Alternative government financing and stochastic endogenous growth

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Abstract

This paper addresses a stochastic endogenous model, in which the stock of securities, both money and bonds, are required to finance the government budget. We study how the expected rate of real growth, the expected inflation rate and economic welfare are influenced by changes in the first and second moments of public spending. The analyses will enable us to assess the merits and demerits of mixed financing case of money and bonds relative to financing one of taxes on wealth examined in Grinols and Turnovsky (J. Econ. Dyn. Control 17 (1993) 1) and Turnovsky (Int. Econom. Rev. 34 (1993) 953). In addition, we construct a model where government relies on a larger ratio of bonds to money to raise the necessary revenue and show how the consequences of mixed financing are affected by the policy decision of government. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

A large number of studies have been made to show that different government financing modes may have different impacts upon the aggregate economy. For example, Turnovsky (1978) postulates the demand functions for real money and consumption in fairly conventional ways and compares the comparative static effects of money-finance and bond-finance on the system in a growing economy. Alogoskoufis and Ploeg (1994) developed a money-in-utility function (MIUF) model with non-interconnected overlapping generations to analyze the effects on the steady-growth rate of lump-sum-tax-financed, money-financed and debt-financed increases in public spending. Palivos and Yip (1995), on the other hand, utilize a cash-in-advance (CIA) model to assess the relative merits of two different modes of financing: tax financing and money financing. However, these models are deterministic; equivalently, these macroeconomists abstract all considerations of stochastic factors from their growth models.

The ratio of real growth is inherently subject to stochastic shocks associated with unforeseen policy or other disturbances. The importance of risk as an influence on the growth rate is widely recognized. As the interaction between the stochastic shocks and real growth should be emphasized in the literature of alternative government financing, the above situation is particularly unfortunate. We can think that there exists a need to extend the models to incorporate richer financial and government sectors.

As seen in Grinols and Turnovsky (1993, 1998) and Turnovsky (1993), recent developments in stochastic endogenous growth provide useful vehicles for reexamining the property of how the real growth rate is affected by a change in public spending under alternative modes of finance. This paper also gives a coherent framework of stochastic endogenous growth and a representative agent with money-in-utility function and yields an alternative to the recent works on money and stochastic endogenous growth. We utilize a stochastic endogenous model to assess the relative merits of different modes of government financing; mixed financing of money and bonds and financing of taxes on wealth. To put it in another way, we explore how the real economy is affected by changes in the first and second moments of public spending under mixed financing of money and bonds, while Grinols and Turnovsky (1993) examine the comparative static effects of the first two moments of public spending under financing by taxes on wealth. Hence, the paper, which describes the endogenous adjustments in money and bonds necessary to finance the government deficit, will shed light on the theoretical interaction between alternative government financing and stochastic endogenous growth. The analysis is supplemented by a set of numerical simulations, measuring the differences in quantity. In addition, this paper illustrates an economy in which government finances the deficits, using a larger ratio of bonds to money.
2. Structure of the model

2.1. Consumers

The representative consumer decides to allocate the wealth between money, government bonds and equities. Hence, the balance sheet constraint is

\[ \frac{M}{P} + \frac{B}{P} + S = W, \tag{1} \]

where \( M \) is the nominal stock of money, \( B \) the nominal stock of bonds, \( P \) the price level of new goods and \( S \) the real stock of equity measured in terms of new output.

He also decides how much to consume output over the instant \( dt \) at the non-stochastic rate \( C dt \) out of income which consists of the real return on these assets.

In other words, his objective is to select his portfolio of assets and rate of consumption to maximize the expected life-time utility, which depends upon consumption \( C_t \) and real money balance \( M_t/P_t \),

\[ E \int_0^\infty U\left(C_t, \frac{M_t}{P_t}\right)e^{-\rho t} dt, \tag{2a} \]

subject to the stochastic wealth accumulation,

\[ dW = W[n_1 dR_M + n_2 dR_B + n_3 dR_S] - C dt - dT, \tag{2b} \]

where \( n_1 \equiv (M_t/P_t) W_t, n_2 \equiv (B_t/P_t) W_t, n_3 \equiv S_t/W_t, \) \( dR_i \) is the stochastic real rate of return on asset \( i \) (\( i = M, B, S \)) and \( dT \) the taxes on wealth paid to government.

The utility function is assumed to be logarithmic:

\[ U = \theta \ln C_t + \gamma \ln \frac{M_t}{P_t}, \gamma + \theta = 1. \]

He expects that the price level evolves according to the geometric Brownian motion process,

\[ \frac{dP}{P} = \pi dt + dp, \tag{3} \]

where \( \pi dt \) is the mean inflation rate over the period \( dt \) and \( dp \) is a temporally independent and normally distributed random variable with zero mean and variance equal to \( \sigma_p^2 dt \).

The rates of return on money and bonds follow from Ito’s calculus:

\[ dR_M = r_M dt - dp \quad \text{where} \quad r_M = -\pi + \sigma_p^2, \tag{4a} \]

\[ dR_B = r_B dt - dp \quad \text{where} \quad r_B = i - \pi + \sigma_p^2. \tag{4b} \]
The real rate of return on equity is defined as
\[ dR_S = r_S \, dt + du, \]
where the mean rate of return \( r_S \) will be determined in the description of the firm. The stochastic component \( du \) is temporally independent and normally distributed with zero mean and variance \( \sigma_u^2 \, dt \) and it too will be specified below.

This representative consumer must pay taxes on his holding of wealth, specified by
\[ dT = \tau W \, dt + W \, dv, \]
where \( dv \) is a temporally independent, normally distributed random variable with zero mean and variance \( \sigma_v^2 \, dt \). \( \tau \) and \( \sigma_v^2 \) will be specified further below.

The stochastic optimization problem can be expressed as being to choose the consumption ratio \( C_t/W_t \) and portfolio share \( n_1, n_2, n_3 \) to
\[ \max E \int_0^\infty [\theta \ln C_t + \gamma \ln n_1 W_t]e^{-\rho t} \, dt \]
\[ \text{s.t.} \quad \frac{dW}{W} = \left[ n_1 r_M + n_2 r_B + n_3 r_S - \frac{C}{W} - \tau \right] dt - (n_1 + n_2) \, dp + n_3 \, du - dv \]
and \( n_1 + n_2 + n_3 = 1. \)

To solve this stochastic optimization problem, we must set up a stochastic Bellman equation. The first-order conditions are derived as
\[ \frac{C}{W} = \theta \rho, \]
\[ \frac{\rho \gamma}{n_1} + r_M = \eta + (n_1 + n_2) \sigma_p^2 - n_3 \sigma_{pu} + \sigma_{pv}, \]
\[ r_B = \eta + (n_1 + n_2) \sigma_p^2 - n_3 \sigma_{pu} + \sigma_{pv}, \]
\[ r_S = \eta - (n_1 + n_2) \sigma_{pu} - n_3 \sigma_u^2 - \sigma_{uv}, \]
where \( \eta \) is the Lagrange multiplier attached to constraint (6c) and
\[ \sigma_{ij} \, dt \equiv \text{cov}(di,dj) \quad i,j = u,v,p. \]

From (4a), (4b), (7b) and (7c), the demand function for money can be derived as
\[ n_1 = \frac{\rho \gamma}{i}. \]
Using (6c), (7c) and (7d), we get the portfolio share of equities
\[ n_3 = \frac{r_S - r_B + \sigma_p^2 + \sigma_{pu} + \sigma_{pv}}{\sigma_u^2 + \sigma_p^2 + 2\sigma_{up}}. \] (8b)

The portfolio share of bonds \( n_2 \) is determined from (6c).

