Dynamic effects of government expenditure in a finance constrained economy

Yoichi Gokan

Faculty of Humanities and Social Sciences, Mie University, 1515, Kamihama-cho, Tsu-shi, Mie 514-8507, Japan

Received 4 March 2003; final version received 8 December 2004
Available online 8 March 2005

Abstract

This paper introduces constant government expenditure in Woodford’s finance constrained model (J. Econ. Theory, 1986) with capital–labor substitution as presented in Grandmont, Pintus and de Vilder (J. Econ. Theory, 1998) and investigates how government expenditure influences local dynamics near multiple steady states, depending upon the elasticity of substitution between capital and labor in production.

© 2005 Elsevier Inc. All rights reserved.

JEL classification: C-62; E-32; O-42

Keywords: Indeterminacy; Local bifurcations; Capital–labor substitution in production; Dynamic effects of government expenditure

1. Introduction

This paper introduces constant government expenditure financed by labor income taxes in Woodford’s finance constrained economy [13] with capital–labor substitution, as studied in Grandmont et al. [8] (noted hereafter G98). It is already well understood that in the one-dimensional version (1D) of that model without productive capital, positive government expenditure generates two non-trivial steady states and the possibility of saddle-node
bifurcation. In this model, and for all feasible values of government expenditure, the low output steady state is indeterminate with an infinite number of nearby sunspot equilibria. Conversely, the high output steady state is determinate and exempt of any stationary endogenous fluctuations close by (see [5,6]). The purpose of the present paper is to examine what happens in the two-dimensional (2D) version of the model with productive capital, as analyzed in G98, in order to uncover similarities and/or differences with the standard 1D case.

Based on the finance-constrained model with productive capital, a significant amount of research has clarified the hypothesis that economic fluctuations are endogenously explained in the absence of persistent exogenous shocks. For example, Grandmont [7] and G98 showed that endogenous fluctuations are possible for low elasticities of substitution between labor and capital, and the frequencies are short enough to mimic actual data. Likewise, Cazzavillan et al. [4] and Barinci and Cherón [2] incorporated increasing returns-to-scale in the finance constrained model and showed that endogenous fluctuations emerge, even if the elasticity of input substitution is arbitrarily large. Barinci [1] similarly demonstrated that a low capital–labor elasticity of substitution is not required for the presence of competitive cycles, if the capitalist’s preference is not logarithmic. De Vilder [12] and Pintus et al. [10] illustrated the existence of chaotic perfect-foresight equilibria. We should note that there co-exists a trivial steady state and a non-trivial steady state in that model with no government expenditure. This body of research shares in common the assumption that government expenditure is set to zero and attention is then restricted to the non-trivial steady state. As seen in the 1D case, the second non-trivial steady state emerges in place of the trivial steady state, if government expenditure is not zero. The fact that government expenditure generates a multiplicity of steady states parallels the 1D case. In this paper, we investigate how government expenditure influences local dynamics near the two non-trivial steady states, depending upon the elasticity of capital–labor substitution. This paper compares the consequences obtained by introducing constant government expenditure in the 2D version of Woodford’s model, and as investigated in G98, to the standard outcomes obtained by the 1D version of the model as examined in Farmer [5] and Farmer and Woodford [6].

In this paper, the tax rate on labor income is endogenously adjusted to the point necessary to finance a given increase in government expenditure. In Schmitt-Grohe and Uribe [11], endogenous labor income taxes are also taken into account as a possible source of indeterminacy. Schmitt-Grohe and Uribe [11] find conditions that are close to those obtained with monopolistic competition as in Benhabib and Farmer [3], but focus instead upon a Cobb–Douglas technology. In addition, they do not investigate saddle-node bifurcation, as here. In contrast, Kuhry [9] examines the possibility of saddle-node bifurcation in a two-sector model with imperfect competition, but focuses on a Cobb–Douglas technology. In this paper, the elasticity of capital–labor substitution is arbitrary and an important variable parameter of the analysis.

The structure of the model is briefly discussed in Section 2 since the framework is the same as in G98, with the exception that the government sector is explicitly introduced in the finance-constrained economy.

