Income taxes and endogenous fluctuations: a generalization

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Abstract

This paper offers an integrated framework of one- and two-sector optimal growth models for the dynamic analysis of the joint effects of distorting taxes and production externalities. We investigate how capital and labor income tax rates influence local dynamics near a steady state depending on the sizes of externalities. Our results clarify how tax rates on factor income affect the range of the sizes of externalities that induce indeterminacy. We find that the possibility of indeterminacy is significantly higher for given values of externalities, the labor supply elasticity and the elasticity of intertemporal substitution in consumption if capital income tax rates are increased from zero to values similar to those in many countries.

JEL classification: C61, C62, H1

Key words: indeterminacy, capital income tax rate, externalities in the investment sector, integrated framework of one- and two-sector growth models

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1 Introduction

It is very important to clarify whether fixed income tax rates can work as an “automatic stabilizer”. This paper examines how fixed tax rates on capital and labor income influence the likelihood of agents' belief-driven aggregate fluctuations, using an integrated framework of one- and two-sector growth models. The integrated framework can allow the scrutiny of both a one-sector Ramsey model and a two-sector real business cycle model in a generalized setting.

As is well known, the steady state is locally indeterminate and thus endogenous fluctuations arise if external effects are introduced in one- and two-sector growth models. Based on an integrated framework of one- and two-sector growth models with production externalities as studied in Benhabib and Farmer (1994, 1996) (note: hereafter B-F.94, 96), this paper explores how labor and capital income tax rates influence local dynamics near a steady state depending on the sizes of externalities. The dynamics imply the following. In the two kinds of optimal growth models, the tax rate on labor income plays no role in the emergence of indeterminacy, but increases in the capital income tax rate significantly reduce the minimum values of externalities leading to indeterminacy and thus enhances the likelihood of indeterminacy.

In the one-sector Ramsey growth model investigated in B-F.94, the minimum sizes of labor externalities are determined independently of the tax rate on capital income; however, the existence of capital income taxes easily reduces to zero the capital externalities necessary for the appearance of indeterminacy. Therefore, the economy is more likely to be in an indeterminate steady state. In the two-sector real business cycle model considered in B-F.96, the range of externalities inducing indeterminacy significantly expands for given values of the labor supply elasticity and the elasticity of intertemporal substitution in consumption if capital income tax rates are increased from zero to values close to those in many countries. Thus, indeterminacy is more likely to emerge at empirically plausi-
ble values of sector-specific externalities, the labor supply elasticity and the elasticity of intertemporal substitution in consumption.

Much recent research has focused on dynamic general equilibrium models with externalities to illustrate that aggregate fluctuations can be endogenously explained without relying on persistently exogenous shocks. B-F.94 showed that an indeterminate steady state can easily arise in the one-sector Ramsey model if the degree of externalities is strong enough that the labor demand curve and the supply curve cross with wrong slopes. This condition is known as a necessary condition for the emergence of indeterminacy. However, this paper will show that it becomes the necessary and sufficient condition for the appearance of indeterminacy only if positive tax rates on capital income are considered.

Basu and Fernald (1995, 1997) and Harrison (1998) demonstrated that empirical estimates of externalities in production are extremely small. Thus, it is important to estimate the degree of externalities required for the emergence of indeterminacy and to explore whether or not the estimated values are compatible with empirical evidence. B-F.96 showed that the minimum values of externalities generating indeterminacy are less stringent in the two-sector real business cycle model than in the one-sector Ramsey model. Unlike the one-sector model, they proved that indeterminacy occurs in the two-sector real business cycle model, even when the labor demand and supply curves have standard slopes. However, we should note that the minimum size of externalities is empirically plausible, only when the elasticity of labor supply is too high to be realistically acceptable. This paper shows that indeterminacy arises at empirically plausible values of the labor supply elasticity and production externalities if capital income tax rates are set at values similar to those in many countries in the two-sector real business cycle model.

Using one- and two-sector growth models, many economists have extensively explored how tax policy can stabilize the economy against agents’ belief-driven aggregate fluctuations. Using the one-sector Ramsey growth model without externalities, for example, Schmitt-Grophe and Uribe (1997) showed that indeterminacy emerges if labor and/or
capital–income tax rates endogenously adjust to satisfy the government budget. In contrast, Guo and Harrison (2005) proved that indeterminacy cannot occur in the one-sector Ramsey model if lump-sum taxes and/or government expenditure endogenously adjust to balance the budget for fixed income tax rates. However, Guo and Harrison do not consider the external effects in production as done in this paper. We offer an integrated framework comprising a one-sector Ramsey model and a two-sector real business cycle model for the analysis of the joint effects of fixed income taxes and production externalities. Unlike the models without externalities, it is shown that fixed income tax rates might become a strong factor destabilizing the economy.

Based on a discrete-time one-sector growth model with production externalities, Guo and Lansing (2002) analyzed how fixed income tax rates affect the minimum size of externalities generating indeterminacy. Because of the analytical complexity arising in the discrete-time growth model, they depended too much on numerical simulations to characterize local stability near a steady state and labor and capital income taxes were not clearly separated. Most importantly, the present paper will show that in a continuous-time version of one-sector Ramsey model, their main implication of tax policy is less significant and is completely overturned if we clearly differentiate capital externalities from labor externalities.

Using a discrete-time two-sector real business cycle model with production externalities, Guo and Harrison (2001) investigated which tax policy that is progressive or regressive with respect to total income leads to indeterminacy for given sizes of externalities. Because of analytical complexity, however, the stability property is not theoretically derived and they did not separate capital and labor income taxes as done in this paper. In contrast, this paper theoretically specifies the range of sizes of externalities inducing indeterminacy for fixed tax rates consistent with data in many countries. Most important thing that we must emphasize is the following. In Guo and Harrison (2001), government expenditure is endogenously adjusted to balance the government budget and then the
level parameter of income tax schedule has no impact on local dynamics,\footnote{Guo and Lansing (2001) emphasized that this property holds in a continuous-time version of one-sector Ramsey model. Therefore, the way of adjusting the government budget is very important in examining local dynamics.} while in the present paper, lump-sum taxes are endogenously determined to satisfy the government budget and then the level parameter of labor income tax schedule unaffe\-cts the local dynamics, but the level parameter of capital income tax schedule significantly influences the emergence of indeterminacy. (It is convenient to add the tables that arrange the existing literature analyzing the relations between indeterminacy and fiscal policy. See Tables 1 and 2.)

Therefore, the main contribution of this paper is to offer an analytically tractable framework for the analysis of the joint effects of distorting taxes and production externalities on the likelihood of indeterminacy.

2 The structure of the model

2.1 Consumers

The economy is populated by an infinitely lived representative agent who maximizes the discounted sum of the utilities derived from consumption and leisure subject to the usual budget constraint. We can consider his/her dynamic optimization problem as follows:

Max

\[ \int_0^\infty \left[ \frac{1 - \phi}{1 - \phi} - \frac{\phi^{1+\gamma}}{1 + \gamma} \right] e^{-\rho t} dt \]  

subject to:

\( (1 - \tau_r) r_t k_t + (1 - \tau_w) w_t l_t - c_t + \tau_t = p_t I_t, \) \hspace{1cm} (2)

\( \dot{k}_t = I_t - \delta k_t, \) \hspace{1cm} (3)

where \( \rho \) is the rate of time preference, \( c_t \) is the level of consumption, \( l_t \) is the quantity of
labor supply, $w_t$ is the real wage rate, which is taxed at the fixed rate $\tau_w$, $k_t$ is the capital stock and depreciates at the rate $\delta$, $r_t$ is the rental rate of capital and is taxed at the fixed rate $\tau_r$, $\tau_t$ is the transfer payment, $I_t$ is the level of private investment, and $p_t$ is the price of investment goods in terms of consumption goods. To solve the above dynamic optimization problem, we set up the current-value Hamiltonian function:

$$H_t = \frac{c_t^{1-\phi}}{1-\phi} - \frac{p_t}{1+\gamma} + \lambda_t [(1 - \tau_r) r_t k_t + (1 - \tau_w) w_t l_t - c_t + \tau_t - p_t I_t] + \mu_t (I_t - \delta k_t).$$

The necessary conditions can be derived as:

$$c_t^{-\phi} = \mu_t / p_t, \quad (4)$$

$$(1 - \tau_w) w_t (\mu_t / p_t) = l_t^\gamma, \quad (5)$$

$$-(1 - \tau_r) \frac{r_t}{p_t} + \delta + \rho = \frac{\dot{\mu}_t}{\mu_t}, \quad (6)$$

where (4) and (5) describe the intratemporal leisure-consumption trade-off and (6) is the Euler equation for consumption. In addition, the following transversality condition must be satisfied:

$$\lim_{t \to \infty} e^{-\rho t} \mu_t \cdot k_t = 0.$$  

2.2 Firms and government

In the consumption goods sector, a typical firm has access to technology that exhibits constant returns-to-scale in privately chosen levels of capital and labor:

$$y_t^c = A_c (k_t^c)^\alpha (l_t^c)^{1-\alpha}, \quad \alpha \in (0, 1), \quad (7)$$
where:

\[ A_c \equiv (K_c^t)^{\alpha - \theta_k^c} (L_c^t)^{(1-\alpha)\theta_c^c} \cdot (K_t)^{\alpha \sigma^k} (L_t)^{(1-\alpha)\sigma^l}. \]

\( k_c^t \) and \( l_c^t \) denote the capital and labor devoted to the consumption goods sector at time \( t \). \( K_c^t \) and \( L_c^t \) denote the average levels of capital and labor allocated to the consumption sector, while \( K_t \) and \( L_t \) denote the total levels of capital and labor in the economy. \( \theta_c^k (\theta_c^l) \) denotes the degree of sector-specific externalities derived from capital (labor) used in this sector. \( \sigma^k (\sigma^l) \) represents the degree of aggregate externalities derived from the sum of capital (labor) used in the two sectors. The variables regarding the externalities are taken as parametric by an individual firm.