### 2.2. Firms

The flow of output \( dY \) is produced from capital \( K \) by means of a stochastic constant return technology:
\[ dY = \alpha K \, dt + \alpha K \, dy, \] (9)
where \( \alpha \) is the marginal physical product of capital and \( dy \) is a temporally independent and normally distributed stochastic process with zero mean and variance \( \sigma_y^2 \, dt \).

Here, we turn to the determination of \( dR \). In general, we define the real rate of return of equities by
\[ dR = \frac{dD}{S} + \frac{ds}{s}, \] (10)
where \( dD \) is the flow of dividend payment and \( s \) the relative price of equities in terms of output.

We assume that capital is adjusted costlessly, that is, \( S = K \). We define the outstanding stock of equities at \( t \) by \( N \) and obtain
\[ sN = K = S. \] (11)

Taking stochastic differential of this relationship yields
\[ (s + ds)dN + N \, ds = dK. \] (12)
We assume that the net output is either paid out as dividends or retained as earning \( (RE) \) to finance new investment:
\[ dD + dRE = dY. \] (13)

Firms are assumed to finance new capital either out of retained earning or by issuing new equities. This is described by the constraint
\[ (s + ds)dN + dRE = dK, \] (14)
where new equities are sold at the current price \( s + ds \). Combining (11)–(14) yields
\[ \frac{ds}{s} = \frac{dY}{S} - \frac{dD}{S}. \] (15)
The substitution of (15) into (10) leads to
\[ dR_s = \frac{dY}{S}. \]  

(16)

Hence, we can specify
\[ r_s = \alpha, \quad du = \alpha \, dy. \]  

(17)

2.3. Government and product market equilibrium

The government budget constraint can be expressed as
\[ d\left(\frac{M}{P}\right) + d\left(\frac{B}{P}\right) = dG - dT + \left(\frac{M}{P}\right)dR_M + \left(\frac{B}{P}\right)dR_B, \]  

(18)

where \( dG \) is the stochastic government expenditure. It is specified as
\[ dG = g\alpha K \, dt + \alpha K \, d\zeta, \]  

(19a)

where \( d\zeta \) is a temporally independent and normally distributed random variable with zero mean and variance \( \sigma^2 \, dt \). This specification means that instantaneous mean government expenditure is a fraction of mean level of output and stochastic disturbance is also a proportional one.

Stochastic monetary policy is
\[ \frac{dM}{M} = \mu \, dt + \alpha \, dx, \]  

(19b)

where \( dx \) is a temporally independent and normally distributed random process with zero mean and variance \( \sigma^2 \, dt \). \( \mu \) and \( \sigma^2 \) will also be specified below.

Government debt policy is specified in terms of maintaining a fixed ratio of government bonds to money
\[ \frac{B}{M} = \lambda, \]  

(19c)

where \( \lambda \) is a policy variable determined by the government.

As the final element of the model, the product market equilibrium can be shown to satisfy
\[ dK = dY - dC - dG, \]  

(20)

which using (7a), (9), (19a), can be expressed as
\[ \frac{dK}{K} = \left[ \alpha(1 - g) - \frac{\theta p}{n_3} \right] dt + \alpha(\alpha Y - \alpha d\zeta). \]  

(21)
The deterministic and stochastic component of (21), $E(dK/K)$ and $dK$, respectively, are

$$E\left(\frac{dK}{K}\right) = \phi dt = \left[\alpha(1 - g) - \frac{\theta \rho}{n_3}\right] dt, \quad (22a)$$

$$dk = \alpha(dy - dz). \quad (22b)$$

3. **Public policies in macroeconomic equilibrium**

As for the elements developed in the previous section, let us combine them to derive the overall macroeconomic equilibrium. The exogenous factors are public expenditure $g$, and debt policy $\lambda$. The exogenous stochastic processes include public expenditure $dz$, productivity $dy$. The remaining deterministic and stochastic factors for money growth, $\mu$ and $dx$, and for tax rates, $\tau$ and $dv$, are given exogenously or determined endogenously, depending on government policy described below.\(^1\)

3.1. **Equilibrium rate of inflation**

Let us derive the equations determining the rate of inflation. Combining the equilibrium conditions in the money and equity markets,

$$\frac{(M/P)}{W} = n_1 \quad \text{and} \quad \frac{K}{W} = n_3,$$

the current price level is expressed as

$$P = \left(\frac{n_3}{n_1}\right)\left(\frac{M}{K}\right).$$

Noting that portfolio shares $n_1$, $n_2$ and $n_3$ are constant through time, the stochastic differentiation of the equation above yields

$$\frac{dP}{P} = \frac{dM}{M} - \frac{dK}{K} - \left(\frac{dM}{M}\right)\left(\frac{dK}{K}\right) + \left(\frac{dK}{K}\right)^2.$$

Substituting for (3), (19b) and (21) into this expression leads to

$$\pi dt + dp = \mu dt + dx - \left[\alpha(1 - g) - \frac{C}{K}\right]$$

$$- \alpha(dy - dz) - \alpha(\sigma_{xy} - \sigma_{xz}) dt + \alpha^2(\sigma_{x}^2 + \sigma_{z}^2) dt.$$
Equating the deterministic and stochastic parts of this equation yields

\[ \pi = \mu - \left[ \alpha(1 - g) - \frac{C}{K} \right] - \alpha(\sigma_{xy} - \sigma_{xz}) + \alpha^2(\sigma_y^2 + \sigma_z^2), \quad (23a) \]
\[ dp = dx - \alpha(dy - dz). \quad (23b) \]

The first of this equation specifies the expected rate of inflation. The second one determines the stochastic rate of inflation in terms of stochastic component of money growth shock, productivity shock and fiscal shock.

3.2. Determination of two alternative adjustments

Substituting for the government expenditure policy (19a), the monetary growth rule (19b) and debt policy (19c) into the government budget constraint (18) and dividing yields

\[ (1 + \lambda)n_1 \frac{d(M/P)}{(M/P)} = \alpha(g \, dt + dz) \frac{K}{W} - \frac{dT}{W} + n_1 \, dR + n_2 \, dB. \]

Next, substituting for the taxes on wealth (5), taking stochastic differential of \( d(M/P) \) and equating the deterministic and stochastic parts leads to the two relations

\[ \alpha n_3 \, g - (1 + \lambda)n_1 \, \mu + n_2 \, i + (1 + \lambda)n_1 \, \sigma_{xp} - \tau = 0, \quad (24a) \]
\[ \alpha n_3 \, dz - (1 + \lambda)n_1 \, dx - dv = 0. \quad (24b) \]

Suppose that exogenous public expenditures (determined by \( g \) and \( dz \)) are financed by adjusting taxes on wealth as needed. Then, (24a) and (24b) describe the endogenously accommodating levels of \( \tau \) and \( dv \) necessary to finance the government budget for the given policy specifications. This public policy, financing by taxes on wealth, corresponds to Grinols and Turnovsky (1993) and Turnovsky (1993).