---

1 The non-zero steady state is generally referred to as the non-trivial steady state, whilst the zero steady state is usually denoted the trivial steady state.
2. Framework

We focus directly on the overlapping generations’ structure arising from the representative agent model of Woodford [13] with an infinite lifetime. It is well understood that the corresponding equations reflect the actual dynamics of the representative agent model near the monetary steady states only if workers discount the future more than capitalists. In that case, identical workers behave like two-period-living agents and participate in the market in two periods. In the early period, they supply a variable quantity of labor hours, save their after-tax income by holding outside money and choose the next period’s consumption. Workers maximize at date \(t\) their utility
\[
c_t^{1-\phi}/(1-\phi) - l_t^{1+\zeta}/(1+\zeta),
\]
subject to the current and anticipated budget constraint,
\[
p_{t+1}c_{t+1} = (1-\tau_{wt})p_t w_t l_t = M_{t+1},
\]
where the assumptions \(0 < \phi < 1\) and \(\zeta > 0\) are required to ensure that the elasticity of labor supply with respect to the real wage is positive. \(c_{t+1}\) is the next period’s consumption, \(l_t\) is the labor supply, \(p_i\) is \(i\)-period price of consumption, \(w_t\) is the real wage rate, which is taxed at the rate \(\tau_{wt}\), and \(M_{t+1}\) is the nominal outside money held at the end of this period. The unique solution is characterized by the budget constraint and the workers’ offer curve, \(c_{t+1} = l_t^\gamma\), where
\[
\gamma \equiv \frac{1-\phi}{1+\zeta} > 1.
\]

On the producer’s side, identical capitalists produce consumption goods \(y_t\) by combining labor \(l_t\) and the capital stock \(k_{t-1}\) according to a constant return technology given by
\[
y_t = l_t f(a_t),
\]
where \(a_t \equiv k_{t-1}/l_t\). The reduced production function \(y_t/l_t = f(a_t)\) is a continuous function of \(a_t\) and has continuous derivatives of all required orders for \(a_t\) with \(f'(a_t) > 0\) and \(f''(a_t) < 0\). The marginal product of capital \(\rho(a_t)\) is negatively related to \(a_t\), while the marginal product of labor \(w(a_t)\) is an increasing function of \(a_t\). Defining the real rental rate of capital as \(\rho_t\), we obtain \(\rho_t = \rho(a_t)\) and \(w_t = w(a_t)\) in equilibrium. Capitalists choose not to hold outside money. This arises from the fact that at steady states and nearby the real return on capital is higher than that of money balances.\(^2\) To simplify the matter, we consider the case that the utility function of capitalists is logarithmic and make the approximation that capitalists do not discount the future at all. The behavior of capitalists is then simple; they do not consume, but reinvest all their capital income. Thus, their optimal choice is
\[
k_t = (1-\delta)k_{t-1} + \rho(a_t)k_{t-1} \equiv R(a_t)k_{t-1},
\]
where \(0 < \delta < 1\) is the depreciation rate of capital stock.

Finally, the behavior of the government sector needs to be stipulated. Government expenditure is kept constant over time, \(g_t = g > 0\), while the tax rate on labor income and the rate of monetary growth \(\mu_t\) are endogenously adjusted necessary to satisfy the budget constraint, \(^3\)
\[
\mu_t (M_t/P_t) + \tau_{wt} w(a_t)l_t = g.
\]

---

\(^2\) As long as the rate of monetary growth is positive, such a situation is also satisfied in this modified Woodford’s model.

\(^3\) To be accurate, government expenditure is financed with a mixture of money and labor income taxes. However, there is no difference in quality between money financing and the financing through labor income taxes, since capitalists hold no money.
3. Multiple steady states and dynamical equations

This paper introduces a constant government expenditure financed by labor income taxes in the version with capital–labor substitution following Woodford’s model. In this case, we derive the dynamical system of \((k_{t-1}, a_t)\) as

\[
k_t = R(a_t)k_{t-1},
\]

\[
w(a_{t+1}) \cdot (k_t/a_{t+1}) = (k_{t-1}/a_t)\gamma + g.
\]

The workers’ budget constraint and (4.2) imply \(w(a_t) \cdot l_t - g = M_t/P_t\), which corresponds to the equilibrium of money market at time \(t\). This relationship implies that real balances are equal to \(c_t\), that is, the consumption of the old workers. The equilibrium condition of a good market, \(l_t f(a_t) = c_t + g + k_t - (1 - \delta)k_{t-1}\) can be ensured by combining (4.1), (4.2) and \(l_t f(a_t) = w(a_t)l_t + \rho(a_t)k_{t-1}\), which is the accounting identity describing the complete distribution of total income between labor and capital.

