_{JI} + G'^J B_{JI} + k^{AB}(K_A E'_{BI} + E_{AI} K'_B)\right.$$
$$\left. +(G^I E'_{AI} + G'^I E_{AI} + k^{BC}(K_B F'_{AC} + F_{AC} K'_B))\mathbf{q}^A\right].$$

Here

$$\bar{\alpha} = (i_G d + d i_G)\alpha' + \langle E - dK, u'\rangle, \quad \bar{u} = i_G du' + \langle F, u'\rangle,$$
$$\bar{G} = [G, G'], \quad \bar{K} = i_G dK' - i_{G'} dK + i_{G'} E + \langle F, K'\rangle,$$
$$\bar{F} = i_G dF' - i_{G'} dF + \langle F, F'\rangle,$$
$$\bar{B} = (d i_G + i_G d)B' - i_{G'} dB + \langle E, E'\rangle + \langle K', dE\rangle$$
$$-\langle dK, E'\rangle + i_{G'} i_G H,$$
$$\bar{E} = (d i_G + i_G d)E' - i_{G'} dE + \langle E, F'\rangle$$
$$-\langle E', F\rangle + \langle dF, K'\rangle - \langle dK, F'\rangle,$$

Current Algebra III

where all the terms are evaluated by σ'. Here $[-,-]$ is a Lie bracket on TM, i_G is an interior product with respect to a vector field G and $\langle -,- \rangle$ is the graded bilinear form on the fiber of E with respect to the metric k^{AB}.

Here local coordinates on $T[1]\Sigma_2$ are (σ,θ) of degree $(0,1)$.

The condition vanishing anomalous terms, $G^I\alpha'_I - k^{AB}K_A u'_B = 0$, $G^J B'_{JI} + G'^J B_{JI} + k^{AB}(K_A E'_{BI} + E_{AI} K'_B) = 0$ and $G^I E'_{AI} + G'^I E_{AI} + k^{BC}(K_B F'_{AC} + F_{AC} K'_B) = 0$ is equivalent that $J_{(i)}$'s are commutative.

Current Algebras of Topological n-Branes I

Let us take a (compact, orientable) manifold Σ_n of dimension n. Choose a super phase space $\mathrm{Map}(T[1]\Sigma_n, T^*[n]M)$.

Let $T^*[n+1]\mathcal{L} = T^*[n+1]T^*[n]M$ be a big graded symplectic manifold of degree $n+1$, where $\mathcal{L} = T^*[n]M$. Take local coordinates (x^I, p_I) of degree $(0, n)$ on $T^*[n]M$, and conjugate Darboux coordinates (ξ_I, χ^I) of degree $(n+1, 1)$ on $T^*[n+1]$.

A graded symplectic structure on $T^*[n+1]\mathcal{L}$ is $\omega_b = \delta x^I \wedge \delta\xi_I + \delta p_I \wedge \delta\chi^I$.

We take a Q-structure function of degree $n+2$:

$$\Theta = \chi^I \xi_I + \frac{1}{(n+2)!} H_{I_1 I_2 \cdots I_{n+2}}(x) \chi^{I_1}\chi^{I_2}\cdots\chi^{I_{n+2}}.$$

Θ is a Q-structure if H is a closed $n+2$-form.

Current Algebras of Topological n-Branes II

Next we consider the space of functions of degree equal to or less than n,
$C_n(T^*[n+1]T^*[n]M) = \{f \in C^\infty(T^*[n+1]T^*[n]M) | |f| \leq n\}$. We concentrate on functions of degree n on $C_n(T^*[n+1]T^*[n]M)$ because the currents constructed from functions of degree less than n have trivial commutation relations. A function of degree n is written as

$$J_{(n)(G,B)} \quad = \quad G^I(x)p_I + \frac{1}{n!}B_{I_1\cdots I_n}(x)\chi^I\cdots\chi^{I_n}.$$

Current Algebras of Topological n-Branes III

Let us consider a local coordinate expression on the mapping space. Let $\mathbf{x}^I$ be a smooth map from $T[1]\Sigma_n$ to M and $\mathbf{p}_I \in \Gamma(T^*[1]\Sigma_n \otimes \mathbf{x}^*(T^*[n]M))$ be a superfield of degree n. A big graded symplectic form $\boldsymbol{\omega}_b$ of degree 1 is defined as

$$\boldsymbol{\omega}_b = \mu_* \mathrm{ev}^* \omega_b = \int_{T[1]\Sigma_n} \mu \, (\delta \mathbf{x}^I \wedge \delta \boldsymbol{\xi}_I + \delta \mathbf{p}_I \wedge \delta \boldsymbol{\chi}^I),$$

where $\boldsymbol{\xi}_I(\sigma, \theta) \in \Gamma(T^*[1]\Sigma_n \otimes \mathbf{x}^*(T^*[n+1]M))$ and $\boldsymbol{\chi}^I(\sigma, \theta) \in \Gamma(T^*[1]\Sigma_2 \otimes \mathbf{x}^*(T^*[n+1]T^*[n]M))$. A Q-structure function is

$$\begin{aligned}\boldsymbol{\Theta} &= \mu_* \mathrm{ev}^* \Theta \\ &= \int_{T[1]\Sigma_n} \mu \, \left(\boldsymbol{\chi}^I \boldsymbol{\xi}_I + \frac{1}{(n+2)!} H_{I_1 I_2 \cdots I_{n+2}}(\mathbf{x}) \boldsymbol{\chi}^{I_1} \boldsymbol{\chi}^{I_2} \cdots \boldsymbol{\chi}^{I_{n+2}} \right).\end{aligned}$$

