Infrastructure, alternative government finance and stochastic endogenous growth

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Abstract

This paper constructs a stochastic version of an endogenously growing economy with a public good that raises the productivity of private capital. We explore how growth and welfare are influenced by changes in the mean and variance of productive public spending under two alternative financing methods, mixed money–bond financing and wealth-tax financing. In addition, to evaluate the differences between money financing and bond financing, we consider mixed money–bond financing, under which a larger ratio of bonds to money is utilized to finance a given increase in public spending.

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1. Introduction

Government expenditure financing is regarded as an important factor explaining variations in economic performance, and many economists have extensively examined the theoretical interaction between aggregate activity and the way that public spending is financed. For example, based on the framework of a nonmonetary...
Ramsey growth model, Turnovsky (1992) examines the steady-state effects of changes in public spending under different financing methods. Using the monetary growth model of Sidrauski (1967), Gokan (2003) investigates how alternative government financing affects the speed at which capital stock converges to a steady state. Ploeg and Alogoskoufis (1994) develop Sidrauski’s model with noninterconnected overlapping generations to compare the growth effects of public spending financed by creating money, issuing bonds and imposing lump-sum taxes. Palivos and Yip (1994) utilize cash-in-advance model of Stockman (1981) to assess the relative merits of tax financing and money financing. However, the present paper differs from the existing literature in two respects.

First, the above economists assume that public spending has no direct impact on the behavior of the private sector. It can be interpreted as being either a real drain on the economy or alternatively as some good that does not influence the productivity of private capital. In the study of public spending, it is convenient to distinguish between government consumption expenditure and government infrastructure expenditure. The economists above restrict their attention to government consumption expenditure, and the concept of government infrastructure expenditure is excluded from their analyses. Clearly, an alternative discussion of the role of government spending is required within this topic. To capture the role of infrastructure capital, this paper incorporates public capital into the production function as an input. However, owing to the analytical complexities, we are restricted to considering a steady-state equilibrium and examining the steady-state effects of alternative government financing.

Second, the stochastic factors are completely ignored in the existing alternative government financing literature. Unforeseen policy shocks occur over time, and the economic activities are inherently subject to the unforeseen risk. Thus, the importance of the interaction between the stochastic factors in the public sector and the real economy should be emphasized in this field. We extend a general equilibrium model to consider a richer, more realistic picture of government sectors. This paper allows us to clarify whether the mean and variance of productive public spending could mitigate the size of business cycle fluctuations in the steady state.

In the present paper, productive public capital is introduced in a stochastic AK model, as studied in Turnovsky (1993) and Grinols and Turnovsky (1993). It should be noted that public spending corresponds to a flow of public capital. There are two different financing methods to raise a given increase in public spending: mixed money–bond financing and financing through taxes on wealth. Hence, we can compare how the level of welfare and the mean and variance of real growth are affected by increasing the first and second moments of productive public spending under the two different financing schemes. In addition, to explore the qualitative differences between money financing and bond financing, this paper considers a mixed money–bond financing under which government relies on a larger ratio of bonds to money to raise the necessary revenue. As for tax and monetary policies, we

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1 An exception is Gokan (2002), who has a stochastic general equilibrium model but focuses on government consumption expenditure.
examine the growth and welfare effects, maintaining fixed levels of the first and second moments of public spending. To put it briefly, the purpose of this paper is to clarify how government should behave to raise a given increase in public deficits or to obtain a given level of public deficits from growth and welfare perspectives.

To express the productive services derived from publicly provided infrastructure, there exist a number of papers analyzing dynamic general equilibrium models where public spending or a public good is introduced in a production function. See Barro (1990), Futagami et al. (1993), Turnovsky and Fisher (1998), Cazzavillan (1996), Glomm and Ravikumbar (1997), Zhang (2000) and Raurich-Puigdevall (2000). In the above literatures, however, the stochastic disturbances are completely ignored and the role of alternative government financing is not emphasized. On the other hands, Turnovsky (1999) considers the economic uncertainty but not the different sources of government revenue. Fisher and Turnovsky (1998, 2004) explore the alternative modes of government financing, but ignore the stochastic factors in the economy.

2. Structure of the model

2.1. Consumers

The representative consumer decides to allocate his or her wealth between money, government bonds and equities. Hence, the balance sheet constraint is

\[ \frac{M}{P} + \frac{B}{P} + S = W, \]  

(1)

where \( M \) is the nominal stock of money, \( B \) the nominal stock of bonds, \( P \) the price level of new goods, and \( S \) the real stock of equity measured in terms of new output.

In addition, the consumer decides how much output to consume over the instant \( dt \) at the nonstochastic rate \( Cdt \) as a proportion of income, which consists of the real return on these assets. In other words, his or her objective is to select a portfolio of assets and rate of consumption to maximize expected lifetime utility, which depends upon consumption \( C_t \) and the real money balance \( M_t/P_t \):

\[ E \int_0^\infty U\left( C_t, \frac{M_t}{P_t} \right) e^{-\rho t} dt, \]  

(2a)

subject to the stochastic wealth accumulation

\[ dW = W[n_1 dR_M + n_2 dR_B + n_3 dR_S] - Cdt - dT, \]  

(2b)

where \( n_1 \equiv (M_t/P_t) / W_t, \ n_2 \equiv (B_t/P_t) / W_t, \ n_3 \equiv S_t / W_t, \ \) \( dR_i = \) stochastic real rate of return on asset \( i \) (\( i = M, B, S \)), and \( dT = \) taxes on wealth paid to government.

The utility function is assumed to be a logarithmic feature:

\[ U = \theta \ln C_t + \gamma \ln \frac{M_t}{P_t}, \quad \gamma + \theta = 1. \]
The consumer expects the price level to evolve according to the geometric Brownian motion process:

\[ \frac{dP}{P} = \pi dt + dp, \]  

(3)

where \( \pi dt \) is the mean inflation rate over the period \( dt \), and \( dp \) is a temporally independent and normally distributed random variable with zero mean and variance equal to \( \sigma_p^2 dt \).

Money has a nominal return of zero, while government pays nonstochastic rates of nominal interest \( i \) over the period \( dt \). Using Ito's calculus, we can obtain the rates of return on money and bonds as

\[ dR_M = r_M dt - dp, \quad \text{where} \quad r_M = -\pi + \sigma_p^2, \]  

(4a)

\[ dR_B = r_B dt - dp, \quad \text{where} \quad r_B = i - \pi + \sigma_p^2. \]  

(4b)

The real rate of return on equity is defined as

\[ dR_S = r_S dt + du, \]  

(4c)

where the mean rate of return \( r_S \) will be determined in the description of the firm. The stochastic component \( du \) is temporally independent and normally distributed with zero mean and variance \( \sigma_u^2 dt \), and it will be specified below.

This representative consumer must pay taxes on his or her wealth holdings, specified by

\[ dT = \tau W dt + W dv, \]  

(5)

where \( dv \) is a temporally independent, normally distributed random variable with zero mean and variance \( \sigma_v^2 dt \). \( \tau \) and \( \sigma_v^2 \) will be specified further below.