In the investment goods sector, the technology of the sector is specified as:

\[ y_I^t = A_I (k_I^t)^{\alpha} (l_I^t)^{1-\alpha}, \] (8)

where:

\[ A_I \equiv (K_I^t)^{\alpha \theta_k^I} (L_I^t)^{(1-\alpha)\theta_I^I} \cdot (K_t)^{\alpha \sigma^k} (L_t)^{(1-\alpha)\sigma^l}. \]

The variables in the investment goods sector can be conjectured from the notations defined in the consumption goods sector.

If we focus on the special case of \( \sigma^k = \sigma^l = 0 \) and \( \theta^i_c = \theta^i_I \) \((i = k, l)\), the production technology in this paper is identical to the one in the two-sector model as investigated in B-F.96 that is regarded as a pioneering paper of two sector real business cycle model analyzing indeterminacy. If the sector specific-externalities are set to zero, i.e. \( \theta^i_C = \theta^i_I = 0 \), the technology in the present paper collapses to the one in the one-sector Ramsey model as studied in B-F.94 that is considered as a pioneering paper of one-sector growth model analyzing indeterminacy. The production technologies (7) and (8) are the same as in Harrison (2001) and Guo and Harrison (2001), when \( \sigma^k = \sigma^l = 0 \) and \( \theta^i_C \neq \theta^i_I \) but \( \theta^j_k = \theta^j_l \) \((j = I, C)\). Thus, it is considered that the production functions (7) and (8) are
the most general in the existing works.

To maximize the profits, capital and labor are hired to satisfy:

\[
\frac{\alpha y^C_t}{k^C_t} = p_t \frac{\alpha y^I_t}{k^I_t} = r_t \quad \text{and} \quad \frac{(1 - \alpha) y^C_t}{l^C_t} = p_t \frac{(1 - \alpha) y^I_t}{l^I_t} = w_t.
\]

(9)

The resource constraints are:

\[
k^C_t + k^I_t = k_t \quad \text{and} \quad l^C_t + l^I_t = l_t.
\]

Finally, we describe the behavior of the government. Following Guo and Lansing (2002), we consider the case that the government distributes the revenue from taxes on labor and capital income to the economic agents as transfers. Then, we can specify the government budget constraint as:

\[
t_{r} r_t k_t + t_{w} w_t l_t = \tau_t.
\]

(10)

Unlike Guo and Lansing (1998), Guo (1999) and Guo and Harrison (2001), the lump-sum transfers \(\tau_t\) endogenously adjust to satisfy the government budget (10) for fixed tax rates on labor and capital income \(\tau_w\) and \(\tau_r\).

3 Equilibrium and local dynamics

In symmetric equilibrium \(k^i_t = K^i_t\), \(l^i_t = L^i_t\) \((i = I, C)\), \(k_t = K_t\) and \(l_t = L_t\), the production functions (7) and (8) can be simplified to:

\[
y^C_t = k_t^{\alpha (\theta^k + \theta^k + 1)} l_t^{(1 - \alpha)(\theta^I + \sigma^I + 1)} (1 - \kappa_t)^{1 + \alpha \theta^k + (1 - \alpha) \theta^I},
\]

(11)

\(^2\)Guo and Lansing (1998), Guo (1999) and Guo and Harrison (2001) consider that government expenditure endogenously adjusts to satisfy the budget. Thus, even if the progressivity of income tax rates is set to zero, their papers cannot be analytically the same as the present paper.
\[
y_t^I = k_t^{\alpha \left( \theta_t^k + \sigma^k + 1 \right)} l_t^{(1-\alpha) \left( \theta_t^k + \sigma^k + 1 \right)} \kappa_t^{1+\alpha \theta_t^k + (1-\alpha) \theta_t^l}
\]

where \( k_t^I = \frac{\nu_t}{\lambda_t} \equiv \kappa_t \).

From Basu and Fernald (1997) and Harrison (1998), the degree of capital externalities is moderately low. It is appropriate to assume the following:

**Assumption 1:** \( \alpha \left( \theta_t^k + \sigma^k + 1 \right) < 1, \ i = I, C. \)

Assumption 1 implies that the level of capital externalities is not large enough to induce endogenous growth. This assumption is imposed by almost all the studies that introduce externalities in a two-sector real business cycle model.

In symmetric equilibrium, the rental rate of capital and the real wage rate are determined to satisfy:

\[
w_t = (1-\alpha) \frac{p_t k_t^{\alpha \left( \theta_t^k + \sigma^k + 1 \right)} l_t^{(1-\alpha) \left( \theta_t^k + \sigma^k + 1 \right)} \kappa_t^{\alpha \theta_t^k + (1-\alpha) \theta_t^l}}{\kappa_t^{\alpha \theta_t^k + (1-\alpha) \theta_t^l}},
\]

\[
r_t = \alpha p_t k_t^{\alpha \left( \theta_t^k + \sigma^k + 1 \right)} l_t^{(1-\alpha) \left( \theta_t^k + \sigma^k + 1 \right)} \kappa_t^{\alpha \theta_t^k + (1-\alpha) \theta_t^l} = \alpha k_t^{\alpha \left( \theta_t^k + \sigma^k + 1 \right)} l_t^{(1-\alpha) \left( \theta_t^k + \sigma^k + 1 \right)} \kappa_t^{\alpha \theta_t^k + (1-\alpha) \theta_t^l}.
\]

Using (13) or (14), the relative price of the investment goods \( p_t \) is expressed as:

\[
p_t = k_t^{\alpha \left( \theta_t^k - \theta_t^l \right)} l_t^{(1-\alpha) \left( \theta_t^k - \theta_t^l \right)} \kappa_t^{\alpha \left( \theta_t^k + (1-\alpha) \theta_t^l \right)} (1-\kappa_t)^{-\left[ \alpha \theta_t^k + (1-\alpha) \theta_t^l \right]}.
\]

The economy is in steady-state equilibrium when an equilibrium path is constant over time. Note that * denotes the steady-state value. The uniqueness of the steady state can be verified. The steady-state value of \( \kappa_t \) can be derived as:

\[
\kappa^* = (1-\tau_r) \frac{\delta}{\delta + \rho} < 1.
\]

\[3\]It is well known that the nonarbitrage conditions (9) imply this relationship.
The local dynamics near the steady state \((\mu^*, k^*)\) can be approximated by:

\[
\begin{bmatrix}
\dot{\mu}_t \\
\dot{k}_t
\end{bmatrix} = \begin{bmatrix}
(\delta + \rho) \left( \frac{J_2}{1} \right) & -\left( \frac{\mu^*}{k^*} \right) (\delta + \rho) \left[ (\alpha J_2 - 1) + \alpha J_1 \left( \frac{J_2}{1} \right) \right] \\
-\left( \frac{\mu^*}{k^*} \right) \delta \left( J_1 + \frac{1 - \kappa^*}{\kappa^*} \right) \left( \frac{1}{\Gamma} \right) & \delta \alpha \left[ (J_1 + \frac{1 - \kappa^*}{\kappa^*}) \left( \frac{J_2}{1} \right) + J_2 - 1 \right]
\end{bmatrix} \begin{bmatrix}
\mu_t - \mu^* \\
k_t - k^*
\end{bmatrix}
\]

where

\[
\Gamma \equiv [\alpha \theta^k_I + (1 - \alpha) \theta^l_I] \left( \frac{1 - \kappa^*}{\kappa^*} \right) - [(\phi - 1) [\alpha \theta^k_c + (1 - \alpha) \theta^l_c] + \phi] \\
+ (1 - \alpha) [\theta^l_I + \sigma^I + 1 + (\phi - 1) (\theta^l_c + \sigma^I + 1)] \frac{(\phi - 1) \left[ \alpha \theta^k_c + (1 - \alpha) \theta^l_c \right] + \phi}{1 + \gamma + (\phi - 1) (1 - \alpha) (\theta^l_c + \sigma^I + 1)}
\]

\[
J_1 \equiv [\alpha \theta^k_I + (1 - \alpha) \theta^l_I] \left( \frac{1 - \kappa^*}{\kappa^*} \right) + (1 - \alpha) (\theta^l_I + \sigma^I + 1) \frac{(\phi - 1) \left( \theta^k_c + \sigma^k + 1 \right)}{1 + \gamma + (\phi - 1) (1 - \alpha) (\theta^l_c + \sigma^I + 1)} - (\theta^l_I + \sigma^k + 1)
\]

\[
J_2 \equiv (1 - \alpha) (\theta^l_I + \sigma^I + 1) \frac{(\phi - 1) \left( \theta^k_c + \sigma^k + 1 \right)}{1 + \gamma + (\phi - 1) (1 - \alpha) (\theta^l_c + \sigma^I + 1)} - [\theta^l_I + \sigma^k + 1 + (\phi - 1) (\theta^l_c + \sigma^k + 1)]
\]

\[
J_3 \equiv (1 - \alpha) [\theta^l_I + \sigma^I + 1 + (\phi - 1) (\theta^l_c + \sigma^I + 1)] \frac{(\phi - 1) \left( \theta^k_c + \sigma^k + 1 \right)}{1 + \gamma + (\phi - 1) (1 - \alpha) (\theta^l_c + \sigma^I + 1)} - \left( \theta^l_I + \sigma^k + 1 + (\phi - 1) (\theta^l_c + \sigma^k + 1) \right)
\]

The Jacobian matrix in (17) has a trace \(T\) and a determinant \(D\) given by:

\[
T = \rho + (\phi - 1) \alpha \delta J_2 + \alpha \delta J_3 \left[ \frac{1 - \kappa^*}{\kappa^*} + (1 + \gamma) \frac{(\phi - 1) \left[ \alpha \theta^k_c + (1 - \alpha) \theta^l_c \right] + \phi}{1 + \gamma + (\phi - 1) (1 - \alpha) (\theta^l_c + \sigma^I + 1)} \right] \cdot \frac{1}{\Gamma} \\
+ (\delta + \rho) (1 + \gamma) \frac{(\phi - 1) \left[ \alpha \theta^k_c + (1 - \alpha) \theta^l_c \right] + \phi}{1 + \gamma + (\phi - 1) (1 - \alpha) (\theta^l_c + \sigma^I + 1)} \cdot \frac{1}{\Gamma};
\]

\[
D = (\delta + \rho) \delta (1 - \alpha J_2) \left( \frac{1 - \kappa^*}{\kappa^*} \right) \cdot \frac{1}{\Gamma}.
\]

As seen in Harrison (2001) analyzing the discrete time version of two-sector model, (17)–(19) imply that local dynamics are unaffected by the sector-specific externalities in
the consumption goods sector \( \theta^i_c \) (\( i = k, l \)), if the utility function is logarithmic in consumption, i.e., \( \phi \to 1 \). It is well understood that this consequence arises if the logarithmic utility in consumption is utilized. In addition, from (17), the following can be immediately derived.