Suppose, in contrast, that in addition to \( g \) and \( dz \), \( \tau \) and \( dv \) are chosen exogenously. Then (24a) and (24b) describe the endogenously accommodating adjustments in money growth, \( \mu \) and \( dx \), necessary to satisfy the government budget constraint. \( \lambda \) is fixed so that government finances by printing money and by issuing bonds in the ratio \( B = \lambda M \). The public policy denotes that government finances the expenditures by printing money and by issuing bonds, i.e., by the mixture of money and bonds.

3.3. Financing of taxes on wealth and monetary policy

Let us consider that public spending are financed by accommodating taxes on wealth, following Grinols and Turnovsky (1993). In this case, it can be assumed
that \(dx, dz\) and \(dy\), which are exogenous stochastic factors here, are mutually uncorrelated, i.e., \(\text{Cov}(d_i, d_j) = 0\) and \(i, j = x, y, z\). Then, the deterministic and stochastic factors can be expressed as

\[
\tau = \alpha n_3 g - (n_1 + n_2)\mu + n_2 i + (n_1 + n_2)\sigma_x^2,
\]
\[
d_v = \alpha n_3 dz - (n_1 + n_2)dx,
\]
\[
d_p = dx - \alpha(dy - dz).
\]

Using the equations above, we can obtain

\[
(I) \quad \sigma_p^2 = \sigma_x^2 + \alpha^2(\sigma_y^2 + \sigma_z^2), \quad \sigma_{uv}^2 = \alpha^2\sigma_y^2, \quad \sigma_{pu} = -\alpha^2\sigma_y^2, \quad \sigma_{uv} = 0, \quad \sigma_{pv} = (n_3 - 1)\sigma_x^2 + \alpha^2 n_3 \sigma_z^2.
\]

Combining the above relations, the macroeconomic equilibrium in this economy can be written as the following set of relationships:

\[
\pi = \mu - \alpha[1 - c(i) - g] + \alpha^2(\sigma_y^2 + \sigma_z^2), \quad (25a)
\]
\[
\mu = \alpha(1 - \alpha \sigma_y^2) = i - \pi + \sigma_y^2, \quad (25b)
\]
\[
c(i) = \frac{C}{\alpha K} = \frac{\rho \theta}{\alpha[1 - \rho \gamma(1 + \lambda)/\gamma]}.
\]

These three equations, which are also provided in Turnovsky (1993), jointly determine the equilibrium values of (i) the expected value of inflation \(\pi\), (ii) the rate of nominal interest \(i\), and (iii) the consumption-mean output ratio \(c(i)\).

Grinols and Turnovsky (1993) also examined the responses of the economy to changes in the first and second moments of fiscal policy shock under financing by taxes on wealth, but they do not derive the formularized conclusions. We derive them here as follows.

The combination of (26a) and (26b) yields

\[
F_\tau \equiv i - \mu - \alpha[c(i) + g] - \alpha^2\sigma_y^2 + \sigma_x^2 = 0,
\]

where the subscript \(\tau\) denotes that we are dealing with the financing case of taxes on wealth.

**Assumption 1.** \(1 > \mu + \alpha g + \alpha^2\sigma_y^2 - \sigma_x^2\),

If we maintain Assumption 1, there exists a unique value of \(i\) that is larger than \((1 + \lambda)\rho \gamma\).

The comparative statics of the first two moments of expenditure policy and monetary policy are summarized in Table 1. The consequences in this table are, of course, described in Grinols and Turnovsky (1993) and Turnovsky (1993).
Table 1  
Effects of fiscal policy and monetary policy

<table>
<thead>
<tr>
<th></th>
<th>Fiscal policy</th>
<th>Monetary policy</th>
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<tbody>
<tr>
<td></td>
<td>$g$</td>
<td>$\sigma^2_z$</td>
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<tr>
<td>Nominal interest rate $i$</td>
<td>$+$</td>
<td>$+$</td>
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<tr>
<td>Expected real growth rate $\phi$</td>
<td>$-$</td>
<td>$+$</td>
</tr>
<tr>
<td>Expected inflation rate $\pi$</td>
<td>$+$</td>
<td>$+$</td>
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</tbody>
</table>

*The proof of this table is described in Appendix A.*

3.4. Mixed financing of money and bonds and tax policy

In contrast with Grinols and Turnovsky (1993), we consider a case where government finances the deficits by printing money as well as by issuing bonds. Then, we cannot assume that $dx$ and $dz$ are mutually uncorrelated any longer, i.e., $\text{Cov}(dx,dz) \neq 0$, while $dz$, $dy$ and $dv$ can be assumed to be mutually uncorrelated, i.e., $\text{Cov}(di,dj) = 0$ and $i,j = v,y,z$.

Noting the above relationships among the stochastic factors, we can obtain

$$
\mu = \frac{\alpha n_3 g + n_2 i - \tau}{n_1 + n_2} + \sigma_{xp},
$$

$$
dx = \frac{\alpha n_3}{n_1 + n_2} dz - \frac{1}{n_1 + n_2} dv,
$$

$$
dp = \frac{\alpha}{n_1 + n_2} dz - \alpha dy - \frac{1}{n_1 + n_2} dv.
$$

Using these relationships, the covariance and variance can be derived as

$$
\text{(II)} \quad \sigma_{xp} = \frac{\alpha^2 n_3 \sigma^2_z}{(n_1 + n_2)^2} + \frac{1}{(n_1 + n_2)^2} \sigma^2_v,
$$

$$
\sigma^2_p = \alpha^2 \left[ \frac{\sigma^2_z}{(n_1 + n_2)^2} + \sigma^2_y \right] + \frac{1}{(n_1 + n_2)^2} \sigma^2_v,
$$

$$
\sigma_{xz} = \frac{\alpha n_3 \sigma^2_z}{(n_1 + n_2)^2}, \quad \sigma_{pv} = -\frac{1}{n_1 + n_2} \sigma^2_v, \quad \sigma_{pu} = -\alpha^2 \sigma^2_y.
$$

Using (II), the macroeconomic equilibrium can be reduced to the following set of relationships:

$$
\pi = \mu - \alpha [1 - c(i) - g] + \alpha^2 \sigma^2_y + \alpha^2 \left( \frac{1}{n_1 + n_2} \right) \sigma^2_z, \quad (26a)
$$
\[ z(1 - z \sigma_z^2) = i - \pi + \alpha^2 \left[ \frac{n_3}{(n_1 + n_2)^2} \right] \sigma_\varepsilon^2 + \frac{1}{(n_1 + n_2)^2} \sigma_v^2, \] (26b)

\[ \mu = \frac{zn_3 g}{n_1 + n_2} + n_2 i - \tau + \alpha^2 \left[ \frac{n_3}{(n_1 + n_2)^2} \right] \sigma_\varepsilon^2 + \frac{1}{(n_1 + n_2)^2} \sigma_v^2, \] (26c)

\[ n_1 + n_2 = \frac{\rho\gamma(1 + \lambda)}{i}, \] (26d)

\[ \lambda n_1 = n_2, \] (26e)

\[ n_1 + n_2 + n_3 = 1, \] (26f)

\[ c(i) \equiv \frac{C}{\alpha K} = \frac{\rho \theta}{\alpha[1 - \rho\gamma(1 + \lambda)/i]}, \] (26g)

These eight equations jointly determine the equilibrium values of (i) expected rate of inflation \( \pi \), (ii) the nominal interest rate \( i \), (iii) the consumption-mean output ratio \( c(i) \), (iv) the portfolio share of money, bonds and equities \( (n_1, n_2, n_3) \) and (v) the mean rate of nominal money growth rate \( \mu \).