The stochastic optimization problem can be expressed as being the choice of a consumption ratio \( C_t/W_t \) and portfolio shares \( n_1, n_2, n_3 \) to

\[ E \int_0^\infty \left[ \theta \ln C_t + \gamma \ln n_1 W_t \right] e^{-\rho t} dt, \]  

(6a)

subject to

\[ dW = \left[ n_1 r_M + n_2 r_B + n_3 r_S - C/W - \tau \right] W dt - (n_1 + n_2) dp + n_3 du - dv, \]  

(6b)

and

\[ n_1 + n_2 + n_3 + 1. \]  

(6c)

In performing the optimization, the representative agent takes \( i, \pi, \tau \), and the relevant variances and covariances as given, though these will be ultimately determined in the stochastic macroeconomic equilibrium. To solve this stochastic optimization problem, we must set up a stochastic Bellman equation. The first-order conditions are derived as \(^2\):

\[ C/W = \theta \rho, \]  

(7a)

\(^2\)The first-order conditions (7a)-(7d) are the same as in the ones in Turnovsky (1993) and Grinols and Turnovsky (1993), where the stochastic dynamic optimization is discussed in detail.
\begin{align*}
\rho'_1/n_1 + r_M &= \eta + (n_1 + n_2)\sigma^2_p - n_3\sigma_{pu} + \sigma_pv, \quad (7b) \\
\phi &= \eta + (n_1 + n_2)\sigma^2_p - n_3\sigma_{pu} + \sigma_pv, \quad (7c) \\
r_S &= \eta - (n_1 + n_2)\sigma_{pu} - n_3\sigma^2_u - \sigma_{uv}, \quad (7d)
\end{align*}

where \( \eta \) is the Lagrange multiplier attached to the constraint (6c) and \( \sigma_{vy} dt = \text{cov}(di, dj), i, j = u, v \) and \( p \).

From (4a), (4b), (7b) and (7c), the demand function for money can be derived as

\[ n_1 = \frac{\rho'_1}{\phi}. \quad (8a) \]

The portfolio share of money depends only upon the nominal interest rate. It is influenced by the stochastic characteristics of the economy only insofar as they affect \( i \).

Using (6c), (7c) and (7d), we obtain the portfolio share of equities:

\[ n_3 = \frac{r_S - r_B + \sigma^2_p + \sigma_{pu} + \sigma_{uv} + \sigma_pv}{\sigma^2_u + \sigma^2_p + 2\sigma_{up}}. \quad (8b) \]

The portfolio share of bonds \( n_2 \) is determined from (6c).

### 2.2. Firms

To characterize the productive services derived from a public good, this paper considers that a flow of output \( dY \) is produced from private capital \( K_c \) and public good \( K_p \) by means of a stochastic constant return technology:

\[ dY = A(K_c)^{(1-\alpha)}(K_p)\alpha(dt + dy), \quad (9) \]

where \( A \) represents the efficiency of the production technology and \( dy \) is a temporally independent and normally distributed stochastic process with zero mean and variance \( \sigma^2_y dt \). For analytical tractability, \( K_p \) is assumed to be a pure public good that is free from congestion. Empirical evidence suggests that \( \alpha \) cannot be large (see Rioja, 2003 and references therein). If we consider that human capital as well as physical one are included in \( K_c \), \( 1-\alpha \) can be moderately high. Thus, we can restrict our attention to the following parameter range:

\[ A.1 : \alpha \leq 1/2. \]

Following Turnovsky (1993), we turn to the determination of \( dR_S \). As the return on equity consists of dividend income and capital gain, we can define the real rate of return on equities by

\[ dR_S = dD/S + ds/s, \quad (10) \]

where, \( dD = \) flow of dividend payments, and \( s = \) relative price of equities in terms of output.

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3This assumption is very useful in verifying the sign of endogenously derived equations. For example, see \( \Gamma^2, \Gamma^{con} (D.2) \) and (D.3) in Appendix A, B and D.
Suppose that capital is adjusted costlessly. Then, we can obtain $S = K_c$ in equilibrium. Defining the outstanding stock of equities at $t$ by $N_t$, we obtain

$$sN = K_c = S.$$  \hfill (11)

Taking the stochastic differential of this relationship yields

$$(s + ds) dN + N ds = dK_c.$$  \hfill (12)

Let us consider that the net output is either paid out as dividends or retained as earnings ($RE$) to finance new investment:

$$dD + dRE = dY.$$  \hfill (13)

If firms finance new capital either out of retained earnings or by issuing new equities, this situation can be expressed as

$$(s + ds) dN + dRE = dK_c,$$  \hfill (14)

where new equities are sold at the current price $s + ds$. Combining (11) through (14) yields

$$\frac{ds}{s} = d \frac{Y}{S} - d \frac{D}{S}.$$  \hfill (15)

Substituting (15) into (10) leads to

$$dR_S = d \frac{Y}{S}.$$  \hfill (16)

Hence, we can specify that

$$r_s = A(K_p/K_c)^{\gamma}, \quad du = A(K_p/K_c)^{\gamma} dy.$$  \hfill (17)

As firms have free use of $K_p$ services, the zero-excess-profit condition also generates (17).

2.3. Government

The government budget constraint can be expressed as

$$d(M/P) + d(B/P) = dK_p - dT + (M/P)dR_M + (B/P)dR_B,$$  \hfill (18)

where $dK_p$ is the stochastic government expenditure. To be compatible with a stochastic version of endogenous growth, let us specify $dK_p$ as

$$dK_p = A(K_c)^{(1-\gamma)}(K_p)^{\gamma}(g dt + dz),$$  \hfill (19a)

where $dz$ is a temporally independent and normally distributed random variable with zero mean and variance $\sigma_z^2 dt$. Eq. (19a) means that the instantaneous mean government expenditure is a fraction of the mean level of output and the stochastic disturbance is proportional.

Stochastic monetary policy is

$$dM/M = \mu dt + dx.$$  \hfill (19b)

\footnote{We assume that capital stock does not depreciate, but the consequences in this paper are not changed, even if capital stock depreciates at a constant rate between zero and one.}
where \( dx \) is a temporally independent and normally distributed random process with zero mean and variance \( \sigma_x^2 dt \). \( \mu \) and \( \sigma_x^2 \) will be specified below.

Government debt policy is specified in terms of maintaining a fixed ratio of government bonds to money:

\[
B/M = \lambda, \quad (19c)
\]

where \( \lambda \) is a policy variable determined by government.\(^5\)

3. Balanced growth equilibrium

In a balanced growth equilibrium the growth rates of capital stock, wealth, consumptions, and so on evolve according to a normal distribution with the same mean and variance, and \( q, n_1, n_2 \) and \( n_3 \) are constant. Let us derive the relationships that are satisfied in a steady state.

3.1. Equilibrium rate of real growth

The product market equilibrium can be shown as

\[
dK_c = dY - dC - dK_p. \quad (20)
\]

Using (7a), (9) and (19a), (20) can also be rewritten as

\[
\frac{dK_c}{K_c} = \left[ A \left( \frac{K_p}{K_c} \right)^z (1 - g) - \frac{\theta_p}{n_3} \right] dt + A \left( \frac{K_p}{K_c} \right)^z (dY - dz). \quad (21)
\]

Let us define the deterministic and stochastic components of (21) as \( E(dK_c/K_c) \) and \( dk_c \), respectively. We obtain

\[
E \left( \frac{dK_c}{K_c} \right) \equiv \phi dt = [Aq^z(1 - g) - c(i)] dt, \quad (22a)
\]

\[
dk_c = Aq^z(dY - dz), \quad (22b)
\]

where \( q \equiv K_p/K_c \), and \( c(i) \equiv C/K_c = \theta_p/[1 - (1 + \lambda)(\rho_y/n)] \).

The above equations describe the rate of capital accumulation as the residual element after consumption and government expenditure. Eqs. (22a) and (22b) are utilized to analyze how the mean and variance of real growth are affected by changes in public policy shocks.