**Proposition 1** In an integrated framework of the one- and the two-sector models, local dynamics are independent of the tax rate on labor income \( \tau_w \). Thus, the existence of labor income taxation does not influence the minimum value of externalities required to be an indeterminate steady state and thus the possibility of indeterminacy.

This consequence is very intuitive and Guo and Lansing (2002) had the same outcome in the discrete-time version of one-sector Ramsey model that \( \phi \to 1 \) is assumed. They showed that a constant tax rate on labor income has no impact on the degree of externalities needed for the appearance of indeterminacy. Thus, Proposition 1 clarifies that this analytical result obtained in Guo and Lansing (2002) also hold in this integrated model of one- and two-sectors that the cases of \( \phi \neq 1 \) are considered. As mentioned in their paper, (4) and (5) imply that the tax rate on labor income affects agents’ intratemporal decisions between labor and consumption at a given date. However, the intertemporal decisions at different dates are very important for agents’ beliefs about future returns on capital to become self-fulfilling.

## 4 One-sector Ramsey optimal growth model: \( \theta_j = 0 \)

\[
(j = I, C)
\]

If the sector-specific externalities are set at \( \theta_j = 0 \) \( (j = I, C) \), (15) shows that the relative price of investment goods becomes one, i.e. \( p_t = 1 \), and the two-sector growth model reduces to the one-sector Ramsey model. Moreover, when we set the elasticity of intertemporal substitution in consumption to be one, i.e., \( \phi \to 1 \), the framework in the
The present paper becomes identical to the one-sector model as studied in B-F.94. As stated in Section 2-2, B-F.94 is the pioneering paper clarifying the condition that indeterminacy emerges in the one-sector Ramsey model. It is important to discover which the existence of capital income taxes tighten or loose the condition for the occurrence of indeterminacy in the one-sector model as studied in B-F.94. Thus, let us assume the following:

**Assumption 2**: \( \theta_j = 0 \) for \( j = I, C \) and \( \phi \to 1 \).

Then, (18) and (19) can be rewritten as:

\[
T = \rho + (\delta + \rho) \left[ 1 - \frac{\sigma^k + 1}{1 - \tau_r} \right] \Gamma^{-1},
\]

(20)

\[
D = (\delta + \rho) \delta \left[ 1 - \alpha (\sigma^k + 1) \right] \left[ \frac{\delta + \rho}{(1 - \tau_r) \alpha \delta} - 1 \right] \Gamma^{-1},
\]

(21)

where \( \Gamma \equiv -1 + \frac{(1-\alpha)(\sigma^l+1)}{1+\gamma} \).

Under Assumption 2, this section investigates how the existence of capital income taxes influences the range of sizes of externalities generating indeterminacy in the one-sector model as investigated in B-F.94. Section 8-1 thinks of the more general constant elasticity of intertemporal substitution in consumption, i.e., \( \phi \neq 1 \) and examines the robustness of the analytical results in this section.

Based on the discrete-time version of the one-sector Ramsey model, Guo and Lansing (2002) examined the relationship between income tax rates and the possibility of indeterminacy. They concluded that the tax rate on labor income has no impact on the appearance of indeterminacy, while increasing the tax rate on capital income raises the minimum values of externalities generating indeterminacy and thus stabilizes the economy against endogenous fluctuations because of agents’ self-fulfilling beliefs. Section 3 proves that the outcome for the labor income tax rate holds in this continuous-time version of the one-sector Ramsey optimal growth model, even if we consider the more general constant
elasticity of intertemporal substitution in consumption, i.e., \( \phi \neq 1 \).

First, we show that the negative relation between the capital income tax rate and the possibility of indeterminacy is not present in this continuous-time version. Guo and Lansing do not differentiate labor and capital in aggregate externalities; more specifically, they restrict attention to the very particular case of \( \sigma^j = \sigma \ (j = k, l) \). Following Guo and Lansing (2002), we also assume the following:

**Assumption 3:** \( \sigma^i = \sigma \ (i = k, l) \).

The determinant (21) is always positive for \( \sigma > \frac{\gamma + \alpha}{1 - \alpha} \), which is the necessary condition for the emergence of indeterminacy as emphasized in B-F.94. We can then obtain the following:

**Proposition 2** Unlike Guo and Lansing (2002), the steady state is always a sink (locally indeterminate) for any tax rates on capital income if \( \sigma^i = \sigma > \frac{\gamma + \alpha}{1 - \alpha} \ (i = k, l) \).

**Proof.** See Appendix A for the analytical proof.

In the discrete-time one-sector growth model, indeterminacy might not arise even if \( \sigma^i = \sigma > \frac{\gamma + \alpha}{1 - \alpha} \). However, only if \( \sigma^i = \sigma > \frac{\gamma + \alpha}{1 - \alpha} \), indeterminacy always emerges for any tax rates on capital income and thus belief-driven aggregate fluctuations are more likely to arise in the continuous-time one-sector growth model.

Though Appendix B clarifies the analytical reason why indeterminacy can emerge at lower sizes of externalities in the continuous-time model than in the discrete-time one, let us discuss the intuition behind which the likelihood of indeterminacy is higher in the continuous-time growth model. For indeterminacy to occur, the marginal product of capital \( r_{t+1} \ [r_t] \) must increase with the capital stock \( k_{t+1} \ [k_t] \) in the discrete-time [continuous-time] case. In the discrete-time case, one considers the rises in \( k_{t+1} \) caused by

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4It is well known that this condition is equivalent to one in which the externalities are so strong that the labor supply and demand curves cross with wrong slopes. To see this, compare the labor supply curve obtained by combining (4) and (5) with the labor demand curve obtained from (13) and (15).
increases in \( k_t \) of one unit. Compare (B-1) with (B-2) in Appendix B. As the good market equilibrium condition is \( k_{t+1} = y_t + (1 - \delta) k_t - c_t \) in the discrete-time model, we can obtain \( \frac{\partial k_{t+1}}{\partial k_t} = r^* + (1 - \delta) \) near the steady state. The next period capital stock \( k_{t+1} \) rises by \( 1 + \rho + \tau r^* (> 1) \) units if \( k_t \) increases by one unit.\(^5\) The capital stock \( k_{t+1} \) increases by more than one unit. From the diminishing law of marginal product of capital, the decreases in the marginal product of capital are larger in the discrete-time model than in the continuous-time one. Thus, larger externalities are necessary in the discrete-time model than in the continuous-time model for an indeterminate steady state.

In place of assumption 3, we consider the more general case as in B-F.94:

**Assumption 3’**: \( \sigma^k \neq \sigma^l \).

In this case, we investigate the dynamic consequences of the capital income tax rate. For the determinant to be positive, the lower bound must be imposed on labor externalities:

\[
\sigma^l > \sigma^l_{\min} \equiv \frac{\gamma + \alpha}{1 - \alpha}, \tag{22}
\]

which is the necessary condition for an indeterminate steady state. From (22), the minimum values of labor externalities inducing indeterminacy are independently determined by the tax rate on capital income. We analyze how rises in the capital income tax rate influence the minimum sizes of capital externalities for given values of labor externalities compatible with (22). Using (20), we can obtain the relationship:

\[
T \gtrless \begin{cases} \ 0 \\
\Leftrightarrow \ 
\sigma^k \gtrless (1 - \tau_r) \left[ \frac{\rho}{\delta + \rho} \right] \left[ -1 + \frac{(1 - \alpha) (1 + \sigma^l)}{1 + \gamma} \right] + 1 \right] - 1 \\
\equiv \ \sigma^k_{\min} (\tau_r) .
\end{cases} \tag{23}
\]

\(^5\)As the Euler equation is \( \frac{c_{t+1}}{c_t} = \frac{1}{1 + \rho} [(1 - \tau_r) r_{t+1} + 1 - \delta] \) in the discrete-time growth model, \( r_{t+1} + 1 - \delta \) is equal to \( 1 + \rho + \tau r^* (> 1) \) in the steady state.
From (23), Figure 1 can be illustrated and we can obtain the following lemma:

**Lemma 1** If \( \sigma^l > \sigma^l_{\text{min}} \), increases in the capital income tax rate stabilize the economy by transforming the steady state from source to sink for \( 0 < \sigma^k < \sigma^k_{\min}(0) \). In contrast, local stability is not qualitatively affected by changes in the capital income tax rate if \( \sigma^l > \sigma^l_{\min} \) and \( \sigma^k > \sigma^k_{\min}(0) \) or \( \sigma^l < \sigma^l_{\min} \).