Eqs. (26a) and (26c) are the portfolio balance equilibrium conditions. Other things being equal, expected inflation depends negatively upon the mean rate of taxes on wealth and the expected rate of capital accumulation. It also depends upon the variances \( \sigma_\varepsilon^2 \), \( \sigma_z^2 \) and \( \sigma_v^2 \). An increase in \( \sigma_z^2 \) raises the expected rate of monetary growth \( \mu \) and a rise in \( \sigma_v^2 \) raises the variance of the growth of capital stock \( \sigma_k^2 \). In addition, increasing \( \sigma_z^2 \) raises \( \mu \) and \( \sigma_k^2 \), while increasing \( \sigma_v^2 \) reduces the covariance \( \sigma_{km} \). These require a higher expected rate of inflation in order for portfolio balance to be maintained. Eq. (26b) describes the equilibrium relationship between the expected real rates of return on equities and bonds. Both these three relationships provide avenues whereby the variances influence on the equilibrium behavior on the economy.

Using (26a)–(26c), we eliminate \( \pi \) and \( \mu \), so we can define the following relationship:

\[ F_{M-B} \equiv i - \alpha[c(i) + g] = \frac{zn_3 g + n_2 i + \alpha^2 \sigma_\varepsilon^2 - \tau}{n_1 + n_2} = 0, \]

where the subscript \( M-B \) describes that we are dealing with the mixed financing case of money and bonds. Here, we assume the following:

**Assumption 2.** \( \rho > (\alpha/\gamma)(1 + \alpha \sigma_z^2) - \tau/\gamma \).

Under Assumption 2, we can find that there exists a unique equilibrium value of \( i > (1 + \lambda)\rho \gamma \). \(^2\)

\(^2\)Noting \( g \in [0,1] \), Assumption 1 implies \( g < \rho \gamma/\alpha - \alpha \sigma_z^2 + \tau/\alpha \). The rearrangement of \( F_{M-B} = 0 \) yields \( \rho \gamma - (\alpha g + \alpha^2 \sigma_\varepsilon^2 - \tau) = (n_1 + n_2)\theta/\rho n_3. \)
Noting that government finances expenditure with a mixture of money and bonds, the comparative statics of the first two moments of expenditure policy and tax policy are summarized in Table 2. Unlike the financing by taxes on wealth, an increase in the mean rate of public spending leads to 100% crowding-out of consumption-output ratio and thus does not affect the expected rate of growth. Nominal interest rate and expected real growth are invariant to the variance of the rate of taxes on wealth. Hence, the first moment of public spending and the second moment of taxes on wealth work in identical directions, while the second moment of public spending and the first moment of taxes on wealth work in opposite directions. Debt policy is super-neutral to the expected rate of capital accumulation. The invariance of the expected capital growth rate to debt policy is very significant, when we think of a mixed financing in which government relies upon a larger ratio of bonds to money as a source of necessary revenue.

### 4. Income-velocity of money

From the perspective of income-velocity of money, we reconsider the comparative static properties of public policy and public policy uncertainty. As is well-known, the quantity equation of money can be expressed as

$$MV = PY,$$

where $V$ is the income-velocity of money.

Thus, central bank can attain a target level of nominal output by controlling the expected rate of monetary growth, if $V$ is maintained in a fixed ratio. Then, if changing public policies have no impact upon the mean rate of monetary growth, the nominal output is not affected This idea applies to, in particular, the financing case of taxes on wealth in which the deterministic and stochastic parts of monetary growth are exogenously given. In our stochastic framework, we can
derive the income-velocity of money as

\[
\frac{1}{\rho} \left[ \frac{1}{\rho^2} - 1 \right] = 1(1 + \lambda)
\]

which increases with the rate of nominal interest. There is actually empirical finding that income-velocity of money varies proportionately with the nominal interest rate of treasury bill (For example, see McGrattan, 1998). As \( V \) cannot also be considered to be maintained in a fixed level analytically, the arguments are not true, even in the case of financing through taxes on wealth.

Let us pick up the mixed financing case of money and bonds in which the overall stochastic disturbance in monetary growth evolves, endogenously, over time. Here, we can show that these effects are just nominal phenomena, even if changes in public policies do affect \( V \) or the expected rate of monetary growth \( \mu \). A rise in the mean rate of public spending \( g \) raises \( V \) and \( \mu \). An increase in the variance of the rate of taxes on wealth \( \sigma^2 \) raises \( \mu \) and on the other hand, has no impact upon \( V \). Thus, increasing any of \( g \) or \( \sigma^2 \) enhances the nominal level of output, but increasing any of them raises the level of inflation and does not influence the level of real output.

5. The comparisons

5.1. Alternative financing effects

In qualitative properties of the comparative statics, there do not exist great difference between money–bond financing and taxes on wealth financing. Here, let us investigate these quantitative differences, using the outcomes obtained in the previous sections. For this purpose, we must recognize the following.

Lemma 1. When \( \rho + \tau = i - \mu + \sigma^2 \) is satisfied, we can obtain \( i^M-B = i^T \).

Proof. Combining \( F_{M-B} = 0 \) and \( F_T = 0 \) yields the relationship. \( \Box \)

Increasing the first two moments of the national income share of government spending, \( g \) and \( \sigma^2 \), tend to raise the rate of nominal interest, but we can compare the enhanced effects under these two modes of government financing.

\( V \) can be easily derived by using \( \lim_{dt \to 0} E(dY)/dt \) and \( V = \mu n_2/n_1 \).
Proposition 1. The rises in the nominal interest rate are larger under mixed money–bond financing than under financing by taxes on wealth, i.e.,

\[
\left( \frac{di}{d\sigma^2_{z}} \right)_{M-B} > \left( \frac{di}{d\sigma^2_{z}} \right)_{T}
\]

\[
\left( \frac{di}{d\sigma^2_{z}} \right)_{M-B} > \left( \frac{di}{d\sigma^2_{z}} \right)_{T}.
\]

Proof. Compare A.1 and A.2 with B.1 and B.2, respectively. □

The crowding-out of the consumption–output ratio \(c(i)\) is caused by a rise in the mean rate of public spending. Let us describe the quantitative difference of the crowding-out of \(c(i)\) between the two methods of financing.