3.2. Equilibrium rate of inflation

Let us derive the equations determining the rate of inflation in a balanced growth equilibrium. Combining the equilibrium conditions in the money and equity markets yields

\[
\frac{(M/P)}{W} = n_1 \quad \text{and} \quad \frac{K_c}{W} = n_3.
\]

\(^5\)As mentioned in Turnovsky (1997, Chapter 10), this policy specification can be empirically supported.
We can express the current price level as

\[ P = \left( \frac{n_3}{n_1} \right) \left( \frac{M}{K_c} \right). \]

As portfolio shares \( n_1, n_2 \) and \( n_3 \) are constant in the steady state, the stochastic differentiation of the above equation yields

\[ \frac{dP}{P} = \frac{dM}{M} - \frac{dK_c}{K_c} - \left( \frac{dM}{M} \right) \left( \frac{dK_c}{K_c} \right) + \left( \frac{dK_c}{K_c} \right)^2. \]

Substituting (3), (19b) and (21) into this expression and equating the deterministic and stochastic parts of this equation yields

\[ \pi = \mu - \phi - Aq^2(\sigma_{xy} - \sigma_{xz}) + (Aq^2)^2(\sigma_{y}^2 - 2\sigma_{yz} + \sigma_{z}^2), \quad (23a) \]

\[ dp = dx - Aq^2(dy - dz). \quad (23b) \]

The first of the equations specifies the expected rate of inflation. The second one determines the stochastic rate of inflation in terms of the stochastic components of the money growth shocks, productivity shocks and fiscal shocks.

3.3. Ratio of public to private capital

The symbol \( q \) is constant in the balanced growth equilibrium, and equivalently \( dq = 0 \) is satisfied. Taking the stochastic differential of \( K_p/K_c \) yields

\[ \frac{dq}{q} = \frac{dK_p}{K_p} - \frac{dK_c}{K_c} - \left( \frac{dK_p}{K_p} \right) \left( \frac{dK_c}{K_c} \right) + \left( \frac{dK_c}{K_c} \right)^2. \]

Substituting (19a) and (21) into the above and equating the deterministic and stochastic parts leads to

\[ gAq^{z-1} - \phi + (Aq^2)^2(\sigma_{y}^2 - 2\sigma_{yz} + \sigma_{z}^2) - (Aq^2)^2[(\sigma_{yz} - \sigma_{z}^2)/q] = 0, \quad (24a) \]

\[ (1 + 1/q)dz = dy. \quad (24b) \]

Here, let us show that the expected growth of private capital equals the expected growth of public capital in the balanced growth equilibrium. From (24b), we can easily obtain

\[ \sigma_{y}^2 = (1 + 1/q)^2 \sigma_{z}^2, \quad \sigma_{yz} = (1 + 1/q) \sigma_{z}^2. \quad (24c) \]

By substituting (24c) into (24a), (24a) can collapse to

\[ \phi = gAq^{z-1}. \quad (25) \]
3.4. Determination of two alternative public financing methods

Substituting the government expenditure policy (19a), the monetary growth rule (19b) and debt policy (19c) into the government budget constraint (18) and dividing by $W$ yields

$$
(1 + \lambda)n_1 \frac{d(M/P)}{M/P} = (g \, dt + dz) \frac{A(K_e)^{1-\beta}(K_p)^z}{W} \frac{dT}{W} + n_1 \, dR_M + n_2 \, dR_B.
$$

Next, substituting for the taxes on wealth (5), taking the stochastic differential of $d(M/P)$ and equating the deterministic and stochastic parts leads to the following two relations:

1. $Aq^z n_3 g - (1 + \lambda)n_1 \mu + n_2 i + (1 + \lambda)n_1 \sigma_{xp} - \tau = 0, \quad (26a)$
2. $Aq^z n_3 dz - (1 + \lambda)n_1 \, dx - dv = 0. \quad (26b)$

If public expenditures $g$ and $dz$ are financed by adjusting the rate of taxes on wealth, $\tau$ and $dv$, respectively, as needed, then (26a) and (26b) describe the endogenous adjustments in taxes necessary to finance the government budget. This public policy corresponds to financing through taxes on wealth.

Next, suppose that $\tau$ and $dv$ are chosen exogenously in addition to $g$ and $dz$. Then, (26a) and (26b) describe the endogenously accommodating adjustments in money growth, $\mu$ and $dx$, required to satisfy the government budget constraint, respectively. Note that $\lambda$ is fixed. Government finances the deficits by printing money as well as by issuing bonds in the ratio $B = \lambda M$. This public policy shows that government finances the expenditure with a mixture of money and bonds.

4. Public policy

We examine how public policy shocks impact on the mean and variance of real growth. The former is (22a), while noting (24b) and (22b), we can express the latter as

$$
\sigma^2_k = (Aq^z)^2 \sigma^2_z.
$$

Changes in government policies affect $\sigma^2_k$ through the effects on $q$. If $\sigma^2_k$ can be reduced by government policy, then the policy is an effective tool for mitigating the size of aggregate fluctuations.

4.1. Financing through taxes on wealth

Suppose that government expenditures are financed through taxes on wealth. Then, the endogenous stochastic disturbances $dv$, $dp$, $du$ and $dy$ can be expressed in terms of the exogenous stochastic factors $dx$ and $dz$, which can be assumed to be
uncorrelated, i.e., $\sigma_{xz} = 0$. In this case, we can derive the following:

\[
\begin{align*}
\mathrm{d}v &= Aq^2 n_3 \, \mathrm{d}z - (1 + \lambda) n_1 \, \mathrm{d}x, \\
\mathrm{d}p &= \mathrm{d}x - Aq^{2-1} \, \mathrm{d}z, \\
\mathrm{d}u &= (1 + 1/q) Aq^2 \, \mathrm{d}z, \\
\mathrm{d}y &= (1 + 1/q) \, \mathrm{d}z.
\end{align*}
\] (27)

Using (27), the variances and covariances can be calculated readily as follows:

\[
\begin{align*}
\sigma_v^2 &= (Aq^2)^2 \sigma_z^2 + (n_1 + n_2)^2 \sigma_x^2, \\
\sigma_p^2 &= (Aq^{2-1})^2 \sigma_z^2 + \sigma_x^2, \\
\sigma_u^2 &= (1 + 1/q)^2 (Aq^2)^2 \sigma_z^2, \\
\sigma_y^2 &= (1 + 1/q)^2 \sigma_z^2. \\
\end{align*}
\] (I)

\[
\begin{align*}
\sigma_{xp} &= \sigma_x^2, \\
\sigma_{pu} &= -(1/q)(1 + 1/q)(Aq^2)^2 \sigma_z^2, \\
\sigma_{aw} &= (1 + 1/q)(Aq^2)^2 n_3 \sigma_z^2, \\
\sigma_{wp} &= -(Aq^{2-1})^2 q n_3 \sigma_z^2 - (n_1 + n_2) \sigma_x^2. \\
\end{align*}
\] (II)

Combining the above relationships, we can express the macroeconomic equilibrium as \(^6\), \(^7\):

\[
\begin{align*}
\pi &= \mu - \phi + (Aq^{2-1})^2 \sigma_z^2, \quad \text{(28a)} \\
Aq^2 &= i - \pi + \sigma_x^2 + (1/q)(1 + 1/q)(Aq^2)^2 \sigma_z^2, \quad \text{(28b)} \\
\phi &= g Aq^{2-1}, \quad \text{(25)} \\
\phi &= [Aq^2(1 - g) - c(i)]. \quad \text{(22a)}
\end{align*}
\]

These four equations determine the equilibrium levels of (i) the expected rate of inflation $\pi$, (ii) the expected rate of capital accumulation $\phi$, (iii) the rate of nominal interest $i$, and (iv) the public–private capital ratio $q$.