From (4.1), a steady state is obtained by solving \(R(a^*) = 1\), so the steady state capital–labor ratio \(a^* = k^*/l^*\) depends only upon the technology, not on government expenditure \(g\) nor on the offer curve \(\gamma\). Thus, the values of \(l^*\) are derived by solving the equation \(w(a^*)l^* - g = (l^*)\gamma\) (see Fig. 1). The graphical solution amounts to the intersection of the line \(c^* = w(a^*)l^* - g\) with the offer curve \(c^* = (l^*)\gamma\) in \((l^*, c^*)\) plane. Under the specification of \(\gamma > 1\), there exist the two non-trivial steady state \(c^*_1 = (l^*_1)\gamma\) \(< c^*_2 = (l^*_2)\gamma\) for \(0 < g < g_2\). These close together when \(g\) goes up to \(g_2\) and disappear for \(g > g_2\). This mimics exactly the 1D argument with no productive capital (e.g., [5,6]). As preparation for local stability analysis, we note the following. When \(g\) goes up to \(g_2\) in the feasible set \([0, g_2]\), private consumption in the low steady state \(c^*_1 = (l^*_1)\gamma\), the capital stock \(k^*_1 = a^*l^*_1\) and the output level \(y^*_1 = f(a^*)l^*_1\), all increase from 0 to \(\tilde{c} = (\tilde{l})\gamma\), \(\tilde{k} = a^*\tilde{l}\) and \(\tilde{y} = f(a^*)\tilde{l}\), while the same item in the high steady state \(c^*_2 = (l^*_2)\gamma\), \(k^*_2 = a^*l^*_2\) and \(y^*_2 = f(a^*)l^*_2\), all decrease to the same values \(\tilde{c}\), \(\tilde{k}\) and \(\tilde{y}\).

Let us now derive the dynamic equations to study the local stability around the two steady states. The procedure usually used to study the local stability of the stationary points is the linear map associated with the Jacobian matrix of (4) evaluated at the fixed points. Let \(e_w \equiv a w'(a)/w(a)\) be the elasticity of the marginal product of labor and \(e_R \equiv a |R'(a)|/R(a)\) the elasticity of the real gross rate of return on capital. The linearized dynamics for the deviations \(dk \equiv k - k^*\) and \(da \equiv a - a^*\), are determined by

\[
dk_t = dk_{t-1} - l^*_1 |e_R| da_t,
\]

\[
da_{t+1} = \frac{1}{l^*_1} \frac{\gamma l^*_1 \frac{\partial}{\partial l} - \frac{\partial}{\partial l} - |e_R|}{e_w - 1} \frac{1}{e_w - 1} da_t,
\]

where \(\gamma^i \equiv \frac{c^*_i}{l^*_i + g} = \frac{(k^*_i/a^*)^{1-1}}{w(a^*)} \) is the steady-state share of private consumption to total consumption \((i = 1, 2)\). The associated Jacobian matrix evaluated at the steady states has trace \(T_i\) and determinant \(D_i\), where

\[
T_i = \frac{e_w + |e_R| - 1}{e_w - 1} - \frac{\gamma^i}{e_w - 1}, \quad (5.1)
\]
Fig. 1. Multiple steady state and government expenditure.

\[ D_i = \alpha_i^w \gamma \left( \frac{|\epsilon_R|}{\epsilon_w} - 1 \right) \]  \tag{5.2}

These equations imply that the expression of \((T, D)\) can be obtained if the elasticity of the offer curve \(\gamma\) obtained for the case of no government expenditure \(g = 0\), as in G98 is replaced by a modified “apparent” elasticity, \(\alpha^w \gamma\). In other words, the solutions of the corresponding characteristic polynomial \(Q(z) \equiv z^2 - Tz + D = 0\) are obtained if we replace \(\gamma\) in the expressions of \(T\) and \(D\) for the case of G98 by \(\alpha^w \gamma\). The implication of this simple fact is as follows. The aim of this paper is to investigate how government expenditure influences the local dynamics near the two steady states. Without actually performing any computations, we can look at how \((T_i, D_i)\) moves by using G98 as a benchmark. Noting that \(\alpha^w\) is determined independently of the steady states, (5.1) and (5.2) mean the following:

**Lemma.** For any values of \(l^*\), the point \((T, D)\) is always located on the line \(\Delta\),

\[ D = -\Theta(a^*)[T - \Pi(a^*)], \]

where \(\Pi(a^*) \equiv \frac{\epsilon_w + |\epsilon_R| - 1}{\epsilon_w - 1}\) and \(\Theta(a^*) \equiv |\epsilon_R| - 1\).

Suppose \(g \to 0\). Then, it can be easily verified that \(D_1 \to 0\) and \(T_1 \to \Pi\), while

\[ D_2 \to \gamma \cdot \left( \frac{|\epsilon_R| - 1}{\epsilon_w - 1} \right) \quad \text{and} \quad T_2 \to \Pi + \gamma \cdot \left( \frac{1}{\epsilon_w - 1} \right). \]

**Proof.** If we combine (5.1) with (5.2) to remove \(\alpha_i^w \gamma\), the line \(\Delta\) is easily obtained. Direct inspection of Fig. 1 and (5) shows that as \(g \to 0\), the low steady-state value of labor supply approaches zero, that is, \(l^*_1 \to 0\), while in the high steady state, the income from labor is
equal to the level of private consumption, that is, \( w(a^*)l^*_2 \rightarrow (l^*_2)^\gamma \). Noting the relationships, we can derive the latter. □

The line \( \Delta \) in this paper is the same as the half line \( \Delta \) extending up to the horizontal axis for \( \gamma = 0 \) in G98. Thus, the location of \((T, D)\) can be easily explained graphically by comparing the two frameworks. From (5), government expenditure influences the local dynamics near the two steady states through its influence on the steady-state shares of private consumption to total consumption \( x^*_i \). When government expenditure \( g \) increases from 0 to \( g_2 \), the share \( x^*_1 \) goes from 0 to \( \tilde{x} = \hat{c}/(\hat{c} + g) \) for the low steady state, while the share \( x^*_2 \) goes down from 1 to the same value \( \tilde{x} \) for the high steady state. We get \( \tilde{x} \gamma = 1 \) as a single consequence of the fact that the slope of \((l^*)\gamma\), that is, \( \gamma(l^*)\gamma-1[= \gamma x^* w(a^*)] \) becomes equal to \( w(a^*) \), when \( g = g_2 \) in Fig. 1. In G98 without government expenditure, the parameter made to vary is \( \gamma \) from 1 to +∞ (the elasticity of the labor supply \( 1/(\gamma - 1) \) moves down from +∞ to 0): the point \((T, D)\) then describes a half line \( \Delta \) as shown in the figure. In this paper, the elasticity of the offer curve \( \gamma \) is set constant and the bifurcation parameter is the share \( x^*_i \) that is made to vary by moving \( g \) up from 0 to \( g_2 \). So in both cases, the parameter that is made to vary is eventually “formally” the same: \( \gamma \) in that case and a modified “apparent” elasticity \( x^*_i \gamma \) in this case. Suppose that \( g \) increases from 0 to \( g_2 \). For the high steady state, the “apparent” elasticity \( x^*_2 \gamma \) moves down from \( \gamma \) to 1 and thereby the corresponding point \((T, D)\) moves on the line \( \Delta \), while for the low steady state, \( x^*_1 \gamma \) moves up from 0 to 1 and thus the point \((T, D)\) moves on the line \( \Delta \). In both cases, the points \((T, D)\) belong to the same extended line \( \Delta \) for a fixed technology. Let us define the part of \( \Delta \) on which the point \((T_i, D_i)\) moves as \( \Delta_i \), when \( g \) changes.