Proposition 2. Wealth-tax financing leads to less than 100% crowding-out of \(c(i)\) and thus a fall in the expected rate of growth, while mixed money–bond financing leads to 100% crowding-out of \(c(i)\), so that the expected rate of growth is not affected. i.e.,

\[
\left( \frac{d\phi}{d\sigma^2_{z}} \right)_{M-B} = 0 > \left( \frac{d\phi}{d\sigma^2_{z}} \right)_{T}
\]

Proof. See Proposition 1 and Tables 1 and 2. We can obtain

\[
- \left( \frac{\hat{c}c}{\hat{c}i} \right) \left( \frac{di}{d\sigma^2_{z}} \right)_{M-B} = 1 > - \left( \frac{\hat{c}c}{\hat{c}i} \right) \left( \frac{di}{d\sigma^2_{z}} \right)_{T} > 0.
\]

A higher variance of public spending enhances the expected rate of real growth by making equity relatively attractive. We must think of which of the two schemes of government financing causes a larger increase in real growth.

Proposition 3. The increase in the expected rate of real growth due to higher government expenditure variability \(\sigma^2_{z}\), is larger under mixed financing of money and bonds than under financing of taxes on wealth, i.e.,

\[
\left( \frac{d\phi}{d\sigma^2_{z}} \right)_{M-B} > \left( \frac{d\phi}{d\sigma^2_{z}} \right)_{T} > 0.
\]

Proof. From Proposition 1, we can easily find that

\[
- \left( \frac{\hat{c}c}{\hat{c}i} \right) \left( \frac{di}{d\sigma^2_{z}} \right)_{M-B} > - \left( \frac{\hat{c}c}{\hat{c}i} \right) \left( \frac{di}{d\sigma^2_{z}} \right)_{T} > 0. \quad \square
\]

Increasing the first two moments of government consumption raise the expected rate of inflation. Let us highlight the differences in quantity that exist
between the expected rates of inflation pertaining to the two regimes of government financing.

**Proposition 4.** The expected rates of inflation achieved under mixed financing of money and bonds are higher than ones achieved under financing of taxes on wealth, i.e.,

\[
\left( \frac{d\pi}{dg} \right)_{M-B} \left( \frac{d\pi}{d\sigma_z} \right)_T > 0 \quad \text{and} \quad \left( \frac{d\pi}{d\sigma_z} \right)_{M-B} > \left( \frac{d\pi}{d\sigma_z} \right)_T > 0.
\]

**Proof.** If \( (\hat{\beta} \xi / \hat{c} i) + \alpha (\hat{c} / \hat{i} i) > 0 \), we can easily give a proof to this proposition, using (A.7) (A.8) (B.5) (B.6) and Proposition 1. \( \xi \) is defined in Appendix B.

### 5.2. Debt policy and mixed financing

Suppose a mixed money–bond financing regime in which government relies on bonds relative to money in order to raise the necessary revenue. We consider how the decision of government affects the comparative static properties of mixed financing that are described in Sections 3 and 4. From Note 2, we can easily obtain Lemma 2.

**Lemma 2.** \( n_1 + n_2 \) (or \( n_3 \)) is invariant to the level of \( \lambda \). Thus, an increase in \( \lambda \) by one unit results in an increase in \( i \) by \( i/(1 + \lambda) \) units, equivalently, \( \hat{c} i / \hat{\lambda} \lambda = i/(1 + \lambda) \).

Using this relationship, we can obtain the following.

**Proposition 5.** As the ratio of bonds to money is larger, the positive effect on \( i \) of an increase in any of \( g \) or \( \sigma_z^2 \) is larger. Even if so, these growth effects are unaffected by the government decision.

**Proof.** Using (B.1)–(B.4) and Lemma 1, we can obtain

\[
\frac{\partial (di/dg)_{M-B}}{\partial \lambda} = \frac{(di/dg)_{M-B}}{1 + \lambda} > 0,
\]

\[
\frac{\partial (di/d\sigma_z^2)_{M-B}}{\partial \lambda} = \frac{(di/d\sigma_z^2)_{M-B}}{1 + \lambda} > 0,
\]

\[
\frac{\partial [c'(i)]}{\partial \lambda} = - \frac{c'(i)}{1 + \lambda}.
\]

Considering the relationships above, we can express

\[
\frac{\partial (d\phi/dg)_{M-B}}{\partial \lambda} = 0 \quad \text{and} \quad \frac{\partial (d\phi/d\sigma_z^2)_{M-B}}{\partial \lambda} = 0.
\]
Next, let us explore how Proposition 5 impacts the effect of policy on the expected rate of inflation.

**Proposition 6.** As the ratio of bonds to money is larger, the positive effect of an increase in any of \( g \) or \( \sigma_z^2 \) on the expected rate of inflation is larger.

**Proof.** Using \( \partial \xi / \partial i = [1 - (\partial \xi / \partial i)]/(1 + \lambda) \), (B.5), (B.6), we can give a proof to this proposition. \( \xi \) is defined in Appendix B. \( \square \)

This outcome is slightly surprising, because a larger \( \lambda \) corresponds to a smaller increase in money stock.

To sum up, if government relies on bonds relative to money as a source of necessary revenue, the rate of nominal interest and the expected rate of inflation are relatively high. However, the government decision has no impact on the growth effects of changes in the mean and variance of government spending.

6. Welfare analysis

In order to assess the consequences of various fiscal policy shocks described above on economic welfare, a welfare criterion must be introduced. For this purpose, we must think of the welfare of the representative agent, as specified by the intertemporal utility function (6a), evaluated at the optimum. By definition, this is equal to the value function used to solve the stochastic optimization problem. The feature of this value function can be described as follows.

**Lemma 3.** In the feature of value function, there exists no difference of the mixed financing case of money and bonds from the financing case of taxes on wealth, that is,

\[
X(W_0, i) = E \int_0^\infty [\theta \ln C_t + \gamma \ln n_t W_t] e^{-\sigma t} dt
\]

\[
= b_0 + b_1 \ln W_0,
\]

\[
b_0 \equiv \left(\frac{\theta}{\rho}\right)(\ln \theta + \ln \rho) + \left(\frac{\gamma}{\rho}\right)\ln n_1 - \frac{\theta}{\rho} + \left(\frac{1}{\rho^2}\right)\left[\alpha(1 - g) - \alpha c + \theta \rho \right]
\]

\[
- \frac{\alpha^2}{2\rho^2} (\sigma_y^2 + \sigma_z^2),
\]

\[
b_1 \equiv \frac{1}{\rho}.
\]
Proof. See Appendix C. □

Here, let us differentiate $X(W, i)$ with respect to the rate of nominal interest $i$, so that we can obtain

$$\frac{\partial X(i, K_0)}{\partial i} = \frac{\gamma [- i^2 + (1 + \lambda)\gamma \rho i + \theta \gamma \rho^2(1 + \lambda)^2]}{\rho i[i - \rho \gamma(1 + \lambda)]^2}.$$ 

We should notice the nominal interest rate that can be obtained by solving the equation

$$- i^2 + (1 + \lambda)\gamma \rho i + \theta \gamma \rho^2(1 + \lambda)^2 = 0.$$ 

Noting $i > \rho \gamma(1 + \lambda)$ in (29), the interest rate $i^*$ is

$$i^* = \frac{\rho(1 + \lambda)(1 - \theta)}{2} \left[ 1 + \sqrt{1 + \frac{4\theta}{1 - \theta}} \right] \geq \rho(1 + \lambda)(1 - \theta),$$

and $\text{sgn} (\partial X/\partial i) = \text{sgn}(i^* - i)$.