Eq. (28a) expresses that the expected inflation rate varies positively with the expected rate of monetary growth and negatively with the expected rate of capital accumulation. In addition, it depends upon the variance $\sigma_z^2$. This is because an increase in $\sigma_z^2$ raises the variance of the growth rate of capital $\sigma_k^2$. This requires a higher expected inflation rate $\pi$ to maintain the portfolio balance. Eq. (28b) describes the equilibrium relationship between the expected real rates of return on bonds and equities. Both these equations provide avenues whereby the variances $\sigma_z^2$ and $\sigma_x^2$ impact on the balanced growth equilibrium.

Table 1 summarizes how the performance of real economy is affected by changes in the first and second moments of the monetary growth rate and productive government spending. Different values of the expected rate of monetary growth $\mu$ have two effects on the expected growth of capital $\phi$. An increase in $\mu$ reduces the equilibrium consumption–capital ratio $c(i)$ through its effect on the nominal

\(^6\)Considering $\sigma_{xz} = 0$ and (27), (28a) can be obtained by substituting (24c) into (23a).

\(^7\)Substitution of (4b), (17), (I) and (II) in Section 4.2 into (8b) yields (28b).
interest rate $i$. With the fraction of output devoted to public spending $g$ remaining fixed, the first effect leads to an increase in $\phi$. However, an increase in $\mu$ reduces the productivity of private capital $Aq^2$ through the positive effect on $n_3$, which leads to a reduction in $\phi$. Table 1 shows that the first effect dominates the second one. An increase in $\mu$ amplifies the stochastic disturbance of real growth $\sigma_k^2$.

In addition, different values of the variance of monetary growth rate $\sigma_x^2$ have two effects on $\phi$. Increasing $\sigma_x^2$ raises $c(i)$, which reduces $\phi$, whereas an increase in $\sigma_x^2$ raises $Aq^2$, which leads to an increase in $\phi$. As the first effect dominates the second one, $\phi$ is positively related to $\sigma_x^2$. In contrast, higher uncertainty associated with monetary policy dampens the degree of $\sigma_k^2$.

A rise in the mean rate of public expenditure $g$ has three impacts on $\phi$. First, an increase in $g$ reduces $\phi$ by raising the real claims of government on the output of economy. Second, increasing $g$ leads to the crowding out of $c(i)$. Third, an increase in $g$ enhances the level of the public good and thus improves the productivity of private capital, which leads to an increase in $\phi$. Table 1 and Appendix A verify that $\phi$ increases with $g$ if $g < g^*$ or $g^* > 1$, whereas $\phi$ declines with $g$, if $g > g^*$ and $g^* < 1$. As in Futagami et al. (1993), we confirm the nonmonotonic relationship between infrastructure and the expected rate of long-run growth. Regardless of the scale of $g$, increasing $g$ has a stabilizing effect on the real economy and, equivalently, $\sigma_k^2$ is negatively related to $g$.

A rise in the variance associated with public expenditure $\sigma_z^2$ has two impacts on $\phi$. A higher $\sigma_z^2$ results in increases in $c(i)$ and $Aq^2$. The former effect dominates the latter. Consequently, increasing $\sigma_z^2$ leads to a reduction in $\phi$ Noting $\sigma_k^2 = (Aq^2-1)\sigma_z^2$, increasing $\sigma_z^2$ has direct and indirect effects on $\sigma_k^2$. If the direct effect is larger (smaller) than the indirect one, a rise in $\sigma_z^2$ destabilizes (stabilizes) the economy.

### 4.2. Mixed money–bond financing

This section considers that government finances the deficits by printing money as well as by issuing bonds. Then, the endogenous stochastic disturbances $dx$, $dp$, $du$ and $dy$ can be expressed in terms of the exogenous stochastic factors $dv$ and $dz$,.

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<thead>
<tr>
<th>Endogenous variables</th>
<th>Monetary policy</th>
<th>Fiscal policy</th>
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<tr>
<td></td>
<td>$\mu$</td>
<td>$\sigma_x^2$</td>
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<tr>
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<tr>
<td>$q$</td>
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<tr>
<td>$\phi$</td>
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(*Proof: See Appendix A.)
which can be assumed to satisfy $\sigma_{vz} = 0$.

\[
\begin{align*}
    dx &= \left[n_3/(n_1 + n_2)\right]Aq^2\,dz - \left[1/(n_1 + n_2)\right]dv, \\
    dp &= \left[n_3/(n_1 + n_2) - 1/q\right]Aq^2\,dz - \left[1/(n_1 + n_2)\right]dv, \\
    du &= (1 + 1/q)Aq^2\,dz, \\
    dy &= (1 + 1/q)\,dz.
\end{align*}
\]

Using the above, we find that the variances and covariances can be derived readily as follows:

\[
\begin{align*}
    \sigma_x^2 &= \left(\frac{n_3}{n_1 + n_2}\right)^2 \left(Aq^2\right)^2 \sigma_z^2 + \left(\frac{1}{n_1 + n_2}\right)^2 \sigma_e^2, \\
    \sigma_y^2 &= \left(1 + \frac{1}{q}\right)^2 \left(Aq^2\right)^2 \sigma_z^2, \\
    \sigma_p^2 &= \left(n_3/n_1 + n_2 - 1/q\right)^2 \left(Aq^2\right)^2 \sigma_z^2 + \left(\frac{1}{n_1 + n_2}\right)^2 \sigma_e^2, \\
    \sigma_{xp} &= \left(n_3/n_1 + n_2\right) \left(\sigma_z^2 + \sigma_e^2\right), \\
    \sigma_{xy} &= \left(n_3/n_1 + n_2\right) \left[\left(1 + \frac{1}{q}\right)^2 \left(Aq^2\right)^2 \sigma_z^2\right], \\
    \sigma_{xz} &= \left(n_3/n_1 + n_2\right) \left(Aq^2\right)^2 \sigma_z^2, \\
    \sigma_{yp} &= -\left(\frac{1}{n_1 + n_2}\right)^2 \sigma_e^2, \\
    \sigma_{pu} &= \left(1 + \frac{1}{q}\right) \left(n_3/n_1 + n_2 - \frac{1}{q}\right) \left(Aq^2\right)^2 \sigma_z^2, \\
    \sigma_{ve} &= 0. \quad (I)
\end{align*}
\]

Considering (III)–(V) and (26a), the macroeconomic equilibrium can be drastically simplified, and the system reduces to the following core set of equilibrium relationships\(^8,9\):

\[
\begin{align*}
    \pi &= \left(n_3/(n_1 + n_2)\right) gAq^2 - \left(\frac{1}{n_1 + n_2}\right) \tau + \left(n_3/n_1 + n_2\right) i + \left(\frac{\sigma_e^2}{(n_1 + n_2)^2}\right) \\
    &\quad - \phi + \left(n_3/n_1 + n_2 - 1/q\right)^2 \left(Aq^2\right)^2 \sigma_z^2, \quad (30a) \\
    Aq^2 + \pi - i + \left(1 + \frac{1}{q}\right) \left(n_3/n_1 + n_2 - \frac{1}{q}\right) \left(Aq^2\right)^2 \sigma_z^2 - \frac{\sigma_e^2}{n_1 + n_2} \\
    &= \frac{n_3}{(n_1 + n_2)^2} \left[\left(Aq^2\right)^2 \sigma_z^2 + \sigma_e^2\right], \quad (30b) \\
    n_1 + n_2 + n_3 &= 1, \quad (30c) \\
    n_1 &= \frac{\rho \gamma}{i}, \quad (30d) \\
    n_2 &= \lambda n_1, \quad (30e)
\end{align*}
\]

\(^8\)If we substitute (24c) and (IV) into (23a), (30a) can be obtained. 
\(^9\)Substituting (4b), (17), (III) and (V) into (8b) leads to (30b).
\[ \phi = gAq^{2-1}, \]  
\[ \phi = [Aq^2(1 - g) - c(i)]. \]  

These seven equations jointly determine the equilibrium levels of (i) the expected rate of inflation \( \pi \), (ii) the nominal interest rate \( i \), (iii) the portfolio share of money, bonds and equities \((n_1, n_2, n_3)\), (iv) the mean rate of capital accumulation \( \phi \), and (v) the public–private capital ratio \( q \).