**Proof.** See Figure 1 and the above arguments. □

Therefore, (22) is the necessary and sufficient condition for the appearance of indeterminacy if \( \sigma^k = \sigma^l \) or \( \tau_r > \hat{\tau}_r \equiv \frac{\sigma^k_{\min}(0)}{1 + \sigma^k_{\min}(0)} \). From the comparative dynamic analysis in lemma 1, we can state:

**Proposition 3** If \( \sigma^l > \sigma^l_{\min} \) and \( \tau_r < \hat{\tau}_r \), increases in the capital income tax rate reduce the minimum sizes of capital externalities leading to indeterminacy for given values of labor externalities \( \sigma^l \in (\sigma^l_{\text{min}}, -) \).

**Proof.** See Figure 1. □

Indeterminacy can arise for the range of values of increasing returns greater than \( 1 + \alpha \sigma^k_{\text{min}}(\tau_r) + (1 - \alpha) \sigma^l \). If \( \tau_r < \hat{\tau}_r \), increases in the capital income tax rate reduce the minimum sizes of increasing returns, and agents’ belief-driven aggregate fluctuations are more likely. As we explore whether or not stochastic sunspot equilibria exist near the steady state, Proposition 3 corresponds to the condition for the emergence of local indeterminacy.

Let us state the following:

**Lemma 2** As \( \sigma^l \to \sigma^l_{\text{min}} \) from the above, \( \sigma^k_{\text{min}}(0) \to 0 \) and thus \( \hat{\tau}_r \to 0 \). Then, (22) is also the sufficient condition for the appearance of indeterminacy only if the capital income tax rate is slightly positive.
Proof. From (23), $\sigma_{\min}^k(0) = \left(\frac{\rho}{\delta + \rho}\right) \left[-1 + \frac{(1-\alpha)(1+\sigma^l)}{1+\gamma}\right]$ is obtained. If we substitute $\sigma^l \rightarrow \sigma_{\min}^l$ into this equation, $\sigma_{\min}^k(0) \rightarrow 0$ is obtained. Therefore, we can verify $\tau_r \rightarrow 0$, because $\tau_r = \frac{\sigma_{\min}^k(0)}{1+\sigma_{\min}^k(0)}$. □

Figure 1 shows that the steady state displays a Hopf-bifurcation at $\sigma_{\min}^k(\tau_r)$ for $\tau_r < \hat{\tau}_r$. In supercritical (subcritical) Hopf-bifurcation, an attracting (a repelling) orbit emerges for capital externalities that are slightly lower (higher) than $\sigma_{\min}^k(\tau_r)$, that is, in the side of unstable (stable) steady state. When Hopf-bifurcation is supercritical, there exists a continuum of perfect-foresight trajectories converging to the invariant circle around the steady state and then we can construct stochastic sunspot equilibria outside the attracting circle that remain away from the steady state. The existence of supercritical Hopf-bifurcation corresponds to the condition for the emergence of global indeterminacy. However, flip bifurcation does not arise in the continuous-time version of growth models as in the present paper. Unlike the discrete-time version of growth model as in Guo and Lansing (2002), global indeterminacy does not arise, when the steady state is locally determinate. Therefore, the public policies designed to generate local determinacy are equivalent to the stabilizing ones.

5 Two-sector real business cycle model; $\sigma^i = 0 \ (i = k, l)$

As long as $\theta_I \neq 0$, this framework can be regarded as a two-sector real business cycle model, even if the aggregate externalities $\sigma^i$ are set to zero. Moreover, Suppose that the elasticity of intertemporal substitution in consumption is set to one, i.e., $\phi \rightarrow 1$, and we do not differentiate the sector-specific externalities from capital and labor, i.e., $\theta^k_j = \theta^l_j \ (j = C, I)$. Then, the framework in the present paper collapses to the two-sector real business cycle model as studied in B-F.96. As stated in Section 2-2, B-F.96 is the pioneering paper clarifying the minimum sizes of externalities leading to indeterminacy in
a two-sector growth model. It is important to investigate how fixed tax rates on capital income influence the likelihood of indeterminacy in the two-sector model as in B-F.96. Thus, we assume the following in place of assumption 2:

**Assumption 2’**: \(\sigma^i = 0 \ (i = k, l), \ \theta^k_j = \theta^l_j \ (j = C, I)\) and \(\phi \to 1\).

To examine how robust the analytical result in this section is, Section 8-2 picks up the more general constant elasticity of intertemporal substitution in consumption, i.e., \(\phi \neq 1\) or the different sizes of sector-specific externalities \(\theta^k_j \neq \theta^l_j \ (j = C, I)\). Considering Assumption 2’, (18) and (19) are rewritten as

\[
T = \rho + (\delta + \rho) \left[1 - \frac{\theta_I + 1}{1 - \tau_r}\right] \Gamma^{-1},
\]

\[
D = (\delta + \rho) \delta \left[1 - \alpha (\theta_I + 1)\right] \left[\frac{\delta + \rho}{(1 - \tau_r) \alpha \delta} - 1\right] \Gamma^{-1}.
\]

where \(\Gamma = -\theta_I \left[1 - \frac{\delta + \rho}{(1 - \tau_r) \alpha \delta}\right] - 1 + \frac{(1 - \alpha)(\theta_I + 1)}{1 + \gamma}\).

First, we provide an analytical characterization of the local dynamics near the steady state in the plane \((\tau_r, \theta_I)\). Put another way, we analyze how the tax rate on capital income influences the local dynamics for a given size of the sector-specific externality \(\theta_I\). To do so, we must obtain diagrams where \(\Gamma = 0\) and \(T = 0\) are satisfied in the plane \((\tau_r, \theta_I)\) by using (24) and (25).

The equation \(\theta_I = \theta_{\max} (\tau_r)\) associated with \(T = 0\) can be expressed as:

\[
\theta_I = \theta_{\max} (\tau_r) \equiv \frac{\Delta}{1 - \frac{1 - \alpha}{1 + \gamma}} \cdot \left[\frac{\delta - 1}{\alpha \delta - (\rho + \delta)}\right] - \frac{1}{\Delta} \left[\frac{\delta - 1}{\alpha \delta - (\rho + \delta)}\right] + \frac{\Delta}{1 - \frac{1 - \alpha}{1 + \gamma}},
\]

where \(\Delta \equiv \left[1 - \frac{1 - \alpha}{1 + \gamma}\right] + \left(\frac{\rho}{\rho + \delta}\right) > 0\).
The equation \( \theta_I = \theta_{\min}(\tau_r) \) associated with \( \Gamma = 0 \) is:

\[
\theta_I = \theta_{\min}(\tau_r) \equiv \frac{1 - \frac{1-\alpha}{1+\gamma}}{-1 + \frac{\rho+\delta}{1-\tau_r\alpha\delta} + \frac{1-\alpha}{1+\gamma}}.
\] (27)

Here, let us assume the following:

Assumption 4: \( \frac{\rho}{\alpha\delta} > 1 \).

### 5.1 Case 1: \( \frac{\rho}{\alpha\delta} - 1 > \frac{1}{\Delta} \)

Then, we can depict (26) and (27) as in Figure 2. We substitute \( \tau_r = 0 \) into (26) and (27) and obtain:

\[
\theta_{\max}(0) \equiv \frac{1 - \frac{1-\alpha}{1+\gamma}}{-1 + \frac{1-\alpha}{1+\gamma} + (\rho + \delta)\left(\frac{1}{\alpha\delta} - \frac{1}{\rho}\right)},
\] (28)

\[
\theta_{\min}(0) \equiv \frac{1 - \frac{1-\alpha}{1+\gamma}}{-1 + \frac{\rho+\delta}{\alpha\delta} + \frac{1-\alpha}{1+\gamma}} < \theta_{\max}(0).
\] (29)

From Figure 2, we can summarize the dynamic effect of the capital income tax as follows.

**Lemma 3** Increases in the tax rate on capital income destabilize the economy if \( \theta_I \in (0, \theta_{\min}(0)) \), while the local dynamics are qualitatively unaffected by changes in the tax rate if \( \theta_I \in (\theta_{\min}(0), \theta_{\max}(0)) \). For \( \theta_I \in \left(\theta_{\max}(0), \frac{1}{\rho/(\alpha\delta) - 1}\right) \), increases in the tax rate on capital income stabilize the economy.

### 5.2 Case 2: \( \frac{\rho}{\alpha\delta} - 1 < \frac{1}{\Delta} \)

\( \theta_I = \theta_{\max}(\tau_r) \) is depicted in Figure 3. From this figure, lemma 3 is rewritten as follows.

**Lemma 4** Increases in the capital income tax rate destabilize the economy if \( \theta_I \in (0, \theta_{\min}(0)) \), while the dynamic behaviors are not qualitatively affected by changes in the tax rate if

\footnote{Note that a representative agent discounts the future less (more) heavily in case 2 than in case 1 (3).}
\[ \theta_1 \in \left( \theta_{\min}(0), \frac{1}{\rho/(\alpha \delta) - 1} \right). \] For \( \theta_1 \in \left( \frac{1}{\rho/(\alpha \delta) - 1}, \theta_{\max}(0) \right) \) increases in the tax rate on capital income transform the steady state from a sink to a source.

From the calibrated models in the existing works such as B-F.94, 96 and Baxter and King (1990), we can regard assumption 4 is satisfied at realistically plausible values of the parameters \( \rho, \alpha \) and \( \delta \). However, we show that the main analytical result are not changed even if the different assumption from the above is imposed. Let us assume the following in place of assumption 4.

**Assumption 4’**: \( \frac{\rho}{\alpha \delta} < 1 \).

### 5.3 Case 3: \( \frac{\rho-1}{1-\frac{\alpha}{\alpha \delta}} < \frac{1}{\Delta} \) is always satisfied under assumption 4’.