The welfare consequences of changes in $\mu, \tau, \sigma_\mu^2$ and $\sigma_\tau^2$ all operate entirely through their effects on $i$. As is well known, increasing $i$ has three effects on the welfare. A higher $i$ lowers the demand for money, which is welfare-deteriorating, and reduces the equilibrium consumption–output ratio and thus accelerates the expected growth of capital, which is welfare-improving, and causes an initial jump in the initial price level, causing a reduction in initial wealth, which is welfare-deteriorating.

Suppose that $i > i^*$. Then, an increase in $\mu$ raises $i$ and thus deteriorates the welfare, even if an increase in $\mu$ stimulates the expected growth of capital. A rise in $\sigma_\mu^2$, on the other hand, reduces $i$ and thus improves the welfare, even if a rise in $\sigma_\mu^2$ is distortionary to the expected growth of capital. In addition, increasing $\tau$ reduces $i$ and improve the welfare, even if increasing $\tau$ results in a fall in the expected growth rate. A higher $\sigma_\tau^2$ does not affect $i$ and is consequently super-neutral to the level of welfare as well as to the expected growth rate.

Let us assume the following, which is consistent with Assumption 2.

Assumption 3. $\beta > 0$, where $\beta \equiv \mu + \tau g + \sigma_\mu^2 \sigma_\tau^2 - 2\theta \rho$.

Using the above, we can get the proposition.

Proposition 7. Unlike the variance of public spending, there does not exist the optimal level of mean public spending.
Proof. Using (B.1), (B.2) and (C.2), we can derive
\[
\left( \frac{\partial X}{\partial i} \right) \frac{d}{d \theta} j - \frac{\partial}{\partial \theta} \left( \frac{n_3}{n_1 + n_2} \left( \frac{1 - n_3 \theta}{\theta} \right) \right) < 0 \quad j = M - B,
\]
\[
= - \left( \frac{\partial^2}{\partial \theta^2} \right) \left( \frac{n_3}{n_3^2 i + \theta \rho (n_1 + n_2)} \right)
\times \left[ (1 - \theta n_3 \rho + in_3 \right] < 0, \quad j = T.
\]
Noting \((\partial X/\partial i)(d/d\sigma^2_j) - \partial/\partial \sigma^2 = 0\), the optimal level of \(\sigma^2\) satisfies the relationships.
\[
(n_3)_j = \frac{3 \theta + 2 \gamma - \sqrt{4 \gamma^2 + \theta^2 + 12 \gamma \theta}}{4 \theta} < 1, \quad j = M - B,
\]
\[
= - \left( \frac{\gamma + 2 \theta \rho + \sqrt{\gamma^2 + 2 \theta \rho^2 + \rho \theta \beta}}{\beta} \right) < 1, \quad j = T.
\]
Moreover, we can add this proposition to the following:

Proposition 8. A rise in the first moment of public spending reduces the level of economic welfare, but mixed financing of money and bonds leads to a larger (resp. a smaller) fall in welfare than financing of taxes on wealth, when \(i > i^*\) (resp. \(i < i^*\)). A rise in the second moment in public spending reduces the level of economic welfare, when \(i > i^*\). Then, mixed financing of money and bonds results in a larger fall in welfare than financing of taxes on wealth.

Proof. See Proposition 1. □

Hence, from a welfare perspective, a smaller fall or a larger increase in the expected real growth rate is not always desirable.

From a growth perspective, we have already obtained Proposition 5 in the case of mixed financing of money and bonds. From a welfare perspective, we can describe the following.

Proposition 9. As the ratio of bonds to money is larger, the enhanced effect of an increase in any of \(g\) or \(\sigma^2\) on \(i\) is larger. The welfare effects are not, however, influenced by the government decision.

---

\(^4\)If the right-hand side in this equation decreases with \(\sigma^2\), it is not inadequate for us to think that there exists a unique optimal level of \(\sigma^2\) when \(j = T\).
Proof. Using (B.1), (B.2), (C.2) and Lemma 2, we can obtain

$$
\frac{\partial (d_i/dg)}{\partial \lambda} = \frac{(d_i/dg)_{M-B}}{1 + \lambda} > 0,
$$

$$
\frac{\partial (d_i/d\sigma^2_{M-B})}{\partial \lambda} = \frac{(d_i/d\sigma^2_{M-B})_{M-B}}{1 + \lambda} > 0,
$$

$$
\frac{\partial (dX/di)}{\partial \lambda} = -\frac{X(W_0,i)}{1 + \lambda}.
$$

Considering the relationships above, we can express

$$
\frac{\partial (dX/dg)_{M-B}}{\partial \lambda} = 0 \quad \text{and} \quad \frac{\partial (dX/d\sigma^2_{M-B})_{M-B}}{\partial \lambda} = 0.5
$$

Thus, even if government relies on bonds relative to money as a source of necessary revenue, the government decision has no impact on the welfare effects of changes in the mean variance of government spending. From the growth and welfare perspectives, it does not matter which public spending is financed by printing money or by issuing bonds.

7. Concluding remarks

As is evident from the above discussion, government expenditure can be interpreted as being either a real drain on the economy or some public good that does not affect the productivity of private capital. We demonstrate an economy where the mixture of money and bonds or the rate of taxes on wealth is needed to finance government expenditure. Let us summarize the merits and demerits of mixed financing of money and bonds relative to financing of taxes on wealth.

(i) An increase in the national income share of government consumption depresses the share of private consumption. In the case of mixed financing of money and bonds, there is 100% crowding-out of private consumption so that the expected rate of real growth is unaffected. In the case of financing of the rate of taxes on wealth, private consumption is not fully crowded out, so the expected

---

5 Unlike Propositions 7 and 8, the arguments in Proposition 9 are equally true in the case that public good is entered into production function for final output. See Gokan (2000).

6 This outcome seems to depend too much upon the debt policy that money and bonds are maintained in a fixed ratio $\lambda$. Many macroeconomists argue that this policy specification is not unrealistic, so we cannot think of the outcomes in this paper to be inappropriate. For example, see in Turnovsky (1995, Chap. 15).
rate of real growth falls. In addition, increasing the national income share of government consumption raises the expected rate of inflation and deteriorates the level of economy welfare. However, mixed financing of money and bonds leads to a higher rate of inflation and a lower level of welfare than financing of taxes on wealth when the nominal interest rate is relatively high.

(ii) A higher uncertainty associated with government spending stimulates the expected rate of real growth, raises the expected rate of inflation and deteriorates the level of economy welfare. However, mixed financing of money and bonds results in a larger increase in real growth, a higher rate of inflation and a lower level of economy welfare than financing of taxes on wealth.

Hence, our analyses suggest that from a growth perspective, mixed financing of money and bonds is preferred, but from a welfare perspective, financing of taxes on wealth is preferred.

We think of a mixed financing case where government finances the deficit in a larger ratio of bonds to money. We can describe how this decision affects the consequences of mixed financing as follows:

(iii) The higher the ratio of bonds to money is, the larger the enhanced effects on the nominal interest rate and on the expected inflation rate of increase in the mean and variance of public consumption are. These growth and welfare effects are not, nevertheless, affected by the government decision.

Thus, the growth and welfare effects make no difference in quantity between money- and bond-financing.