Eq. (30a) describes the condition of portfolio balance equilibrium. The expected rate of inflation \( \pi \) is positively related to \( \sigma_v^2 \) and \( \sigma_z^2 \), and is negatively related to \( \tau \) and \( \phi \). In particular, a rise in \( \sigma_v^2 \) raises \( \mu \) through its positive effect on \( \sigma_{xp} \) and consequently results in a higher \( \pi \) in order to maintain the portfolio balance. Increasing \( \sigma_z^2 \) leads to a higher \( \pi \) by raising \( \mu \) and \( \sigma_k^2 \), whereas an increase in \( \sigma_z^2 \) leads to a lower \( \pi \) by raising \( \sigma_{MK} \). The former effect dominates the latter, and thus \( \pi \) increases. Eq. (30b) expresses the equilibrium relationship between the expected real rates of return on equities and bonds. Both of these equations provide avenues whereby the variances \( \sigma_v^2 \) and \( \sigma_z^2 \) impact on the balanced growth equilibrium. Eq. (30c) is simply the portfolio adding-up condition, whereas (30d) and (30e) are the demand for money and the proportionality of bonds to money, respectively.

Let us examine how macroeconomic performance is influenced by changes in public policies. An increase in the mean rate of taxes on wealth \( \tau \) has two opposite effects on \( \phi \), but we see that \( \phi \) decreases with \( \tau \) and \( \sigma_k^2 \) also decreases with \( \tau \). Noting \( \sigma_p^2, \sigma_pv, (30a) \) and \( (30b) \), the uncertainty about the rate of taxes on wealth \( \sigma_v^2 \) is not included in the balanced growth equilibrium. As a change in \( \sigma_v^2 \) has no impact upon \( i \) and \( q \), both \( \phi \) and \( \sigma_k^2 \) are independent of the level of \( \sigma_v^2 \). An increase in \( g \) has the same three effects on \( \phi \) as in the previous section. Unlike the case of financing through taxes on wealth, the positive effects always dominate the negative effect for any value of \( g \in (0, 1) \). As shown in Section 6, this is because mixed money–bond financing leads to more crowding out of \( c(i) \) than does financing via taxes on wealth, and \( q \) is not be sensitive to a change in \( g \) under either of the two financing modes. Hence, a rise in \( g \) enhances \( \phi \). Increasing \( g \) stabilizes the rate of real growth, and equivalently, an increase in \( g \) reduces \( \sigma_k^2 \). The property of the effects of \( \sigma_z^2 \) is exactly the same as in Section 4.1. See Appendix B.

5. Welfare analysis

In order to assess the consequences of public policy shocks on economic welfare, we must consider the welfare of the representative agent as specified by the intertemporal utility function (6a), evaluated at the optimum. This is equal to the value function used to derive the necessary conditions (7a)–(7d). The feature of this value function can be described as follows.

**Lemma 1.** There exists no difference in feature between the value functions that are endogenously solved in the case of mixed money–bond financing and financing via taxes on wealth.
Proof. For the logarithmic utility function, the optimized level of utility starting from an initial stock of wealth $W_0$ is given by

$$X(W_0) = E \int_0^\infty \left[ \theta \ln C_t^* + \gamma \ln n_t^* W_t^* \right] e^{-\rho t} dt$$

$$= b_0 + b_1 \ln W_0,$$

where $*$ denotes the optimized value.

Substituting this function and the optimized values into the stochastic Bellman equation yields

$$b_0 = \left( \frac{\theta}{\rho} \right) (\ln \theta + \ln \rho) + \left( \frac{\gamma}{\rho} \right) \ln n_t^* - \frac{\theta}{\rho} + \left( \frac{1}{\rho^2} \right) \psi - \frac{\delta}{2\rho^2},$$

$$b_1 = \frac{1}{\rho},$$

where

$$\psi \equiv n_t^* r_M + n_t^* r_B + n_t^* r_S - \tau,$$

$$\delta \equiv (n_t^* + n_t^* )^2 \sigma_p^2 + n_t^* \sigma_u^2 + \sigma_v^2 - 2(n_t^* + n_t^*) n_t^* \sigma_{up} - 2 n_t^* \sigma_{up} + 2(n_t^* + n_t^*) \sigma_{vp}.$$
Thus, we can obtain the same value function as (32) in the previous case. However, note that endogenous variables such as $i$ and $q$ differ between the two financing cases.

Hence, Lemma 1 shows that the welfare consequences of changes in $\mu, \tau, \sigma^2_x$ and $\sigma^2_v$ all operate entirely through their effects on $i$ and $q$. The level of welfare increases with the expected real growth $\phi$ and decreases with the variance of the growth of capital stock $\sigma^2_k$. We can obtain the welfare gain by reducing the steady-state scale of aggregate fluctuations. Increasing $q$ has two effects on welfare, which are welfare improving. In contrast, the effect of an increase in $i$ is ambiguous and increasing $i$ has three effects on the welfare. A higher $i$ lowers the demand for money, which reduces welfare and $c(i)$, thus stimulating $\phi$, which improves welfare. This causes an initial jump in the initial price level, which in turn causes a reduction in initial wealth, which reduces welfare.

Which of these five effects is the strongest is ambiguous. Thus, the welfare effect of an increase in $g$ or $\sigma^2_{z}$ is unclear. To assess the effects of public policies on welfare, let us differentiate (32) with respect to $i$. We can obtain

$$\frac{\partial X}{\partial i} = \gamma[-\hat{\phi}^2 + (1 + \lambda)\gamma\rho i + \theta\rho^2(1 + \lambda)^2] \rho[i - \gamma\rho(1 + \lambda)].$$

To derive the welfare implications of public policies, we must find the level of $i$ that can be obtained by solving the equation

$$-\hat{\phi}^2 + (1 + \lambda)\gamma\rho i + \theta\rho^2(1 + \lambda)^2 = 0.$$

Let us define the interest rate that satisfies the above as $i^*$. Noting $i > \rho\gamma(1 + \lambda)$ in (29), the level of $i^*$ is

$$i^* = \rho(1 + \lambda)(1 - \theta) \frac{1}{2} \left[ 1 + \sqrt{1 + \frac{4\theta}{1 - \theta}} \right] \geq \rho(1 + \lambda)(1 - \theta),$$

and $\text{sgn}(\partial X/\partial i) = \text{sgn}(i^* - i)$.

Suppose that the interest rate is relatively high, and equivalently, $i > i^*$. An increase in $\mu$ reduces $q$ and raises $i$. Consequently, increasing $\mu$ reduces welfare even if a rise in $\mu$ stimulates $\phi$. An increase in $\sigma^2_x$ raises $q$ and reduces $i$. As a result, a higher $\sigma^2_x$ improves welfare even if it is distortionary to $\phi$.10 Increasing $\tau$ raises $q$ and reduces $i$. Thus, a rise in $\tau$ improves welfare even if $\tau$ is injurious to $\phi$. As both $q$ and $i$ are independent of the value of $\sigma^2_v$, welfare is invariant to the level of $\sigma^2_v$.