Note that \( \theta_{\min}(0) > 0 \) and \( \theta_{\max}(0) < 0 \) are satisfied in case 3. Thus, Figure 4 can be obtained. As a result, lemma 4 is rewritten as follows.

**Lemma 5** The economy is destabilized by increases in \( \tau_r \), if \( \theta_1 \in (0, \theta_{\min}(0)) \), while the dynamic behaviors are not qualitatively affected by changes in \( \tau_r \) if \( \theta_1 \in (\theta_{\min}(0), -) \).

Irrespective of the parameter values, increases in the capital income tax rate destabilize the economy for the range of low externalities. Depending upon the parameter values, in contrast, there are the regions of moderate externalities in which indeterminacy arises for most values of the capital tax rate and the regions of high externalities in which increases in the capital tax rate can stabilize the economy. The above comparative dynamic analysis implies:

**Proposition 4** For any values of the rate of time preference, the existence of capital income taxes reduces the minimum sizes of increasing returns inducing indeterminacy to \( 1 + \theta_{\min}(\tau_r) \).

**Proof.** Compare \( \theta_{\min}(0) \) with \( \theta_{\min}(\tau_r) \) for \( \tau_r \neq 0 \) in Figures 2, 3 and 4. ■
Therefore, increases in the capital income tax rate destabilize the economy by making agents’ belief-driven aggregate fluctuations more likely.\footnote{The reason behind this result is described in Section 6.} The implication of the income tax rate for the possibility of indeterminacy is the same as in the one-sector Ramsey growth model.

### 6 Quantitative effects and interpretations

Sections 4 and 5 introduce constant tax rates on capital and labor income into an integrated framework of one- and two-sector models as studied in B-F.94 and 96. If the tax rate on capital income is set to zero, the model in Section 4 (5) collapses into the one in B-F.94 (96). This section estimates to what extent increases in the capital income tax rate reduces the minimum sizes of increasing returns inducing indeterminacy. To facilitate the comparison, we employ almost the same parameter values utilized in B-F.96. As for the sector-specific externalities in consumption sector $\theta_I$, the same value is used as in Harrison (2001). See Table 3.

First, the one-sector Ramsey growth model is considered. As for the minimum values of labor externalities, (22) means $\sigma^l_{\text{min}} = 1.857$. Let us set the value of labor externalities at $\sigma^l = 1.86(2.0)$, which is minimally (moderately) higher than the lower bound $\sigma^l_{\text{min}}$. When the tax rate on capital is equal to zero, i.e., $\tau_r = 0$, we can get $\sigma^k_{\text{min}}(0) = 0.7 \cdot 10^{-3} (3.3 \cdot 10^{-2})$. Then, the steady state is a source for $0 < \sigma^k < 0.7 \cdot 10^{-3} (3.3 \cdot 10^{-2})$, exhibits a Hopf-bifurcation at $\sigma^k = 0.7 \cdot 10^{-3} (3.3 \cdot 10^{-2})$ and is locally indeterminate for $0.7 \cdot 10^{-3} (3.3 \cdot 10^{-2}) < \sigma^k < 2.333$.\footnote{This upper bound comes from assumption 1.} Suppose that the capital tax rate is set at the same rate of $\tau_r = 0.375$ as in Hendricks (1999), who uses IRS data to compute the value for $\tau_r$.\footnote{Rivas (2003) also uses the capital tax rate from Hendricks (1999).} As $\hat{\tau}_r = 6.99 \cdot 10^{-4} (3.22 \cdot 10^{-2})$, the minimum sizes of capital externalities can be easily reduced from $0.7 \cdot 10^{-3} (3.3 \cdot 10^{-2})$ to 0. See Figure 1. This outcome is compatible
with lemma 2. If we set the capital income tax rate at realistically plausible values, the capital externalities become unnecessary and thus (22) is also the sufficient condition for the appearance of indeterminacy.

Next, we estimate to what extent the minimum sizes of externalities are reduced by increasing the capital income tax rate in a two-sector real business cycle model. If we employ the parameter values shown in Table 3, the parameter values are consistent with case 1 in the previous section. If $\tau_r = 0$, we can obtain $\theta_{\text{min}}(0) = 0.14943$ and $\theta_{\text{max}}(0) = 0.48148$. Thus, the steady state is a saddle (locally determinate) for $0 < \theta_I < 0.14943$, is a sink (locally indeterminate) for $0.14943 < \theta_I < 0.48143$, exhibits a Hopf bifurcation at $\theta_I = 0.48143$ and is a source for $0.48143 < \theta_I < 2.333$. In contrast, if $\tau_r = 0.375$, we can obtain $\theta_{\text{min}}(0.375) = 0.088435$ and $\theta_{\text{max}}(0.375) = 0.96078$ and, thus, the steady state is locally indeterminate for $0.088 < \theta_I < 0.96$. If the capital income tax rates are similar to those in many countries, the minimum size of externalities is significantly reduced.\footnote{Harrison (1998) provided an empirically realistic value of $\theta_I$ of 0.108. Thus, the steady state is a saddle (locally determinate) at empirically plausible values of externalities if $\tau_r = 0$. In contrast, indeterminacy can result at empirically plausible values of the sector-specific externality if $\tau_r = 0.375$.} Indeterminacy can result for a sufficiently wide range of values of the sector-specific externality.

Let us clarify the intuition behind the two-sector growth model. For our purpose, we explain the mechanism under which indeterminacy emerges in that model. Note that the return on the capital stock is positively related to its capital gains and the marginal product of capital. When the return on the capital stock is expected to increase tomorrow, consumption is sacrificed for investment by reallocating resources across sectors, and tomorrow’s capital stock and consumption increase. As a result, the relative price will first fall and then rise. The scale of capital gains is larger because the degree of $\theta_I$ is larger. Increases in $\theta_I$ raise the marginal product of capital and thus the returns on capital. The

\footnote{When $\theta_I = 0.108$, the steady state is a saddle (locally determinate) for $0 < \tau_r < 0.2501$, and it is a sink (locally indeterminate) for $0.2501 < \tau_r < 1$. Empirical evidence suggested by Hendricks (1999) implied that reducing the capital income tax rate is necessary in many countries to stabilize the economy by eliminating the indeterminacy of equilibria.}
expected path of return on capital can be self-fulfilling if the effects of $\theta_I$ on these processes are strong enough to dominate the agents’ desire to smooth consumption.\textsuperscript{12}

As for the one-sector Ramsey growth model, the price level of investment goods is constant (see (15)). There exists no capital gain effect generated by reallocating resources across two sectors. Thus, the minimum sizes of externalities inducing indeterminacy are much higher than in the two-sector real business cycle model. Increases in $\tau_r$ reduce the after-tax marginal product of capital returns, leading to less incentive for capital investment and, in contrast, enhance the countercyclical movement of consumption by lowering agents’ income. As seen in the two-sector model, increases in $\tau_r$ reduce the minimum sizes of externalities because the income effect is stronger than the substitution effect.

7 Labor supply elasticity and capital income taxes

It is well understood that the minimum values of externalities leading to indeterminacy are smaller in the one- and two-sector growth models because the labor supply elasticity (the inverse of labor supply elasticity $\gamma$) is higher (lower).\textsuperscript{13} This section considers the robustness of the results obtained in Section 6 for different values of the labor supply elasticity.

Lemma 1 verifies that the existence of a positive tax rate on capital income makes capital externalities unnecessary for the appearance of indeterminacy, if labor externalities are set at values slightly higher than the minimum size leading to the emergence of indeterminacy. Table 4 verifies the following. As $\tau_r < \tau_r = 0.375$ are obtained even for labor externalities moderately higher than the minimum size, the capital externalities

\textsuperscript{12}Considering the mechanism through which indeterminacy arises in that model, it is plausible that the parameter range generating indeterminacy increases because the representative agent discounts the future less heavily; i.e., $\rho$ is lower. See Figures 2–4.

\textsuperscript{13}This property is easily understood by noting that an increase in labor raises the marginal product of capital.

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needed for an indeterminate steady state are easily reduced to zero by a slight increase in the capital income tax rate from zero. However, this does not mean that the existence of a slightly positive tax rate on capital income allows the steady state to be locally indeterminate at empirically plausible sizes of increasing returns, because the values of $\sigma_{\text{min}}^L$ in Table 4 are completely outside the range of empirical estimates.

Next, we focus on the two-sector real business cycle model. B-F.96 showed that the minimum magnitude of $\theta_I$ is 0.07 (i.e., $\theta_{\text{min}}(0) = 0.07$) if the labor supply elasticity is close to infinity (i.e., $\gamma \to 0$). From B-F.96, indeterminacy is impossible at empirically plausible levels of $\theta_I$ if the labor supply elasticity is relatively low. However, the wage-elasticity of labor supply is not extremely high, but rather is relatively low. See Pencavel (1986) and Killingsworth and Heckman (1986).\footnote{Pencavel showed that most estimates of the male labor supply elasticity are between 0 and 0.45. As for the labor supply elasticity of women, Killingsworth and Heckman presented a wide range of estimates from −0.3 to 14.00 and concluded that the elasticity is probably somewhat higher for women than for men.}

Suppose that capital income tax rates are similar to those in many countries. Then, we show that increases in capital income taxes significantly expand the range of values of externalities inducing indeterminacy $\theta_I$, even if the labor supply elasticity is lower than in Section 6. Specially, we investigate to what extent the existence of capital income taxation reduces the minimum values of $\theta_I$ leading to an indeterminate steady state when the labor supply elasticity is relatively low. Put differently, we compare $\theta_{\text{min}}(0)$ with $\theta_{\text{min}}(0.375)$ when $\gamma$ is relatively high. Note that the steady state is locally indeterminate for $\theta_{\text{min}}(\tau_r) < \theta_I < \theta_{\text{max}}(\tau_r)$.