Finally, we state the properties of comparative static effects of changes in the first two moments of the rate of monetary growth and the rate of taxes on wealth.

As the mean (resp. the variance) of money growth rate is higher, the mean rate of capital growth is larger (resp. smaller), while the level of economy welfare is smaller (resp. larger). However, the relationships are also confirmed in Turnovsky (1993).

(iv) As the mean rate of taxes on wealth is higher, the mean rate of capital growth is smaller and on the other hand, the level of economy welfare is larger. However, the mean rate of capital growth and the level of economy welfare are invariant to the variance of the rate of taxes on wealth.

Suppose that the government deficits in this period are given, i.e., \( g \) is fixed in period \( t \). Then, from a growth (resp. a welfare) perspective, it is more desirable that government relies on the rate of monetary growth (resp. the rate of taxes on wealth) to obtain the given revenue.

8. For further reading

Appendix A. Proof of Table 1

Using $F_T = 0$, let us think of the comparative static effects of changes in the first two moments $g, \sigma_z^2$ on the nominal interest rate $i$:

\[
\left( \frac{di}{dg} \right)_T = \frac{\alpha}{D_1} > 0, \quad (A.1)
\]

\[
\left( \frac{di}{d\sigma_z^2} \right)_T = \frac{\alpha^2}{D_1} > 0, \quad (A.2)
\]

where

\[
D_1 \equiv 1 + \frac{\theta \rho (n_1 + n_2)}{n_3^2 i} > 0.
\]

From (26c), (A.4), (B.1) and (B.2), we can express the effects of changes in $g$ and $\sigma_z^2$ on the equilibrium consumption–output ratio $c(i)$ as

\[
\left[ \frac{dc(i)}{dg} \right]_T = - \left[ \frac{\theta \rho (n_1 + n_2)}{x n_3^2 i} \right] \left( \frac{di}{dg} \right)_T < 0, \quad (A.3)
\]

\[
\left[ \frac{dc(i)}{d\sigma_z^2} \right]_T = - \left[ \frac{\theta \rho (n_1 + n_2)}{x n_3^2 i} \right] \left( \frac{di}{d\sigma_z^2} \right)_T < 0. \quad (A.4)
\]

We can describe the effects of changes in $g$ and $\sigma_z^2$ on the expected rate of real growth $\phi$ as

\[
\left( \frac{d\phi}{dg} \right)_T = - x \left[ \frac{dc(i)}{dg} \right]_T = - x = - \frac{\alpha}{D_1} < 0, \quad (A.5)
\]

\[
\left( \frac{d\phi}{d\sigma_z^2} \right)_T = - x \left[ \frac{dc(i)}{d\sigma_z^2} \right]_T = \left[ \frac{\theta \rho (n_1 + n_2)}{x n_3^2 i} \right] \left( \frac{\alpha}{D_1} \right)_T > 0. \quad (A.6)
\]

Finally, we will investigate how increasing $g$ and $\sigma_z^2$ have influences on the expected rate of inflation $\pi$. From (26a), we can obtain

\[
\left( \frac{d\pi}{dg} \right)_T = - \left( \frac{d\phi}{dg} \right)_T > 0, \quad (A.7)
\]

\[
\left( \frac{d\pi}{d\sigma_z^2} \right)_T = - \left( \frac{d\phi}{d\sigma_z^2} \right)_T + \alpha^2 = \frac{\alpha^2}{D_1} > 0. \quad (A.8)
\]

The effects of monetary policy and the policy randomness on economic growth and other endogenous variable are, in detail, discussed in Turnovsky (1993) and Grinols and Turnovsky (1993).
Appendix B. Proof of Table 2

Suppose that $\tau$ is moderately small. Noting $F_{M-B} = 0$, (26d), (26f) and (26g), the comparative static effects of changes in $g$ and $\sigma_{z}^{2}$ on $i$ are

$$
\frac{d i}{d g}_{M-B} = \frac{\lambda/(n_1 + n_2)}{D_1 - D_2} > 0, \quad (B.1)
$$

$$
\frac{d i}{d \sigma_{z}^{2}}_{M-B} = \frac{\lambda^2/(n_1 + n_2)}{D_1 - D_2} > 0, \quad (B.2)
$$

where

$$
D_2 \equiv \frac{\lambda(g + \lambda \sigma_{z}^{2}) + \lambda \rho \gamma - \tau}{\rho \gamma(1 + \lambda)} > 0.
$$

Using (B.1) and $F_{M-B} = 0$, the comparative static effects of fiscal policy, $g$ on $\phi$ is

$$
\frac{d \phi}{d g}_{M-B} = -\lambda \left[\frac{dc(i)}{d g}_{M-B}\right] - \lambda,
$$

$$
= \frac{\lambda D_2 - i - ((1/\gamma) + \lambda) \rho \gamma}{D_1 - D_2} = 0, \quad (B.3)
$$

which, by contrast with tax-financing, asserts that a rise in $g$ leads to 100% crowding-out of $c(i)$ and thus unaffects $\phi$.

We must think of the property of comparative static effect of fiscal policy randomness $\sigma_{z}^{2}$ on $\phi$, which can be expressed as

$$
\frac{d \phi}{d \sigma_{z}^{2}}_{M-B} = -\lambda \left[\frac{dc(i)}{d \sigma_{z}^{2}}_{M-B}\right] = \left[\frac{\theta \rho}{n_1^2 i}\right] \left(\frac{\lambda^2}{D_1 - D_2}\right)_{M-B} > 0. \quad (B.4)
$$

As seen in the financing case of taxes on wealth, increasing randomness of public spending raises $\phi$ by making equity relatively more attractive.

Let us examine how $\pi$ is affected by changes in $g$ and $\sigma_{z}^{2}$. Combining (26a) and (26c) yields

$$
\pi = \bar{\zeta} + \phi + \lambda^2 \sigma_{z}^{2},
$$

where

$$
\bar{\zeta} = \frac{(2n_3 g + n_2 i - \tau + (\lambda^2 \sigma_{z}^{2} + \sigma_{z}^{2})/(n_1 + n_2))}{n_1 + n_2}.
$$
Hence, the properties of comparative statics are described as

\[
\left( \frac{d\pi}{dq} \right)_{M-B} = \left[ \left( \frac{\partial \xi}{\partial i} \right) + x \left( \frac{\partial c}{\partial i} \right) \right] \left( \frac{di}{dg} \right)_{M-B} + \frac{x}{n_1 + n_2} > 0, \tag{B.5}
\]

\[
\left( \frac{d\pi}{d\sigma^2} \right)_{M-B} = \left[ \left( \frac{\partial \xi}{\partial i} \right) + x \left( \frac{\partial c}{\partial i} \right) \right] \left( \frac{di}{d\sigma^2} \right)_{M-B} + \frac{x^2}{(n_1 + n_2)^2} > ?, \tag{B.6}
\]

where

\[
\left( \frac{\partial \xi}{\partial i} \right) = \frac{[xg + n_2 i + 2(x^2 \sigma^2 + \sigma^2)/(n_1 + n_2) - \tau]}{(1 + \lambda)\rho\gamma} > 0.
\]

Noting (26a) and (B.3), Eq. (B.5) reveals that the expected rate of inflation must rise at a higher rate than the expected rate of monetary growth, in order to maintain the portfolio balance. Considering (26a) and (B.4), Eq. (B.6) implies that the expected rate of inflation will be raised or reduced, depending on the response to the expected rate of monetary growth of a change in \(\sigma^2\).