---

10The analytical result that a higher monetary uncertainty improves welfare might sounds counter-intuitive. The reason behind the result is that the positive welfare effects of increases in $i$ dominate the negative one for $i > i^*$. 
6. Comparisons

6.1. Consequences of different public financing methods

Let us explore the quantitative differences between mixed money–bond financing and financing via taxes on wealth, using the outcomes obtained in the previous sections. First, we must state the following lemma.

Lemma 2. If the relationship, \( i - \mu + \sigma_x^2 = \rho + \tau \) is satisfied, we can obtain

\[
(i^T, q^T) = (i^{M-B}, q^{M-B}),
\]

where \( T \) and \( M-B \) denote the cases of financing through taxes on wealth and mixed money–bond financing, respectively.

Proof. See Appendix C.

Suppose that \( g < g^* \) or \( g^* > 1 \) is satisfied. Then, a higher mean rate of government spending \( g^T \) enhances the expected growth rate \( \phi \). Let us compare the enhanced effects of \( g^T \) and \( g^{M-B} \).

Proposition 1. Mixed money–bond financing leads to a larger increase in the expected rate of capital accumulation than does financing through taxes on wealth.

Proof. As shown in Appendix D, this is because the crowding out of \( c(i) \) is larger under mixed money–bond financing than it is under financing through taxes on wealth, and \( q \) is not sensitive to a change in \( g \).

From a growth perspective, mixed money–bond financing is always preferred.

Regardless of government financing methods, a higher \( g \) mitigates the stochastic disturbance of real growth \( \sigma_k \). We compare the stabilizing effects under the different financing regimes. \( \square \)

Proposition 2. The stabilizing effect on real growth is less effective under mixed money–bond financing than under financing through taxes on wealth.

Proof. See Appendix D.

In comparing the welfare effects of alternative government financing, let us concentrate on the range of values of the interest rate larger than \( i^* \).11 Then, we can state the following proposition. \( \square \)

Proposition 3. The positive (negative) welfare effect is smaller (larger) under mixed money–bond financing than under financing through taxes on wealth.

\[\text{Footnote: Over the periods 1967–1985, the ratio of publicly held debts to M3 in the United States averages 0.24 with a standard deviation of only 0.055. See Turnovsky (1997, Chapter 10). We substitute empirically plausible values of } \rho = 0.03, \theta = 0.6 \text{ and } \lambda = 0.24 \text{ into (34) and obtain } i^* = 2.7124\%. \text{ Therefore, it is probable that the welfare consequences hold in a real economy, if the nominal interest rate exceeds 2.7\%.}\]
Proof. Appendix D shows that increasing $g^T$ enhances welfare through the effects on $i$ and $q$ more than does an increase in $g^{M-B}$.

From a welfare perspective, wealth-tax financing is always preferred. Greater uncertainty about the occurrence of a fiscal policy shock $\sigma^2_z$ is distortionary to $\phi$. Let us compare the negative effects for the different types of financing. □

**Proposition 4.** The decrease in the expected rate of growth is larger under mixed money–bond financing than under financing through taxes on wealth.

Proof. Appendix E proves that mixed money–bond financing results in a larger fall in $\phi$ because it makes the equity relatively less attractive than under wealth-tax financing.

As mentioned in Section 4, the qualitative effect on $\sigma^2_k$ of a rise in $\sigma^2_z$ is ambiguous. However, we can summarize the quantitative effects of the two financing regimes as follows. □

**Proposition 5.** The amplified (dampened) effect on the stochastic disturbance of real growth is smaller (larger) under mixed money–bond financing than under wealth-tax financing.

Proof. See Appendix E.

Suppose $i \geq i^*$. Then, we can reveal the welfare effects of an increase in $\sigma^2_z$ under the two financing regimes. □

**Proposition 6.** The negative (positive) effect on welfare is smaller (larger) under mixed money–bond financing than under wealth-tax financing.

Proof. Appendix E derives the welfare effects of a rise in $\sigma^2_z$ through $i$ and $q$ and shows that the negative (positive) effect is smaller (larger) under mixed money–bond financing.

From a growth (welfare) perspective, the instability of government spending under mixed money–bond financing is more (less) serious than under wealth-tax financing. □

6.2. Debt policy

To examine whether the qualitative difference between money financing and bond financing exists, Section 6.2 considers how the consequences obtained in Section 4.2 are modified if government relies more on bonds relative to money in order to raise the necessary revenue.

First, we summarize how the growth and welfare effects of an increase in $g^{M-B}$ are affected by such a government decision.

**Proposition 7.** As the ratio of bonds to money increases, the positive effect of an increase in $g$ on $i$ becomes larger. However, the growth and welfare effects are unaffected by the government decision.
Proof. See Appendix F.

Finally, let us state how the growth and welfare effects of an increase in $\sigma_z^{2M-B}$ are affected by the decision. □

Proposition 8. As the ratio of bonds to money increases, the negative effect of an increase in $\sigma_z^2$ on $i$ becomes smaller. However, the growth and welfare effects are independent of the government decision.

Proof. See Appendix F.

Therefore, the macroeconomic performance is not influenced by whether public spending is financed by creating money or by issuing bonds. □

7. Concluding remarks

In this paper, taxes on wealth or both money and bonds are used to finance a given increase in productive public spending. Let us summarize the merits and demerits of mixed money–bond financing relative to wealth-tax financing in a stochastically growing economy.

(i) As the mean rate of public spending is increased, the expected rate of real growth becomes larger, whereas the variance of real growth becomes smaller. On the one hand, the increase in the expected rate of growth is larger under mixed money–bond financing, whereas on the other hand, the decrease in the variance is larger under wealth-tax financing. It is ambiguous whether the level of welfare improves or deteriorates as a result of increasing the mean rate of public spending. However, the positive (negative) effect is smaller (larger) under mixed money–bond financing than under wealth-tax financing. Therefore, from a growth (welfare) perspective, mixed money–bond financing (wealth-tax financing) is preferred.

(ii) As uncertainty about government spending increases, the expected rate of real growth becomes smaller, whereas the effect on the variance of real growth is indeterminate. Mixed money–bond financing leads to a larger fall in the expected rate of growth, whereas wealth-tax financing, on the other hand, leads to a larger increase (a smaller fall) in the variance. In addition, how the level of welfare is affected by a greater uncertainty regarding government spending is ambiguous, but mixed money–bond financing results in a smaller fall (larger increase) in the level of welfare. Therefore, from a growth (welfare) perspective, instability of government spending under mixed money–bond financing is more (less) serious than under wealth-tax financing.

To explore the qualitative difference between money financing and bond financing, we consider mixed money–bond financing when a larger ratio of bonds to money is utilized to finance a given increase in public spending. We can summarize how such a decision affects the consequences of mixed money–bond financing as follows:
(iii) As the ratio of bonds to money increases, the positive effect of an increase in the mean rate of public spending on the rate of nominal interest becomes larger. However, the growth and welfare effects are unaffected by government’s decision.

(iv) As the ratio of bonds to money increases, the negative effect of an increase in the uncertainty about public spending on the rate of nominal interest becomes smaller. However, even so, the growth and welfare effects are not influenced. From growth and welfare perspectives, it is not significant whether the government prints money or issues bonds to raise the necessary revenue in this stochastic AK model.