Table 5 shows that indeterminacy arises for a relatively narrow range of values of $\theta_I$ if the capital income tax rate is set at $\tau_r = 0$. Unlike $\theta_{\text{min}}(0)$, however, the values of $\theta_{\text{min}}(0.375)$ can fall into the standard errors of the empirical estimate obtained in Harrison (1998); that is, $\theta_I = 0.108$. Note that the labor supply elasticity is low enough to be compatible with empirical evidence. Table 5 shows the following. For given values of
\( \gamma; \theta_{\text{min}}(0.375) \) is much lower than \( \theta_{\text{min}}(0) \) and indeterminacy can emerge at a sufficiently wide range of values of sector-specific externalities when we set \( \tau_r = 0.375 \). Therefore, the possibility of indeterminacy significantly expands in a two-sector real business cycle model, as the fixed tax rate on capital income is higher.\(^{15}\) Belief-driven aggregate fluctuations are more likely to arise at realistically plausible values of externalities and the labor supply elasticity.

8 More general cases

From Sections 4 and 5, the main analytical results can be summarized as follows. In the one-sector Ramsey model, fixed income tax rates have no impact on the minimum sizes of labor externalities leading to indeterminacy. However, the existence of slightly positive tax rates easily reduces to zero the capital externalities unnecessary for the occurrence of indeterminacy. As for a two-sector real business cycle model, as fixed tax rates on capital income are higher, the degree of increasing returns in investment sector needed for indeterminacy is lower and thus the parameter region producing indeterminacy significantly expands. Irrespective of one- and two-sectors, the existence of fixed capital income taxes enhances the likelihood of indeterminacy. The purpose of this section is to show the robustness of these conclusions if we take account of the more general preference and technology.

8.1 One-sector Ramsey model \( \phi \neq 1 \)

Section 4 sets the utility function to be logarithmic in consumption, i.e., \( \phi \rightarrow 1 \). In contrast, this section considers the more general constant elasticity of intertemporal substitution in consumption, i.e., \( \phi \neq 1 \) and thus assumes Assumption 2" instead of Assumption

\(^{15}\)Note that the parameter values used in the numerical simulations belong to case 1, as mentioned in Section 6.
**Assumption 2**: \( \theta_i = 0 \) for \( i = I, C \) and \( \phi \neq 1 \).

Then, (20) and (21) are rewritten as

\[
T = \rho - \alpha \delta J' \cdot \frac{(\phi - 1)^2 (1 - \alpha) (\sigma^k + 1)}{\phi (\phi - 1) (1 - \alpha) (\sigma^k + 1)} + (\delta + \rho) \phi \Pi \cdot \frac{1 - \frac{1 + \sigma^k}{1 - \tau_r}}{\Gamma} \tag{30}
\]

\[
D = -\delta (\delta + \rho) \left( \alpha J'_2 - 1 \right) \left( \frac{1 - \kappa^*}{\kappa^*} \right) \cdot \frac{1}{\Gamma} \tag{31}
\]

where \( \Pi \equiv \frac{1 + \gamma}{\phi + (\phi - 1)(1 - \alpha)(\sigma^k + 1)} \), \( \Gamma \equiv \left[ (1 - \alpha) (\sigma^l + 1) - (1 + \gamma) \right] \frac{\phi + (\phi - 1)(1 - \alpha)(\sigma^k + 1)}{1 + \gamma + (\phi - 1)(1 - \alpha)(\sigma^l + 1)} \) and \( J'_2 \equiv (\sigma^k + 1) \cdot \frac{1 + \gamma}{1 + \gamma + (\phi - 1)(1 - \alpha)(\sigma^l + 1)} \).

Consider the elasticity of intertemporal substitution in consumption lower than one, i.e., \( \phi > 1 \). From (31), the determinant \( D \) is positive for \( \sigma^l > \sigma^l_{\min} (\equiv \frac{\gamma + \alpha}{1 - \alpha}) \). When the labor externalities \( \sigma^l \) is slightly above the minimum value \( \sigma^l_{\min} \), (30) shows that the trace \( T \) is negative for non zero tax rates \( \tau_r \neq 0 \), even if the capital externalities are absent, i.e., \( \sigma^k = 0 \). In contrast, suppose the elasticity of intertemporal substitution in consumption higher than one, i.e., \( \phi < 1 \). If \( \frac{\phi + (\phi - 1)(1 - \alpha)(\sigma^k + 1)}{1 + \gamma + (\phi - 1)(1 - \alpha)(\sigma^l + 1)} > 0 \) and \( \alpha J'_2 - 1 < 0 \) are assumed\(^{16}\), the same arguments are true of the case of \( \phi < 1 \). Therefore, the main analytical results are not changed even if we take account of the more general constant elasticity of intertemporal elasticity in consumption, i.e, \( \phi \neq 1 \).

### 8.2 Two-sector real business cycle model \( \phi \neq 1 \)

This section examines the robustness of the analytical results in Section 5. In other words, we show that the main analytical conclusions are not changed in a two-sector real business cycle model even if we consider the more general constant elasticity of intertemporal elasticity in consumption, i.e., \( \phi \neq 1 \).

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\(^{16}\) If \( \phi \) does not take the unrealisitically values that are sufficiently close to zero, we can easily see that the two assumptions are satisfied at empirically plausible values of parameters.
substitution, i.e., \( \phi \neq 1 \) or even if we differentiate the sector-specific externalities between capital and labor, i.e., \( \theta^c_j \neq \theta^l_j \) \((j = I, C)\).

First, let us generalize only the logarithmic utility function in consumption. Thus, we assume A.2'” in place of A.2’:

**Assumption 2’”:** \( \sigma^i = 0 \) \((i = k, l)\), \( \theta^k_j = \theta^l_j \) \((j = C, I)\) and \( \phi \neq 1 \).

Then, \( \Gamma \) in (18) and (19) can be expressed as

\[
\Gamma \equiv \left[ 1 - \kappa^* \right] + (1 - \alpha) \frac{\phi + (\phi - 1)(1 - \alpha)(\theta^c + 1)}{1 + \gamma + (\phi - 1)(1 - \alpha)(\theta^c + 1)} \right] \theta_I - (\alpha + \gamma) \frac{\phi + (\phi - 1)(1 - \alpha)(\theta^c + 1)}{1 + \gamma + (\phi - 1)(1 - \alpha)(\theta^c + 1)}
\]

Solving \( \Gamma = 0 \) for \( \theta_I \) yields the equation \( \theta_I = \theta_{\min}(\tau_r) \cdot (25) \) can be rewritten as

\[
\theta_I = \theta_{\min}(\tau_r) \equiv \frac{(\alpha + \gamma) \frac{\phi + (\phi - 1)(1 - \alpha)(\theta^c + 1)}{1 + \gamma + (\phi - 1)(1 - \alpha)(\theta^c + 1)}}{(1 - \tau \rho) \alpha \delta - 1 + (1 - \alpha) \cdot \frac{\phi + (\phi - 1)(1 - \alpha)(\theta^c + 1)}{1 + \gamma + (\phi - 1)(1 - \alpha)(\theta^c + 1)}}
\]

This equation shows \( \theta'_{\min}(\tau_r) < 0 \) if \( \phi > 1 \). When \( \phi + (\phi - 1)(1 - \alpha)(\theta^c + 1) > 0 \) when \( \phi < 1 \) is satisfied for \( \phi < 1 \). Moreover, if the sector-specific externalities \( \theta_I \) is slightly higher than \( \theta_{\min}(\tau_r) \), the trace \( T \) is negative for \( \tau_r > \frac{\alpha(\frac{\delta}{\pi r})}{1 + \alpha(\frac{\delta}{\pi r})} \). (See Appendix C for the proof.) Thus, we can regard \( \theta_{\min}(\tau_r) \) as the minimum sizes of \( \theta_I \) leading to indeterminacy and the minimum sizes of externalities are reduced by increasing the capital income tax rates. We can conclude that higher tax rates on capital income enhance the probability of indeterminacy even if the more general utility function is used. The quantitative effects on \( \theta_{\min}(\tau_r) \) are summarized in Table 6. This table shows the following. As the elasticity of intertemporal substitution in consumption \( \phi^{-1} \) is lower, the minimum sizes of externalities

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17 Let us assume \( 1 + \gamma + (\phi - 1)(1 - \alpha)(\theta^c + 1) > 0 \) and \( \phi + (\phi - 1)(\theta^c + 1) > 0 \) when \( \phi < 1 \). As mentioned in footnote 16, the two assumptions are easily satisfied at empirically plausible values of parameters if \( \phi \) does not take the value that is unrealistically close to zero. For example, these assumptions are consistent with the parameter values chosen in Harrison (2002). Then, \( \phi + (\phi - 1)(1 - \alpha)(\theta^c + 1) > 0 \), \( \Theta > 0 \) and \( \Omega > 0 \) are satisfied. \( \Theta \) and \( \Omega \) appear in Appendix C, where the definitions of \( \Theta \) and \( \Omega \) are described.

18 If the parameter values in Table 1 are used, we can obtain the value of \( \frac{\alpha(\frac{\delta}{\pi r})}{1 + \alpha(\frac{\delta}{\pi r})} = 4.76 \), which is significantly close to zero. Thus, this inequality is not restrictive.

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in investment sector is larger. This outcome is the same as in Harrison (2001) analyzing the discrete-time version of two-sector growth model with no government sector. The extent to which increases in $\tau_r$ reduce the sizes of $\theta_{\min}(\tau_r)$ is relatively large even if $\phi \neq 1$. However, indeterminacy is difficult to emerge for sufficiently high values of $\phi$ even if we take account of the realistically plausible levels of $\tau_r$.