4. Local bifurcation and stability

To analyze the dynamic effects of government expenditure, we investigate whether the segment \( \Delta_1 \) for the low steady state and the half-line \( \Delta_2 \) for the high steady state cross the stability triangle ABC in the diagram, segment [BC] in its interior for a Hopf-bifurcation or the line (AB) for a Flip-bifurcation. As the position of \( \Delta \) depends on \( \sigma \), but not on \( g \), we must explore how \( \Delta_i \) locates on the plane, when \( \sigma \) increases from 0 to +∞. From G98, we can see that in the case of no government expenditure, there co-exists the non-trivial steady state and the trivial steady state. G98 examines how the stability and bifurcation for the non-trivial steady state are affected when \( \gamma \) increases from 1 to +∞. Thus, the position of the half-line \( \Delta_2 \) as a function of the technology is directly available from G98.

Here, let us give the reader concise information on the saddle-node bifurcation that takes place when \( g \) goes up through \( g_2 \). There the two steady states coalesce together with one local eigenvalue going to 1 from below for one steady state and one local eigenvalue going to 1 from above for the other steady state, and both steady states disappear. Put differently, the points \((T, D)\) corresponding to the two steady states move on \( \Delta_1 \), and \( \Delta_2 \) towards the line (AC) and disappear when \( g \) moves up through \( g_2 \).

As mentioned in the introduction, the dynamic effects of government expenditure are extremely simple in the 1D version of the model without productive capital. By using the same notations \( \gamma_F, \gamma_H, \sigma_H, \sigma_F \) and \( \sigma_1 \) as in G98, we explore how the consequences for the
stability and bifurcation of each steady state are modified when the two-dimensional version of the model with productive capital is utilized. 4, 5

For the high steady state, when \( g \) goes up from 0 to \( g_2 \), the apparent elasticity of the offer curve \( x_2^*\gamma \) moves down from \( \gamma \) (assumed to be constant and greater than \( \gamma_H \) and \( \gamma_F \)) to 1. For a fixed technology, the corresponding point \((T, D)\) moves on the half-line \( \Delta_2(\sigma) \) toward the line (AB), so we get:

- if \( 0 < \sigma < \sigma_F \), the high steady state is a source for \( \gamma > x_2^*\gamma > \gamma_H \), undergoes a Hopf-bifurcation at \( x_2^*\gamma = \gamma_H \), and is a sink (locally indeterminate) for \( \gamma_H > x_2^*\gamma > 1 \) (see Fig. 2 in G98 and Fig. 2 in this paper).
- if \( \sigma_F < \sigma < \sigma_H \), the high steady state is a saddle (locally determinate) for \( \gamma > x_2^*\gamma > \gamma_F \), a source for \( \gamma_H > x_2^*\gamma > \gamma_F \), and is a sink (locally indeterminate) for \( \gamma_H > x_2^*\gamma > 1 \). A Flip-bifurcation occurs at \( x_2^*\gamma = \gamma_F \) and a Hopf-bifurcation arises at \( x_2^*\gamma = \gamma_H \) (see Fig. 3 in G98 and Fig. 3 in this paper).
- if \( \sigma_H < \sigma < \sigma_1 \), \( \sigma \neq \delta(1 - s) \), the high steady state is always a saddle (locally determinate), when \( \gamma > x_2^*\gamma > 1 \). (see Figs. 2–5 in this paper).

For the low steady state, when \( g \) increases from 0 to \( g_2 \) with fixed technology and \( \gamma \) constant, the apparent elasticity of the offer curve \( x_1^*\gamma \) increases from 0 to 1. The corresponding point \((T, D)\) moves on the segment \( \Delta_1(\sigma) \) toward the line (AB), so we obtain:

- if \( 0 < \sigma < \sigma_1 \), the low steady state is always a saddle (locally determinate) for \( 0 < x_1^*\gamma < 1 \). (see Figs. 2–5 in this paper).

---

4 Noting that \( s \) denotes the share of capital in total output, let us follow the assumption made throughout in G98 that \( \delta(1 - s)/s < 1 \), where \( s = a^\ast \rho(a^\ast)/f(a^\ast) \).

5 We can confirm \( \ell_w = s/\sigma \) and \( |\epsilon_R| = \delta(1 - s)/\sigma \), where \( \sigma \) is the elasticity of capital–labor substitution that can be considered as a function of \( a^\ast \).
if $\sigma_1 < \sigma < \sigma_H'$, the low steady state is a saddle (locally determinate) for $0 < x_1^* \gamma < \gamma_F$, undergoes a Flip-bifurcation at $x_1^* \gamma = \gamma_F$, and is a source for $\gamma_F < x_1^* \gamma < 1$ (see Figs. 6 and 7 in this paper).