Finally, let us state the growth and welfare effects of changes in policy shocks, such as the mean and variance of the rate of monetary growth and the rate of taxes on wealth.

(v) As the mean (variance) of the money growth rate increases, the expected rate of real growth becomes larger (smaller), as does the variance of real growth, whereas the welfare level is smaller (larger).

(vi) As the mean rate of taxes on wealth increases, the expected rate of real growth becomes smaller, as does the variance of real growth, whereas the welfare level increases. However, the variance of the rate of taxes on wealth has no impact upon the first and second moments of real growth and the level of welfare.

Suppose that the government deficits in this period are given, i.e., $g$ is fixed in period $t$. Then, from a growth (welfare) perspective, it is more desirable that government relies on the rate of monetary growth (the rate of taxes on wealth) to obtain the given revenue.

Acknowledgment

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Appendix A. (Proof of Table 1)

Using (28a)–(28d) to eliminate $\varphi$ and $\pi$, we can obtain

$$ (1 - g)Aq^2 - gAq^{2-1} = \theta \rho + \frac{(1 + \lambda)\rho_i}{i - (1 + \lambda)\rho_i}, \quad (A.1) $$

$$ i = Aq^2 - gAq^{2-1} + \mu - \sigma_x^2 - \left(\frac{1}{q^2}\right)(Aq^2)^2\sigma_x^2. \quad (A.2) $$

Thus, we can depict (A.1) and (A.2) as in Fig. 1. Noting $z \leq 1/2$, Fig. 1 reveals that positive values of $q$ and $i$ exist that can generate $0 < n_1, n_2$ and $n_3 < 1$. 
Differentiating (28a), (28b), (25) and (22a) with respect to $q, i$ and $\phi$ yields

\[
\begin{bmatrix}
\Gamma^T & -1 & -1 \\
\Phi^T & 0 & -1 \\
\Theta^T & c'(i) & 1 
\end{bmatrix}
\begin{bmatrix}
dq \\
di \\
d\phi 
\end{bmatrix} = \begin{bmatrix}
-d\mu + d\sigma^2_x + (1/q)(AQ^x)^2 d\sigma^2_z \\
-AQ^{x-1} dg \\
-AQ^x dg 
\end{bmatrix},
\]

\[
\Gamma^T \equiv \alpha Aq^{x-1} - (2\alpha - 1)(AQ^{x-1})^2 \sigma_z^2 > 0, \\
\Phi^T \equiv - (1 - \alpha)AQ^{x-2} < 0, \\
\Theta^T \equiv - \alpha(1 - g)AQ^{x-1} < 0, \\
c'(i) \equiv - [\theta \rho (n_1 + n_2)/(n_3^2 \delta)] < 0.
\]

The comparative static effects of public policy shocks are shown in Table 3.

Appendix B. (Proof of Table 2)

\[
(1 - g)AQ^x - gAQ^{x-1} = \theta \rho + \frac{(1 + \lambda)\rho\gamma}{i - (1 + \lambda)\rho\gamma}, \quad (B.1)
\]

\[
AQ^x - gAQ^{x-1} - \left(\frac{n_1}{n_1 + n_2}\right)i - \left(\frac{1}{n_1 + n_2}\right)\tau \\
+ \left(\frac{n_3}{n_1 + n_2}\right)gAQ^x - \left(\frac{1}{n_1 + n_2}\right)\left(\frac{1}{q}\right)(AQ^x)^2 \sigma_z^2 = 0. \quad (B.2)
\]

From Fig. 2, it is easily verified that there exist positive steady-state levels of $q$ and $i$ that can generate $0 < n_1, n_2$ and $n_3 < 1$ (Table 2).
Considering (30c)-(30e) and differentiating (30a), (30b), (25) and (22a) with respect to $q$, $i$ and $f$ yields

\[
\begin{bmatrix}
\Gamma_{M^B} & -\frac{\phi}{t} & -1 \\
\phi_{M^B} & 0 & -1 \\
\Theta_{M^B} & c'(i) & 1
\end{bmatrix}
\begin{bmatrix}
\frac{d\gamma}{d\varphi} \\
\frac{di}{d\varphi} \\
\frac{d\varphi}{d\varphi}
\end{bmatrix}
= \begin{bmatrix}
-\frac{1}{n_1+n_2} d\tau + \frac{n_3}{n_1+n_2} Aq^x dg + \frac{1}{\sigma_z}(Aq^x)^2 d\sigma_z^2 \\
-Aq^{x-1} dg \\
-Aq^x dg
\end{bmatrix},
\]

\[
\Gamma_{M^B} \equiv \alpha Aq^{x-1} + \frac{n_3}{n_1+n_2} g \alpha Aq^{x-1} - \frac{1}{n_1+n_2} (2\alpha - 1)(Aq^{x-1})^2 \sigma_z^2 > 0,
\]

\[
\phi_{M^B} \equiv -(1-\alpha)Aq^{x-2} < 0,
\]

\[
\Theta_{M^B} \equiv -\alpha(1-\gamma)Aq^{x-1} < 0.
\]

The comparative static effects are shown in Tables 3 and 4.

![Fig. 2. Uniqueness of steady-state equilibrium.](image)

Table 2
Comparative statics

<table>
<thead>
<tr>
<th>Endogenous variables</th>
<th>Tax rate</th>
<th>Public spending</th>
</tr>
</thead>
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<tr>
<td>$\tau$</td>
<td>$\sigma_z^2$</td>
<td>$g$</td>
</tr>
<tr>
<td>$i$</td>
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<td>$+$</td>
</tr>
<tr>
<td>$q$</td>
<td>$+$</td>
<td>$+$</td>
</tr>
<tr>
<td>$\phi$</td>
<td>$-$</td>
<td>$+$</td>
</tr>
</tbody>
</table>

(*) Proof: See Appendix B.
Appendix C. (Proof of Lemma 2)

Eq. (B.2) can be also expressed as

\[ n_1 + n_2 = \frac{pr^2 - gAQ^2 + r + (1/q)(AQ^{-1})^2\sigma_z^2}{(1 - g)AQ^2 - gAQ^{-1}}. \]  

(C.1)

Combining (28c) and (28d) yields

\[ (1 - g)AQ^2 - gAQ^{-1} = \frac{\theta p}{1 - (n_1 + n_2)}. \]  

(C.2)
Substituting (C.1) into (C.2), we obtain

\[ Aq^2 - gAq^{z-1} - (1/q)(Aq^{z-1})^2 \sigma_z^2 = \tau + \rho. \]  

(C.3)

Comparing (A.2) with (C.3), we can easily see that Lemma 2 is satisfied.