Finally, let us consider the more general production technology such that the capital externalities are differentiated from the labor externalities. We consider how the analytical results are affected if A.2’ is replaced by A.2'”:

**Assumption 2'”**: $\sigma^i = 0$ ($j = k, l$), $\theta_j^k \neq \theta_j^l$ ($j = C, I$) and $\phi \to 1$.

Then, we can rewrite $\Gamma$ determining the minimum sizes of sector-specific externalities as

$$
\Gamma \equiv (1 - \alpha) \left( \frac{1 - \kappa^*}{\kappa^*} + \frac{1}{1 + \gamma} \right) \theta_I^l + \alpha \left( \frac{1 - \kappa^*}{\kappa^*} \right) \theta_I^k - \frac{\alpha + \gamma}{1 + \gamma}.
$$

Defining the combinations of $\theta_I^i$ and $\tau_r$ satisfying $\Gamma = 0$ as $\theta_{\min}^i(\tau_r)$ ($i = l, k$), we can easily verify $\theta_{\min}^i(\tau_r) < 0$. Moreover, the trace $T$ is

$$
T = \rho - (\delta + \rho) \left( \frac{\theta_I^k + 1}{1 - \tau_r} \right) \cdot \frac{1}{\Gamma}
$$

(33) show that the trace $T$ is negative if $\theta_I^i$ is slightly above $\theta_{\min}^i(\tau_r)$. Thus, we can regard that $\theta_{\min}^i(\tau_r)$ is the minimum sizes of sector-specific externalities in investment sector generating indeterminacy. We can get the same analytical result as in Section 5 that higher capital income taxes raise the probability of indeterminacy even if we differentiate the sector-specific externalities of capital and labor, i.e., $\theta_i^k \neq \theta_i^l$ ($i = C, I$).
9 Conclusion

Using an integrated framework of the continuous-time versions of one- and two-sector optimal growth models, this paper analyzed the dynamic effects of capital and labor income tax rates for given values of externalities and explores how income tax rates affect the minimum values of externalities required for the occurrence of indeterminacy. Unlike the labor income tax rate, the capital income tax rate can have a major effect on the set of parameters for which indeterminacy emerges. The analytical relations between capital income taxes and the possibility of indeterminacy can be summarized as follows.

For the one-sector Ramsey optimal growth model, the likelihood of indeterminacy is unaffected by increases in capital income tax rates if we do not discriminate aggregate externalities between capital and labor. For any tax rates on capital income, the steady state is locally indeterminate. This result is strikingly different from in Guo and Lansing (2002). However, if we differentiate capital from labor externalities, the minimum values of labor externalities inducing indeterminacy are determined independently of the capital income tax rate, but the existence of slightly positive tax rate on capital income easily makes capital externalities unnecessary for the appearance of indeterminacy.

Regarding the two-sector real business cycle model, increases in the capital income tax rate significantly reduces the minimum values of externalities required for an indeterminacy for given values of the labor supply elasticity and the elasticity of intertemporal substitution in consumption. If capital tax rates are similar to those in many countries, the range of externalities inducing indeterminacy greatly expands and it is more likely to emerge at empirically plausible values of externalities, the labor supply elasticity and the elasticity of intertemporal substitution in consumption.

Irrespective of the one- and two-sector optimal growth models, therefore, the existence of capital income taxation makes indeterminacy more likely in the range of parameter values typically explored. It is analytically shown that fixed capital income tax rates
cannot work as an “automatic stabilizer”.

Appendix A (Proof of proposition 1)

When \( \sigma^i > \frac{\gamma + \alpha}{1 - \alpha} \), (20) implies that the trace is negative for any tax rate on capital income if:

\[
\rho \left[ -1 + \frac{(1 - \alpha)(1 + \sigma^i)}{1 + \gamma} \right] - (\delta + \rho) \left( \frac{\tau_r + \sigma^h}{1 - \tau_r} \right) < 0. \tag{A-1}
\]

Noting assumption 1, (A.1) is rewritten as:

\[
\left[ \rho \left( 1 - \frac{1 - \alpha}{1 + \gamma} \right) - \frac{\delta}{1 - \tau_c} \right] \sigma < (\delta + \rho) \left( \frac{\tau_r}{1 - \tau_r} \right) + \rho \left( \frac{\alpha + \gamma}{1 + \gamma} \right). \tag{A-2}
\]

As the left hand (the right hand) in (A-2) is negative (positive), (A-2) is satisfied.

For any tax rates on capita income, therefore, the steady state is always a sink, when \( \sigma^i > \frac{\gamma + \alpha}{1 - \alpha} \) and \( \sigma^i = \sigma \) (i = K, L).

Appendix B (The reason why the required externalities are lower in the continuous-time case than in the discrete-time case)

If we express the local dynamics of one-sector growth model in the plane \((k_t, c_t)\), (17) becomes:

\[
\begin{bmatrix}
\dot{k}_t \\
\dot{c}_t
\end{bmatrix} = \begin{bmatrix}
\lambda_1 - 1 & \left( \frac{k^*}{c^*} \right) \lambda_2 \\
\left( \frac{c^*}{k^*} \right) (1 + \rho) \lambda_3 & (1 + \rho) (1 - \lambda_4)
\end{bmatrix}
\begin{bmatrix}
k_t - k^* \\
c_t - c^*
\end{bmatrix}, \tag{B-1}
\]

where

\[
\lambda_1 \equiv 1 - \delta - \frac{(1 + \gamma)(\rho + \delta)(1 + \sigma^k)}{[(1 - \alpha)(1 + \sigma^i) - (1 + \gamma)](1 - \tau_r)},
\]

\[
\lambda_2 \equiv 1 + \frac{(1 + \gamma)(\rho + \delta)}{[(1 - \alpha)(1 + \sigma^i) - (1 + \gamma)](1 - \tau_r) \alpha},
\]

\[
\lambda_3 \equiv -\left( \frac{\rho + \delta}{1 + \rho} \right) \frac{(1 + \gamma) \alpha (1 + \sigma^k) + (1 - \alpha)(1 + \sigma^i) - (1 + \gamma)}{[(1 - \alpha)(1 + \sigma^i) - (1 + \gamma)]},
\]

29
and

$$\lambda_4 \equiv 1 - \left( \frac{\rho + \delta}{1 + \rho} \right) \frac{(1 - \alpha) \sigma^l}{(1 - \alpha) \sigma^l - (1 + \gamma)}. $$

In the discrete-time one-sector Ramsey model, the counterpart of (B-1) is expressed as:

$$
\begin{bmatrix}
  k_{t+1} - k^* \\
  c_{t+1} - c^*
\end{bmatrix} =
\begin{bmatrix}
  \lambda_1 & \left( \frac{k^*}{c^*} \right) \lambda_2 \\
  \left( \frac{c^*}{k^*} \right) \left( \frac{\lambda_1 \lambda_3}{\lambda_4} \right) & \frac{1 + \lambda_2 \lambda_3}{\lambda_4}
\end{bmatrix}
\begin{bmatrix}
  k_t - k^* \\
  c_t - c^*
\end{bmatrix}.
$$

(B-2)

The trace ($T$) and determinant ($D$) can be derived as:

$$T = \lambda_1 + \frac{1 + \lambda_2 \lambda_3}{\lambda_4},$$

$$D = \frac{\lambda_1}{\lambda_4}.$$  

These equations are almost the same as in Guo and Lansing (2002) except that we set $\sigma^k \neq \sigma^l$ and $\tau_r \neq \tau_w$. The Euler equation is the equation that determines the growth rate of consumption, but the timing of the interest rate in the Euler equation is different between the continuous-time and discrete-time cases. The interest rate in this period (i.e., the marginal product of capital in this period) determines the growth rate in the continuous case, while the interest rate in the next period (i.e., the marginal product of capital in the next period) determines the growth rate in the discrete case. As the next period capital stock $k_{t+1}$ is a function of current period consumption $c_t$ and current period capital stock $k_t$, equation (B-2) is more complicated than (B-1).

In the numerical simulations in Guo and Lansing (2002), the parameters were chosen such that $T < 0$ and $0 < D < 1$ are satisfied for $0 \leq \tau_r < 1$. They considered the combination of the tax rates and the sizes of externalities that satisfy $D + T + 1 = 0$. Noting that indeterminacy arises in the range $D + T + 1 > 0$, they concluded that larger externalities are needed (i.e., $D + T + 1 > 0$ is more likely to hold), as $\tau_r$ is higher. We should note that the externalities of capital and labor are not separated in their numerical
simulations (i.e., $\sigma^k = \sigma^l$ is assumed).

Then, let us consider why we can obtain Proposition 2 in contrast with Guo and Lansing (2002). We examine the continuous-time growth model. Suppose that the sizes of labor externalities are fixed in the range $\sigma^l > (1 + \gamma) / (1 - \alpha)$. Then, the determinant is positive. If the trace is negative, indeterminacy arises. We consider the case of $\tau_r = 0$. The trace is positive (negative) if $\sigma^k = 0$ ($\sigma^k = \sigma^l$). Noting that the trace is negatively related to $\sigma^k$ and the trace is zero at $\sigma^k = \sigma^k_{\text{min}}(0)$, we can easily verify $\sigma^k_{\text{min}}(0) < (1 + \gamma) / (1 - \alpha)$. As the trace is negative for $\sigma^k_{\text{min}}(0) < \sigma^k < 1/\alpha - 1$, indeterminacy emerges at $\sigma^k = \sigma^l \in ((1 + \gamma) / (1 - \alpha), 1/\alpha - 1)$.\(^{19}\) Suppose that $\tau_r$ is increased from 0 to some value within the range $(0,1)$. From $\frac{\partial \sigma^k_{\text{min}}(\tau_r)}{\partial \tau_r} < 0$, the trace is negative at smaller sizes of $\sigma^k$. Thus, indeterminacy always emerges for $\sigma^k = \sigma^l \in ((1 + \gamma) / (1 - \alpha), 1/\alpha - 1)$ and $\tau_r \in [0,1]$.