---

6 The condition that the line $\Delta$ goes through $B$ generates a second degree equation with two solutions $\sigma_H$, $\sigma_H'$ and we can verify $\sigma_H < s < \sigma_H'$. 
Fig. 6. $s_1 < s < s$.

Fig. 7. $s < s < s'_H$.

- if $s'_H < s < s'_F \equiv s + \delta (1 - s)/2$ the low steady state is a saddle (locally determinate), when $0 < z_1 \gamma < \gamma_F$, a sink (locally indeterminate) when $\gamma_F < z_1 \gamma < \gamma_H$, and is a source when $\gamma_H < z_1 \gamma < 1$. A Flip-bifurcation occurs at $z_1 \gamma = \gamma_F$ and a Hopf-bifurcation emerges at $z_1 \gamma = \gamma_H$ (see Fig. 8 in this paper).
- if $s > s'_F$, the low steady state is a sink (locally indeterminate), when $0 < z_1 \gamma < \gamma_H$, displays a Hopf-bifurcation at $z_1 \gamma = \gamma_H$, and is a source when $\gamma_H < z_1 \gamma < 1$. (see Fig. 9 in this paper).

Unlike the high steady state, we can obtain these consequences without any restrictions on the value of the wage-elasticity of labor supply $1/(\gamma - 1)$.

When $g$ varies in $(0, g_2)$, therefore, only one steady state among the two is locally indeterminate; hence, with stationary sunspot equilibria nearby for some strict sub-interval, or generates local endogenous cycles through Flip and/or Hopf bifurcations for some specific values of government expenditure. It is the high steady state for very small

---

7 If $s$ exists within this range, the line $\Delta$ crosses (AB) between B and the point $B_0 = (-1, 0)$.
elasticities of capital–labor substitution $\sigma < \sigma_1$ (exactly as in G98) and it is the low steady state for $\sigma > \sigma_1$. On the other hand, the high (resp. the low) steady state is determinate for all feasible government expenditure $g$ in $(0, g_2)$ for large elasticities $\sigma > \sigma_1$ (resp. for small elasticities $\sigma < \sigma_1$). For elasticities of the capital–labor substitution larger than $\sigma_1$, endogenous stochastic or deterministic fluctuations emerge for some sub-range of values of government expenditure near the low steady state. Therefore, in this respect, for the most relevant range of $\sigma > \sigma_1$, the 2D case does resemble the 1D configuration with the exception of the possibility of Flip and Hopf bifurcations that could not occur in the case of the latter.

The outlook appears somewhat different if one considers the range of values of government expenditure below, but near, the saddle node bifurcation $g_2$ where the two steady states close and coalesce together. Then, the high steady state is indeterminate, while the low steady state is a saddle for very small elasticities of substitution $\sigma < \sigma_1$. For the range of $\sigma > \sigma_1$, the high steady state is always a saddle, but the low steady state is a source.

5. Conclusion

This paper examines how government expenditure affects the local dynamics near the multiple steady states in Woodford’s finance constrained model [13] depending upon the
elasticity of capital–labor substitution. We compare the consequences to those obtained in the existing literature, particularly Farmer [5] and Farmer and Woodford [6], in which the 1D version of that model without productive capital is used.

The similarity is as follows. If the elasticity of capital–labor substitution is high, where \( \sigma > \sigma_1 \), the high steady state is always determinate and the low steady state is indeterminate for this sub-interval of government expenditure. Unlike the 1D version, however, the low steady state displays local endogenous cycles through Flip and/or Hopf bifurcations for some specific values of government consumption. The high elasticity of labor supply is not required for the emergence of such fluctuations.

The essential difference emerges when government expenditure is below, but close to, the saddle-node bifurcation. The high steady state is indeterminate and the low steady state is a saddle for very small elasticities of substitution \( \sigma < \sigma_1 \), while the high steady state is a saddle and the low steady state is a source for the range of \( \sigma > \sigma_1 \).

Acknowledgments

The author is grateful to the associate editor for several insightful comments and valuable suggestions that were extremely useful in revising an earlier version of this paper. He also thanks an anonymous referee for constructive comments.

References