Using \(F_{M-B} = 0\), let us derive the qualitative effects of changes in \(\tau\) and \(\sigma^2\) on \(i\).

\[
\left( \frac{di}{d\tau} \right) = - \left[ \frac{1/(n_1 + n_2)}{D_1 - D_2} \right] < 0, \tag{B.7}
\]

\[
\left( \frac{di}{d\sigma^2} \right) = 0. \tag{B.8}
\]

As seen in Eq. (B.8), the nominal interest rate is invariant to the level of \(\sigma^2\). The property of comparative statics is very important, when we examine the growth and welfare effects.

Hence, these growth effects can be described as

\[
\left( \frac{d\phi}{d\tau} \right) = - \left( \frac{\theta \rho}{\sigma^2 i} \right) \left( \frac{1}{D_1 - D_2} \right) < 0. \tag{B.9}
\]

\[
\left( \frac{d\phi}{d\sigma^2} \right) = 0. \tag{B.10}
\]

Unlike the relationship between \(\sigma^2\) and \(g\), we can highlight the contrast that exists between the responses of \(\phi\) to changes in \(\sigma^2\) and in \(\tau\).

Finally, we can summarize these effects on \(\pi\) as

\[
\left( \frac{d\pi}{d\tau} \right) = \left[ \left( \frac{\partial \xi}{\partial i} \right) + x \left( \frac{\partial c}{\partial i} \right) \right] \left( \frac{di}{d\tau} \right) - \frac{1}{n_1 + n_2} = ?, \tag{B.11}
\]

\[
\left( \frac{d\pi}{d\sigma^2} \right) = \frac{x^2}{n_1 + n_2} > 0. \tag{B.12}
\]
Suppose that \([(\hat{c}c/\hat{c}i) + z(\hat{c}c/\hat{c}u)]\] is positive, but is moderately small, then, we can obtain the signs of (B.6) and (B.11) as shown in Table 2. By contrast with monetary policy, a rise in the first moment of tax rate has the opposite effect on \(\pi\) to that of the second moment.

Appendix C. Proof of Lemma 3

For the logarithmic utility function, the optimized level of utility, starting from an initial stock of wealth \(W_0\), is given by

\[
X(W_0) = E \int_0^\infty [\theta \ln C_t^* + \gamma \ln n_t^* W_t^*] e^{-\rho t} dt
\]

\[
= b_0 + b_1 \ln W_0
\]

where \(*\) denotes the optimized value.

Substituting this function and the optimized values into the stochastic Bellman equation yields

\[
b_0 = \left(\frac{\theta}{\rho}\right)(\ln \theta + \ln \rho) + \left(\frac{\gamma}{\rho}\right) \ln n_0^* - \frac{\theta}{\rho} \phi - \frac{\delta^*}{2\rho^2},
\]

\[
b_1 = \frac{1}{\rho},
\]

where

\[
\phi \equiv n_1^r r_M + n_2^r r_B + n_3^r r_s - \tau,
\]

\[
\delta \equiv (n_1^r + n_2^r)^2 \sigma_p^2 + n_3^r \sigma_u^2 + \sigma_v^2 - 2(n_1^r + n_2^r)n_3^r \sigma_{up} - 2n_3^r \sigma_{uv} + 2(n_1^r + n_2^r) \sigma_{vp}.
\]

The substitution of the variances obtained in Sections 3 and 4 into \(\sigma_p^2, \sigma_u^2, \sigma_v^2, \sigma_{uv}, \sigma_{vp}\) and \(\sigma_{up}\) leads to

\[
\phi = z(1 - g) - zc + \theta \rho,
\]

\[
\delta = z^2(\sigma_v^2 + \sigma_z^2).
\]

Using, \(W_0\) can be expressed in terms of the equilibrium interest rate as

\[
W_0 = \frac{iK_0}{i - \rho \gamma (1 + \lambda)}. \quad (C.1)
\]

Therefore, the welfare criterion can be shown as

\[
X(i, K_0) = b_0 + \frac{1}{\rho} [\ln i - \ln (i - \rho \gamma (1 + \lambda))] + \frac{1}{\rho} \ln K_0, \quad (C.2)
\]
Table 3
Comparative static effects in $i$ and $\pi^*$

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</tbody>
</table>

$^*\rho = 0.03, \; \alpha = 0.03, \; \gamma = 0.12, \; \sigma_i^2 = 0.025, \; \tau = 0.01(0.0125), \; \sigma_i^2 = 0.005, \; \gamma = 0.4, \; \theta = 0.6, \; \lambda = 1.2, \; \mu - \sigma_i^2 = 0.0122577(0.0091613)$. 

Table 4
Comparative static effects in $\phi$ and $X$

<table>
<thead>
<tr>
<th></th>
<th>$\frac{d\phi}{dg}$</th>
<th>$\frac{d\phi}{d\sigma_i^2}$</th>
<th>$\frac{dX}{dg}$</th>
<th>$\frac{dX}{d\sigma_i^2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T. Finance</td>
<td>$-0.0175365$</td>
<td>0.000373905</td>
<td>$-35.3799$</td>
<td>$-0.561396$</td>
</tr>
<tr>
<td></td>
<td>($-0.0156475$)</td>
<td>(0.000430575)</td>
<td>($-33.9357$)</td>
<td>($-0.520252$)</td>
</tr>
<tr>
<td>M–B. Finance</td>
<td>0</td>
<td>0.000899999</td>
<td>$-38.2549$</td>
<td>$-0.647782$</td>
</tr>
<tr>
<td></td>
<td>(0)</td>
<td>(0.000900001)</td>
<td>($-34.9583$)</td>
<td>($-0.548747$)</td>
</tr>
</tbody>
</table>

where

$$b_0 \equiv \frac{\theta}{\rho} \ln \theta + \frac{\gamma}{\rho} \ln \gamma + \frac{1}{\rho} \ln \rho - \frac{\gamma}{\rho} \ln i + \frac{\alpha}{\rho^2} [1 - c(i) - g] - \frac{\gamma^2}{2 \rho^2} (\sigma_i^2 + \sigma_i^2)$$

This value function is the same feature as in the financing case of taxes on wealth examined in Turnovsky (1993).

Appendix D. Numerical simulations

As seen in Section 3, there are no great qualitative differences between mixed financing of money and bonds and financing of taxes on wealth. In Section 5, we compare the quantitative differences between these two financing cases. Here let us measure the differences in quantity, using the numerical simulations. Foot note to Table 3 lists the values of parameters that I have used in the simulations. Most of these are relatively standard, and are similar to the values that are used in other calibrated models.

Table 3 reports that the impacts upon $i$ and $\pi$ are moderately sensitive to the different ways of government financing, equivalently, there are relatively great disparities in $i$ and $\pi$ arising from these two cases. However, Table 4 reveals that
there exist the quantitative disparities in $\phi$ and $X$ pertaining to the two cases, as seen in Section 5, but such disparities are negligibly small.

References