We consider the discrete-time growth model (B-2) investigated in Guo and Lansing (2002). Note that they focused only on the case of $\sigma^k = \sigma^l (= \sigma)$. Defining the size of $\sigma \in ((1 + \gamma) / (1 - \alpha), 1/\alpha - 1)$, the following is satisfied in their numerical simulations. When $\tau_r = 0$, the steady state is a saddle for $\sigma < 1/\alpha - 1$ and is sink (locally indeterminate) for $\sigma < (1 + \gamma) / (1 - \alpha)$. Moreover, $\sigma$ is positively related to $\tau_r$ (see Fig. 1 in their paper). Unlike the continuous case, the steady state might not be locally indeterminate in the discrete-time growth model even if $\sigma > (1 + \gamma) / (1 - \alpha)$. Larger externalities are needed in the discrete-time model. Phrased differently, indeterminacy is more likely to arise in the continuous-time growth model than in the discrete-time one.

Appendix C (The case of $\theta_j^c = \theta_j^k$, $\sigma^i = 0$ and $\phi \neq 1$)

Appendix C proves that the trace $T$ is negative for $\tau_r > \frac{\alpha (\frac{\theta_j^k}{\tau_r})}{1 + \alpha (\frac{\theta_j^k}{\tau_r})}$ only if the sector-specific externalities $\theta_j^k$ is slightly above the minimum sizes of externalities $\theta_{\text{min}}(\tau_r)$. In the numerical simulations in Guo and Lansing (2002), the parameter values are chosen such that $\frac{1 + \gamma}{1 - \alpha} < \frac{1 - \alpha}{\alpha}$ is satisfied. We also assume $\frac{1 + \gamma}{1 - \alpha} < \frac{1 - \alpha}{\alpha}$.

\(^{19}\) In the numerical simulations in Guo and Lansing (2002), the parameter values are chosen such that $\frac{1 + \gamma}{1 - \alpha} < \frac{1 - \alpha}{\alpha}$ is satisfied. We also assume $\frac{1 + \gamma}{1 - \alpha} < \frac{1 - \alpha}{\alpha}$.
the case of $\theta_j^c = \theta_j^k$, $\sigma^i = 0$ and $\phi \neq 1$, let us write the trace $T$ as

$$ T = \rho - (\phi - 1) \alpha \delta (\theta_c + 1) \cdot \Theta $$

$$ -\alpha \delta \left[ \theta_I + 1 + (\phi - 1) (\theta_c + 1) \right] \cdot \Theta \left[ \frac{1 - \kappa^*}{\kappa^*} + (1 + \gamma) \cdot \Omega \right] \cdot \frac{1}{\Gamma} $$

$$ + (\delta + \rho) (1 + \gamma) \cdot \Omega \cdot \frac{1}{\Gamma} $$

where $\Theta \equiv \frac{1 + \gamma}{1 + \gamma + (\phi - 1)(1 - \alpha)(\theta_c + 1)} > 0$ and $\Omega \equiv \frac{(\phi - 1)(\theta_c + 1)}{1 + \gamma + (\phi - 1)(1 - \alpha)(\theta_c + 1)} > 0$ (As for the signs of $\Theta$ and $\Omega$, see footnote 17).

(C-1) is rewritten as

$$ T = \rho - (\phi - 1) \alpha \delta (\theta_c + 1) \cdot \Theta $$

$$ - (\delta + \rho) \cdot \Theta \left[ \theta_I + 1 + (\phi - 1) (\theta_c + 1) \right] \cdot \frac{1 - \kappa^*}{1 - \tau_r} - [1 + (\phi - 1) (\theta_c + 1)] \cdot \frac{1}{\Gamma} $$

$$ -\alpha \delta \left[ \theta_I + 1 + (\phi - 1) (\theta_c + 1) \right] \cdot \Theta (1 + \gamma) \cdot \Omega \cdot \frac{1}{\Gamma} $$

Noting the second row in (C-2) and $\frac{1 - \kappa^*}{1 - \tau_r} - 1 < 0 \Leftrightarrow \tau_r > \frac{\alpha (\frac{\delta}{\tau_r})}{1 + \alpha (\frac{\delta}{\tau_r})}$, we can obtain

$$ T < \rho - (\phi - 1) \alpha \delta (\theta_c + 1) \cdot \Theta $$

$$ - (\delta + \rho) \cdot \Theta \left[ 1 + (\phi - 1) (\theta_c + 1) \right] \left( \frac{1 - \kappa^*}{1 - \tau_r} - 1 \right) \cdot \frac{1}{\Gamma} $$

$$ -\alpha \delta \left[ \theta_I + 1 + (\phi - 1) (\theta_c + 1) \right] \cdot \Theta (1 + \gamma) \cdot \Omega \cdot \frac{1}{\Gamma} < 0, $$

only if $\theta_I$ is slightly higher than $\theta_{\min} (\tau_r)$, and equivalently $\Gamma$ is sufficiently close to zero.

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References


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<th>One- or Two- Sector</th>
<th>Log or CRRA utility</th>
<th>External Effects</th>
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Table 3. Parameter values.

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<th>Parameter</th>
<th>Value</th>
<th>Description</th>
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<tr>
<td>$\alpha$</td>
<td>0.3</td>
<td>Capital share</td>
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<tr>
<td>$\rho$</td>
<td>0.05</td>
<td>Pure time preference rate</td>
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<tr>
<td>$\delta$</td>
<td>0.1</td>
<td>Capital depreciation rate</td>
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<tr>
<td>$\gamma$</td>
<td>1</td>
<td>Inverse of labor supply elasticity</td>
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<tr>
<td>$\phi$</td>
<td>1</td>
<td>Inverse of the elasticity of intertemporal substitution in consumption</td>
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<tr>
<td>$\theta_I$</td>
<td>0.0774</td>
<td>Externalities in consumption good sector</td>
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<tr>
<td>$\tau_r$</td>
<td>0 or 0.375</td>
<td>Capital income tax rate</td>
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Table 4: Calibration results in the one-sector Ramsey growth model

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>$\sigma^L_{\min}$</th>
<th>$\sigma^L$ ($=\sigma^L_{\min} + 0.2$)</th>
<th>$\sigma^K_{\min}$ ($0$)</th>
<th>$\hat{r}_r$</th>
<th>$\sigma^K_{\min}$ (0.375)</th>
</tr>
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<tr>
<td>1</td>
<td>1.8571</td>
<td>2.0571</td>
<td>0.04676</td>
<td>4.4671 × 10^{-2}</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>3.2857</td>
<td>3.4857</td>
<td>0.0311</td>
<td>3.0162 × 10^{-2}</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>4.7143</td>
<td>4.9143</td>
<td>0.0233</td>
<td>2.2769 × 10^{-2}</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>6.1429</td>
<td>6.3429</td>
<td>0.0184</td>
<td>1.8068 × 10^{-2}</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>7.5714</td>
<td>7.7714</td>
<td>0.0156</td>
<td>0.01536</td>
<td>0</td>
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Note: The remaining parameter values are the same as in Table 1.

Table 5: Calibration results in the two-sector real business cycle model

<table>
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<th>$\gamma$</th>
<th>$\theta_{\min}$ (0)</th>
<th>$\theta_{\max}$ (0)</th>
<th>$\theta_{\min}$ (0.375)</th>
<th>$\theta_{\max}$ (0.375)</th>
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<tr>
<td>1</td>
<td>0.14943</td>
<td>0.48148</td>
<td>0.088435</td>
<td>0.96078</td>
</tr>
<tr>
<td>2</td>
<td>0.1811</td>
<td>1.0548</td>
<td>0.10599</td>
<td>0.62162</td>
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<tr>
<td>3</td>
<td>0.1976</td>
<td>0.70213</td>
<td>0.11498</td>
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<td>4</td>
<td>0.20773</td>
<td>0.75439</td>
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<tr>
<td>5</td>
<td>0.21457</td>
<td>0.79104</td>
<td>0.12412</td>
<td>1.1583</td>
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Note: The remaining parameter values are the same as in Table 1.
Table 6: The elasticity of intertemporal substitution and indeterminacy

<table>
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<th>$\phi$</th>
<th>$\theta_{\min}(0)$</th>
<th>$\theta_{\min}(0.375)$</th>
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<tr>
<td>$\frac{3}{2}$</td>
<td>0.028612</td>
<td>0.016458</td>
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<td>$\phi=2$</td>
<td>0.076205</td>
<td>0.0444325</td>
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<td>$\phi=1$</td>
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<td>$\phi=1.5$</td>
<td>0.22493</td>
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<td>$\phi=2$</td>
<td>0.27660</td>
<td>0.16883</td>
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<td>$\phi=3$</td>
<td>0.34272</td>
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<td>$\phi=5$</td>
<td>0.41064</td>
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Note: The remaining parameter values are the same as in Table 1. Even if $\tau_r = 0$, that is $\frac{\alpha \left( \frac{\delta}{\delta + \rho} \right)}{1 + \alpha \left( \frac{\delta}{\delta + \rho} \right)}$ is not satisfied, we can easily get $T < 0$, when $\theta_{\ell}$ is slightly above $\theta_{\min}(0)$. 


Figure 1: Case of \( \sigma^i > \sigma^i_{\min} \equiv \frac{\gamma + \alpha}{1 - \alpha} \)

Figure 2: Case 1

Sink (locally in determinate)

Source

Sink (locally in determinate)

Saddle

Tr. = 0

\( \tau_r \)

\( \theta_{\min}(0) \)

\( \theta_{\max}(0) \)

\( \theta_1 \)
Figure 3: Case 2

Figure 4: Case 3