**Appendix D. (Proofs of Propositions 1–3)**

We can obtain

\[ \Delta^{M-B} = \Delta^T/(n_1 + n_2) = -(\Phi + \Theta)/(n_1 + n_2) > 0. \]  

(D.1)

To prove Proposition 1, let us show that \(D_{11}^T \Delta^{M-B} - D_{11}^{M-B} \Delta^T < 0.\) Using (D.1) and \(\Gamma^{M-B} = -\Theta + [g\xi Aq^{z-1} - (2\xi - 1)(Aq^{z-1})^2 \sigma_z^2],\) we obtain

\[ D_{11}^T \Delta^{M-B} - D_{11}^{M-B} \Delta^T = (Aq^{z-1})^2 \left[ -c'(i)/(n_1 + n_2) \right] \times \left[ \left( g - (2\xi - 1)Aq^{z-1} \sigma_z^2 \right)(\Phi + \Theta) - (g - \xi)(g\xi - (2\xi - 1)Aq^{z-1} \sigma_z^2)Aq^{z-1} \right]. \]
Noting $\alpha \leq 1/2$ and $Aq^{x-1}(g-x) - (\Phi + \Theta) = Aq^{x-1}(1-x)g(1+1/q) > 0$, we can easily find that

$$[	ext{d}\phi / \text{d}q]_{M-B} > [\text{d}\phi / \text{d}g]_T > 0.$$ 

Let us prove Proposition 2. We can see that

$$D^T_{21} A^{M-B} - D^T_{21} A^{M-B} = Aq^{x-1} \left( -\frac{c'(i)}{n_1 + n_2} \right) \left[ g\alpha - (2\alpha - 1)Aq^{x-1} \sigma_z^2 - Aq^{x} c'(i) \left( T^{M-B} - \frac{\Phi + \Theta}{n_1 + n_2} \right) \right] > 0.$$

Hence, we can obtain

$$[\text{d}q / \text{d}g]_T > [\text{d}q / \text{d}g]_{M-B} > 0.$$ 

(D.2)

Noting that $\sigma_z^2$ varies negatively with $q$, (D.2) corresponds to the proof of Proposition 2.

As for Proposition 3, we can obtain the following:

$$(D^T_{31} A^{M-B} - D^T_{31} A^{M-B})(n_1 + n_2) = Aq^{x-1}(Aq^x + Aq^{x-1}) \times \left[ (2\alpha - 1)Aq^{x-1} \sigma_z^2(\Phi + \Theta) - n_3 g\alpha Aq^{x-1} A^T \right] + n_3 Aq^x \Phi \Delta$$

$$+ Aq^x \Phi(\Phi + \Theta) + \alpha(A^{M-B} - A^T)(Aq^{x-1})^2(g + q) > 0.$$

Hence, we can understand that

$$[\text{d}i / \text{d}g]_{M-B} > [\text{d}i / \text{d}g]_T > 0.$$ 

(D.3)

As shown in Section 4, increasing $i$ reduces economic welfare if $i > i^*$, whereas increasing $q$ always improves economic welfare. Thus, (D.2) and (D.3) mean that Proposition 3 is satisfied.

Appendix E. (Proofs of Propositions 4–6)

Concerning Proposition 4, we can derive the following:

$$D^T_{12} A^{M-B} - D^T_{12} A^{M-B} = (Aq^{x-1})^3 c'(i)(1 - x)g \left( \frac{\Phi + \Theta}{n_1 + n_2} \right) > 0.$$

Thus, we find that

$$[\text{d}\phi / \text{d}r_z^2]_{M-B} < [\text{d}\phi / \text{d}r_z^2]_T < 0.$$ 

Regarding Proposition 5, we can obtain

$$D^T_{22} A^{M-B} - D^T_{22} A^{M-B} = - \left( \frac{c'(i)}{q} \right) (Aq^{x-1})^3(1 - x)g \left( \frac{\Phi + \Theta}{n_1 + n_2} \right) < 0.$$
Accordingly, we can see that
\[
[dq/d\sigma^2_{z}]_{M-B} > [dq/d\sigma^2_{z}]_{T} > 0. \tag{E.1}
\]
Noting \( \sigma^2_k = (Aq^{k-1})^2 \sigma^2_z \), (E.1) implies that Proposition 5 is proved. As for the proof of Proposition 6, we can express the following:
\[
D_{32}^M A^T - D_{32}^T A^{M-B} = - (1/q)(Aq^3)^2(\Phi + \Theta)^2 < 0.
\]
As a result, we can obtain
\[
[dq/d\sigma^2_{z}]_{M-B} < [dq/d\sigma^2_{z}]_{T} < 0. \tag{E.2}
\]
Noting \( i \geq i^* \), (E.1) and (E.2) mean that we can prove Proposition 6.

Appendix F. (Proofs of Propositions 7 and 8)

From Appendix C, we can easily find that \( q, n_1 + n_2 \) and \( n_3 \) are invariant to the level of \( \lambda \), so we can obtain
\[
\begin{align*}
\frac{\partial i}{\partial \lambda} &= i/(1 + \lambda), \\
\frac{\partial c'(i)}{\partial \lambda} &= - \frac{1}{1 + \lambda} c'(i), \\
\frac{\partial (c(i)/i)}{\partial \lambda} &= - \frac{1}{1 + \lambda} \left\{ \frac{c(i)}{i} \right\}, \\
\frac{\partial (\partial X/\partial i)}{\partial \lambda} &= - \frac{1}{1 + \lambda} \left\{ \frac{\partial X}{\partial i} \right\}. \tag{F.1}
\end{align*}
\]
Let us prove Proposition 7. Using the above relationships, we can derive the following:
\[
\begin{align*}
\frac{\partial A^{M-B}}{\partial \lambda} &= - A^{M-B}/(1 + \lambda), \\
\frac{\partial D_{11}^{M-B}}{\partial \lambda} &= - D_{11}^{M-B}/(1 + \lambda), \\
\frac{\partial D_{21}^{M-B}}{\partial \lambda} &= - D_{21}^{M-B}/(1 + \lambda), \\
\frac{\partial D_{31}^{M-B}}{\partial \lambda} &= 0. \tag{F.2} - (F.5)
\end{align*}
\]
From (F.2) and (F.5), we can obtain
\[
\frac{\partial (di/dg)_{M-B}}{\partial \lambda} = \frac{1}{1 + \lambda} \left( \frac{di}{dg} \right)_{M-B} > 0. \tag{F.6}
\]
Following (F.2) and (F.4), we can understand that
\[
\frac{\partial (dq/dg)_{M-B}}{\partial \lambda} = 0. \tag{F.7}
\]
Noting (F.2) and (F.3), we can obtain
\[
\frac{\partial (d\phi/dg)_{M-B}}{\partial \lambda} = 0. \quad (F.8)
\]
Considering (F.1), (F.6) and (F.7), we can show that
\[
\frac{\partial (dX/dg)_{M-B}}{\partial \lambda} = 0. \quad (F.9)
\]
Eqs. (F.6), (F.8) and (F.9) correspond to the proof of Proposition 7.

Let us prove Proposition 8. We can obtain
\[
\frac{\partial D_{12}^{M-B}}{\partial \lambda} = -D_{12}^{M-B}/(1 + \lambda),
\frac{\partial D_{22}^{M-B}}{\partial \lambda} = -D_{22}^{M-B}/(1 + \lambda),
\frac{\partial D_{32}^{M-B}}{\partial \lambda} = 0, \quad (F.10) - (F.12)
\]
From (F.2) and (F.12), we can see that
\[
\frac{\partial (di/d\sigma_z^2)_{M-B}}{\partial \lambda} = \frac{1}{1 + \lambda} \left( \frac{di}{d\sigma_z^2} \right)_{M-B} > 0. \quad (F.13)
\]
Following (F.2) and (F.11), we can verify that
\[
\frac{\partial (dq/d\sigma_z^2)_{M-B}}{\partial \lambda} = 0. \quad (F.14)
\]
Noting (F.2) and (F.10), we can understand that
\[
\frac{\partial (d\phi/d\sigma_z^2)_{M-B}}{\partial \lambda} = 0. \quad (F.15)
\]
Considering (F.1), (F.13) and (F.14), we can show that
\[
\frac{\partial (dX/d\sigma_z^2)_{M-B}}{\partial \lambda} = 0. \quad (F.16)
\]
Eqs. (F.13), (F.15) and (F.16) correspond to the proof of Proposition 8.

References