Current Algebras of Topological n-Branes IV

The canonical transformation function is given by

$$S_s = \iota_{\hat{D}}\mu_* \mathrm{ev}^* \vartheta_s = \int_{T[1]\Sigma_n} \mu \left((-1)^{n+1} \mathbf{p}_I \mathbf{dx}^I \right).$$

Take the space of the pullback
$\mathcal{CA}_n(T^*[n+1]T^*[n]M)) = \{\mathcal{J} \in C^\infty(\mathrm{Map}(T[1]\Sigma_n, T^*[n+1]T^*[n]M))|\mathcal{J} = \mu_* \epsilon\, \mathrm{ev}^* J, J \in C_n(T^*[n+1]T^*[n]M)\}$.

Then the twisted pullback of a function $J_{(n)}$ is

$$\mathbf{J}_{(n)(G,B)}(\sigma, \theta) = \int_{T[1]\Sigma_n} \mu \epsilon_{(0)} \left(G^I(\mathbf{x})\mathbf{p}_I + \frac{1}{n!} B_{I_1 \cdots I_n}(\mathbf{x}) \mathbf{dx}^I \cdots \mathbf{dx}^{I_n} \right).$$

Current Algebras of Topological n-Branes V

The derived brackets define the current algebra as follows:

$$\{\mathbf{J}_{(n)(G,B)}(\epsilon), \mathbf{J}_{(n)(G',B')}(\epsilon')\}_{P.B.} = -\mathbf{J}_{(n)([J_1,J_2]_D)}(\epsilon\epsilon')$$
$$-e^{\delta_{S_s}} \int_{T[1]\Sigma_n} \mu(\mathbf{d}\epsilon_{(0)})\epsilon'_{(0)} \mathrm{ev}^* \langle J_{(n)(G,B)}, J_{(n)(G',B')}\rangle.$$

Here $[J_1, J_2]_D$ is the higher Dorfman bracket on $TM \oplus \wedge^n T^*M$ defined by

$$[(u,a),(v,b)]_D = [u,v] + L_u b - \iota_v da,$$

Current Algebras of Topological n-Branes VI

for $u, v \in \Gamma(TM)$ and $a, b \in \Gamma(\wedge^n T^*M)$, and $\langle J_1 , J_2 \rangle = i_u b + i_v a$ is a pairing $TM \oplus \wedge^n T^*M \times TM \oplus \wedge^n T^*M \to \wedge^{n-1} T^*M$. The classical part of the equations

$$\{\mathbf{J}_{(n)(G,B)}|_{cl}(\epsilon), \mathbf{J}_{(n)(G',B')}|_{cl}(\epsilon')\}_{P.B.} = -\mathbf{J}_{(n)([J_1,J_2]_D)}|_{cl}(\epsilon\epsilon')$$
$$-\frac{1}{(n-1)!}\int_{\Sigma_n} d\epsilon_{cl(0)}\epsilon'_{cl(0)}\langle J_{(n)(G,B)} , J_{(n)(G',B')}\rangle dx^I \wedge \cdots \wedge dx^{I_{n-1}},$$

coincides with the generalized current algebra of the topological n-brane theory.

Poisson algebra on small bracket

$Comm_{n-1}(T^*[n]\mathcal{L})$: a subspace of functions which **commute** under $\{-,-\}_b$.

$(Comm_{n-1}(T^*[n]\mathcal{L}), \{-,-\}_s)$ is a Poisson algebra on the derived bracket, where $\{-,-\}_s = \{\{-,\Theta\}_b, -\}_b$.

cf.) Admissible functions on the Dirac structure Courant '90

Example 20

A simplest example is
$B_{n-1}(T^*[n]\mathcal{L}, 0) := \{f \in C_{n-1}(T^*[n]\mathcal{L}) | f|_{T^*[n]} = 0\}$.
It is isomorphic to the Poisson algebra, $(C_{n-1}(\mathcal{L}), \{-,-\}_s)$.

Small Poisson Algebras II

Example 21

If we choose a canonical function α, we can construct the other subspaces of commutative functions,

$B_{n-1}(T^*[n]\mathcal{L}, \alpha) = \{e^{\delta_\alpha} f | f \in B_{n-1}(T^*[n]\mathcal{L}, 0)\}.$

Because $\{e^{\delta_\alpha} f, e^{\delta_\alpha} g\}_b = e^{\delta_\alpha}\{f, g\}_b = 0$, $B_{n-1}(T^*[n]\mathcal{L}, \alpha)$ with $\{-, -\}_s$ is the Poisson algebra.

cf.) This leads to the current algebras in AKSZ sigma models.

V-Data

Definition 22

Voronov '05

$(L, \mathfrak{a}, P, \Delta)$ is called the V-data, where L is a graded Lie algebra, $\mathfrak{a}$ is an abelian Lie subalgebra of L, P is a projection $L \to \mathfrak{a}$, and $\Delta \in \mathrm{Ker}(P)$ is an operator of degree 1 such that $\Delta^2 = 0$.

The V-data $(L, \mathfrak{a}, P, \Delta)$ to a QP pair.

L corresponds to $\mathcal{M} = T^*[n]\mathcal{L}$.

$\mathfrak{a}$ corresponds to a Lagrangian submanifold $\mathcal{L}$.

P corresponds to $T^*[n]\mathcal{L} \to \mathcal{L}$.

Δ is Θ.